Caps on all-pay auction with stochastic abilities

Yu Zhou[1,2,3]

[1] *Institution: Wuhan University, Wuhan, China*
[2] *E-mail address: econyz1216@gmail.com*
[3] *Website: kelvinzhou.github.io/home/*

Abstract

We study an all-pay auction in a private and independent information setting in which n bidders bid for an indivisible prize. Each bidder's cost is a linear function of his bid and ability. Bids are bound by a common bid cap. We shows that, a bid cap lowers the bids of high-ability bidders and increases the bids of memium-ability bidders. The expected total bids increase the bid cap. As a result, the organizer prefers not to set a bid cap if he wants to maximize is expected revenue.

Contents

1 The model

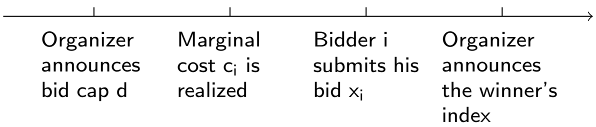
We consider n bidders complete for an indivisible prize. The set of bidders $\{1, \ldots, n\}$ is denoted by N. The value of prize is normalized to 1. Bidders simutaneously excert their effort(bid) $0 \leq x_i \leq d$, where $d \in (0, +\infty)$ is a common known bid cap. And the prize is given to only one bidder with the highest bid.(Ties are broken randomly).

Bidder i bears a marginal cost c_i, which is private information to i. All bidders other than i perceive c_i as a random selection out of a support $[\underline{c}, \overline{c}] \in (0, \infty)$, governed by the cumulative distribution function F, and independent of others' marginal costs. We assume that F is continuous differentiable, and we denote the associated probability density function by f. We also assume that $f(c) > 0$ for all $c \in [\underline{c}, \overline{c}]$.

We regard the marginal cost c_i as a measure of bidder's ability, because a lower c_i means a lower cost when the same effort is excerted. The higher marginal cost a bidder bears, the lower ability he has.

The organizer announces the bid cap d, before c_i is realized. Nature then determines bidders' ability profiles $\mathbf{c} = (c_1, c_2, \ldots, c_n)$. And bidders simutaneously submit their effort entries $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ after their abilities realized. The timeline of this game is figured out below:

FIGURE 1: Timeline



We denote m to be the number of bidders submitting the highest effort in $\mathbf{x}$, and let $w_i = \mathbb{1}\{x_i \geq x_j, \forall j \neq i\}$ indicates whether his effort is highest($w_i = 1$) or not ($w_i = 0$). Then the realized payoff to bidder i is given by

$$u(c_i, \mathbf{x}) = \frac{w_i}{m} - c_i \cdot x_i \tag{1}$$

And the expected payoff to bidder i when he makes decision is given by

$$EV(c_i, x_i) = E\{\frac{w_i}{m} - c_i \cdot x_i \mid c_i, x_i\} \tag{2}$$

We denote a symmetric bidding strategy as $\beta(c_i, d)$
$(\forall i \in N)$, where

$$\begin{aligned} \beta : [\underline{c}, \overline{c}] \times \mathbf{R}_+ &\to \mathbf{R}_+ \\ (c_i, d) &\to x_i \end{aligned} \tag{3}$$

2 Characterization of equilibria

We first consider the case with redundant cap where bid cap is too large to have actrual constraint on any bidders. A classic incomplete-information all-pay auction without bid cap arises.

Lemma 1. Consider an incomplete-information all-pay auction without bid cap, there exists an unique symmetric equilibrium in which bidding strategy for bidder i is

$$\widetilde{\beta}(c_i) = \int_{c_i}^{\overline{c}} \frac{1}{y}(n-1)(1-F(y))^{n-2} f(y)\,\mathrm{d}y \tag{4}$$

and the expected revenue for organizer is

$$\widetilde{ER} == n \int_{\underline{c}}^{\overline{c}} \frac{1}{y}(n-1)(1-F(y))^{n-2} f(y) F(y)\,\mathrm{d}y \tag{5}$$

and the expected payoff for bidder i is

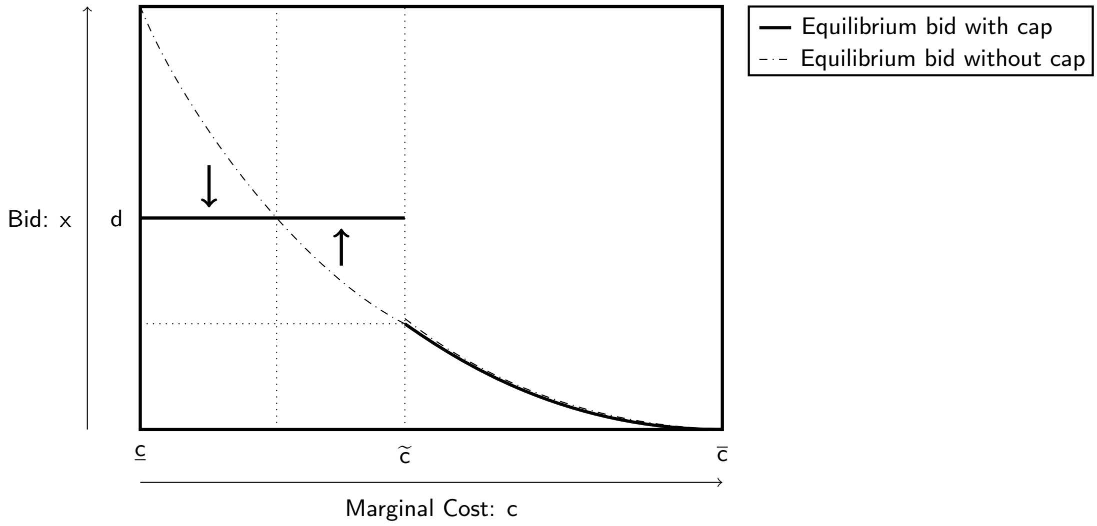
$$\widetilde{EV}(c_i) = (1-F(c_i))^{n-1} \quad - c_i \int_{c_i}^{\overline{c}} \frac{1}{y}(n-1)(1-F(y))^{n-2} f(y)\,\mathrm{d}y \tag{6}$$

Proof. See the Appendix. □

Proposition 1. Consider an all-pay auction with a bid cap $d \geq \int_{\underline{c}}^{\overline{c}} \frac{1}{y}(n-1)(1-F(y))^{n-2} f(y)\,\mathrm{d}y$. Then the bid cap is redundant, and there exists an unique symmetric equilibrium where bidding strategy is given by

$$\begin{aligned} \beta(c_i, d) &= \widetilde{\beta}(c_i) \\ &= \int_{c_i}^{\overline{c}} \frac{1}{y}(n-1)(1-F(y))^{n-2} f(y)\,\mathrm{d}y \end{aligned} \tag{7}$$

FIGURE 2: Equilibrium Bid with respective to Marginal Cost



and the ex ante expected revenue for organizer is given by

$$\begin{aligned} ER(d) &= \widetilde{ER} \\ &= n\int_{\underline{c}}^{\overline{c}} \frac{1}{y}(n-1)(1-F(y))^{n-2}f(y)F(y)\,\mathrm{d}y \end{aligned} \tag{8}$$

and expected payoff for bidder i is

$$\begin{aligned} EV(c_i,d) &= \widetilde{EV}(c_i) \\ &= (1-F(c_i))^{n-1} - c_i\int_{c_i}^{\overline{c}} \frac{1}{y}(n-1)(1-F(y))^{n-2}f(y)\,\mathrm{d}y \end{aligned} \tag{9}$$

Proof.

$$\begin{aligned} \max_{c\in[\underline{c},\overline{c}]} \widetilde{\beta}(c) &= \widetilde{\beta}(\underline{c}) \\ &= \int_{\underline{c}}^{\overline{c}} \frac{1}{y}(n-1)(1-F(y))^{n-2}f(y)\,\mathrm{d}y \end{aligned}$$

Thus, if $d > \int_{\underline{c}}^{\overline{c}} \frac{1}{y}(n-1)(1-F(y))^{n-2}f(y)\,\mathrm{d}y$, then the bid cap is ineffective. According to lemma 1, the symmetric equilibrium is unique, and

$$\begin{aligned} &\beta(c,d) = \widetilde{\beta}(c) \\ &ER(d) = \widetilde{ER} \\ &EV(c,d) = \widetilde{EV}(c) \end{aligned}$$

□

We then consider the case with effective cap.

Proposition 2. Consider an all-pay auction with a bid cap $0 < d < \int_{\underline{c}}^{\overline{c}} \frac{1}{y}(n-1)(1-F(y))^{n-2}f(y)\,\mathrm{d}y$. Then the bid cap is effective, and there exists an unique symmetric monotone pure-strategy Nash equilibrium where bidding strategy is given by

$$\beta(c_i,d) = \begin{cases} d & \text{if}\, \underline{c} \le c_i < \widetilde{c} \\ \widetilde{\beta}(c_i) & \text{if}\, \widetilde{c} \le c_i \le \overline{c} \end{cases} \tag{10}$$

and the ex ante expected total effort is given by

$$(11) \qquad ER(d) = n[\int_{\widetilde{c}}^{\overline{c}} \frac{1}{y}(n-1)(1-F(y))^{n-2} f(y)F(y)\,\mathrm{d}y + F(\widetilde{c})(\frac{1-(1-F(\widetilde{c}))^n}{nF(\widetilde{c})\widetilde{c}} - \frac{(1-F(\widetilde{c}))^{n-1}}{\widetilde{c}})]$$

where the critical value $\widetilde{c} = \widetilde{c}(d)$ is strictly monotonic decreasing, and defined by

$$d = \int_{\widetilde{c}}^{\overline{c}} \frac{1}{y}(n-1)(1-F(y))^{n-2} f(y)\,\mathrm{d}y + \frac{1-(1-F(\widetilde{c}))^n - nF(\widetilde{c})(1-F(\widetilde{c}))^{n-1}}{nF(\widetilde{c})\widetilde{c}}$$

Proof. See the Appendix. □

Proposition 3. The expected revenue of organizer if an strictly increasing function of the bid cap d, which means organizer will never use a cap.

Proof. See the Apendix. □

Propositin 3 states that the organizer perfers no cap policy, regardless of the marginal cost distribution and the number of bidders. With a bid cap, some middle-ability-level bidders will perfer a higher bid since there is a upper bound to limit bids submitted by higher-ability bidders. However, this gain is relatively small for organizer to offset lose from decrease of bid submitted by higher-ability bidders.

Appendices

A Proofs of propositions

Proof of lemma 1. First, we suppose there exist some symmetric equilibrium bidding strategies, and we can deduce some properties implied by "equilibrium":

1. *Weakly decreasing* $\widetilde{\beta}(\cdot)$ is weakly decreasing in $[\underline{c}, \overline{c}]$.
2. *Atomless bid* There is no subset $E \subseteq [\underline{c}, \overline{c}]$ having positive probability measure according to F, such that $\forall c, c' \in E,\ \widetilde{\beta}(c) = \widetilde{\beta}(c')$.
3. *Interval bid* $\widetilde{\beta}([\underline{c}, \overline{c}])$ is an interval.

These three properties also impies:

4. *Strictly decreasing* $\widetilde{\beta}(\cdot)$ is strictly decreasing in $[\underline{c}, \overline{c}]$.
5. *Continuous* $\widetilde{\beta}(\cdot)$ is continuous in $[\underline{c}, \overline{c}]$.

What is more, there is only one $\widetilde{\beta}(\cdot)$ satisfying the above properties, so uniqueness has been proved. Next, we will figure out one special symmetric bidding strategy, and verify it to be the best response for each bidder.

Proof of weakly decreasing. Pick any $c,\ c' \in [\underline{c}, \overline{c}]$. Since $\widetilde{\beta}(\cdot)$ is the best reponse, the bidder who bears c as his marginal cost will never be better when he selects any effort than following $\widetilde{\beta}(c)$. This implies that he will get no more compensation if he select other's bidding strategy according to $\widetilde{\beta}$, which displayed by following relations:

$$(12) \qquad \begin{cases} \text{Prob}_{\widetilde{\beta}}(\text{win} \mid x = \widetilde{\beta}(c)) - c \cdot \widetilde{\beta}(c) \geq \text{Prob}_{\widetilde{\beta}}(\text{win} \mid x = \widetilde{\beta}c') - c \cdot \widetilde{\beta}(c') \\ \text{Prob}_{\widetilde{\beta}}(\text{win} \mid x = \widetilde{\beta}(c')) - c' \cdot \widetilde{\beta}(c') \geq \text{Prob}_{\widetilde{\beta}}(\text{win} \mid x = \widetilde{\beta}c) - c' \cdot \widetilde{\beta}(c) \end{cases}$$

We call equation(12) the incentive compatibility condition. It can be transformed to be

$$(c' - c)(\widetilde{\beta}(c) - \widetilde{\beta}(c')) \geq 0$$

If c' is larger than c, then $\widetilde{\beta}(c')$ must be no larger than $\widetilde{\beta}(c)$ to make the inquality hold, which means $\widetilde{\beta}(\cdot)$ is weakly decreasing. ■

Proof of atomless bid. Suppose there exists a subset $E \subseteq [\underline{c}, \overline{c}]$ satisfying Prob$(E) > 0$ and $\widetilde{\beta}(E) = \{\hat{x}\}$. If there is one bidder whose marginal cost is $c \in E$, he can set his effort to be $\hat{x} + \epsilon$ where ϵ is small enough such that Prob$_{\widetilde{\beta}}$(win $\mid x = \hat{x}) - c \cdot \hat{x} >$ Prob$_{\widetilde{\beta}}$(win $\mid x = \hat{x} + \epsilon) - c \cdot (\hat{x} + \epsilon)$.

This results from the function $\text{Prob}_{\widetilde{\beta}}(\text{win} \mid x)$ is discontinuous at $\hat{x}$. As a result, he prefers $\hat{x} + \epsilon$ to $\hat{x}$, which generating a contradiction. Atomless bid is proved. ∎

Proof of interval bid. Suppose $\widetilde{\beta}([\underline{c}, \overline{c}])$ is not an interval. Then there must exist a point $\hat{c} \in [\underline{c}, \overline{c}]$, such that $\lim_{c \to \hat{c}} \widetilde{\beta}(c) > \widetilde{\beta}(\hat{c})$ (limit exists since $\widetilde{\beta}(\cdot)$ is monotonic). However, the bidder who select $\lim_{c \to \hat{c}} \widetilde{\beta}(c) + \epsilon$ will auctually adjust his effort to $\widetilde{\beta}(\hat{c})$, since cost will decrease a lot while $\text{Prob}_{\widetilde{\beta}}(\text{win} \mid x$ will just change relatively small. So there is no such a point $\hat{c}$. That is to say $\widetilde{\beta}([\underline{c}, \overline{c}])$ is an interval. ∎

Proof of strictly decreasing. Suppose $\widetilde{\beta}(\cdot)$ is not strictly decreasing, then there must exist an interval $[a, b] \in [\underline{c}, \overline{c}]$ such that $\widetilde{\beta}([a, b]) = \hat{x}$. However, $\text{Prob}([a, b]) > 0$, which contradicts *atomless bid*. This implies $\widetilde{\beta}(\cdot)$ is strictly decreasing. ∎

Proof of continuous. Suppose $\widetilde{\beta}(\cdot)$ is discontinuous at $\hat{c}$, and $\lim_{c \to \hat{c}^-} \widetilde{\beta}(c) > \widetilde{\beta}(\hat{c})$ ($\lim_{c \to \hat{c}^+} \widetilde{\beta}(c) < \widetilde{\beta}(\hat{c})$ will be proved in the same way). Since $\widetilde{\beta}(\cdot)$ is strictly decreasing and is an interval, we can easily find a contradictory. So $\widetilde{\beta}(\cdot)$ is continuous. ∎

Proof of uniqueness Equation(12) can be transformed to be:

$$\begin{cases} \frac{\widetilde{\beta}(c) - \widetilde{\beta}(c')}{c - c'} \leq \frac{1}{c} \cdot \frac{\text{Prob}_{\widetilde{\beta}}(\text{win} \mid x = \widetilde{\beta}(c)) - \text{Prob}_{\widetilde{\beta}}(\text{win} \mid x = \widetilde{\beta}(c'))}{c - c'} \\ \frac{\widetilde{\beta}(c) - \widetilde{\beta}(c')}{c - c'} \geq \frac{1}{c'} \cdot \frac{\text{Prob}_{\widetilde{\beta}}(\text{win} \mid x = \widetilde{\beta}(c)) - \text{Prob}_{\widetilde{\beta}}(\text{win} \mid x = \widetilde{\beta}(c'))}{c - c'} \end{cases}$$

Since $\widetilde{\beta}(\cdot)$ is strictly decreasing, $\text{Prob}_{\widetilde{\beta}}(\text{win} \mid x = \widetilde{\beta}(c)) = (1 - F(c))^{n-1}$. Then we have:

$$\begin{cases} \frac{\widetilde{\beta}(c) - \widetilde{\beta}(c')}{c - c'} \leq \frac{1}{c} \cdot \frac{(1 - F(c))^{n-1} - (1 - F(c'))^{n-1}}{c - c'} \\ \frac{\widetilde{\beta}(c) - \widetilde{\beta}(c')}{c - c'} \geq \frac{1}{c'} \cdot \frac{(1 - F(c))^{n-1} - (1 - F(c'))^{n-1}}{c - c'} \end{cases}$$

Let $c' \to c$, we can get:

$$\widetilde{\beta}'(c) = -\frac{1}{c}(n-1)(1 - F(c))^{n-2} f(c) \tag{13}$$

What is more, cross-section condition must satisfy:

$$\widetilde{\beta}(\overline{c}) = 0 \tag{14}$$

Since his $\text{Prob}_{\widetilde{\beta}}(\text{win} \mid x = \widetilde{\beta}(\overline{c}))$ is always 0. As a result, if there exist symmetric equilibrium bidding strategies, they must be this single one:

$$\widetilde{\beta}(c) = \int_c^{\overline{c}} \frac{1}{y}(n-1)(1 - F(y))^{n-2} f(y) \, \mathrm{d}y \tag{15}$$

∎

Proof of existence. We are going to verify $\widetilde{\beta}(c) = \int_c^{\overline{c}} \frac{1}{y}(n-1)(1 - F(y))^{n-2} f(y) \, \mathrm{d}y$ is the best response for each bidder.

For bidder i who bears marginal cost c_i, he will select his own effort believing other bidders all follow bidding strategy $\widetilde{\beta}(\cdot)$.

$$\begin{aligned} & \max_{x_i} \; EV_{\widetilde{\beta}}(c, x) = \text{Prob}_{\widetilde{\beta}}(\text{win} \mid x = x_i) - c_i x_i \\ \iff & \max_{x_i} \; EV_{\widetilde{\beta}}(c, x) = (1 - F(\widetilde{\beta}^{-1}(x_i)))^{n-1} - c_i x_i \\ \implies & D_1 EV_{\widetilde{\beta}}(c, x) = -(n-1)(1 - F(\widetilde{\beta}^{-1}(x_i)))^{n-2} \cdot f(\widetilde{\beta}^{-1}(x_i)) \frac{1}{\widetilde{\beta}'(\widetilde{\beta}^{-1}(x_i))} - c_i \end{aligned}$$

By equation (13) and the fact $\widetilde{\beta}'(\cdot) < 0$, we obtain

$$D_1 EV_{\widetilde{\beta}}(c, x) = \widetilde{\beta}^{-1}(x_i) - c_i \begin{cases} > 0 & \text{if } x_i < \widetilde{\beta}(c_i) \\ = 0 & \text{if } x_i = \widetilde{\beta}(c_i) \\ < 0 & \text{if } x_i > \widetilde{\beta}(c_i) \end{cases}$$

Thus, $\widetilde{\beta}(c_i)$ is optimal choose for bidder i, provided that others bid according to $\widetilde{\beta}$. We have verified that $\widetilde{\beta}$ is the best response for each bidder. ∎

In conclusion, without cap, there exists an unique symmetric equilibrium.

Expected revenue for organizer & expected payoff for bidder.

$$\begin{aligned}
\widetilde{ER} &= \sum_{i=1}^{n} \int_{\underline{c}}^{\overline{c}} \widetilde{\beta}(c_i)\,\mathrm{d}F(c_i) \\
&= n \int_{\underline{c}}^{\overline{c}} \widetilde{\beta}(c_i)\,\mathrm{d}F(c_i) \\
&= n \int_{\underline{c}}^{\overline{c}} \int_{c_i}^{\overline{c}} \frac{1}{y}(n-1)(1-F(y))^{n-2} f(y)\,\mathrm{d}y\mathrm{d}F(c_i) \\
&= n \int_{\underline{c}}^{\overline{c}} \int_{\underline{c}}^{y} \frac{1}{y}(n-1)(1-F(y))^{n-2} f(y)\,\mathrm{d}F(c_i)\mathrm{d}y \\
&= n \int_{\underline{c}}^{\overline{c}} \frac{1}{y}(n-1)(1-F(y))^{n-2} f(y)F(y)\,\mathrm{d}y
\end{aligned}$$

$$\begin{aligned}
\widetilde{EV}(c_i) &= \text{Prob}_{\widetilde{\beta}}(\text{win} \mid x = \widetilde{\beta}(c_i)) - c_i\widetilde{\beta}(c_i) \\
&= (1-F(c_i))^{n-1} - c_i \int_{c_i}^{\overline{c}} \frac{1}{y}(n-1)(1-F(y))^{n-2} f(y)\,\mathrm{d}y
\end{aligned}$$

∎□

Proof of proposition 2. In this case, $\widetilde{\beta}(\cdot)$ is never a symmetric equilibrium bidding strategy, since $\widetilde{\beta}(\underline{c})$ is larger than cap d, which is forbidden. We claim equation (10) and (12) is the best response for bidder i against others' strategies.

Relation between d and $\widetilde{c}$.

$$\begin{aligned}
&\lim_{\widetilde{c}\to\underline{c}} d(\widetilde{c}) = \widetilde{\beta}(\underline{c}) \\
&\lim_{\widetilde{c}\to\overline{c}} d(\widetilde{c}) = 0
\end{aligned}$$

What is more, d is strictly decreasing with $\widetilde{c}$

$$\begin{aligned}
d'(\widetilde{c}) = &-\frac{2(1-F(\widetilde{c}))f(\widetilde{c}) + \cdots + (n-1)(1-F(\widetilde{c}))^{n-2}f(\widetilde{c})}{n\widetilde{c}} \\
&- \frac{1 + (1-F(\widetilde{c}))^2 + \cdots + (1-F(\widetilde{c}))^{n-1} - n(1-F(\widetilde{c}))^{n-1}}{n\widetilde{c}^2} \\
&< -\frac{1 + (1-F(\widetilde{c}))^2 + \cdots + (1-F(\widetilde{c}))^{n-1} - n(1-F(\widetilde{c}))^{n-1}}{n\widetilde{c}^2} \\
&< 0
\end{aligned}$$

Remark:

$$\begin{aligned}
\frac{1-(1-F(\widetilde{c}))^n}{F(\widetilde{c})} &= \frac{1-(1-F(\widetilde{c}))^n}{1-(1-F(\widetilde{c}))} \\
&= 1 + (1-F(\widetilde{c}) + \cdots + (1-F(\widetilde{c})^{n-1}
\end{aligned}$$

∎

Proof of $\beta(\cdot)$ is the best reponse for $c \geq \widetilde{c}$. $\forall c_i \in [\widetilde{c}, \overline{c}]$, $x_i \in (\beta(\widetilde{c}), d)$ will never be the best response, which is dominated by $\beta(\widetilde{c})$. Since $\beta([\underline{c}, \overline{c}]) = [0, d] \setminus (\beta(\widetilde{c}), d)$, we could regard choice of bidder i as selecting $\hat{c}$ and submit $\beta(\hat{c})$. The expected payoff generated by his chioce $\hat{c}$ is

$$EV(c_i, \hat{c}_i) = \begin{cases} (1-F(\hat{c}_i))^{n-1} - c_i \int_{\hat{c}_i}^{\overline{c}} \frac{1}{y}(n-1)(1-F(y))^{n-2} f(y)\,\mathrm{d}y & \text{if } \hat{c}_i \geq \widetilde{c} \\ \frac{1}{nF(\widetilde{c})}(1-(1-F(\widetilde{c}))^n) - c_i d & \text{if } \hat{c}_i < \widetilde{c} \end{cases}$$

For $\hat{c}_i \geq \widetilde{c}$:

$$D_2EV(c_i, \hat{c}_i) = (\frac{c_i}{\hat{c}_i} - 1)(n-1)(1-F(\hat{c}_i))^{n-2} f(\hat{c}_i)$$

Thus, we could obtain

$$D_2EV(c_i, \hat{c}_i) = \begin{cases} > 0 & \text{if } \widetilde{c} \leq \hat{c}_i < c_i \\ = 0 & \text{if } \hat{c}_i = c_i \\ < 0 & \text{if } c_i < \hat{c}_i \leq \overline{c} \end{cases}$$

So bidder i's optimal choice $\hat{c}_i \in [\widetilde{c}, \overline{c}]$ is c_i.
Next, we compare expected payoff when $x_i = d$ with $x_i = \beta(c_i)$.

$$\begin{aligned} \Delta(c_i) &= EV(c_i, x_i = \beta(c_i)) - EV(c_i, x_i = d) \\ &= (1-F(c_i))^{n-1} + c_i \int_{\widetilde{c}}^{c_i} \frac{1}{y}(n-1)(1-F(y))^{n-2} f(y)\, \mathrm{d}y - \frac{1}{nF(\widetilde{c})}(1-(1-F(\widetilde{c}))^n) \\ &\quad + c_i \cdot \frac{1-(1-F(\widetilde{c}))^n - nF(\widetilde{c})(1-F(\widetilde{c}))^{n-1}}{nF(\widetilde{c})\widetilde{c}} \end{aligned}$$

$\Delta(\widetilde{c})$ is equal to 0, and if $c_i > \widetilde{c}$

$$\begin{aligned} \Delta'(c_i) &= \int_{\widetilde{c}}^{c_i} \frac{1}{y}(n-1)(1-F(y))^{n-2} f(y)\, \mathrm{d}y + \frac{1-(1-F(\widetilde{c}))^n}{nF(\widetilde{c})\widetilde{c}} - \frac{(1-F(\widetilde{c}))^{n-2}}{\widetilde{c}} \\ &> \frac{1-(1-F(\widetilde{c}))^n}{nF(\widetilde{c})\widetilde{c}} - \frac{(1-F(\widetilde{c}))^{n-2}}{\widetilde{c}} \\ &= \frac{1}{n\widetilde{c}}(1 + (1-F(\widetilde{c})) + \cdots + (1-F(\widetilde{c}))^{n-1} - n(1-F(\widetilde{c}))^{n-1}) \\ &> 0 \end{aligned}$$

which means $\Delta(\widetilde{c}) > 0$. That is to say, $\beta(\cdot)$ is the best response for bidder i with marginal cost in $[\widetilde{c}, \overline{c}]$, and at critical point $\widetilde{c}$, bidder is exactly indifferent between submitting $\widetilde{\beta}(\widetilde{c})$ and d.

For $\hat{c}_i < \widetilde{c}$, it can also be verified that $x = d$ is the best response using same method. ∎

Expexted revenue for organizer.

$$\begin{aligned} ER(d) &= \sum_{i=1} ndF(\widetilde{c}) + \int_{\widetilde{c}}^{\overline{c}} \beta(c_i)\, \mathrm{d}F(c_i) \\ &= n[dF(\widetilde{c}) + \int_{\widetilde{c}}^{\overline{c}} \int_{c_i}^{\overline{c}} \frac{1}{y}(n-1)(1-F(y))^{n-2} f(y)\, \mathrm{d}y \mathrm{d}F(c_i)] \\ &= n[dF(\widetilde{c}) + \int_{\widetilde{c}}^{\overline{c}} \int_{\widetilde{c}}^{y} \frac{1}{y}(n-1)(1-F(y))^{n-2} f(y)\, \mathrm{d}F(c_i)\mathrm{d}y] \\ &= n[dF(\widetilde{c}) + \int_{\widetilde{c}}^{\overline{c}} (F(y) - F(\widetilde{c}))\frac{1}{y}(n-1)(1-F(y))^{n-2} f(y)\, \mathrm{d}y] \\ &= n[\int_{\widetilde{c}}^{\overline{c}} \frac{1}{y}(n-1)(1-F(y))^{n-2} f(y)F(y)\, \mathrm{d}y + F(\widetilde{c})(\frac{1-(1-F(\widetilde{c}))^n}{nF(\widetilde{c})\widetilde{c}} - \frac{(1-F(\widetilde{c}))^{n-1}}{\widetilde{c}})] \end{aligned}$$

∎ □

Proof of proposition 3. As equation(11) shows, the expected revenue is a function of the bid cap d adopted by organizer. In the following content, we will show that it is strictly increasing with respective to d.

Differentiating $EV(d)$ with respective to $\widetilde{c}$ gives

$$\begin{aligned} D_{\widetilde{c}}EV(d(\widetilde{c})) &= \frac{(1-F(\widetilde{c}))^n}{n\widetilde{c}^2} + \frac{F(\widetilde{c})(1-F(\widetilde{c}))^{n-1}}{\widetilde{c}^2} - \frac{1}{n\widetilde{c}^2} \\ &= -\frac{1}{n\widetilde{c}^2}(1-(1-F(\widetilde{c}))^n) + \frac{F(\widetilde{c})(1-F(\widetilde{c}))^{n-1}}{\widetilde{c}^2} \\ &= -\frac{1}{n\widetilde{c}^2} F(\widetilde{c})(1 + (1-F(\widetilde{c})) + \cdots + (1-F(\widetilde{c}))^{n-1} - n(1-F(\widetilde{c}))^{n-1}) \\ &< 0 \end{aligned}$$

That is $EV(d(\widetilde{c}))$ is a strictly decreasing function in $\widetilde{c}$, while $d(\widetilde{c})$ is also a a strictly decreasing function in $\widetilde{c}$. So $EV(d)$ is a strictly increasing function with respective to the bid cap d. $\square$

B Nomenclature

c	Marginal cost
F (resp. f)	Distribution (resp. density) function of marginal cost
$[\underline{c}, \overline{c}]$	The support of F
x	Effort(bid)
v	Prize valuation(normalized to 1)
n	Numbers of bidders
N	The set of bidders
i	Index of bidders
m	The number of bidders submitting the highest effort
d	Bid cap
w	whether his effort is highest
β	Symmetric equilibrium bidding strategy with bid cap
ER	Expected revenue for organizer with bid cap
EV	Expected payoff for a bidder with bid cap
$\widetilde{\beta}$	Symmetric equilibrium bidding strategy without bid cap
$\widetilde{ER}$	Expected revenue for organizer without bid cap
$\widetilde{EV}$	Expected payoff for a bidder without bid cap
D	Derivative
Prob(win)	Probability of winning
$\widetilde{c}$	Citical marginal cost where bidder is indifferent between submitting d and $\widetilde{\beta}(\widetilde{c})$

Bibliography

Arieh Gavious, Benny Moldovanu and Aner Sela (2002). "Bid Costs and Endogenous Bid Caps". In: *The RAND Journal of Economics* 33.4, pp. 709–722. URL: `http://www.jstor.org/stable/3087482`.

Qiang Fu Qian Jiao, Jingfeng Lu (2014). "Disclosure policy in a multi-prize all-pay auction with stochastic abilities". In: *Economics Letter* 125.3, pp. 376–380. URL: `http://doi.org/10.1016/j.econlet.2014.10.014`.