Undergraduate Research Opportunities
Programme in Science

An Introduction to the Representation Theory of Groups

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AY2018/19

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1 Introduction

In this report, we aim to provide a concise treatment to a few fundamental ideas and results of the representation theory of finite groups. The main founders of representation theory were William Burnside, Heinrich Maschke, Issai Schur, and Ferdinand Georg Frobenius. Many notions/results (including the original ones by the founders and their contemporaries) in modern treatises of representation theory are cast in a somewhat more powerful form, one which makes use of the notion of a group algebra and the slew of tools available to investigate it. We shall not follow that path. Instead, we shall pursue the line of thinking of its originators, in the hopes that the ensuing theory does not appear overly abstract/unmotivated. To this end, we refer mainly to the outstanding book by Emmanuel Kowalski.

However, for the uninitiated Kowalski may be overly terse. It is our goal to present the material more thoroughly, and at the same time organize the material in a systematic way, noting that Kowalski also ventures off to talk about other algebraic objects besides finite groups, which is our focus.

Moreover, there are a few elements in this text which are not from Kowalski, nor any of the references listed below. These include the second, third and fourth isomorphism theorems for representations, the theorem in the section on matrix representations, etc. They came into being either from discussions with my supervisor (Prof. Lee Soo Teck) or from some thinking of my own. A few important results that are 'left to the reader'/ left as exercises in Kowalski are listed as propositions/lemmas in this text (e.g. Lemma 2.21, Proposition 4.11).

The prerequisites to understanding this report are modest - a solid grasp of undergraduate linear algebra (NUS MA2101), abstract algebra (NUS MA2202 and MA3201) and an acquaintance with a few slightly more advanced concepts of linear algebra, namely dual spaces and multilinear algebra.

We split the overall body into 3 sections. In the first, we introduce the fundamental objects, their properties, and the relationships between them. In the second, we discuss a few major results in finite representations, which build a foundation on which we develop the third section, which deals with character theory.

2 Fundamental Notions

2.1 Basic Objects

2.1.1 Representations

Definition (Linear Representation)**.** Let G be a group, and K a field. A **linear representation** of G, defined over K, is simply a group homomorphism

$$\rho : G \longrightarrow GL(V)$$

where V is a K-vector space, called the **representation space** of ρ .The dimension of V is called the **dimension** of ρ, denoted $\dim(\rho)$.

Remark. In this text, we shall only consider *finite-dimensional* representations.

Remark. Note that $GL(V) = \mathrm{Aut}(V)$, where $\mathrm{Aut}(V)$ is the group of (vector space) automorphisms from V to itself. For every representation ρ, we have a corresponding group action of G on V, namely:

$$\cdot : G \times V \longrightarrow V$$
$$(g, v) \mapsto g \cdot v = \rho(g)(v)$$

This is entirely similar to what we have encountered in group theory, that for every group homomorphism from G to S_A (the symmetric group of A), there is a corresponding group action of G on the set A. Here, instead of considering mere sets we consider vector spaces, which have more structure. Thus instead of S_A, we have $GL(V)$.

Henceforth, we shall liberally switch between $\rho(g)(v)$ and $g \cdot v$. As we shall see, once we define more representations 'derived' from the original representation ρ, this notation will be handy.

Before further proceeding, let us give a few examples of representations that appear often. In particular, the ‘regular representation’ will be important when we construct the theory of characters.

Example.

1. **Permutation Representation** Recall from group theory: given any set X and an action of G on X, we have the group homomorphism

$$G \longrightarrow S_X$$
$$g \mapsto (x \mapsto g \cdot x).$$

Denote the *free vector space generated by* X by $K(X)$. This is the K-vector space generated by the basis vectors $\{e_x\}$ indexed by the elements of X. Refer to [3] (page 354) for its construction. (Note that [3] considers the general case of modules.)

If X is a finite set (of size n) we can simply think of the elements of $K(X)$ as n-tuples, and where X is embedded into $K(X)$ by $x_1 \mapsto (1, 0, 0, ..., 0)$, $x_2 \mapsto (0, 1, 0, ..., 0), \ldots, x_n \mapsto (0, 0, 0, ..., 1)$.

Now there is always an associated linear representation of G by $K(X)$, which we call the **permutation representation**. It is given by

$$\pi_G : G \longrightarrow GL(K(X))$$
$$g \mapsto (\sum_{x \in X} a_x e_x \mapsto \sum_{x \in X} a_x e_{g \cdot x}).$$

In particular, $\pi_G(g)e_x = e_{g \cdot x}$. The verification that the map π_G is indeed a representation is straightforward, and exploits the action of G on X, i.e. $g \cdot (h \cdot x) = (gh) \cdot x$. Also, $\dim(\pi_G) = |X|$, by construction.

As a special case of the permutation representation, we consider the (finite) group G, and let G itself play the role of X, with the group action being $g \cdot h = gh$. This representation is also denoted by π_G.

2. **Regular Representation** Denote by $C_K(G)$ the vector space of all functions

$$f : G \longrightarrow K$$

with addition and scalar multiplication defined by the corresponding pointwise operations. The **regular representation** $\rho_G(g)$ is defined by

$$\underline{\rho_G(g)f}(x) = f(xg)$$

for all $f \in C_K(G)$.

Again, it is straightforward to verify that ρ_G is a representation. For example, let us verify that $\rho_G(gh) = \rho_G(g)\rho_G(h)$. To do so, note that since for each group element g, $\rho_G(g)$ is itself a map, so we are to prove the equivalence of maps. Pick an arbitrary $f \in C_K(G)$. We have

$$\underline{\rho_G(gh)f}(x) = f(xgh).$$

Next, since $\rho_G(h)f$ is the function $f_1 : y \mapsto f(yh)$, $\rho_G(g)\rho_G(h)f = \rho_G(g)f_1$ maps x to

$$f_1(xg) = f((xg)h) = f(xgh).$$

What we have shown is that $\rho_G(gh)f = \rho_G(g)\rho_G(h)f$ are equal as functions on G. Since f was arbitrary, we have $\rho_G(gh) = \rho_G(g)\rho_G(h)$.

3. The following two examples are variants of the regular representation.

   (a) $\underline{\lambda_G(g)f}(x) = f(g^{-1}x)$ also defines a representation of G. We shall call this the *left regular representation*. With this in mind, should ambiguities arise, we shall refer to the ‘normal’ regular representation as the *right regular representation*. The term ‘regular representation’ will always refer to the right regular representation.

(b) In a similar vein though slightly more complicated, is the representation of $G \times G$ by $C_K(G)$ given by

$$\underline{\rho(g,h)f}(x) = f(g^{-1}xh).$$

Often, ρ turns out to be either injective or trivial. We give names to these representations with these characteristics:

Definition (Faithful and Trivial Representations)**.**

1. A representation ρ of G is **faithful** if ρ is injective, i.e. $\ker(\rho) = \{1\}$ in G.
2. A representation ρ of G is **trivial** if ρ is the trivial homomorphism, i.e. $\ker(\rho) = G$.

2.1.2 Morphisms

Remark. Sometimes only the trivial representation of dimension 1 is called *the trivial representation*. We shall not use this convention; instead we shall denote this representation by $\mathbf{1}$.

Now we ask ourselves, as we do upon encountering any algebraic object: when are two objects considered ‘the same’, and how does a ‘structure-preserving’ map look like? Intuitively for representations, suppose we are given two representations

$$\rho_1 : G \longrightarrow GL(V_1)$$

and

$$\rho_2 : G \longrightarrow GL(V_2),$$

where we have a vector space isomorphism $\Phi : V_1 \longrightarrow V_2$. For any g, there is no reason to consider $\rho_1(g)$ and $\rho_2(g)$ different if $\rho_2(g) = \Phi\rho_1(g)\Phi^{-1}$. What this says is $\rho_1(g)$ and $\rho_2(g)$ are basically 'behave' the same way as functions, but the names of their elements differ, and this is captured by the 'name-changer' Φ. We could be less stringent in our demands, however. Instead of an isomorphism from V to W, we ask for just a (vector space) homomorphism, also called a linear map. This approach simply mirrors that taken for groups/rings/modules and it turns out, we have a corresponding set of the four Isomorphism Theorems for representations as well!

Now, let us translate intuition into formalism.

Definition (Morphisms of Representations)**.** Let $\rho_1(g)$ and $\rho_2(g)$ be given as above. A **morphism** between ρ_1 and ρ_2 is a K-linear map

$$\Phi : V_1 \longrightarrow V_2$$

such that

$$\Phi\rho_1(g) = \rho_2(g)\Phi$$

for all $g \in G$, or equivalently,

$$\underline{\Phi\rho_1(g)}v = \underline{\rho_2(g)\Phi}v$$

for all $v \in V_1$, $g \in G$.

One also calls Φ an intertwiner. We shall use the terms 'morphism' and 'intertwiner' interchangeably. We shall also write $\Phi(g \cdot v) = g \cdot \Phi(v)$, where $g \in G$, $v \in V_1$.

This definition can be captured succinctly in a commutative diagram:

$$\begin{array}{ccc} V_1 & \xrightarrow{\Phi} & V_2 \\ {\scriptstyle \rho_1(g)}\downarrow & & \downarrow{\scriptstyle \rho_2(g)} \\ V_1 & \xrightarrow{\Phi} & V_2 \end{array}$$

for all $g \in G$.

Proposition 2.1. The set of morphisms from ρ_1 to ρ_2 form a vector space, and is in fact a vector subspace of $\mathrm{Hom}_K(V_1, V_2)$, the space of linear maps between V_1 and V_2. We denote our space of morphisms by $\mathrm{Hom}_G(\rho_1, \rho_2)$.

Proof. We only have to show that for any $c \in K$ and $\Phi, \Psi \in \mathrm{Hom}_G(\rho_1, \rho_2)$, $c\Phi + \Psi \in \mathrm{Hom}_G(\rho_1, \rho_2)$ as well. This is straightforward:

$$\begin{aligned}(c\Phi + \Psi)\rho_1(g) &= c\Phi\rho_1(g) + \Psi\rho_1(g) \\ &= c\rho_2(g)\Phi + \rho_2(g)\Psi \\ &= \rho_2(g)(c\Phi + \Psi).\end{aligned}$$

□

Proposition 2.2 (Functoriality)**.**

1. For any representation ρ of G and a vector space V, the identity map on V is a homomorphism from $\rho \longrightarrow \rho$.

2. Given representations ρ_1, ρ_2, ρ_3 on V_1, V_2, V_3 respectively, and morphisms

$$V_1 \xrightarrow{\Phi_1} V_2 \xrightarrow{\Phi_2} V_3,$$

the composite $V_1 \xrightarrow{\Phi_2 \circ \Phi_1} V_3$ is a morphism between ρ_1 and ρ_3.

Remark. In the language of category theory, which we will only use incidentally in this text, this proposition states that the representations of a given group G over a given field K are the *objects* of a category, with the *morphisms* (the term 'morphism' has a technical meaning itself within category theory, and it is this meaning we refer to here) given by the intertwiners. [2]

Remark. If a morphism Φ is bijective, then its inverse Φ^{-1} is also a morphism (between ρ_1 and ρ_2). We then call Φ an **isomorphism**. In this scenario, the condition $\Phi\rho_1(g) = \rho_2(g)\Phi$ is equivalent to $\rho_2(g) = \Phi\rho_1(g)\Phi^{-1}$. We also say that ρ_1 is **equivalent/isomorphic** to ρ_2, and write $\rho_1 \cong \rho_2$.

Example ($\pi_G \cong \rho_G$)**.** Let G be a finite group. Consider the regular representation π_G and the special case of the permutation representation when $G = X$, π_G. We claim that π_G (with representation space $K(G)$) is isomorphic to ρ_G (with representation space $C_K(G)$). To see this, define $\Phi : K(G) \longrightarrow C_K(G)$ by

$$\Phi(\underbrace{\sum_{g \in G} a_g e_g}_{v})(h) = a_{h^{-1}}$$

In particular,

$$\Phi(e_g)(h) = \begin{cases} 1 & \text{if } h = g^{-1} \\ 0 & \text{otherwise} \end{cases}$$

It is not hard to establish the linearity and bijectivity of Φ. We now claim Φ is a morphism of representations, i.e. $\Phi\pi_G(g) = \rho_G(g)\Phi$, or $\Phi\pi_G(g)v = \rho_G(g)\Phi v$ for all $v \in K(G)$. We point out here that it is enough to establish the latter only on the basis vectors $\{e_g\}$ of $K(G)$ - the general case follows from the linearity of Φ, $\pi_G(g)$ and $\rho_G(g)$.

Proof. In $K(G)$, we have $g \cdot e_h = e_{gh}$, and that

$$\Phi(g \cdot e_h)(x) = \Phi(e_{gh})(x) = \begin{cases} 1 & \text{if } xg = h^{-1}, \text{ i.e. if } x = (gh)^{-1} \\ 0 & \text{otherwise} \end{cases}$$

Next, in $C_K(G)$ we have

$$g \cdot \Phi(e_h)(x) = \rho_G(g)\Phi(e_h)(x) = \begin{cases} 1 & \text{if } xg = h^{-1}, \text{ i.e. if } x = (gh)^{-1} \\ 0 & \text{otherwise} \end{cases}$$

which is precisely the same as $\Phi(g \cdot e_h)(x)$, thus establishing the equality $\Phi(g \cdot e_h) = g \cdot \Phi(e_h)$. $\square$

Having gained some familiarity with the basic objects, we now discuss how to construct new representations based on existing ones, e.g. subrepresentations, quotient representations, direct sum and tensor products, duals and combinations thereof.

2.2 New Things from Old

2.2.1 Subrepresentations and Quotient Representations

We start off with the easiest construction - by considering just a part of the whole. Remember that for groups/vector spaces, the notion of a subgroup/subspace is slightly more involved than that of a subset; namely, we have to consider the closure of the sets under the algebraic operations to ensure that the subgroup/subspace really is a group/vector space. The same goes for representations.

Definition (Subrepresentation)**.** A vector subspace $W \leq V$ is said to be **G-invariant** if it is $\rho(g)$-invariant for all $g \in G$, i.e. if $\rho(g)(w) \in W$ for all $g \in G$, $w \in W$.

Given a G-invariant subspace W, the restriction of $\rho(g)$ to W defines a representation of G by $GL(W)$: $\rho|_W : G \longrightarrow GL(W)$ defined by

$$\rho|_W(g) = \rho(g)|_W$$

for all $g \in G$. $\rho|_W$ is called a **subrepresentation** of ρ. **We shall use the term 'subrepresentation' and 'G-invariant subspace' interchangeably** as, while not completely precise technically, this seems to be a norm in the literature.

Remark. Given a subrepresentation $\rho|_W$, the inclusion map

$$i : W \hookrightarrow V$$

becomes a morphism of representations. We illustrate below the associated commutative diagram:

$$\begin{tikzcd} W \arrow[r, "i"] \arrow[d, "\rho|_W(g)"'] & V \arrow[d, "\rho(g)"] \\ W \arrow[r, "i"] & V \end{tikzcd}$$

We have not yet proved that ρ_W really is a representation. We do it now:

Proof. That $\rho|_W$ is a group homomorphism, and that $\rho(g)|_W$ is linear is clear. It remains to show that $\rho(g)|_W$ is really an automorphism of W. However the extra condition of bijectivity comes directly from the fact that $\rho|_W$ is defined for all $g \in G$, thus for any particular $g \in G$, $\rho(g)|_W$ has an inverse, namely $\rho(g^{-1})|_W$, and thus is bijective. □

Remark. If $W_1, W_2 \leq V$ are G-invariant, so is $W_1 + W_2$ and $W_1 \cap W_2$.

Example (**Space of G-invariants**).

1. Consider the set V^G of all vectors $v \in V$ which are pointwise invariant under G: $v \in V^G$ iff $g \cdot v = v$ for all $g \in G$. V^G is not only a subspace of V, it is also a (trivial) subrepresentation of ρ.

   Proof. V^G is obviously a subrepresentation. Its triviality is also direct:

   $$\rho|_{V^G}(g)(v) = \rho(g)|_{V^G}(v) = \rho(g)(v) = v$$

   for all $g \in G$ and $v \in V^G$. □

   We note that by construction, V^G is the largest subrepresentation of ρ which is trivial.

2. In fact, if we take the vectors invariant under a *normal subgroup* $H \trianglelefteq G$, $V^H = \{\, v \in V \mid \rho(h)(v) = v \text{ for all } h \text{ in } H \,\}$ is still a subrepresentation of ρ, though usually not a trivial one.

Next, we consider quotient representations.

Definition. Given a G-invariant subspace W of V, the group G also has a natural representation by the quotient space V/W, called the **quotient representation**. It is defined by:

$$\begin{aligned} \bar{\rho} : G &\longrightarrow GL(V/W) \\ g &\mapsto (\bar{v} \mapsto \overline{\rho(g)(v)}), \end{aligned}$$

i.e. $\bar{\rho}(g)(\bar{v}) = \overline{\rho(g)(v)}$. (Remember the bar notation: $\overline{v} \in V/W$ refers to $v + W$.) The proof that the $\bar{\rho}$ is indeed a representation is tedious but straightforward, in the same style as that for subrepresentations, so we shall omit it.

Remark. Given a quotient representation $\bar{\rho}$, the projection

$$\pi : V \twoheadrightarrow V/W$$

becomes a morphism of representations. We illustrate below the associated commutative diagram:

$$\begin{tikzcd} V \arrow[r, "\pi"] \arrow[d, "\rho(g)"'] & V/W \arrow[d, "\bar{\rho}(g)"] \\ V \arrow[r, "\pi"] & V/W \end{tikzcd}$$

Remark. Note that this bar notation for quotient representations is somewhat vague, since it makes no reference as to which G-invariant subspace is involved. In practice this is clear from the context, but should ambiguities arise, we shall also adopt the following notation: $\bar{\rho} = \rho/\rho|_W$ for the G-invariant subspace W, where $\rho/\rho|_W$ is the associated subrepresentation.

2.2.2 Direct Sums

Definition.

Given $\rho_1 : G \longrightarrow GL(V_1)$ and $\rho_2 : G \longrightarrow GL(V_2)$, the **external direct sum**

$$\rho_1 \oplus \rho_2 : G \longrightarrow GL(V_1 \oplus V_2)$$
$$g \mapsto \rho_1(g) \oplus \rho_2(g)$$

is a representation of G.

In short:

$$\underline{(\rho_1 \oplus \rho_2)(g)}(v_1, v_2) = \underline{\rho_1(g) \oplus \rho_2(g)}(v_1, v_2) = (\rho_1(g)v_1, \rho_2(g)v_2)$$

Remark. Given $\rho : G \longrightarrow GL(V)$ with $V = V_1 \oplus V_2$ where V_1, V_2 are G-invariant. Note that here the '$\oplus$' refers to the internal direct sum of V_1 and V_2. For notational simplicity, we denote $\rho|_{V_1}$ by ρ_1 and $\rho|_{V_2}$ by ρ_2.

From linear algebra,we have the following commutative diagram:

$$\begin{tikzcd} V_1 \oplus V_2 \arrow[r, "(\rho_1 \oplus \rho_2)(g)"] \arrow[d, "\Phi"'] & V_1 \oplus V_2 \arrow[d, "\Phi"] \\ V \arrow[r, "\rho(g)"] & V \end{tikzcd}$$

where the bijective intertwiner Φ is given by

$$\Phi : (\sum_{i=1}^{m} a_i v_i \, , \sum_{j=m+1}^{n} b_j v_j) \longmapsto \sum_{i=1}^{m} a_i v_i + \sum_{j=m+1}^{n} b_j v_j$$

We thus have

$$\rho \cong \rho_1 \oplus \rho_2$$

*There is potentially an ambiguity arising from whether a direct sum is external or internal. In practice this is clear from the context; if not, we will make any necessary clarifications.

Finally, we can generalize the external direct sum to n factors:

$$(\bigoplus_{i=1}^{n} \rho_i) : G \longrightarrow GL(\bigoplus_{i=1}^{n} V_i)$$
$$g \mapsto \bigoplus_{i=1}^{n} \rho_i(g)$$

Next, we discuss a few useful results.

Proposition 2.3.

1. $\dim(\bigoplus_{i=1}^{n} \rho_i) = \sum_{i=1}^{n} \dim\rho_i$

2. $\mathrm{Hom}_G(\rho, \bigoplus_{i=1}^{n} \rho_i) \cong \bigoplus_{i=1}^{n} \mathrm{Hom}_G(\rho, \rho_i)$ as vector spaces.

3. $\mathrm{Hom}_G(\bigoplus_{i=1}^{n} \rho_i, \rho) \cong \bigoplus_{i=1}^{n} \mathrm{Hom}_G(\rho_i, \rho)$ as vector spaces.

These results generalize directly from linear algebra.

Proposition 2.4.

1. $\rho_1 \oplus \rho_2 \cong \rho_2 \oplus \rho_1$

2. $\rho_1 \oplus \rho_2 \oplus \rho_3 \cong \rho_1 \oplus (\rho_2 \oplus \rho_3) \cong (\rho_1 \oplus \rho_2) \oplus \rho_3$

3. If $\rho_1 \cong \rho_2$, then $\rho_1 \oplus \rho_3 \cong \rho_2 \oplus \rho_3$.

These results generalize directly from linear algebra.

Lemma 2.5. Let G be a group, V a vector space and $\rho : G \longrightarrow GL(V)$ a representation. Let V_1 be a subrepresentation of V. Denote by V_2 a complementary subspace of V_1, i.e. $V = V_1 \bigoplus V_2$.

V_2 is a subrepresentation of V if and only if the linear projection map

$$\Phi : V \longrightarrow V$$
$$v_1 + v_2 \mapsto v_1$$

(where $v_1 \in V_1$ and $v_2 \in V_2$) with image V_1 and kernel V_2 is an intertwiner, i.e. $\Phi \in \mathrm{Hom}_G(\rho, \rho)$.

Proof. ($\Longleftarrow$) If Φ is an intertwiner, it will be established below (c.f. Proposition 2.6) that its kernel is a subrepresentation of its domain. So here, $\ker(\Phi)$ is a subrepresentation of V.

($\Longrightarrow$) Given that V_2 is a subrepresentation. Pick an arbitrary $v \in V$. v has the form $v_1 + v_2$ for some $v_1 \in V_1$ and $v_2 \in V_2$. Since $\rho(g)$ is a linear map, we have $\Phi\rho(g)v = \rho(g)v_2 = \rho(g)\Phi(v)$. □

2.2.3 The Isomorphism Theorems

In this section, we develop the four isomorphism theorems (which are prevalent for basic algebraic objects) for representations. The proofs are very much the same as that for modules/vector spaces, but with the added condition of G-invariance. The following two results are useful so we list them in a proposition.

Proposition 2.6. Given a morphism $\Phi : V_1 \longrightarrow V_2$ of representations of G, the following standard subspaces are themselves representations of G:

1. $\ker(\Phi) \subseteq V_1$ is a subrepresentation of V_1

2. $\mathrm{Im}(\Phi) = \Phi(V_1) \subseteq V_2$ is a subrepresentation of V_2

Proof. Standard, omitted. □

Theorem 2.7 (**1st Isomorphism Theorem**)**.** Given a morphism $\Phi : V_1 \longrightarrow V_2$ of representations of G, the quotient map

$$\bar{\Phi} : V_1/\ker(\Phi) \longrightarrow \Phi(V_1)$$

is an isomorphism of *representations*, i.e.

$$\bar{\rho_1} \cong \rho_2|_{\Phi(V_1)}$$

Proof. We simply verify the intertwining property:

$$\begin{aligned}\underline{\bar{\Phi}\bar{\rho}_1(g)}(\bar{v}) &= \bar{\Phi}(\overline{\rho_1(g)(v)}) \\ &= \Phi\rho_1(g)(v) \\ &= \rho_2(g)\Phi(v) \\ &= \rho_2|_{\Phi(V_1)}(g)\Phi(v) \\ &= \underline{\rho_2|_{\Phi(V_1)}(g)\bar{\Phi}}(\bar{v})\end{aligned}$$

□

Remark. At this point, we are ready to collate in a commutative diagram several previous results, including the 1st Isomorphism Theorem:

$$\begin{array}{ccccccccc} V_1 & \xrightarrow{\pi_1} & E_1/\ker(\Phi) & \xrightarrow{\bar{\Phi}} & \Phi(V_1) & \xrightarrow{i} & V_2 & \xrightarrow{\pi_2} & V_2/\Phi(V_1) \\ {\scriptstyle \rho_1(g)}\downarrow & & {\scriptstyle \bar{\rho}_1(g)}\downarrow & & {\scriptstyle \rho_2|_{\Phi(V_1)}(g)}\downarrow & & \downarrow{\scriptstyle \rho_2(g)} & & \downarrow{\scriptstyle \bar{\rho}_2(g)} \\ V_1 & \xrightarrow[\pi_1]{} & E_1/\ker(\Phi) & \xrightarrow[\bar{\Phi}]{} & \Phi(V_1) & \xrightarrow[i]{} & V_2 & \xrightarrow[\pi_2]{} & V_2/\Phi(V_1) \end{array} .$$

Theorem 2.8 (2nd Isomorphism Theorem). Let V_1, V_2 be G-invariant subspaces of V. Let $\rho_1, \rho_2, \rho_{1+2}$ and $\rho_{1\cap 2}$ denote the representations of G with representations spaces $V_1, V_2, V_1 + V_2$ and $V_1 \cap V_2$ respectively. Then we have the isomorphism of representations

$$\rho_{1+2}/\rho_2 \cong \rho_1/\rho_{1\cap 2}$$

Proof. From linear algebra, the linear map

$$\begin{aligned}\Phi : V_1 + V_2 &\longrightarrow V_1/V_1 \cap V_2 \\ v_1 + v_2 &\mapsto \bar{v_1}\end{aligned}$$

is surjective linear with $\ker(\Phi) = V_2$. We show Φ is an intertwiner:

$$\begin{aligned}
\underline{\Phi\rho_{1+2}(g)}(v_1 + v_2) &= \Phi(\rho_{1+2}(g)v_1 + \rho_{1+2}(g)v_2) \\
&= \overline{\rho_{1+2}(g)v_1} \\
&= \overline{\rho_1(g)v_1} \\
&= (\rho_1/\rho_{1\cap 2})(g)(\bar{v_1}) \\
&= \underline{(\rho_1/\rho_{1\cap 2})(g)}\Phi(v_1 + v_2)
\end{aligned}$$

Then by the 1st Isomorphism Theorem applied to Φ, our result immediately follows.

□

Remark. For the special case where $V_1 \cap V_2 = \{0\}$, we have $V_1/\{0\} \cong V_1$, so the second isomorphism theorem simplifies to

$$\rho_{1\oplus 2}/\rho_2 \cong \rho_1$$

Theorem 2.9 (3rd Isomorphism Theorem). Let U, V be G-invariant subspaces such that $U \leq W \leq V$. Then

$$\rho_{V/U}/\rho_{W/U} \cong \rho_{V/W}.$$

Proof. From linear algebra, the linear map

$$\begin{aligned}
\Phi : V/U &\longrightarrow V/W \\
v + U &\mapsto v + W
\end{aligned}$$

is surjective linear with $\ker(\Phi) = W/U$. We show Φ is an intertwiner:

$$\begin{aligned}
\underline{\Phi\rho_{V/U}(g)}(v + U) &= \Phi(\rho_V(g)v + W \\
&= \rho_{V/W}(g)(v + W) \\
&= \underline{\rho_{V/W}(g)}\Phi(v + U)
\end{aligned}$$

Then by the 1st Isomorphism Theorem applied to Φ, our result immediately follows. $\square$

Lemma 2.10.

$$U \supseteq W \text{is a } G\text{-inv subspace of V} \iff U/W \text{is a } G\text{-inv subspace of V/W}$$

Proof. ($\Longrightarrow$) U is G-invariant $\Longrightarrow \rho(g)(u) \in U \Longrightarrow \bar{\rho}(g)(\bar{u}) = \overline{\rho(g)(u)} \in U/W$. That is, U/W is G-invariant.

($\Longleftarrow$) U/W is G-invariant $\Longrightarrow \bar{\rho}(g)(\bar{u}) = \overline{\rho(g)(u)} \in U/W \Longrightarrow \rho(g)(u) - u' = w \in W \subseteq U \Longrightarrow \rho(g)(u) = w + u' \in U$.That is, U is G-invariant. $\square$

The following remark is going to come in handy in our proof of the 4th Isomorphism Theorem. It is a fact from algebra, and we shall state it without proof.

Remark. Given $A, B \in \{\, C \,|\, W \leq C \leq V \,\}$. We have

$$A = B \iff A/W = B/W$$

Theorem 2.11 (**4th Isomorphism Theorem/ Correspondence Theorem**)**.** Let W be a (not necessarily G-invariant) subspace of V. We have the following bijection of sets:

$$\begin{aligned}\{\, U \,|\, W \leq U \leq V,\, U \ G\text{-invariant} \,\} &\longleftrightarrow \{\, U/W \,|\, W \leq U \leq V,\, U \ G\text{-invariant} \,\} \\ &= \{\text{subrepresentations of } V/W\}\end{aligned}$$

Proof. Recall from the 'normal' 4th Isomorphism Theorem for vector spaces that we have

$$\{\, A \,|\, W \leq A \leq V \,\} \longleftrightarrow \{\, A/W \,|\, W \leq A \leq V \,\}$$

The idea is to simply cut out the G-invariant A's. To do so however, we need to ensure that cutting out an element from the first set corresponds to cutting out one and only one from the other, and vice versa. This is exactly what Lemma 2.10 and Remark 2.2.3 assert. Thus we are done. $\square$

Whenever a representation can be written (up to isomorphism) as a direct sum of smaller representations, this gives useful information about the representation itself - by investigating the direct sum constituents of the representation, its so called 'building blocks', we can deduce properties of the representation itself. Thus it is in our interest to be able to decompose representations. Typically, one wishes to perform as many decompositions as possible, but the limitation is that the representation of concern might not have that many non-trivial subrepresentations to 'peel off'.

This topic of decompositions is our next subject of study. We introduce a few definitions first.

2.3 Irreducibility, Semisimplicity, Complete Reducibility

Definition (Irreducibility, Semisimplicity, Complete Reducibility).

1. A representation ρ of G acting on $V \neq \{0\}$ is **irreducible/simple** if the only subrepresentations of ρ are 0 and ρ itself. In other words, $\{0\}$ and V are the only G-invariant subspaces of V.

2. A representation ρ of G is **semisimple** if $V \neq \{0\}$ can be written as a direct sum of irreducible subrepresentations, i.e.

$$V = \bigoplus_{i=1}^{n} V_i$$
$$\rho \cong \bigoplus_{i=1}^{n} \rho_i$$

3. A semisimple representation ρ of G is **isotypic** if its irreducible constituents are all isomorphic; if they are all isomorphic to a representation π, then we say ρ is π-isotypic.

4. A representation ρ of G is **completely reducible** if $V \neq \{0\}$ and for *any* subrepresentation ρ_1 of ρ, we can find a complementary subrepresentation ρ_2 of ρ such that $\rho \cong \rho_1 \oplus \rho_2$.

Remark.

1. Another term is sometimes used in the literature: a representation is called **decomposable** if there exist non-zero proper G-invariant subspaces V_1, V_2 such that $V = V_1 \oplus V_2$. Otherwise the representation is **indecomposable**.

2. An irreducible representation is in particular semisimple, completely reducible and indecomposable.

3. It is common to abuse language and refer to the representation space V itself as irreducible/semisimple etc and we shall do that in this text.

Now we discuss a few main results. First, it should be intuitive clear that the properties mentioned above are preserved under isomorphisms. Indeed, many texts assume this without further elaboration. We shall state this fact as a theorem, but will not lay down the proof, for it is straightforward (albeit somewhat tedious), but offers no new insights nor techniques.

Theorem 2.12. If ρ is equivalent to an *irreducible/semisimple/completely reducible/decomposable* representation, then it is also *irreducible/semisimple/ completely reducible/decomposable.*

We will see later that up to isomorphisms of representations, the irreducible summands of a semisimple representation are uniquely determined (up to permutation, of course). This is a major result, called the Jordan-Hölder-Noether Theorem.

We now show that if two irreducible representations are non-isomorphic, then they 'do not interact', in the sense that there is no homomorphism between them (besides the trivial homomorphism, of course).

Lemma 2.13 (**Schur's Lemma I**)**.**

1. Given an irreducible representation π of G and an arbitrary representation ρ of G, any morphism $\Phi : \pi \longrightarrow \rho$ is either 0 or injective, and any morphism $\Phi : \rho \longrightarrow \pi$ is either 0 or surjective.

2. Given irreducible representations ρ and π of G, a morphism between ρ and π is either 0 or an isomorphism. In particular, if ρ and π are not isomorphic, then $\mathrm{Hom}_G(\rho, \pi) = \{0\}$.

Proof. 1. Given a morphism $\Phi : \pi \longrightarrow \rho$, we know that its kernel is a subrepresentation of π. If π is irreducible, the only possibilities are that $\ker(\Phi) = 0$, i.e. Φ is injective; or that $\ker(\Phi) = \pi$, i.e. Φ is 0 (the trivial map). Similarly for $\Phi : \rho \longrightarrow \pi$, either $\mathrm{Im}(\Phi) = 0$, i.e. Φ is 0, or $\mathrm{Im}(\Phi) = \pi$, i.e. Φ is surjective.

2. From 1., if Φ is non-zero then it must be both injective and surjective, i.e. an isomorphism. $\square$

A reminder: in this text, we only consider finite-dimensional representations.

Lemma 2.14. Any non-zero representation ρ has at least an irreducible subrepresentation.

Proof. Simply select a non-zero subrepresentation of minimal dimension, which exists: in the 'worst case scenario', we are reduced to a 1-dimensional subrepresentation, which is irreducible. Also note that if ρ itself is already irreducible, then its irreducible subrepresentation is itself. $\square$

Theorem 2.15 (**Semisimple $\Longleftrightarrow$ Completely Reducible**)**.** A representation ρ is semisimple if and only if it is completely reducible.

Proof. ($\Longrightarrow$) Suppose ρ is semisimple, i.e. $\rho = \bigoplus_{i=1}^{n} \rho_i$, with ρ_i all irreducible. Pick a subrepresentation W of V, and consider the maximal (by inclusion) subrepresentation

W' of V such that $W \cap W' = \{0\}$. In other words, W and W' are in direct sum. We note at this point that since we are only considering finite dimensional representations, such a W' always exists. Also, note that it is possible that W be V itself and W' be 0. Now after obtaining W', it remains to show that in fact $V = W \oplus W'$, and doing so establishes complete reducibility.

For every i, consider the intersection $(W \oplus W') \cap V_i \subseteq V$. Since sums and intersections preserve G-invariance, this intersection is a subrepresentation of V contained in V_i. V_i is irreducible, so the intersection is either 0 or V_i itself. We claim that it it the latter.

Claim: $(W \oplus W') \cap V_i = \{0\} \implies W \cap (W' + V_i) = \{0\}$. Proof: Assume the hypothesis. Pick $a \in W \cap (W' + V_i)$. $a \in W$ and $a \in W' + V_i$, so $a = w = w' + v_i \implies w - w' \in W \oplus W'$ is equal to $v_i \in V_i$. Since $(W \oplus W') \cap V_i = \{0\}$, $w - w' = v_i = 0 \implies w = w'$. Since $W \cap W' = \{0\}$, $w = 0$. Thus $a = 0$, i.e. $W \cap (W' + V_i) = \{0\}$.

Now it becomes clear why $(W \oplus W') \cap V_i$ cannot be $\{0\}$. For if it is, then $W \cap (W' + V_i) = \{0\}$, immediately contradicting the fact that W' is the maximal subrepresentation in direct sum with W. Finally, since $(W \oplus W') \cap V_i = V_i$, we have $(W \oplus W') \supseteq V_i$, and this holds for all V_i. That is, $V = \bigoplus_{i=1}^{n} V_i \subseteq W \oplus W'$, so $V = W \oplus W'$.

($\Longleftarrow$) Now suppose ρ is completely reducible. If V is irreducible then it is semisimple and we are done, so assume it it not irreducible. We then write $V = V_1 \oplus W_1$ where V_1 is an irreducible (proper) subrepresentation, which by Lemma 2.14 exists, and W_1 its complementary subrepresentation. Note that $0 < \dim V_1 < \dim V$ and $0 < \dim W_1 < \dim V$. Next we proceed to check if W_1 itself is irreducible or not. If yes then we are done, and if not then $W_1 = V_2 \oplus W_2$, where V_2 is an irreducible (proper) subrepresentation of W_1, and W_2 its complementary subrepresentation in W_1.

We repeat this process: if W_i is irreducible then the process terminates and we are done, and if it is not irreducible we write it as $W_i = V_{i+1} \oplus W_{i+1}$. But this process must terminate, i.e. there is some j such that $W_j = V_{j+1} \oplus W_{j+1}$ with both $V_{j+1}, W_{j+1} = V_{j+2}$ irreducible. Because if not, we have the unending chain $\dim V > \dim V_1 > \dim V_2 > ... > 0$, contradicting the finite-dimensionality of V. Finally, we have $V = \bigoplus_{i=1}^{j+2}$ is semisimple. □

Lemma 2.16 (Stability of Semisimplicity)**.**

1. Let ρ be a semisimple representation of G. Any subrepresentation of ρ is also semisimple; any quotient representation of ρ is also semisimple.
2. Let $\rho_1, \rho_2, ..., \rho_n$ be semisimple representations. Then their (external) direct sum $\bigoplus_{i=1}^{n} \rho_i$ is semisimple.
3. Let ρ be an arbitrary representation of G, and let V_1, V_2 be semisimple subrepresentations of V. The sum (whether direct or not) $V_1 + V_2 \subseteq V$ is semisimple.

Proof. We shall make use of the equivalent condition of complete reducibility.

1. Let ρ act on V, and let $W \subseteq V$ be a subrepresentation. We are going to check that W is completely reducible. Let W_1 be any subrepresentation of W - so W_1 is also a subrepresentation of V. Since V is semisimple = completely reducible by hypothesis, there exists a subrepresentation $W_2 \subseteq V$ such that $V = W_1 \oplus W_2$. We finally claim that $W = W_1 \oplus (W \cap W_2)$. Indeed, W_1 and $W \cap W_2$ are in direct sum; if $w \in W \subseteq V$ and we write $w = v_1 + v_2$ with $v_1 \in W_1$ and $v_2 \in W_2$, we also get $v_2 \in W \cap W_2$ since v_1 is also in W.
   Next, given a subrepresentation W of V and the corresponding quotient subrepresentation V/W. Since V is completely reducible, $V = W \oplus W'$, where W' is a subrepresentation, which is also semisimple (We just proved that!).

Now $V/W = (W \oplus W')/W \cong W'$ by the (special case of the) 2nd Isomorphism Theorem. Since W' is semisimple and semisimplicity is preserved under isomorphisms, we conclude that V/W is semisimple.

2. The proof is tedious due to the notation, but the idea is straightforward. We consider the case when $n = 2$, but the technique generalizes to general n as well. If $V_1 = \bigoplus_{i=1}^m V_{1i}$ and $V_2 = \bigoplus_{j=1}^n V_{2j}$, then we can show that $V_1 \oplus V_2 = \bigoplus_{i=1}^m (V_{1i} \times \{0_2\}) \oplus \bigoplus_{j=1}^n (\{0_1\} \times V_{2j})$, with all the summands irreducible.

3. From linear algebra, the sum $V_1 + V_2$ is isomorphic (as vector spaces) to $(V_1 \oplus V_2)/\ker(\Phi)$ where Φ defined by

$$\Phi : V_1 \oplus V_2 \longrightarrow V_1 + V_2$$
$$(v_1, v_2) \mapsto v_1 + v_2$$

is a surjective intertwiner. From the 2nd isomorphism theorem for representations, we have that the representations are isomorphic (as opposed to mere vector space isomorphisms). Since $(V_1 \oplus V_2)/\ker(\Phi)$ is semisimple, so is $V_1 + V_2$.

□

We now consider even more constructions, namely the tensor product of representations and the dual of a representation.

2.4 More New Things

2.4.1 Tensor Products

Definition.

Given $\rho_1 : G \longrightarrow GL(V_1)$ and $\rho_2 : G \longrightarrow GL(V_2)$, the **tensor product**

$$\rho_1 \otimes \rho_2 : G \longrightarrow GL(V_1 \otimes V_2)$$
$$g \mapsto \rho_1(g) \otimes \rho_2(g)$$

is a representation of G.

In short:

$$\begin{aligned}\underline{(\rho_1 \otimes \rho_2)(g)}(\sum_i a_i v_i \otimes w_i) &= \underline{\rho_1(g) \otimes \rho_2(g)}(\sum_i a_i v_i \otimes w_i) \\ &= (\sum_i a_i \rho_1(g) v_i \otimes \rho_2(g) w_i).\end{aligned}$$

Proposition 2.17.

1. $\rho_1 \otimes \rho_2 \cong \rho_2 \otimes \rho_1$
2. $\rho_1 \otimes \rho_2 \otimes \rho_3 \cong \rho_1 \otimes (\rho_2 \otimes \rho_3) \cong (\rho_1 \otimes \rho_2) \otimes \rho_3$
3. If $\rho_1 \cong \rho_2$, then $\rho_1 \otimes \rho_3 \cong \rho_2 \otimes \rho_3$.
4. $\mathbf{1} \otimes \rho \cong \rho \otimes \mathbf{1} \cong \rho$, where $\mathbf{1}$ is the one-dimensional trivial representation.
5. $\dim(\bigotimes_{i=1}^n \rho_i) = \prod_{i=1}^n \dim \rho_i$
6. $\rho \otimes \bigoplus_{i=1}^n \rho_i \cong \bigoplus_{i=1}^n (\rho \otimes \rho_i)$

These results generalize directly from linear algebra. We shall prove the last statement as an illustration.

Proof. This is established by constructing the commutative diagram below:

$$\begin{tikzcd}
V \otimes (V_1 \oplus V_2 \oplus ... \oplus V_n) \arrow[r, "\Phi"] \arrow[d, "(\rho \otimes \bigoplus_{i=1}^n \rho_i)(g)"'] & (V \otimes V_1) \oplus (V \otimes V_2) \oplus ... \oplus (V \otimes V_n) \arrow[d, "(\bigoplus_{i=1}^n \rho \otimes \rho_i)(g)"] \\
V \otimes (V_1 \oplus V_2 \oplus ... \oplus V_n) \arrow[r, "\Phi"] & (V \otimes V_1) \oplus (V \otimes V_2) \oplus ... \oplus (V \otimes V_n)
\end{tikzcd}$$

Φ is defined the obvious way: $\Phi(v \otimes (v_1, ..., v_n)) = (v \otimes v_1, ..., v \otimes v_n)$ on the elementary tensors and then extending by linearity. $\square$

2.4.2 The Dual Representation

Let ρ be a representation of G, acting on the vector space V. Recall the definition of the dual space of V, $V^* = \mathrm{Hom}_K(V, K)$. We can define a representation of G on the dual space V^*.

Definition. The **dual/contragredient** representation of ρ, denoted by ρ^*, is defined by

$$\rho^*(g)(\lambda) = \lambda \circ \rho(g^{-1}),$$

which is equivalent to saying that

$$\rho^*(g) = \rho(g^{-1})^*$$

where we use the standard notation L^* for the dual map of L.

A commonly used notation is

$$< g \cdot \lambda, v > = < \lambda, g^{-1} \cdot v >$$

For completeness, we verify that ρ^* is indeed a representation (i.e. a group homomorphism from G to $GL(V^*)$.

Proof.

$$\begin{aligned} \underline{\rho^*(gh)(\lambda)}(v) &= \lambda(\rho(h^{-1})\rho(g^{-1})(v)) \\ &= \underline{\rho^*(h)\lambda}(\rho(g^{-1})v) \\ &= \underline{\rho^*(g)\rho^*(h)(\lambda)}(v) \end{aligned}$$

for all $g, h \in G$, $v \in V$ and $\lambda \in V^*$. □

Proposition 2.18.

1. Given representations ρ_1 and ρ_2 acting on V_1 and V_2 respectively, and intertwiner $L : V_1 \longrightarrow V_2$, the dual map of L

$$L^* : V_2^* \longrightarrow V_1^*$$

is also an intertwiner between ρ_2^* and ρ_1^*. In particular, if $\rho_1 \cong \rho_2$, then $\rho_1^* \cong \rho_2^*$.

2. $(\rho_1 \oplus \rho_2 \oplus ... \oplus \rho_n)^* \cong \rho_1^* \oplus \rho_2^* \oplus ... \oplus \rho_n^*$

3. $(\rho_1 \otimes \rho_2 \otimes ... \otimes \rho_n)^* \cong \rho_1^* \otimes \rho_2^* \otimes ... \otimes \rho_n^*$

4. $\rho \cong \rho^{**}$

Proof.

1. We have to show $L\rho_1(g) = \rho_2(g)L \Longrightarrow L^*\rho_2^*(g) = \rho_1^*(g)L^*$.

   Pick an arbitrary $\lambda \in V_2^*$. We have

$$\begin{aligned}
\underline{\rho_1^*(g)L^*}(\lambda) &= \rho_1^*(g)(L^*(\lambda)) \\
&= \rho_1^*(g)(\lambda \circ L) \\
&= \rho_1(g^{-1})^*(\lambda \circ L) \\
&= \lambda \circ L \circ \rho_1(g^{-1}) \\
&= \lambda \circ \rho_2(g^{-1}) \circ L \\
&= (\rho_2(g^{-1})L)^*(\lambda) \\
&= (L^*\rho_2(g^{-1})^*)(\lambda) \\
&= \underline{L^*\rho_2^*(g)}(\lambda)
\end{aligned}$$

2. We prove $(\rho_1 \oplus \rho_2)^* \cong \rho_1^* \oplus \rho_2^*$. The general case is similar in proof.

   Define $\Phi : V_1^* \oplus V_2^* \longrightarrow (V_1 \oplus V_2)^*$ by

$$\underline{\Phi(f_1, f_2)}(v_1, v_2) = f_1(v_1) + f_2(v_2).$$

From the linear algebra of dual spaces/dual modules, Φ is an isomorphism. (Again, note that we consider only finite-dimensional spaces/ representations). See [5].

Now we show that Φ is an intertwiner, i.e. $\Phi(\rho_1^* \oplus \rho_2^*)(g) = (\rho_1 \oplus \rho_2)^*(g)\Phi$. We shall make use of the fact $(\rho_1 \oplus \rho_2)^*(g) = ((\rho_1 \oplus \rho_2)(g^{-1}))^* = (\rho_1(g^{-1}) \oplus \rho_2(g^{-1}))^*$. Pick arbitrary $f_1, f_2 \in V_1^* \oplus V_2^*$ and $v_1, v_2 \in V_1 \oplus V_2$.

$$\begin{aligned}
(\underline{\Phi(\rho_1^* \oplus \rho_2^*)(g)}(f_1, f_2))(v_1, v_2) &= \underline{\Phi(\rho_1^*(g) f_1, \rho_2^*(g) f_2}(v_1, v_2) \\
&= \underline{\rho_1^*(g) f_1}(v_1) + \underline{\rho_2^*(g) f_2}(v_2) \\
&= \underline{f_1 \circ \rho_1(g^{-1})}(v_1) + \underline{f_2 \circ \rho_2(g^{-1})}(v_2)
\end{aligned}$$

and

$$\begin{aligned}
(\underline{(\rho_1 \oplus \rho_2)^*(g)}\Phi(f_1, f_2))(v_1, v_2) &= \underline{(\rho_1 \oplus \rho_2)^*(g)(\Phi(f_1, f_2))}(v_1, v_2) \\
&= \underline{\Phi(f_1, f_2) \circ [\rho_1(g^{-1}) \oplus \rho_2(g^{-1})]}(v_1, v_2) \\
&= \Phi(f_1, f_2)(\rho_1(g^{-1})v_1, \rho_2(g^{-1})v_2) \\
&= \underline{f_1 \circ \rho_1(g^{-1})}(v_1) + \underline{f_2 \circ \rho_2(g^{-1})}(v_2).
\end{aligned}$$

which establishes the equivalence. This gives us the commutative diagram:

$$\begin{tikzcd}
V_1^* \oplus V_2^* \arrow[r, "\Phi"] \arrow[d, "(\rho_1^* \oplus \rho_2^*)(g)"'] & (V_1 \oplus V_2)^* \arrow[d, "(\rho_1 \oplus \rho_2)^*(g)"] \\
V_1^* \oplus V_2^* \arrow[r, "\Phi"] & (V_1 \oplus V_2)^*
\end{tikzcd}$$

3. The proof for tensor products is entirely the same as that for direct sums, with the changes $(v_1, v_2) \longleftrightarrow v_1 \otimes v_2$ and $+ \longleftrightarrow \cdot$. Also note that we only have to evaluate the maps on elementary tensors, since we can extend by linearity to all tensors.

We prove $(\rho_1 \otimes \rho_2)^* \cong \rho_1^* \otimes \rho_2^*$. The general case is similar in proof.

Define $\Phi : V_1^* \otimes V_2^* \longrightarrow (V_1 \otimes V_2)^*$ by

$$\underline{\Phi(f_1 \otimes f_2)}(v_1 \otimes v_2) = f_1(v_1)f_2(v_2).$$

From the linear algebra of dual spaces/dual modules, Φ is an isomorphism. See [5].

Now we show that Φ is an intertwiner, i.e. $\Phi(\rho_1^* \otimes \rho_2^*)(g) = (\rho_1 \otimes \rho_2)^*(g)\Phi$. We shall make use of the fact $(\rho_1 \otimes \rho_2)^*(g) = ((\rho_1 \otimes \rho_2)(g^{-1}))^* = (\rho_1(g^{-1}) \otimes \rho_2(g^{-1}))^*$. Pick arbitrary $f_1 \otimes f_2 \in V_1^* \otimes V_2^*$ and $v_1 \otimes v_2 \in V_1 \otimes V_2$.

$$\begin{aligned}
(\Phi\underline{(\rho_1^* \otimes \rho_2^*)(g)}(f_1 \otimes f_2))(v_1 \otimes v_2) &= \underline{\Phi(\rho_1^*(g)f_1 \otimes \rho_2^*(g)f_2}(v_1 \otimes v_2) \\
&= \underline{\rho_1^*(g)f_1}(v_1)\underline{\rho_2^*(g)f_2}(v_2) \\
&= f_1(\rho_1(g^{-1})v_1) \cdot f_2(\rho_2(g^{-1})v_2)
\end{aligned}$$

and

$$\begin{aligned}
(\underline{(\rho_1 \otimes \rho_2)^*(g)\Phi}(f_1 \otimes f_2))(v_1 \otimes v_2) &= \underline{(\rho_1 \otimes \rho_2)^*(g)(\Phi(f_1 \otimes f_2))}(v_1 \otimes v_2) \\
&= \underline{\Phi(f_1 \otimes f_2) \circ [\rho_1(g^{-1}) \otimes \rho_2(g^{-1})]}(v_1 \otimes v_2) \\
&= \Phi(f_1 \otimes f_2)(\rho_1(g^{-1})v_1 \otimes \rho_2(g^{-1})v_2) \\
&= f_1(\rho_1(g^{-1})v_1) \cdot f_2(\rho_2(g^{-1})v_2).
\end{aligned}$$

which establishes the equivalence. This gives us the commutative diagram:

$$\begin{array}{ccc}
V_1^* \otimes V_2^* & \xrightarrow{\Phi} & (V_1 \otimes V_2)^* \\
{\scriptstyle (\rho_1^* \otimes \rho_2^*)(g)}\downarrow & & \downarrow{\scriptstyle (\rho_1 \otimes \rho_2)^*(g)} \\
V_1^* \otimes V_2^* & \xrightarrow{\Phi} & (V_1 \otimes V_2)^*
\end{array}$$

4. Given $\rho : G \longrightarrow GL(V)$. $\rho^{**} : G \longrightarrow GL(V^{**})$ can also be written as

$$\rho^{**}(g) = (\rho^*(g^{-1}))^* = \rho(g)^{**}$$

where ** denotes the double dual.

We recall that (c.f. [5]) since V is finite-dimensional, every element of V^{**} can be written in the form ev_v (ev for evaluate), where for some $v \in V$, $ev_v(\lambda) = \lambda(v)$ for all $\lambda \in V^*$. Also, there is a natural isomorphism (called the double duality isomorphism) given by

$$\begin{aligned} \Phi : V &\longrightarrow V^{**} \\ v &\longmapsto ev_v. \end{aligned}$$

It is precisely this double duality isomorphism that shall serve as our intertwiner between ρ and ρ^{**}.

Furthermore, for $L :\longrightarrow W$, its double dual L^{**} works like this:

$$L^{**}(ev_v) = ev_v \circ L^* = ev_{L(v)}$$

Finally, we show that Φ is an intertwiner.

$$\begin{aligned} \underline{\rho^{**}(g)\Phi}(v) &= \rho(g)^{**}(ev_v) \\ &= ev_{\rho(g)v} \\ \underline{\Phi\rho(g)}(v) &= \Phi(\rho(g)v) \\ &= ev_{\rho(g)v} \end{aligned}$$

We thus have the commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\Phi} & V^{**} \\ {\scriptstyle \rho(g)}\big\downarrow & & \big\downarrow{\scriptstyle \rho^{**}(g)=\rho(g)^{**}} \\ V & \xrightarrow{\Phi} & V^{**} \end{array}$$

□

Theorem 2.19. ρ is irreducible $\Longleftrightarrow$ ρ^* is irreducible. In particular, there is a bijection between the set of subrepresentations W of ρ, and the set of subrepresentations W^* of ρ^*, given by

$$W \longmapsto W^\perp = \{\lambda \in V^* \mid \lambda(w) = 0 \,\text{for all}\, w \in W\}$$
$$W^* \longmapsto W^{*\perp} = \{v \in V \mid \lambda(v) = 0 \,\text{for all}\, \lambda \in W^*\}$$

Proof. Let W be a subrepresentation of V. We claim that $W^\perp$ as defined above is a subrepresentation of V^*, i.e. $\rho^*(g)(\lambda) \in W^\perp$ for all $g \in G$, $\lambda \in W^\perp$. This is easy to establish: pick an arbitrary $w \in W$. Then $\underline{\rho^*(g)(\lambda)}(w) = \lambda(\rho(g^{-1})w) = \lambda(w') = 0$, since $w' \in W$ as well. Thus $\rho^*(g)(\lambda) \in W^\perp$. Since g, λ was arbitrary, $W^\perp$ is G-invariant.

Now let W^* be a subrepresentation of V^*. We claim $W^{*\perp}$ as defined above is a subrepresentation of V, i.e. $\rho(g)(v) \in W^{*\perp}$ for all $g \in G$, $v \in W^{*\perp}$. Pick an arbitrary $\lambda \in W^*$. $\lambda(\rho(g)v) = \underbrace{\rho^*(g^{-1})(\lambda)}_{\in W^*}(v) = 0$. Hence $W^{*\perp}$ is G-invariant.

It remains to show that there is a bijection between the set of subrepresentations W of ρ, and the set of subrepresentations W^* of ρ^*. We omit the details, but the idea is to show that the two maps defined above are inverses of each other, i.e.

$$W \longmapsto W^\perp \longmapsto W^{\perp\perp} = W$$

This would establish the bijection, and as a consequence, ρ is irreducible if and only if ρ^* is irreducible. $\square$

Next, we discuss a very important concept - the idea that hom-spaces themselves carry a representation of G.

2.5 Actions on Hom Spaces

Theorem 2.20. Let $\rho : G \longrightarrow GL(V)$ and $\tau : G \longrightarrow GL(W)$ be representations. Then the map $\Gamma : G \longrightarrow GL(\mathrm{Hom}_K(V, W))$ defined by

$$\Gamma(g)(\phi) = \tau(g)\phi\rho(g^{-1}) \quad \text{(where } \phi \in \mathrm{Hom}_K(V, W))$$

is a representation.

Furthermore,

1. $\mathrm{Hom}_K(V, W)^G = \mathrm{Hom}_G(\rho, \tau)$
2. $\Gamma \cong \rho^* \otimes \tau$

Proof. We confirm that Γ is indeed a group homomorphism:

$$\begin{aligned}
\underline{\Gamma(gh)}(\phi) &= \tau(g)\tau(h)\phi\rho(h)^{-1}\rho(g)^{-1} \\
&= \tau(g)\Gamma(h)(\phi)\rho(g)^{-1} \\
&= \underline{\Gamma(g)\Gamma(h)}(\phi)
\end{aligned}$$

Then,

$$\begin{aligned}
\mathrm{Hom}_K(V, W)^G &= \{\, \phi \in \mathrm{Hom}_K(V, W) \mid \Gamma(g)\phi = \phi \quad \text{for all } g \,\} \\
&= \{\, \phi \in \mathrm{Hom}_K(V, W) \mid \tau(g)\phi\rho(g^{-1}) = \phi \quad \text{for all } g \,\} \\
&= \mathrm{Hom}_G(\rho, \tau).
\end{aligned}$$

Finally, we show $\Gamma \cong \rho^* \otimes \tau$. Consider the map $T : V^* \otimes W \longmapsto \mathrm{Hom}_K(V, W)$ defined by

$$T(\psi \otimes v)(w) = \psi(w)(v)$$

where $\psi \in V^*$, $v \in V, w \in W$. From linear algebra [5], T is an isomorphism. We claim that T intertwines Γ and $\rho^* \otimes \tau$:

$$\begin{aligned}
(\underline{T(\rho^* \otimes \tau)(g)}(\psi \otimes w))(v) &= T(\rho^*(g)\psi \otimes \tau(g)w)(v) \\
&= \underline{\rho^*(g)(\psi)}(v) \cdot \tau(g)w \\
&= \psi(\rho(g^{-1})v) \cdot \tau(g)w
\end{aligned}$$

and

$$\begin{aligned}
(\underline{\Gamma(g)T}(\psi \otimes w))(v) &= \underline{\tau(g)T(\psi \otimes w)\rho(g^{-1})}(v) \\
&= \tau(g)[\psi(\rho(g^{-1})v)w] \\
&= \psi(\rho(g^{-1})v) \cdot \tau(g)w.
\end{aligned}$$

In a commutative diagram,

$$\begin{array}{ccc}
V^* \otimes W & \xrightarrow{T} & \mathrm{Hom}_K(V, W) \\
{\scriptstyle (\rho^*\otimes\tau)(g)} \big\downarrow & & \big\downarrow {\scriptstyle \Gamma(g)} \\
V^* \otimes W & \xrightarrow{T} & \mathrm{Hom}_K(V, W)
\end{array}$$

□

Remark. It is also common to write $(g \cdot \phi)(v) = g \cdot \phi(g^{-1} \cdot v)$ in place of $\Gamma(g)(\phi) = \tau(g)\phi\rho(g^{-1})$.

Consider the representation above in the special case that ρ is the trivial representation, i.e. $\rho(g) = I_V$ for all $g \in G$. This will come in handy when we discuss Burnside's results.

Lemma 2.21. Consider the special case of Theorem 2.20, when $\rho(g) = I_V$, so that $\Gamma(g)(\phi) = \tau(g)\phi$. Let $\dim(V) = n$, and construct the (external) direct sum of τ:

$$\tau^{\oplus n} : G \longrightarrow GL(W^{\oplus n}).$$

Claim: $\Gamma \cong \tau^{\oplus n}$.

Proof. Pick any basis for V, say $(v_1, ..., v_n)$. Define the map $\Phi : W^{\oplus n} \longrightarrow \mathrm{Hom}_K(V, W)$ by

$$\Phi(w_1, ..., w_n)(v = \sum_i a_i v_i) = \sum_i a_i w_i$$

where in particular, $\Phi(w_1, ..., w_n)(v_j) = w_j$ for any basis vector v_j. Note that the w_i's are arbitrary vectors of W. Φ is bijective linear, and we omit the proof. We show that Φ is an intertwiner between $\tau^{\oplus n}$ and Γ:

$$\begin{aligned}
(\underline{\Phi\tau^{\oplus n}(g)}(w_1, ...w_n))(v) &= \Phi(\tau(g)w_1, ...\tau(g)w_n)(\sum_i a_i v_i) \\
&= \sum_i a_i \tau(g) w_i \\
&= \tau(g)(\sum_i a_i w_i) \\
&= \underline{\tau(g) \circ \Phi(w_1, ...w_n)}(v) \\
&= (\underline{\Gamma(g)}\Phi(w_1, ...w_n))(v)
\end{aligned}$$

In a commutative diagram,

$$\begin{array}{ccc}
W^{\oplus n} & \xrightarrow{\Phi} & \mathrm{Hom}_K(V, W) \\
{\scriptstyle \tau^{\oplus n}(g)}\big\downarrow & & \big\downarrow{\scriptstyle \Gamma(g)} \\
W^{\oplus n} & \xrightarrow{\Phi} & \mathrm{Hom}_K(V, W)
\end{array}$$

Note that $\dim(\mathrm{Hom}_K(V, W)) = \dim(V) \cdot \dim(W) = \dim(W^{\oplus n})$. $\square$

2.6 Matrix Representations

In this subsection, we briefly talk about the notion of matrix representations. So far, we have taken the abstract point of view where a representation maps g to a linear map. In practice, when one wishes to perform computations or obtain certain insights which may be obscured by abstraction, one associates the linear map $\rho(g)$ to its matrix representation, after making a choice of basis. A useful example comes in the next section in Lemma 3.10, where we consider the matrix representation of $\rho(g)$ to prove that 'if ρ is reducible, then $\rho(G)$ does not span Ω'.

Definition. Given a representation $\rho : G \longrightarrow GL(V)$. Choose a basis B for V. The **matrix representation** associated with ρ is simply a group homomorphism

$$\begin{aligned} \rho_B : G &\longrightarrow GL_n(K) \\ g &\longmapsto [\rho(g)]_B \end{aligned}$$

Suppose for the same representation ρ, we pick 2 different bases, say B and B'. This gives us two matrix representations associated to ρ, namely ρ_B and $\rho_{B'}$. It is a result in elementary linear algebra that for all $g \in G$, $\rho_{B'}(g) = P\rho_B(g)P^{-1}$ with P being the change-of-basis matrix from B to B'. This motivates the following definition:

Definition. (Isomorphisms of Matrix Representations) Two matrix representations ρ_B and $\rho_{B'}$ associated to ρ are called **equivalent/isomorphic** if $\exists P \in GL_n(K)$ such that $\rho_{B'}(g) = P\rho_B(g)P^{-1}$ for all $g \in G$, or equivalently, $P\rho_B(g) = \rho_{B'}(g)P$ for all $g \in G$. This is similar in flavour to isomorphisms between representations.

Now we discuss a theorem that hopefully further justifies the interest in matrix representations. Namely, two representations are isomorphic if and only if their matrix representations (independent of the bases chosen) are isomorphic. This is useful because in practice it may be easier to tell whether a pair of matrix representations are isomorphic or not, than for the original representations themselves. The proof is a straightforward exercise in linear algebra.

Theorem 2.22. Given two representations $\rho : G \longrightarrow GL(V)$ and $\tau : G \longrightarrow GL(W)$ where $\dim(V) = \dim(W) = n$. Pick any basis for V and likewise for W. Denote the bases by R and T respectively. Construct the matrix representations $\rho_R : G \longrightarrow GL_n(K)$ and $\tau_T : G \longrightarrow GL_n(K)$. Then,

$$\rho \cong \tau \iff \rho_R \cong \tau_T.$$

Proof.

1. ($\Longrightarrow$) Suppose $\rho \cong \tau$, i.e. $\exists \Phi : V \longrightarrow W$ s.t. $\tau(g) = \Phi\rho(g)\Phi^{-1}$ for all $g \in G$. Then taking matrix representations of both sides of the equation gives

$$\begin{aligned} \tau_T(g) = [\tau(g)]_T &= [\Phi\rho(g)\Phi^{-1}]_T \\ &= [\Phi]_R^T[\rho(g)]_R[\Phi^{-1}]_T^R \\ &= [\Phi]_R^T\rho_R(g)[\Phi^{-1}]_T^R, \end{aligned}$$

so taking $P = [\Phi]_R^T$, we have $\rho_R \cong \tau_T$.

2. ($\Longleftarrow$) Suppose $\rho_R \cong \tau_T$, i.e. $P \in GL_n(K)$ s.t. $\tau_T(g) = P\rho_R(g)P^{-1}$ for all $g \in G$. Choose $\Phi : V \longrightarrow W$ s.t. $[\Phi]_R^T = P$, or equivalently,

$$P = [[\Phi(r_1)]_T \; ... \; [\Phi(r_n)]_T]$$

where $R = \{r_1, ..., r_n\}$. Since P is invertible, Φ is an isomorphism. Then

$$\begin{aligned} [\tau(g)]_T = \tau_T(g) &= P\rho_R(g)P^{-1} \\ &= [\Phi]_R^T[\rho(g)]_R[\Phi^{-1}]_T^R \\ &= [\Phi\rho(g)\Phi^{-1}]_T, \end{aligned}$$

so $\tau(g) = \Phi\rho(g)\Phi^{-1}$, i.e. $\rho \cong \tau$.

□

For a (external) direct sum of representations $\rho_1 \oplus \rho_2$, we may concatenate the bases $\{v_1, ..., v_n\}$ of ρ_1 and $\{w_1, ..., w_m\}$ of ρ_2 to obtain a basis $\{(v_1, 0), ..., (v_n, 0), (0, w_1), ...(0, w_m)\}$ in which the matrix representation of $\rho_1 \oplus \rho_2$ takes the form of a block-diagonal matrix

$$[(\rho_1 \oplus \rho_2)(g)]_{B_1 B_2} = \left[\begin{array}{c|c} [\rho_1(g)]_{B_1} & O \\ \hline O & [\rho_2(g)]_{B_2} \end{array}\right].$$

This tells us that if a representation ρ is decomposable (c.f. Remark 2.3), then with the obvious choice of bases its matrix representation sends g to block-diagonal matrices. In particular, this holds for semisimple representations as well. Note, however, that if a given matrix representation sends g to non-block-diagonal matrices, this does not imply that the representation itself is indecomposable/not semisimple, simply because it could be that a 'bad' basis was chosen.

With these, we end our short discussion on matrix representations and conclude our second section. In the next, we discuss a few major results.

3 Important Results

3.1 The Jordan-Hölder-Noether Theorem

Theorem 3.1. Let G be a group and $V \neq \{0\}$ be a K-vector space. Let $\rho : G \longrightarrow GL(V)$ be a representation of G.

1. (Existence) There exists a finite sequence of subrepresentations

$$\{0\} = V_0 \subset V_1 \subset V_2 \subset ... \subset V_{n-1} \subset V_n = V$$

of V such that for all $i, 1 \leq i \leq n$, the quotient representations V_i/V_{i-1} are irreducible. Such sequences are called **composition series**, and the irreducible representations V_i/V_{i-1} are called the **composition factors** (of V).

2. (Uniqueness) Any two composition series are equivalent in the following sense: the number of terms are the same, and the irreducible composition factors are isomorphic, up to a permutation.
More precisely, suppose we have the 2 series

$$\{0\} = V_0 \subset V_1 \subset V_2 \subset ... \subset V_{n-1} \subset V_n = V$$
$$\{0\} = W_0 \subset W_1 \subset W_2 \subset ... \subset W_{n-1} \subset W_m = V,$$

then $m = n$. Given any V_{i+1}/V_i, there exists a (unique) j such that $V_{i+1}/V_i \cong W_{j+1}/W_j$, and we have the following bijection:

$$\{V_{i+1}/V_i\}_{0\leq i\leq n-1} \longleftrightarrow \{W_{j+1}/V_j\}_{0\leq j\leq n-1}$$

In particular, for any irreducible representation π of G, the **multiplicity of π in** ρ, $n_\pi(\rho) = |\{\, i \mid V_{i+1}/V_i \cong \pi \,\}|$ is independent of the choice of composition series (V_i).

Proof. (Existence)

Recall from the 4th Isomorphism Theorem the bijection of sets

$$\{\, U \,|\, W \leq U \leq V,\, U\ G\text{-invariant}\,\} \longleftrightarrow \{\, U/W \,|\, W \leq U \leq V,\, U\ G\text{-invariant}\,\}$$
$$= \{\text{subrepresentations of } V/W\}.$$

If V is irreducible then the composition series is $0 \subset V$ and we are done. If not, select an irreducible subrepresentation $0 \subset V_1 \subset V$ (which exists, by Lemma 2.14). Note that $V_1/V_0 \cong V_1$ is irreducible.

Consider $V/V_1 \neq 0$. If V/V_1 is irreducible then we are done, with the composition series $0 \subset V_1 \subset V$ and composition factors $V_1/V_0, V/V_1$. If not, select an irreducible subrepresentation $0 \subset V_2/V_1 \subset V/V_1$. Note that V_2 is a subrepresentation of V and we now have $0 \subset V_1 \subset V_2 \subset V$, and $V_1/V_0, V_2/V_1$ irreducible.

Consider $V/V_2 \neq 0$. Proceed with the same argument above and repeat for $V/V_3, V/V_4 \ldots$ if necessary. Now for every reducible quotient representation V/V_j, we end up 'squeezing' another subrepresentation V_{j+1} in between V_j and V, i.e. $V_j \subset V_{j+1} \subset V$. This process must terminate because $\dim(V)$ is finite.

Thus we end up having the composition series

$$\{0\} = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_{n-1} \subset V_n = V$$

and composition factors $\{V_1/V_0, V_2/V_1, \ldots, V_n/V_{n-1}\}$. □

Before proving the uniqueness part of the theorem, we state an important lemma which will be useful in the proof. This lemma is the representation theoretic analogue of the Zassenhaus Lemma from group theory.

Lemma 3.2. (**Zassenhaus Lemma for Representations**) Given a representation $\tilde{V}$, consider the following subrepresentations $V \subseteq V' \subseteq \tilde{V}$ and $W \subseteq W' \subseteq \tilde{V}$. We have the following isomorphism of representations:

$$\underline{V + V' \cap W'}/\underline{V + V' \cap W} \cong \underline{W + V' \cap W'}/\underline{W + V \cap W'}.$$

Proof. Define the following map:

$$\begin{aligned}
\Phi : V + V' \cap W' &\longrightarrow W + V' \cap W' / W + V \cap W' \\
v + k &\longmapsto \bar{k} \quad (= k + (W + V \cap W'))
\end{aligned}$$

It is not too hard to show that Φ is well-defined, linear and surjective.

Now we claim that Φ intertwines the representations associated to its domain and codomain. Namely, that we have the commutative diagram:

$$\begin{array}{ccc}
V + V' \cap W' & \xrightarrow{\Phi} & W + V' \cap W'/W + V \cap W' \\
{\scriptstyle \rho(g)}\big\downarrow & & \big\downarrow{\scriptstyle \bar{\rho}(g)} \\
V + V' \cap W' & \xrightarrow{\Phi} & W + V' \cap W'/W + V \cap W'
\end{array}$$

Indeed, this is true because

$$\begin{aligned}
\underline{\Phi\rho(g)}(v + k) &= \Phi(\rho(g)v + \rho(g)k) \\
&= \overline{\rho(g)k} \\
&= \bar{\rho}(g)(\bar{k}) \\
&= \underline{\bar{\rho}(g)\Phi}(v + k)
\end{aligned}$$

Next we show $\underline{\ker(\Phi) = V + V' \cap W}$. First note that by definition, $\ker(\Phi) = \{\, v + k \mid k \in V' \cap W' \text{ and } k \in W + V \cap W'\}$.

($\supseteq$) Pick an arbitrary $v + p \in V + V' \cap W$. In particular, $p \in V' \cap W'$ because $W \subseteq W'$, and $p \in W + V \cap W'$ because $p \in W$, so $v + p \in \ker(\Phi)$.

($\subseteq$) Pick $v + k \in \ker(\Phi)$. $k = s + t$ for some $s \in W$ and $t \in V \cap W'$. We have $v + k = v + s + t$. Now

$$\begin{aligned}
v \in V, t \in V &\implies v + t \in V \\
k \in V', t \in V' &\implies s = k - t \in V'.
\end{aligned}$$

Thus $v + k = \underbrace{(v+t)}_{\in V} + \underbrace{s}_{\in V' \cap W} \in V + V' \cap W$.

Finally, from the 1st Isomorphism Theorem applied to Φ, the Zassenhaus Lemma holds. $\square$

Now we prove uniqueness.

Proof. (Uniqueness)
Assume we have two composition series

$$\{0\} = V_0 \subset V_1 \subset V_2 \subset ... \subset V_{n-1} \subset V_n = V$$
$$\{0\} = W_0 \subset W_1 \subset W_2 \subset ... \subset W_{n-1} \subset W_m = V,$$

and respective composition factors $\{V_{i+1}/V_i\}_{0\leq i\leq n-1}$, $\{W_{j+1}/V_j\}_{0\leq j\leq n-1}$. We use the second sequence to insert steps between the successive subrepresentations of the first sequence, and vice versa. That is, for $0 \leq i \leq n-1$, let

$$V_{ij} = V_i + (V_{i+1} \cap W_j) \quad 0 \leq j \leq m$$

and for $0 \leq j \leq m-1$, let

$$W_{ji} = W_j + (W_{j+1} \cap V_i) \quad 0 \leq i \leq n.$$

Then we have for each i,

$$V_i = V_{i0} \subseteq V_{i1} \subseteq V_{i2} \subseteq ... \subseteq V_{i,m-1} \subseteq V_{im} = V_{i+1}$$

and for each j,

$$W_j = W_{j0} \subseteq W_{j1} \subseteq W_{i2} \subseteq ... \subseteq W_{i,n-1} \subseteq W_{in} = W_{j+1}.$$

By construction, each V_{ij} and W_{ji} is a subrepresentation of V. Now for each $i, 0 \leq i \leq n-1$, there is no proper intermediate subrepresentation between V_i and V_{i+1}

(because if this were not the case, i.e. a proper subrepresentation exists, this would contradict the irreducibility of V_{i+1}/V_i by the 4th Isomorphism Theorem). Thus there exists a *unique* index j (where $0 \leq j \leq m-1$),the 'cut-off' point, for which $V_{ij} = V_i$ and $V_{i,j+1} = V_{i+1}$.

Now, for each i, we have

$$\begin{aligned} V_{i+1}/V_i &= V_{i,j+1}/V_{ij} \\ &= (V_i + (V_{i+1} \cap W_{j+1}))/(V_i + (V_{i+1} \cap W_j)) \\ &\cong (W_j + (V_{i+1} \cap W_{j+1}))/(W_j + (V_i \cap W_{j+1})) \\ &= W_{j,i+1}/W_{ji} \\ &= W_{j+1}/W_j, \end{aligned}$$

where Zassenhaus' Lemma was used, and where j is unique for each i.
But applying the same reasoning to the second sequence instead, we obtain for each j,

$$W_{j+1}/W_j = V_{i+1}/V_i,$$

where i is unique for each j. This establishes a bijection

$$\begin{gathered} \{V_{i+1}/V_i\}_{0 \leq i \leq n-1} \longleftrightarrow \{W_{j+1}/V_j\}_{0 \leq j \leq m-1} \\ i \longleftrightarrow j, \end{gathered}$$

and in particular,

$$\begin{gathered} \{0, 1, ..., n-1\} \longleftrightarrow \{0, 1, ..., m-1\} \\ i \longleftrightarrow j, \end{gathered}$$

thus completing the proof. □

Corollary 3.3. Let ρ be a semisimple representation: $\rho \cong \rho_1 \oplus \rho_2 \oplus ... \oplus \rho_n$ with all ρ_i's irreducible. (Remember that ρ_i is shorthand for $\rho|_{V_i}$, the restriction of ρ to the corresponding G-invariant subspace V_i). Then a composition series (E_i) is given by

$$E_0 = 0 \qquad E_i \cong \rho_1 \oplus ... \oplus \rho_i \quad 1 \leq i \leq n,$$

with $V_i/V_{i-1} \cong \rho_i$.

If for another set of irreducible subrepresentations we have $\rho \cong \rho'_1 \oplus \rho'_2 \oplus ... \oplus \rho'_n$, then due to the isomorphism between composition factors, we have a bijection between $\{\rho_i\}$ and $\{\rho'_j\}$. **Thus the irreducible summands of a semisimple representation are unique up to isomorphism**.

Proposition 3.4. Let $\rho : G \longrightarrow GL(V)$ be a (nonzero) representation of G, and $\tau : G \longrightarrow GL(W)$ an irreducible representation of G. If

$$\mathrm{Hom}_G(\rho, \tau) \neq 0 \qquad \text{or} \qquad \mathrm{Hom}_G(\tau, \rho) \neq 0,$$

then τ is among the Jordan-Hölder-Noether composition factors of ρ.

Proof. ($\mathrm{Hom}_G(\rho, \tau) \neq 0$) By assumption, we have a nonzero intertwiner $\rho \xrightarrow{\Phi} \tau$. Since $\Phi(V)$ is a non-zero subrepresentation of τ, which is irreducible, it must be W itself, i.e. Φ is surjective. Thus $\ker(\Phi)$ is a proper subrepresentation of V. By the 1st Isomorphism Theorem, we have the following isomorphism of representations:

$$V/\ker(\Phi) \cong \tau.$$

Now consider a composition series for $\ker(\Phi)$, say

$$0 = V_0 \subset V_1 \subset ... \subset V_{n-1} = \ker(\Phi)$$

and defining $V_n = V$, we obtain a composition series for ρ, in which τ is one of the composition factors.

($\mathrm{Hom}_G(\tau, \rho) \neq 0$) The proof for this case is similar to the one above. By assumption, we have a nonzero intertwiner $\tau \xrightarrow{\Phi} \rho$. Since $\ker(\Phi)$ is a proper subrepresentation of τ, which is irreducible, it must be 0, i.e. Φ is injective. Thus $W \cong \Phi(W) \subseteq V$, so $\tau \cong \rho|_{\Phi(W)}$. Since τ is irreducible, so is $\Phi(W)$.

Now consider a composition series for $V/\Phi(W)$, say

$$0 = \Phi(W)/\Phi(W) \subset V_1/\Phi(W) \subset ... \subset V_{n-1}/\Phi(W) \subset V/\Phi(W) \quad (*)$$

By the 4th Isomorphism Theorem, we have a corresponding sequence of representations:

$$0 \subset \Phi(W) \subset V_1 \subset ... \subset V_{n-1} \subset V,$$

which we proceed to show *is* a composition series.

Pick an arbitrary quotient V_i/V_{i-1}. By the 3rd Isomorphism Theorem, this is isomorphic to its counterpart in (*), i.e.

$$V_i/V_{i-1} \cong (V_i/\Phi(W))/(V_{i-1}/\Phi(W)).$$

But because (*) is a composition series, $(V_i/\Phi(W))/(V_{i-1}/\Phi(W))$ is irreducible, and so is V_i/V_{i-1}.

Finally, $\tau \cong \Phi(W) \cong \Phi(W)/\{0\}$ is a composition factor of ρ. $\square$

Remark.

1. In the first case, $\ker(\Phi)$ itself could be 0, in which case we are immediately done, since $\tau \cong V/\{0\}$ is a composition factor.

2. In the second case, $V/\Phi(W)$ itself could be 0, or equivalently $\Phi(W) = V$, in which case we are immediately done, since we have the two-term composition series $0 \subseteq V = \Phi(W)$, and $\tau \cong \Phi(W) = V \cong V/\{0\}$ is a composition factor.

Remark. If $\rho : G \longrightarrow GL(V)$ is semisimple, it is natural to ask what its irreducible summands look like. We clarify here that while we know the isomorphism classes of the summands and their multiplicities (Corollary 3.3), the actual summands themselves are not determined unless we further impose some constraints.

Indeed, this is already evident from the example where G is a trivial group. In this case, any representation is semisimple, with one-dimensional summands. The summands are determined by a choice of basis, which are of course not unique.

3.2 A Result on Isotypic Components

Theorem 3.5. Let $\rho : G \longrightarrow GL(V)$ be a semisimple representation of G.

1. Consider an irreducible representation π of G. Let $V = \bigoplus_{i=1}^{n} V_i$ be a decomposition of V into irreducible subrepresentations. The subrepresentation of V defined by

$$\bigoplus_{V_i \cong \pi} V_i \subseteq V,$$

   where i runs over all indices such that the subrepresentation V_i is isomorphic to π, is independent of the decomposition. Indeed, it is equal to the internal direct sum of all subrepresentations of V isomorphic to π.

   $\bigoplus_{V_i \cong \pi} V_i$, which is the largest subrepresentation of ρ which is π-isotypic, is called the **π-isotypic component** of ρ, and is denoted by $M(\pi)$.

2. In particular, if all the irreducible summands ρ_i of ρ occur with multiplicity 1 (so each ρ_i is the ρ_i-isotypic component of ρ), then the <u>irreducible</u> <u>summands of ρ are unique</u> (not simply unique up to isomorphism). That is,

$$V = \bigoplus_{i=1}^{n} V_i$$

is the only decomposition of ρ into irreducible components. In particular, any subrepresentation of ρ is of the form

$$\bigoplus_{i \in S} V_i$$

where $S \subseteq \{1, 2, ..., n\}$.

Note that in general this is not true, as we mentioned in Remark 3.1.

3. Suppose we have semisimple representations $\rho : G \longrightarrow GL(V)$ and $\tau : G \longrightarrow GL(W)$, and an intertwiner $\rho \xrightarrow{\Phi} \tau$. Then

$$\Phi(M_\rho(\pi)) \subseteq M_\tau(\pi)$$

Proof.

1. First let us denote by $M(\pi)$ the internal direct sum of all subrepresentations of V isomorphic to π. This is a well-defined subspace of V, and it is clear that for the fixed decomposition $V = \bigoplus_{i=1}^n V_i$, we have $\bigoplus_{V_i \cong \pi} V_i \subseteq M(\pi)$. It remains to show that the converse inclusion holds, i.e. $M(\pi) \subseteq \bigoplus_{V_i \cong \pi} V_i$.

   Pick $v \in M(\pi)$. By the definition of $M(\pi)$, $v = f_1 + f_2 + ... + f_n$, where for each i, $f_i \in F_i \cong \pi$. Now from the decomposition, consider those V_j's where $V_j \ncong \pi$. Pick an arbitrary F_i. The projection map

$$p_j : F_i \longrightarrow V_j$$

   is in $\mathrm{Hom}_G(F_i, V_j)$, so it must be the zero map by Schur's Lemma. That is, for any element $f_i \in F_i$, its component along V_j is zero, for all j. This precisely means that $f_i \in \bigoplus_{V_i \cong \pi} V_i$, and since i was arbitrary, this holds for all i. Thus by the vector space closure of $\bigoplus_{V_i \cong \pi} V_i$, $v = f_1 + f_2 + ... + f_n \in \bigoplus_{V_i \cong \pi} V_i$. That is, $M(\pi) \subseteq \bigoplus_{V_i \cong \pi} V_i$.

2. Suppose $V = \bigoplus_{i=1}^{n} W_i$ is another decomposition of ρ into irreducible summands. Since we must have

$$\bigoplus_{W_i \cong V_i} W_i = V_i,$$

and since V_i is irreducible, there must be only one term in the sum, i.e. W_j for some j - because if there are more than one term, this contradicts the irreducibility of V_i. Thus there is a bijection via identification, between the sets $\{V_i\}$ and $\{W_j\}$.

Finally, given any subrepresentation ρ' of ρ, we know that ρ' is also semisimple, and that its own irreducible subrepresentations must be irreducible subrepresentations of ρ, i.e. they must come from the set $\{V_i\}$.

3. Let $V = \bigoplus_{i=1}^{n} V_i$ be a a decomposition of V into irreducible summands. If $M_\rho(\pi) = \{0\}$, i.e. no component of ρ is isomorphic to π, then we are immediately done. So assume $M_\rho(\pi) \neq \{0\}$. Also, label the components such that if k of them are isomorphic to π, then they are labelled $V_1, ..., V_k$.

Now pick $v \in M_\rho(\pi)$. By the definition of $M_\rho(\pi)$, $v = v_1 + v_2 + ... + v_k$, where for each i, $v_i \in V_i \cong \pi$. $\Phi(v) = \Phi(v_1) + \Phi(v_2) + ... + \Phi(v_k)$, and $\Phi(v_i) \in \Phi(V_i)$.

Now pick an arbitrary $i \in \{1, 2, ...k\}$. Consider the restriction of Φ to V_i. $\Phi|_{V_i} : V_i \longrightarrow W$ is still an intertwiner. From now on, We shall for the sake of convenience abuse notation and simply denote $\Phi|_{V_i}$ by Φ. We claim that if Φ is not the zero map, then it is injective. But this is clear from Schur's Lemma I (Lemma 2.13) applied to $\pi = V_i$ and $\rho = \tau$.

Next, collect all those $\Phi|_{V_i}$ which are not zero maps. We thus have $V_i \cong \Phi(V_i)$, so $\Phi(V_i) \cong \pi$, i.e. $\Phi(V_i) \subseteq M_\tau(\pi)$. In other words, $\Phi(v_i) \in M_\tau(\pi)$. For those $\Phi|_{V_j}$'s which are zero maps, $\Phi(v_j)$ is simply 0. It is now clear that $\Phi(v) \in M_\tau(\pi)$.

Hence, $\Phi(M_\rho(\pi)) \subseteq M_\tau(\pi)$.

□

3.3 Schur's Lemma

Theorem 3.6 (Schur's Lemma). Let K be an algebraically closed field, and π_1 and π_2 be irreducible K-representations of G.
If π_1 and π_2 are not isomorphic, then $\mathrm{Hom}_G(\pi_1, \pi_2) = 0$, or equivalently,

$$\dim(\mathrm{Hom}_G(\pi_1, \pi_2)) = 0.$$

If π_1 and π_2 are isomorphic, then

$$\dim(\mathrm{Hom}_G(\pi_1, \pi_2)) = 1$$

and in particular, if $\pi_1 = \pi_2 = \pi$, then $\mathrm{Hom}_G(\pi, \pi) = \mathrm{Hom}_K(\pi, \pi)^G = K \cdot I_\pi$. In a concise form, we write

$$\dim(\mathrm{Hom}_G(\pi_1, \pi_2) = \delta_{\pi_1 \pi_2} = \begin{cases} 1 & \text{if } \pi_1 \cong \pi_2 \\ 0 & \text{otherwise} \end{cases}$$

Note that the algebraic closure of K was only used in the isomorphic case. In the non-isomorphic case, Schur's Lemma I does not require algebraic closure.

Proof.

1. The part when π_1 and π_2 are not isomorphic was already dealt with in Schur's Lemma I (Lemma 2.13).

2. Consider the special case first, where $\pi_1 = \pi_2 = \pi$. Let Φ be an intertwiner. Then because K is algebraically closed, it has an eigenvalue, $\lambda \in K$. By the definition of an eigenvalue, the map $\lambda I - \Phi$ is not invertible, which is equivalent to saying it is not an isomorphism. Now clearly $I \in \mathrm{Hom}_G(\pi, \pi)$. Because

$\mathrm{Hom}_G(\pi, \pi)$ is a vector space, we must have $\lambda I - \Phi \in \mathrm{Hom}_G(\pi, \pi)$ as well. From Schur's Lemma I (Lemma 2.13 (2)), $\lambda I - \Phi$ is the zero map, i.e. $\Phi = \lambda I$. We have shown that $\mathrm{Hom}_G(\pi, \pi) \subseteq K \cdot I_\pi$. Clearly the reverse inclusion holds, so $\mathrm{Hom}_G(\pi, \pi) = K \cdot I_\pi$.

3. In the general case where π_1 and π_2 are isomorphic, let Φ denote their intertwiner, i.e. $\Phi\pi_1(g) = \pi_2(g)\Phi$ for all $g \in G$. Let $T \in \mathrm{Hom}_G(\pi_1, \pi_2)$. $T\pi_1(g) = \pi_2(g)T$, so we have $(\Phi^{-1}T)\pi_1(g) = \pi_1(g)(\Phi^{-1}T)$, i.e. $\Phi^{-1}T \in \mathrm{Hom}_G(\pi_1, \pi_1)$. Thus, $\Phi^{-1}T = \lambda I_{\pi_1}$ (from the proof in part 2.), i.e. $T = \lambda\Phi$, for some $\lambda \in K$.

   This shows that $\mathrm{Hom}_G(\pi_1, \pi_2) = \mathrm{Span}\{\Phi\}$, i.e. $\dim(\mathrm{Hom}_G(\pi_1, \pi_2)) = 1$.

□

Proposition 3.7. If $\pi : G \longrightarrow GL(V)$ is a semisimple representation of G such that $\dim(\mathrm{Hom}_G(\pi, \pi)) = 1$, then π is irreducible.

Proof. Let W be an irreducible subrepresentation of V and W' its complementary subrepresentation, such that $V = W \oplus W'$. Consider the projection $\Phi : V \longrightarrow V$ onto W with kernel W'. $\Phi \in \mathrm{Hom}_G(\pi, \pi)$, since W' is a subrepresentation (c.f. Lemma 2.5).

By the assumption that $\dim(\mathrm{Hom}_G(\pi, \pi)) = 1$, Φ λI for some nonzero λ. Now pick an arbitrary $v \in V$, which will be of the form $v = w + w'$. $\Phi(v) = w + 0$ by definition, and $\Phi(v) = \lambda w + \lambda w'$ by what we just proved. By the property of direct sums, we must have $\lambda = 1$ and $w' = 0$, i.e. $\Phi = I$. Hence $V = W$ is irreducible, completing the proof. □

Proposition 3.8. (Multiplicities) Let K be an algebraically closed field, ρ a semisimple K-representation of G and π an irreducible K-representation of G. We have

$$n_\pi(\rho) = \dim(\mathrm{Hom}_G(\pi, \rho)) = \dim(\mathrm{Hom}_G(\rho, \pi)).$$

Proof. We first perform the decomposition of ρ into irreducible subrepresentations, i.e.

$$\rho \cong \bigoplus_{i=1}^{n} \rho_i.$$

From Proposition 2.3, we have

$$\mathrm{Hom}_G(\pi, \rho) = \mathrm{Hom}_G(\pi, \bigoplus_{i=1}^{n} \rho_i) \cong \bigoplus_{i=1}^{n} \mathrm{Hom}_G(\pi, \rho_i)$$

The space on the RHS has dimension equal to the number of indices for which $\rho_i \cong \pi$, by Schur's Lemma. Specifically we make use of

$$\dim(\mathrm{Hom}_G(\pi_1, \pi_2)) = \delta_{\pi_1\pi_2} = \begin{cases} 1 & \text{if } \pi_1 \cong \pi_2 \\ 0 & \text{otherwise.} \end{cases}$$

The result is of course $n_\pi(\rho)$. An identical argument applies to $\mathrm{Hom}_G(\rho, \pi)$. $\square$

3.4 Burnside's Irreducibility Criterion and Consequences

3.4.1 Burnside's Irreducibility Criterion

Theorem 3.9. Let K be an algebraically closed field and let $\rho : G \longrightarrow GL(V)$ be a representation. The following statements are equivalent:

1. ρ is irreducible.

2. $\rho(G)$ spans Ω. (where $\Omega := \mathrm{End}_K(V) = \mathrm{Hom}_K(V, V)$)

3. $R = \{ \lambda \in \Omega^* \mid < \lambda, \rho(g) >= 0 \text{ for all } g \in G \}$ is equal to $\{0\}$.

The proof we give is that of Burnside's original argument, expressed in modern mathematical jargon. But first, let us establish a few lemmas.

Lemma 3.10. If ρ is reducible, then $\rho(G)$ does not span Ω.

Proof. By definition, if ρ is reducible, then there exists a proper, nonzero subrepresentation ρ'. Consider the basis for V comprising the basis vectors for V' extended to V, i.e.

$$B = \{\underbrace{e_1, e_2, ..., e_k}_{basis\, for\, V'}, e_{k+1}, ..., e_n\}$$

In this basis, for every $g \in G$, we have the matrix representation of $\rho(g)$ looking like

$$[\rho(g)]_B = \left[\begin{array}{c|c} A & B \\ \hline O & C \end{array}\right]$$

where of course A, B, C varies with g. Note that A is exactly $[\rho'(g)]_B$. Since the set of all such matrices cannot span the matrix space associated with Ω, $\rho(G)$ does not span Ω. $\square$

Lemma 3.11. Let U be a finite-dimensional vector space and S be a subset of U. Define

$$R(S) = \{\, \lambda \in U^* \mid \lambda(s) = 0 \text{ for all } s \in S\}$$

Then S spans U if and only if $R(S) = \{0\}$.

Proof.

1. ($\Longrightarrow$) It is clear that $\{0\} \subseteq R(S)$. We show the reverse inclusion. Pick an arbitrary $\lambda \in R(S)$. Let B be a basis for U and choose an arbitrary element $b \in B$. Since S spans U by assumption,

   $$b = c_1 s_1 + ... = c_m s_m$$

   for some $s_1, ..., s_n \in S$. Then

   $$\lambda(b) = c_1\lambda(s_1) + c_2\lambda(s_2) + c_m\lambda(s_m) = 0.$$

   Since λ is 0 on a basis of U, by linearity it is the zero map on U. Since λ was arbitrary, $R(S) \subseteq \{0\}$.

2. Now we assume $R(S) = 0$. Let $\text{Span}\{S\} = W$ be a subspace of U. Suppose for a contradiction that $W \neq U$. Let $\{u_1, ...u_k\}$ be a basis for W and extend it to a basis $\{u_1, u_2, ..., u_k, u_{k+1}, ..., u_n\}$ for U. Let $\{u_1^*, u_2^*, ..., u_k^*, u_{k+1}^*, ..., u_n^*\}$ be the corresponding dual basis for U^*. It is direct from the definition of a dual basis that
$$u_{k+1}^*(u_i) = 0 \text{ for all } 1 \leq i \leq k.$$
Extending by linearity, it follows that $u_{k+1}^*(w) = 0$ for all $w \in W$, and in particular, $u_{k+1}^*(s) = 0$ for all $s \in S$, since $S \subseteq W$. Thus $u_{k+1}^* \in R(S)$ - but u_{k+1}^* is not a zero map, contradicting $R(S) = 0$.

□

Lemma 3.12. Let $\rho : G \longrightarrow GL(V)$ be irreducible. Let G act on $\Omega = \text{End}_K(V)$ in the sense of Theorem 2.20, i.e.

$$g \cdot T = \rho(g) \circ T$$

for all $T \in \Omega$. Then, let G act on Ω^* by the dual of the action above, i.e. for any $\lambda \in \Omega^*$ and $T \in \Omega$,

$$(g \cdot \lambda)(T) = \lambda(g^{-1} \cdot T) = \lambda(\rho(g^{-1}) \circ T)$$

or even more concisely,

$$< g \cdot \lambda, T >=< \lambda, g^{-1} \cdot T >=< \lambda, \rho(g^{-1}) \circ T > .$$

Then under this action, $R \subseteq \Omega^*$ is a subrepresentation of Ω^*.

Proof. Pick an arbitrary $g \in G$ and $\phi \in R$. We have to show that $g \cdot \phi \in R$. Now for any $h \in G$,

$$\begin{aligned} < g \cdot \phi, \rho(h) > &=< \phi, g^{-1} \cdot \rho(h) > \\ &=< \phi, \rho(g^{-1}h) >= 0, \end{aligned}$$

so by definition of R, $g \cdot \phi \in R$. $\therefore$ R is a subrepresentation of Ω^*. $\square$

Lemma 3.13. There is a vector space isomorphism given by

$$\begin{aligned} \alpha : V &\longrightarrow \mathrm{Hom}_G(V^*, \Omega^*) \\ v &\longmapsto \alpha_v \end{aligned}$$

where α_v is defined by

$$< \alpha_v(\lambda), T > = < \lambda, T(v) >$$

for $\lambda \in V^*$ and $T \in \Omega$.

Proof. The proof that each α_v is indeed in $\mathrm{Hom}_G(V^*, \Omega^*)$ and that the map $(\alpha : v \mapsto \alpha_v)$ is linear is simply routine symbol manipulation, so we shall omit it.

It remains to show that α is bijective. We do so by showing α is injective and that $\dim(V) = \dim(\mathrm{Hom}_G(V^*, \Omega^*))$.

1. (Injectivity) Suppose $v \in \ker(\alpha)$. By definition, $\alpha(v) = \alpha_v = 0$ is the zero map from V^* to Ω^*. This in turn implies that for any $\lambda \in V^*$, $\alpha_v(\lambda) = 0$ is the zero map on Ω^*, i.e. $< \alpha_v(\lambda), T >=< \lambda, Tv >= 0$ for all $T \in V$. In particular, when $T = I_V$, we have $< \lambda, v >= 0$, where remember that λ is arbitrary. Thus $v = 0$, and $\ker(\alpha) = 0$.

2. (Equal Dimensions)

$$\begin{aligned} \dim(\mathrm{Hom}_G(V^*, \Omega^*)) &= \dim(\mathrm{Hom}_G(V^*, V^{*\oplus \dim(V)})) \\ &= \dim(V) \cdot \dim(\mathrm{Hom}_G(V^*, V^*)) = \dim(V) \end{aligned}$$

   where Schur's Lemma was used in the last equality. Note that Schur's Lemma is applicable because K is algebraically closed here.

$\square$

Now we begin the proof of Burnside's Irreducibility Criterion, which by now has mostly become a matter of collating the above few lemmas.

Proof.

1. ($2 \iff 3$) This is Lemma 3.11. applied to $S = \rho(G)$ and $U = \Omega$.

2. ($2 \implies 1$) This is the contrapositive of Lemma 3.10.

3. ($1 \implies 3$) It remains to show this. Assume ρ is irreducible. For the sake of contradiction, assume $R \neq \{0\}$.

   First, Ω with the action described in Lemma 3.12 is isomorphic to a direct sum of $\dim(V)$ copies of ρ by Lemma 2.21. That is,

$$\Omega \cong \underbrace{\rho \oplus \ldots \oplus \rho}_{\dim(V) \text{ times}}$$

   By Proposition 2.18 (1),(2),

$$\Omega^* \cong (\underbrace{\rho \oplus \ldots \oplus \rho}_{\dim(V) \text{ times}})^* \cong \underbrace{\rho^* \oplus \ldots \oplus \rho^*}_{\dim(V) \text{ times}}$$

   where we note that ρ^* is also irreducible by Theorem 2.19. At this point, we have established the semisimplicity of both Ω and Ω^*.

   By assumption, $R \neq 0$, so it contains an irreducible subrepresentation which we denote by π. In particular, π is an irreducible subrepresentation of Ω^*, since R is a subrepresentation of Ω^*.

   By Corollary 3.3, the irreducible summands of a semisimple representation are unique up to isomorphism, so $\pi \cong \rho^*$. Let us call their intertwiner θ. That is,

$$\theta : V^* \longrightarrow \pi \subseteq R \subseteq \Omega^*.$$

   In particular, θ can be regarded as intertwining V^* and Ω^*, i.e. $\theta \in \mathrm{Hom}_G(V^*, \Omega^*)$.

Now, recall the vector space isomorphism from Lemma 3.13. Thus, $\theta = \alpha_v$ for some v which is nonzero (otherwise $\theta = \alpha_0$ will be the zero map, and that is a contradiction). Thus we have $\theta(V^*) = \alpha_v(V^*) \subseteq R$. But this means for all $\lambda \in V^*$

$$< \alpha_v(\lambda), \rho(g) >=< \lambda, \rho(g)v >= 0$$

for all $g \in G$. In particular, considering $g = 1$, we have for all λ, $< \lambda, v >= 0$, and this is none other than saying that $v = 0$, which is a contradiction to what we asserted just a few lines above.

$\therefore$ Our assumption that $R \neq 0$ is false. $R = 0$, and we have completed the proof.

$\square$

3.4.2 Matrix Coefficients and their Properties

Given a representation ρ. Fix a basis $B = (v_i)$ of V and and consider the matrix representation of ρ in the given basis,

$$\rho^B : g \longrightarrow GL_{\dim(V)}(K).$$

For each entry of $\rho^B(g)$, we can define a function mapping g to that entry. More precisely, we define $\dim(V)^2$ functions ρ^B_{ij} such that

$$\begin{aligned} \rho^B_{ij} : G &\longrightarrow K \\ g &\longmapsto < \lambda_i, \rho(g)v_j > \end{aligned}$$

where (λ_i) are the dual basis vectors corresponding to (v_i).

Corollary 3.14. Let K be algebraically closed and ρ a K-representation. If ρ is irreducible, and we pick any basis B for its representation space, then the functions ρ^B_{ij} are linearly independent as elements in $C_K(G)$.

Proof. Let the following linear combination equal zero: $\sum_{ij} \alpha_{ij}\rho^B_{ij}$. Define a map ϕ in Ω^*:

$$< \phi, T >= \sum_{ij} \alpha_{ij} < \lambda_i, T(v_j) > .$$

We note that $< \phi, \rho(g) >$ is precisely $\sum_{ij} \alpha_{ij}\rho^B_{ij}(g)$. But by assumption if $\sum_{ij} \alpha_{ij}\rho^B_{ij}$ is the zero map, then on any g, $\sum_{ij} \alpha_{ij}\rho^B_{ij}(g) = 0$, i.e. $< \phi, \rho(g) >= 0$. In other words, $\phi \in R$ where R is the relation space as defined when we discussed Burnside's Irreducibility Criterion. But by this very criterion, since ρ is irreducible, R must be the trivial space $\{0\}$. Thus ϕ is the zero map, and the α_{ij}'s are all zero, implying linear independence. □

This motivates us to make the following definition:

Definition. (Matrix Coefficient) Let G be a group, K a field and $\rho : G \longrightarrow GL(V)$ a representation of G. A **matrix coefficient** of ρ is any function on G of the type

$$\begin{aligned} f_{v\lambda} : G &\longrightarrow K \\ g &\longmapsto \lambda(\rho(g)v) =< \lambda, \rho(g)v > \end{aligned}$$

for some fixed $v \in V$ and $\lambda \in V^*$.

We have just shown that for an irreducible representation, the matrix coefficients corresponding to a fixed basis (v_i) and its dual basis (λ_i) are linearly independent. In fact, more can be said. The matrix coefficients for non-isomorphic (irreducible) representations are linearly independent as well, as stated in the theorem below.

Theorem 3.15. (Linear Independence of Matrix Coefficients)

1. For any finite collection (ρ_i) of pairwise non-isomorphic, irreducible representations of G, with representation spaces V_i, let

$$\rho = \bigoplus_i \rho_i \quad \text{act on} \quad V = \bigoplus_i V_i.$$

Then the span of the elements $\rho(g)$ in $\mathrm{End}_K(V)$ is equal to

$$\bigoplus_i \mathrm{End}_K(V_i).$$

2. The matrix coefficients of irreducible representations of G are linearly independent in the following sense. For any finite collection (ρ_i) of pairwise non-isomorphic, irreducible representations of G, with representation spaces V_i, and for any choice $(v_{ij})_{1\leq j\leq \mathrm{dim}V_i}$ of bases of V_i and the corresponding dual bases (λ_{ij}), the family of functions

$$\{f_{v_{ij}\lambda_{ik}}\}_{ijk}$$

on G are linearly independent as elements of the vector space $C_K(G)$ of K-valued functions on G.

Note that there are

$$\sum_i (\mathrm{dim}V_i)^2$$

functions in this family, given by

$$G \longrightarrow K$$

$$g \longmapsto < \lambda_{ik}, \rho_i(g)v_{ij} >_{V_i}$$

for $1 \leq j, k \leq \mathrm{dim}V_i$.

Proof.

1. Denote $\mathrm{End}_K(V)$ by Ω, and $\mathrm{Hom}_K(V_j, V_i)$ by Ω_{ij}. First, for notational simplicity, we are going to make an identification between Ω_{ij} and the subspace of Ω comprising all maps $T : V \longrightarrow V$ that map the summand V_j to V_i and all other summands V_l, $l \neq j$ to 0. This identification was actually implicit in the statement of the theorem. With this identification, we can write

$$\Omega = \bigoplus_{i,j} \Omega_{ij}$$

and taking the dual gives a dual counterpart

$$\Omega^* = \bigoplus_{i,j} \Omega_{ij}^*.$$

For an illustration of how the direct sum works, consider the simple case where we just have two summands in V: V_1 and V_2. A general map is given by

$$T : V_1 \oplus V_2 \longrightarrow V_1 \oplus V_2$$
$$(v_1, v_2) \longmapsto (w_1, w_2).$$

Denote by T_1 the maps that maps (v_1, v_2) to w_1, i.e. the first component of $T(v_1, v_2)$ and similarly for T_2. T_1 and T_2 are the component functions of T, so to speak.

Now for T_1 define $T_{11} : V_1 \times \{0\} \longrightarrow V$ and $T_{12} : V_1 \times \{0\} \longrightarrow V$ where $T_1(v_1, 0) = (T_{11}(v_1, 0), T_{12}(v_1, 0))$. Similarly, for T_2 define $T_{21} : \{0\} \times V_2 \longrightarrow V$ and $T_{22} : \{0\} \times V_2 \longrightarrow V$ where $T_2(0, v_2) = (T_{21}(0, v_2), T_{22}(0, v_2))$. From this, a general $T \in \Omega$ can be written as $T = T_{11} + T_{12} + T_{21} + T_{22}$.

Now define

$$R = \{ \ \phi \in \Omega^* \mid \ < \phi, \rho(g) >= 0 \quad \forall g \in G \ \}$$

and

$$S = \{ \ \phi \in \Omega^* \mid \ < \phi, T >= 0 \quad \forall T \in \bigoplus_i \mathrm{End}_K(V_i) \ \}.$$

By duality, if $R = S$, then $\rho(G)$ spans $\bigoplus_i \mathrm{End}_K(V_i)$. Indeed, $R = S$. We omit the proof as it is a bit involved, but we shall outline the general idea. We cite ([1], pg 103-104) for reference.

Outline: We are able to get the decomposition

$$\Omega^* \cong \bigoplus_i (\dim \rho_i^*),$$

i.e. V^* is semisimple and is decomposed into a direct sum of ρ_i^*'s. Since $R, S \subseteq \Omega^*$, they are semisimple and thus have irreducible components among $\{\rho_i^*\}$ as well, by Corollary 3.3. It is shown that these irreducible components of R and S are exactly the same.

2. Now we use part 1 of the theorem to prove part 2. But since $\rho(G)$ spans $\bigoplus_i \mathrm{End}_K(V_i)$, the proof is exactly the same as that for Corollary 3.14.

$\square$

Theorem 3.16. (ρ-isotypic component of $C_K(G)$ spanned by the matrix coefficients of ρ) Let G be a group, K an algebraically closed field and ρ an irreducible K-representation of G. Let $M(\rho)$ (the notation to be justified immediately below) be the subspace of $C_K(G)$ spanned by all the matrix coefficients $f_{v\lambda}$ of ρ.

1. The space $M(\rho)$ is depends on ρ up to isomorphism, i.e. if $\rho \cong \tau$, then $M(\rho) = M(\tau)$.

2. $M(\rho)$ is a subrepresentation of the regular representation of G acting on $C_K(G)$; moreover, $M(\rho)$ is semisimple and isomorphic to $\rho^{\oplus \dim \rho}$.

3. $M(\rho)$ is the ρ-isotypic component of $C_K(G)$ (thus justifying the notation), i.e. any subrepresentation of $C_K(G)$ isomorphic to ρ is contained in $M(\rho)$.

Proof.

1. Let $\rho : G \longrightarrow GL(V)$ and $\tau : G \longrightarrow GL(W)$ by isomorphic representations. Let $T : V \longrightarrow W$ be their intertwiner. Then for any $w \in W$ and $\lambda \in W^*$, writing

$w = T(v)$ for some $v \in V$, we have

$$\begin{aligned} f_{w\lambda}(g) &=< \lambda, \tau(g)w >_W \\ &=< \lambda, \tau(g)T(v) >_W \\ &=< \lambda, T\rho(g)v >_W \\ &=< T^*(\lambda), \rho(g)v >_V \\ &= f_{v,T^*(\lambda)}(g). \end{aligned}$$

That is, every matrix coefficient for τ is also one for ρ. By symmetry, every matrix coefficient for ρ is also one for τ. Since $M(\rho), M(\tau)$ are by definition spanned by matrix coeffients, they are equal for both ρ and τ, i.e. $M(\rho) = M(\tau)$.

2. We first show that $M(\rho) \subseteq C_K(G)$ is a subrepresentation of the regular representation: for $v \in V, \lambda \in V^*$ and $g \in G$, we have

$$\begin{aligned} \rho_G(g)f_{v\lambda}(x) &= f_{v\lambda}(xg) \\ &=< \lambda, \rho(xg)v > \\ &= f_{\rho(g)v,\lambda}(x) \in M(\rho). \end{aligned}$$

In fact, from this formula, we notice that if we define the linear map

$$\begin{aligned} T_\lambda : V &\longrightarrow M(\rho) \\ v &\longmapsto f_{v\lambda}, \end{aligned}$$

we have

$$T_\lambda(\rho(g)v) = f_{\rho(g)v,\lambda}(x) = \rho_G(g)f_{v\lambda}(x) = \rho_G(g)T_\lambda(v).$$

That is, T_λ intertwines ρ and $M(\rho)$. Now choose a basis (v_i) for V and let (λ_i) denote its dual basis. By construction $M(\rho) = \text{Span}\,\{f_{v\lambda}\} = \text{Span}\,\{f_{v_i\lambda_j}\}$, where $1 \leq i, j \leq \dim(\rho)$. Furthermore, as we have shown that the matrix coefficients corresponding to any chosen basis and its dual basis are linearly independent (c.f. Theorem 3.15 (2)), these functions thus form a basis for $M(\rho)$. In particular, $\dim M(\rho) = \dim(\rho)^2$.

Now consider the intertwiner

$$T = \bigoplus_i T_{\lambda_i} : \bigoplus_i V \longrightarrow M(\rho).$$

T is surjective (because every element of $M(\rho)$ is spanned by $\{f_{v_i\lambda_j}\}$, and these can be 'traced back' to $\bigoplus_i V$. Surjectivity, plus the clear equality of dimensions of $\bigoplus_i V$ and $M(\rho)$, implies that T is an isomorphism intertwining $M(\rho)$ and $\rho^{\oplus \dim \rho}$.

3. Let $E \subseteq C_K(G)$ be a subrepresentation of the regular representation that is isomorphic to ρ. To show $E \subseteq M(\rho)$, we show that all the elements f of E are in fact matrix coefficients of ρ, so that $f \in M(\rho)$.

   Let $\delta \in C_K(G)^*$ be the linear form defined by

$$\delta(f) = f(1)$$

   for any $f \in C_K(G)$. Then consider its restriction to E^*, $\delta|_E \in E^*$. Now for any $f \in E$ and $x \in G$, we have

$$< \delta_E, \rho_G(x)f >= \rho_G(x)f(1) = f(x),$$

   where the last equality comes from the definition of a regular representation. Thus we can interpret f as a matrix coefficient of ρ_G, and since $\rho_G|_E \cong \rho$, we have $f \in M(\rho)$, by part 1 of this very theorem.

□

In the next section, we shall derive further consequences of the linear independence of matrix coefficients, and build up character theory.

4 Character Theory of Finite Group Representations

4.1 Characters

Definition. The **character** of the representation ρ is the function $\chi : G \longrightarrow K$ defined by:

$$\chi(g) = \operatorname{Tr} \rho(g)$$

A character is called **irreducible** if its associated representation is irreducible.

Remark. Note that characters are non-zero maps, since $\chi(1) = \operatorname{Tr} \rho(1) = \dim \rho$.

Proposition 4.1. Let ρ_1 and ρ_2 be representations.

1. $\rho_1 \cong \rho_2 \Longrightarrow \chi_{\rho_1} = \chi_{\rho_2}$.
2. Let K be algebraically closed, ρ_1 and ρ_2 be irreducible representations. $\rho_1 \cong \rho_2 \Longleftrightarrow \chi_{\rho_1} = \chi_{\rho_2}$.
3. The characters of irreducible, pairwise non-isomorphic representations are linearly independent in $C_K(G)$.
4. Let K be algebraically closed and of characteristic zero, ρ_1 and ρ_2 be semisimple representations. $\rho_1 \cong \rho_2 \Longleftrightarrow \chi_{\rho_1} = \chi_{\rho_2}$.

Proof.

1. By definition, we have $\rho_2(g) = \Phi\rho_1(g)\Phi^{-1}$ for all $g \in G$. The trace function is invariant under conjugation of its argument, so

$$\chi_{\rho_1}(g) = \operatorname{Tr} \rho_1(g) = \operatorname{Tr} \rho_2(g) = \chi_{\rho_2}(g)$$

for all $g \in G$, thus the characters are equal (as functions).

2. The forward assertion is part 1. To prove the converse, we note that

$$\operatorname{Tr}\rho(g) = \sum_i < \lambda_i, \rho(g)v_i > .$$

That is, $\chi_\rho = \sum_i f_{v_i\lambda_i}$ as functions on G. Having an equality

$$\chi_{\rho_1} = \chi_{\rho_2}$$

is equivalent to having

$$\sum_i f_{v_{1i}\lambda_{1i}} = \sum_j f_{v_{2j}\lambda_{2j}},$$

i.e. a linear relation between certain matrix coefficients of ρ_1 and ρ_2. By Theorem 3.15 (2), if ρ_1 and ρ_2 are irreducible but not isomorphic, then the matrix coefficients should be linearly independent, contradicting the above. Thus $\rho_1 \cong \rho_2$.

3. Given $\sum_{\pi=1}^{n} \alpha_\pi \chi_\pi = 0$ (where π labels the isomorphism classes of the representations), expanding each χ_π in terms of diagonal matrix coefficients and using Theorem 3.15 (2) shows that each $\alpha_\pi = 0_K$.

4. The forward assertion is part 1. To prove the converse, first express both semisimple representations in their irreducible decompositions, i.e.

$$\rho_1 \cong \bigoplus_\pi n_\pi(\rho_1)\pi$$
$$\rho_2 \cong \bigoplus_\pi n_\pi(\rho_2)\pi$$

where again, π labels the isomorphism classes, and $n_\pi(\rho)\pi$ is shorthand for

$\pi^{\oplus n_\pi(\rho)}$. Taking the traces for both representations gives us[1]

$$\chi_{\rho_1} = \sum_\pi n_\pi(\rho_1)\chi_\pi$$
$$\chi_{\rho_2} = \sum_\pi n_\pi(\rho_2)\chi_\pi.$$

By assumption, $\chi_{\rho_1} = \chi_{\rho_2}$, i.e.

$$\sum_\pi n_\pi(\rho_1)\chi_\pi = \sum_\pi n_\pi(\rho_2)\chi_\pi.$$

From part 3 of this very proposition, we have that $n_\pi(\rho_1) = n_\pi(\rho_2)$, but as values of K (c.f. the footnote), i.e.

$$n_\pi(\rho_1) \cdot 1_K = n_\pi(\rho_2) \cdot 1_K.$$

This is where we make use of the zero-characteristic of K - because K having characteristic zero implies that $n_\pi(\rho_1) = n_\pi(\rho_2)$ *as integers*, i.e.

$$\rho_1 \cong \bigoplus_\pi n_\pi(\rho_1)\pi = \bigoplus_\pi n_\pi(\rho_2)\pi \cong \rho_2.$$

□

Example. (Character of the Regular Representation and Permutation Representation) Let K be any field, G a finite group, so that the regular representation ρ_G (with representation space $C_K(G)$) is finite-dimensional, the dimension given by $|G|$ (c.f. Examples 2.1.1, 2.1.2). Its character is given by

$$\chi(g) = \begin{cases} |G| & \text{if } g = 1 \\ 0 & \text{if } g \neq 1. \end{cases}$$

[1]Note that here for the characters, $n_\pi(\rho)$ is no longer an integer. It is in fact shorthand for $n_\pi(\rho) \cdot 1_K = \underbrace{1_K + 1_K + \ldots + 1_K}_{n_\pi(\rho) \text{ times}} \in K$. We shall make this abuse of notation throughout this section.

Indeed, consider the canonical basis of $C_K(G)$ comprising the functions $\{\delta_x\}_{x\in G}$ where

$$\delta_x = \begin{cases} 1 & \text{if } g = x \\ 0 & \text{if } g \neq x. \end{cases}$$

Then

$$\rho_G(g)\delta_x = \delta_{xg^{-1}}$$

for all $g, x \in G$. This means that $\rho_G(g)$ acts on the basis vectors by permuting them, and the corresponding matrix representations are permutation matrices (we recall here that permutation matrices have entries either 1 or 0). The trace of $\rho_G(g)$ is thus the number of fixed points of the map, or equivalently, the number of 1's on the diagonal of its matrix representation.

But $\rho_G(g)\delta_x = \delta_x$ if and only if $g = 1$, and this holds for all $x \in G$. For the other g's, every single δ_x gets 'rotated out of place', so to speak. Thus the character is given by the above expression.

From Example 2.1.2, the permutation representation π_G is isomorphic to ρ_G, thus this result holds as well.

Next, we give the expressions for the characters of representations constructed from the existing ones, e.g. direct sums, tensor products etc. The results are fairly evident, so we omit the proof.

Proposition 4.2. Let G be a group, K any field and ρ a K-representation of G.

1. $\chi_\rho(gxg^{-1}) = \chi_\rho(x)$, or equivalently, $\chi_\rho(gx) = \chi_\rho(xg)$ for all $g, x \in G$.

2. $\chi_\rho(1) = \dim \rho$

3. $\chi_{\rho_1 \oplus \rho_2} = \chi_{\rho_1} + \chi_{\rho_2}$

4. $\chi_{\rho_1 \otimes \rho_2} = \chi_{\rho_1}\chi_{\rho_2}$

5. $\chi_{\rho^*}(g) = \chi_\rho(g^{-1})$

Before moving on to Maschke's Theorem, we introduce a special kind of function which appears frequently in the subsequent sections.

Definition. A function $f : G \longrightarrow K$ is called a **class function** if it is constant on the conjugacy classes of G, i.e.

$$f(g) = f(hgh^{-1}) \qquad \text{for all } g, h \in G.$$

Note that characters are class functions.

Proposition 4.3. The subset of all class functions in $C_K(G)$ form a vector space, which we denote as $c_K(G)$.

Proof. This is straightforward. Let $f_1, f_2 \in C_K(G)$ and $c \in K$. Then

$$(cf_1 + f_2)(hgh^{-1}) = cf_1(hgh^{-1}) + f_2(hgh^{-1}) = cf_1(g) + f_2(g) = (cf_1 + f_2)(g)$$

□

4.2 Maschke's Theorem and Applications

From this subsection onward, we shall consider only finite groups.

Theorem 4.4 (**Maschke's Theorem**)**.** Let G be a finite group, and K be a field of characteristic 0. Then any representation

$$\rho : G \longrightarrow GL(V)$$

is semisimple.

Thus, in some sense, when Maschke's Theorem applies, the classification of the representations of G reduces to classifying the irreducible ones.

Proof. This proof makes use of Lemma 2.5. We establish complete reducibility, which is equivalent to semisimplicity.

Pick a subrepresentation $W \subseteq V$. This subrepresentation has a complementary subspace, which we denote by W', i.e. $V = W \oplus W'$. The idea is to construct an intertwining projector $P : V \longrightarrow V$ where $P(V) = W$ and $\ker(P) = W'$. If we have this, then Lemma 2.5 states that W' itself is a subrepresentation of V.

From linear algebra, there is a projector $P_0 \in \mathrm{End}_K(V)$ where $P(V) = W$. But P_0 is not necessarily an intertwiner, i.e. it is not necessarily in $\mathrm{Hom}_G(\rho, \rho)$.

Now recall that $\mathrm{Hom}_G(\rho, \rho) = \mathrm{End}_K(V)^G$, where the action on $\mathrm{End}_K(V)$ is the action on hom-spaces, defined in Theorem 2.20. Thus we want a projector P such that $P \in \mathrm{End}_K(V)^G$. The trick to construct such a P is to 'average out' P_0.

Thus, we let

$$P = \frac{1}{|G|} \sum_{g \in G} g \cdot P_0$$

where as a reminder, $g \cdot P_0 = \rho(g) P_0 \rho(g^{-1})$. One thing we should immediately take note of is the numbers 1 and $|G|$ here are elements of K, which is why we require K to have characteristic 0 for division by $|G|$ in K to be defined (in fact, this would work as long as K is a field of characteristic not dividing $|G|$).

It is indeed the case that $P \in \mathrm{End}_K(V)^G$: for any $h \in G$,

$$\begin{aligned} h \cdot P = \frac{1}{|G|} \sum_{g \in G} h \cdot (g \cdot P_0) &= \frac{1}{|G|} \sum_{g \in G} (hg \cdot P_0) \\ &= \frac{1}{|G|} \sum_{x \in G} x \cdot P_0 \\ &= P \end{aligned}$$

It remains to show that $P(V) = W$ and that P is indeed a projector, i.e. $P^2 = P$.

1. ($P(V) \subseteq W$) Since W is a subrepresentation, each summand of P has the

property that for any $v \in V$,

$$(g \cdot P_0)v = \rho(g)(P_0(\rho(g^{-1})v)) \in W,$$

so $P(v) \in W$.

2. ($W \subseteq P(V)$) Pick an arbitrary $w \in W$. We have

$$\begin{aligned} P(w) &= \frac{1}{|G|} \sum_{g \in G} \rho(g) P_0(\rho(g^{-1})w) \\ &= \frac{1}{|G|} \sum_{g \in G} \rho(gg^{-1})w = w. \end{aligned}$$

That is, $w = P(w) \in P(W) \subseteq P(V)$.

3. ($P^2 = P$) Pick an arbitrary $v \in V$. $P^2(v) = P(P(v)) = P(v)$, since $P(v) \in W$.

Hence we are done. □

At this point, we introduce the notation $\hat{G}$ **for the set of representatives of isomorphism classes of irreducible representations of** G. We will show that $\hat{G}$ can be identified with the set of characters of irreducible representations of G.

Applying Maschke's Theorem to the regular representation ρ_G on $C_K(G)$, we deduce from Theorem 3.16 the following corollary.

Corollary 4.5. (Decomposition of the regular representation) Let G be a finite group, and K be an algebraically closed field of characteristic 0. Then

$$\rho_G \cong \bigoplus_{\rho \in \hat{G}} \rho^{\oplus \dim \rho}.$$

In particular, we have

$$\sum_{\rho \in \hat{G}} (\dim \rho)^2 = |G|.$$

Thus we have obtained an upper bound on the number of isomorphism classes of irreducible representations of G, namely $|G|$ (in the case that all the irreducible representations have dimension 1).

Proof. As usual, to prove statements on semisimplicity we prove complete reducibility instead. We first assert that $C_K(G) = \bigoplus_{\rho \in \hat{G}} M(\rho)$ as vector spaces, where here '$\bigoplus$' is the internal direct sum.

Suppose this were not the case, i.e. the equality does not hold. Since each $M(\rho)$ is a subspace (in fact, a subrepresentation!) of $C_K(G)$ (c.f. Theorem 3.16 (2)), we have $\bigoplus_{\rho \in \hat{G}} M(\rho) \subset C_K(G)$. Since by Maschke's Theorem ρ_G is now semisimple and thus completely reducible, we squeeze in a subrepresentation $\pi \neq 0$ such that $C_K(G) = (\bigoplus_{\rho \in \hat{G}} M(\rho)) \oplus \pi$.

But π itself is semisimple too, since subrepresentations of semisimple representations are themselves semisimple. Thus

$$\pi \cong \pi_1 \oplus \pi_2 \oplus \ldots \oplus \pi_k$$

where the π's belong to $\hat{G}$. We now have, for some isomorphism classes ρ, a situation where $M(\rho)$ is in direct sum with a subrepresentation of $C_K(G)$ isomorphic to ρ, contradicting Theorem 3.16 (3). Hence $C_K(G) = \bigoplus_{\rho \in \hat{G}} M(\rho)$, and because each $M(\rho)$ is isomorphic to $\rho^{\oplus \dim \rho}$, our assertion that

$$\rho_G \cong \bigoplus_{\rho \in \hat{G}} \rho^{\oplus \dim \rho}$$

holds. To prove $\sum_{\rho \in \hat{G}} (\dim \rho)^2 = |G|$, simply take the dimensions on both sides of the equation. □

4.3 Decomposing Representations

In this subsection, K shall denote an algebraically closed field of characteristic 0, unless explicitly stated otherwise.

4.3.1 Orthogonality of Matrix Coefficients and Characters

Proposition 4.6. (Projection onto Invariant Subspace) Let G be a finite group, and K be an algebraically closed field of characteristic 0. Then for any representation $\rho : G \longrightarrow GL(V)$, the map

$$P = \frac{1}{|G|}\sum_{g\in G}\rho(g) : V \longrightarrow V$$

is a projection onto V^G, and is an intertwiner, i.e. $P \in \mathrm{Hom}_G(\rho, \rho)$. Moreover,

$$\frac{1}{|G|}\sum_{g\in G}\chi_\rho(g) = \dim(V^G)\cdot 1_K.$$

Proof. The proof is similar to that in Maschke's Theorem. Pick any $h \in G$.

$$\begin{aligned}(\rho(h)P)v = \rho(h)(\frac{1}{|G|}\sum_{g\in G}\rho(g)v) &= \frac{1}{|G|}\sum_{g\in G}\rho(hg)(v)\\ &= \frac{1}{|G|}\sum_{x\in G}\rho(x)\\ &= \frac{1}{|G|}\sum_{g\in G}\rho(gh)(v)\\ &= (\frac{1}{|G|}\sum_{g\in G}\rho(g)\rho(h)v) = (P(\rho(h))v\end{aligned}$$

This shows not just that $P(V) \subseteq V^G$, but also the intertwining property of P. Next pick an arbitrary $v \in V^G$. Since P acts as an identity map on V^G, immediately $v = P(v) \in P(V^G) \subseteq P(V)$, so $V^G \subseteq P(V)$.

Finally, the trace of a projection is equal to the dimension of its image. By choosing the obvious set of basis vectors, its matrix representation looks like

$$[P] = \left[\begin{array}{c|c} I & O \\ \hline O & O \end{array}\right].$$

□

Consider G and K as before. Let $\pi_1 : G \longrightarrow GL(V_1)$ and $\pi_2 : G \longrightarrow GL(V_2)$ be irreducible representations. By Schur's Lemma the invariant space (under the natural action of G on $\Omega_{12} = \mathrm{Hom}_K(V_1, V_2)$) Ω_{12}^G has dimension 0 or 1 depending on whether π_1 and π_2 are isomorphic. Let us investigate both cases.

We begin by considering the case when π_1 and π_2 are not isomorphic.

Lemma 4.7. For any $\Phi \in \Omega_{12}$, we have $\frac{1}{|G|}\sum_{g\in G} g \cdot \Phi = 0$.

Proof. For any $h \in G$, $h \cdot (\frac{1}{|G|}\sum_{g\in G} g \cdot \Phi) = \frac{1}{|G|}\sum_{g\in G} h \cdot (g \cdot \Phi) = \frac{1}{|G|}\sum_{x\in G} x \cdot \Phi$, i.e. $\frac{1}{|G|}\sum_{g\in G} g \cdot \Phi \in \Omega_{12}^G$. But since $\Omega_{12}^G = \{0\}$, the map must be the zero map. □

Proposition 4.8. (Orthogonality) Let K be algebraically closed and of characteristic 0, and π_1, π_2 be non-isomorphic irreducible K-representations of G. Then for any $v \in V_1, w \in V_2, \lambda \in V_1^*$ and $\mu \in V_2^*$, we have

$$\frac{1}{|G|}\sum_{g\in G} f_{v\lambda}(g) f_{w\mu}(g^{-1}) = \frac{1}{|G|}\sum_{g\in G} < \lambda, g \cdot v >_{V_1} < \mu, g^{-1} \cdot w >_{V_2} = 0$$

Proof. Consider a rank-1 map $\Phi : V_1 \longrightarrow V_2$ of the form:

$$\Phi : v \longmapsto < \lambda, v > w$$

where $w \in V_2, \lambda \in V_1^*$ are fixed. From Lemma 4.7, $\frac{1}{|G|}\sum_{g\in G} g \cdot \Phi = 0 \Longrightarrow$

$$\begin{aligned} 0 &= \frac{1}{|G|}\sum_{g\in G} \pi_2(g)(< \lambda, \pi_1(g^{-1})v > w) \\ &= \frac{1}{|G|}\sum_{g\in G} f_{v\lambda}(g)\pi_2(g^{-1})w \in V_2 \end{aligned}$$

for all v (note that in the last equality, we have replaced g by g^{-1}, but this doesn't matter since we sum over all $g \in G$).

Finally, apply μ on both sides of the equation to give

$$\frac{1}{|G|}\sum_{g\in G} f_{v\lambda}(g) f_{w\mu}(g^{-1}) = \frac{1}{|G|}\sum_{g\in G} < \lambda, g \cdot v >_{V_1} < \mu, g^{-1} \cdot w >_{V_2} = 0.$$

□

These type of sums pop up frequently, so we make the following definition:

Definition. (Inner product on $C_K(G)$) Let G be a finite group and K be a field (need not be algebraically closed) of characteristic zero. For $\varphi_1, \varphi_2 \in C_K(G)$, we define their inner product in $C_K(G)$ to be

$$[\varphi_1, \varphi_2] = \frac{1}{|G|} \sum_{g \in G} \varphi_1(g)\varphi_2(g^{-1}).$$

This is a nondegenerate, symmetric inner product.

Thus using the language of inner products, Proposition 4.8 says that the matrix coefficients for non-isomorphic representations are orthogonal in $C_K(G)$, or, $[f_{v\lambda}, f_{w\mu}] = 0$.

Finally, since $\Omega_{12}^G = 0$, from Proposition 4.6 (the 'moreover' part), we have $\sum_{g \in G} \chi_{\Omega_{12}}(g)$. On the other hand, since $\Omega_{12} \cong V_1^* \otimes V_2$, from Proposition 4.1 (1), we have

$$\chi_{\Omega_{12}}(g) = \chi_{\pi_1}(g^{-1})\chi_{\pi_2}(g).$$

Thus we have the formula

$$[\chi_{\pi_1}, \chi_{\pi_2}] = \frac{1}{|G|} \sum_{g \in G} \chi_{\pi_1}(g)\chi_{\pi_2}(g^{-1}) = 0. \qquad (*)$$

Now we consider the case when π_1 and π_2 are isomorphic. But notice that since isomorphic representations have the same characters, as far as investigations of characters are concerned we can assume $\pi_1 = \pi_2$. So let us denote $\pi_1 = \pi_2 = \pi$ and $\Omega = \mathrm{End}_K(V)$.

First, as with (*), we have

$$[\chi_\pi, \chi_\pi] = \frac{1}{|G|} \sum_{g \in G} \chi_\pi(g)\chi_\pi(g^{-1}) = 1. \qquad (**)$$

The reasoning is entirely the same as that for (*), except now $\Omega^G = K \cdot I_\pi$ or equivalently, $\dim \Omega^G = 1$.

Next we consider Proposition 4.8 again for an irreducible representation.

Proposition 4.9. (Orthogonality v.2.) Let K be an algebraically closed field of characteristic 0, and π an irreducible K-representation of G. For any $v, w \in V$ and $\lambda, \mu \in V^*$, we have

$$[f_{v\lambda}, f_{w\mu}] = \frac{1}{|G|}\sum_{g\in G} < \lambda, g\cdot v >< \mu, g^{-1}\cdot w >= \frac{< \lambda, w >< \mu, v >}{\dim V}.$$

Proof. Consider the rank-1 map in Proposition 4.8 again, i.e. here we have

$$\Phi : V \longrightarrow V$$
$$v \longmapsto < \lambda, v > w.$$

where $\lambda \in V^*$ and $w \in V$ are fixed. By Schur's Lemma (this is why the algebraic closure of K is required here) we have $\Omega^G = K \cdot I_\pi$,

$$\frac{1}{|G|}\sum_{g\in G} g\cdot\Phi = \alpha \cdot I_V$$

for some $\alpha \in K$. To determine α, we compute the trace. Since

$$\mathrm{Tr}(g\cdot\Phi) = \mathrm{Tr}(\pi(g)\Phi\pi(g^{-1})) = \mathrm{Tr}(\Phi),$$

we get

$$(\dim V)\alpha = \mathrm{Tr}\,(\alpha \cdot I_\pi) = \mathrm{Tr}\,(\Phi).$$

But how do we compute $\mathrm{Tr}\,(\Phi)$? Simply consider a basis of V of which w is an element of (remember that w is given and fixed). Then by considering the matrix representation of Φ in this basis, we see that the only nonzero term on the diagonal is exactly $< \lambda, w >$ (where λ is also given and fixed). Thus,

$$\alpha = \frac{< \lambda, w >}{\dim V}.$$

Thus,

$$\frac{1}{|G|}\sum_{g\in G} g\cdot\Phi = \frac{< \lambda, w >}{\dim V}\cdot I_V.$$

Applying this to a vector $v \in V$ and applying further a linear form $\mu \in V^*$, we get the desired result. $\square$

At this point, let us collate the results $(*)$ and $(**)$ into a theorem for easy reference.

Theorem 4.10. (Orthogonality of Characters) Let ρ and π be irreducible representations of G. We have

$$[\chi_\rho, \chi_\pi] = \frac{1}{|G|}\sum_{g \in G} \chi_\rho(g)\chi_\pi(g^{-1}) = \begin{cases} 1 & \text{if } \rho \cong \pi \\ 0 & \text{if } \rho \ncong \pi. \end{cases}$$

As an application of what we have developed so far, we are going to state and prove a major theorem. But first, a lemma.

Proposition 4.11. Let X be a finite set with an action of G and let π_G be the associated permutation representation. We have

$$\dim K(X)^G = |\{\ \mathcal{O}_x \mid x \in X\ \}|$$

where $\mathcal{O}_x$ denotes the orbit of x under the action of G.

Proof. Recall that the basis vectors for $K(X)$ are the embedding of X into $K(X)$, i.e. $\{e_x\}_{x \in X}$. Denote the element $\sum_{g \in G} e_{g \cdot x}$ by f_x. Note that for x's lying in the same orbit, their corresponding f_x's are equal, and the number of distinct f_x's are precisely the number of orbits of X. We show that $\{f_x\}_{x \in X}$ forms a basis for $K(X)^G$, and this immediately implies the theorem.

1. ($f_x \in K(X)^G$) This is clear: For any $h \in G$, $h \cdot f_x = h \cdot \sum_{g \in G} e_{g \cdot x} = \sum_{g \in G} e_{h \cdot (g \cdot x)} = f_x$.

2. (Linear Independence) Given $\sum_x a_x f_x = 0$. Simply expand f_x in terms of its e_x-components, and note that each x belongs to a unique orbit. Then because e_x itself is linearly independent, all a_x's must be zero.

3. (Span) Pick an arbitrary $v \in K(X)^G$. By definition, $v = \sum_x a_x e_x$ has the property that $g \cdot v = v$ for all $g \in G$. In other words, $\sum_x a_x e_{g\cdot x} = \sum_x a_{g^{-1}\cdot x} e_x = \sum_x a_x e_x$. That is, for all g, $a_{g^{-1}\cdot x} = a_x$ or equivalently; for all g, $a_{g\cdot x} = a_x$. Then collecting the x's in the same orbit, one can make the notational simplification $v = \sum_x a_x f_x$.

□

Corollary 4.12. Recall the space of class functions, $c_K(G) \subseteq C_K(G)$. Claim:

$$\dim c_K(G) = |\{\text{conjugacy classes of G}\}|.$$

Proof. Recall from Example 2.1.1 the right and left regular representations, where for the right regular representation, we have

$$(g \cdot f)(x) = f(xg)$$

and for the left regular representation we have

$$(g \cdot f)(x) = f(g^{-1}x).$$

Here we define a 'mixed' representation:

$$(g \cdot f)(x) = f(g^{-1}xg).$$

One can easily show that this is a representation (simply repeat the proofs for the right and left regular representations), and that this representation is isomorphic to the permutation representation π_G (because both the right regular and left regular representations are also isomorphic to the permutation representation).

Now that we have $c_K(G) \cong K(X)$, we also have $c_K(G)^G \cong K(X)^G$. But in our special case, $X = G$, and the action of G on G is simply conjugation by $g \in G$. And we know that conjugacy classes are simply orbits under the action of conjugation. Thus from the dimension result from Proposition 4.11, we are done. □

Theorem 4.13. (Number of Irreducible Characters) Let G be a finite group and K an algebraically closed field of characteristic 0. **The number $|\hat{G}|$ of isomorphism classes of irreducible representations of G is equal to the number of distinct characters of G is equal to the number of conjugacy classes in G.**

Proof. Recall from Proposition 4.1 (3) that the characters of irreducible (pairwise non-isomorphic) representations of G are linearly independent class functions in $C_K(G)$. The subspace they span in $C_K(G)$ is thus a subspace of $c_K(G)$, and the number of distinct characters is hence at most $\dim c_K(G)$. What we will claim now is that the characters actually form a basis of $c_K(G)$, i.e. they also span $c_K(G)$.

First pick an arbitrary $\varphi \in c_K(G) \subseteq C_K(G)$, i.e. φ is a class function. For each irreducible representation π, fix a basis (v_i^π) for the representation space V_π, and denote by (λ_j^π) its dual basis in V_π^*. Let

$$f_{ij}^\pi \in C_K(G) \qquad 1 \leq i, j \leq \dim(\pi)$$

denote the corresponding matrix coefficients. Theorem 3.16 and Corollary 4.5 tell us that

$$\varphi = \sum_\pi \sum_{ij} \alpha_{ij}^\pi f_{ij}^\pi$$

We now show that in fact, we can simply write φ as a linear combination of just the characters. This is where we make use of the orthogonality relations Proposition 4.8 and Proposition 4.9. These two propositions show that

$$[f_{ij}^\pi, f_{ij}^\rho] = \begin{cases} 0 & \text{if } \pi \neq \rho, \text{ or } (i,j) \neq (l,k) \\ \frac{1}{\dim(\rho)} & \text{if } \pi = \rho \quad \text{and} \quad (i,j) = (l,k). \end{cases}$$

Taking the inner product with some f^ρ_{kl} on both sides of $\varphi = \sum_\pi \sum_{ij} \alpha^\pi_{ij} f^\pi_{ij}$ gives us

$$\begin{aligned}
[\varphi, f^\rho_{kl}] &= \frac{1}{|G|} \sum_{g\in G} \varphi(g) f^\rho_{kl}(g^{-1}) \\
&= \sum_\pi \sum_{ij} \alpha^\pi_{ij} [f^\pi_{ij}, f^\rho_{ij}] \\
&= \frac{\alpha^\rho_{lk}}{\dim(\rho)}.
\end{aligned}$$

Remember that $f^\rho_{kl}(g)$ is the (l,k)-th entry of the matrix representation of $\rho(g)$ in the basis (v^ρ_l) (c.f. Subsection 3.4.2). Also note that $\varphi(g) \in K$ is just a scalar. Thus $[\varphi, f^\rho_{kl}]$ is simply the (l,k)-th entry of the matrix representing

$$A_\varphi = \frac{1}{|G|} \sum_{g\in G} \varphi(g)\rho(g^{-1}).$$

We show that $A_\varphi \in \mathrm{End}_K(\rho) = K \cdot I_\rho$ (by Schur's Lemma):

$$\begin{aligned}
A_\varphi(\rho(h)v) &= \frac{1}{|G|} \sum_{g\in G} \varphi(g)\rho(g^{-1}h)v \\
&= \frac{1}{|G|} \sum_{g\in G} \varphi(g)\rho(h(h^{-1}g^{-1}h))v \\
&= \frac{1}{|G|} \sum_{g\in G} \varphi(hgh^{-1})\rho(h)\rho(g^{-1})v \\
&= \rho(h)A_\varphi(v)
\end{aligned}$$

where in the third line, we rewrote hgh^{-1} for g, which again doesn't matter since we are summing over all $g \in G$; and in the last line we use the fact that φ is a class function.

Thus, $A_\varphi \in K \cdot I_\rho$ is a scalar matrix. Its off-diagonal entries are all 0 and the diagonal entries are identical, i.e.

$$[\varphi, f^\pi_{ji}] = \delta_{ji} c_\pi.$$

Substituting this into $\varphi = \sum_\pi \sum_{ij} \alpha_{ij}^\pi f_{ij}^\pi$ gives us

$$\begin{aligned}
\varphi &= \sum_\pi \sum_{ij} \alpha_{ij}^\pi f_{ij}^\pi \\
&= \sum_\pi \sum_{ij} [\varphi, f_{ji}^\pi] \cdot \dim(\pi) \cdot f_{ij}^\pi \\
&= \sum_\pi \sum_{ij} \delta_{ji} c_\pi \cdot \dim(\pi) \cdot f_{ij}^\pi \\
&= \sum_\pi \sum_{i} c_\pi \cdot \dim(\pi) \cdot f_{ii}^\pi \\
&= \sum_\pi \underbrace{c_\pi \cdot \dim(\pi)}_{\alpha_\pi} \chi_\pi .
\end{aligned}$$

Note that we have used $\chi_\pi = \sum_i f_{ii}^\pi$.
That is, we have shown that the characters actually spans $c_K(G)$, and thus is a basis for $c_K(G)$. From Corollary 4.12, $\dim c_K(G)$ is equal to the number of conjugacy classes of G. Therefore, thinking of the each character as tagging the isomorphism class it is associated with, we have

$$\begin{aligned}
\sharp|\hat{G}| = \sharp\text{distinct characters} &= \sharp\text{basis vectors for } c_K(G) \\
&= \dim c_K(G) \\
&= \sharp\text{conjugacy classes of } G,
\end{aligned}$$

finishing the proof.

□

Before discussing how character theory helps in deducing multiplicities of the irreducible summands of a representation, we state an important theorem; but without proof, for that requires some algebraic number theory.

Theorem 4.14. Let G be a finite group and K an algebraically closed field of characteristic 0. The dimension of any irreducible representation of G divides $|G|$.

4.3.2 Multiplicities

Lemma 4.15. (Multiplicity Formula) Let G be a finite group and K an algebraically closed field of characteristic 0. For any representation ρ and any irreducible representation $\pi$of G, the multiplicity of π in ρ is given by

$$n_\pi(\rho) = [\chi_\rho, \chi_\pi] = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g)\chi_\pi(g^{-1}).$$

Proof. Since ρ is semisimple by Maschke, its character is given by

$$\chi_\rho = \sum_{\tau \in \hat{G}} n_\tau(\rho)\chi_\tau.$$

Taking the inner product with χ_π and using the orthogonality of characters in Theorem 4.10, we have

$$\begin{aligned} [\chi_\rho, \chi_\pi] &= \sum_{\tau \in \hat{G}} n_\tau(\rho)[\chi_\tau, \chi_\pi] \\ &= \sum_{\tau \in \hat{G}} n_\tau(\rho)\delta_{\tau\pi} \\ &= n_\pi(\rho). \end{aligned}$$

This is actually an equality in K, since both LHS and RHS are elements of K. But since K is of characteristic 0, this becomes an equality of integers as well. □

Now we arrive at a major theorem, which gives a criterion for irreducibility for a representation in terms of its character.

Theorem 4.16. Let G be a finite group, K an algebraically closed field of characteristic 0 and ρ a representation. We have

$$[\chi_\rho, \chi_\rho] = \sum_{\pi \in \hat{G}} n_\pi(\rho)^2.$$

In particular, ρ is irreducible if and only if $[\chi_\rho, \chi_\rho] = 1$.

Proof. By linearity and orthogonality,

$$[\chi_\rho, \chi_\rho] = \sum_{\pi_1, \pi_2} n_{\pi_1}(\rho) n_{\pi_2}(\rho)[\chi_{\pi_1}, \chi_{\pi_2}] = \sum_{\pi} n_\pi(\rho)^2.$$

If this is equal to 1 (as in equality of integers), the only possibility is that one of the multiplicities $n_\pi(\rho)$ is equal to 1 and all the others are 0, i.e. $\rho \cong \pi$ is irreducible. The converse is clear. □

This section may seem disorganized, with various definitions and assertions appearing in an unsystematic manner. We give a short summary of the work done, outlining the main milestones while noting that many of the results are stepping stones toward these goals.

4.4 A Brief Summary

First note that for most of this section on character theory we consider only finite groups. Maschke's Theorem asserts that any representation of a finite group (over a field of characteristic 0) is semisimple. Thus to investigate and classify a (finite) group's representations, one need only look at the irreducible representations.

Then we introduced the concept of a character, and the inner product of characters, leading to a notion of orthogonality between characters. It turns out that for any two characters, their inner product is either 0 or 1 depending on whether their associated representations are isomorphic or not.

After that, we showed that the number of isomorphism classes of irreducible representations of G (which is equal to the number of distinct characters of G) is actually equal to the number of conjugacy classes in G, thus reducing a problem in representation theory to one in group theory. This result is only possible because we showed that the characters could be used to 'tag' the isomorphism classes of representations.

Finally we showed that a representation is irreducible if and only if the 'norm' of its character is 1. Along the way, we stated without proof that the dimension of an irreducible representation of G must divide $|G|$. The list of results stated greatly helps in analyzing, characterizing and classifying the representations of a finite group over 'nice' fields, in particular the field of complex numbers.

5 Conclusion

Thus we finish the report. The next stage would be to study a formulation of representation theory based on the group algebra, and after that to move on to general groups and other algebraic objects. In particular, the Representation Theory of Lie Groups and Lie Algebras plays an enormous role in many branches of mathematics, most notably differential geometry. The nice thing is, a few ideas/results from the character theory of finite group representations serves as a blueprint when it comes to characterizing these infinite groups, provided we impose further restrictions (namely, a certain notion of compactness).

We hope that this report has satisfied its purpose. I would also like to express my utmost gratitude to Prof. Lee for giving me the opportunity to pursue what started as a mathematical curiosity in my math methods class, for the consultation sessions, and for the general conversations.

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