# Could bid cap control bankruptcy rate?

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## Abstract

We study a first-price auction with budget-constrained bidders in a setting that allows borrowing and declaring bankruptcy. Bids are bounded by a common bid cap determined by the organizer. We show that a large bid cap is not effective in lowering bankruptcy rate. A small bid cap is effective in lowering bankruptcy rate but also reduces the expected revenue.

## Contents



## 1 The model

We consider $n \geq 2$ bidders complete for an indivisible prize through a first-price auction. The set of bidders $\{1, \ldots, n\}$ is denoted by $N$. The value of prize for organizer is zero. For every bidder, prize's value (denoted by $v$) is uncertain which follows a binomial distribution, where $\overline{v}$ with probability $1-\theta$ and $0$ with probability $\theta$. This distribution is commonly known by each bidder, that is to say, $\overline{v}$ and $\theta$ are common knowledge. The actual value is not revealed until it is sold to the winner.

Instead of "deep pockets" setting, each bidder has a budget constraint denoted by $e$, which is private information to themselves. All bidders other than $i$ perceive $e_i$ as a random selection out of support $[\underline{e}, \overline{e}] \in [0, +\infty)$, governed by the cumulative distribution $F$, and independent of others' budgets. For convenience, we assume $F$ is continuous differentiable on its support, and its associated probability density function is $f$.

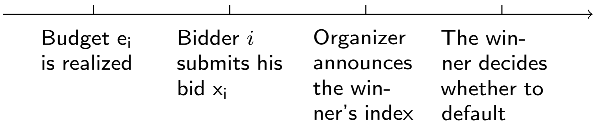
In the first stage, bidders simultaneously submit their bids $x \geq 0$. Submitting a bid larger than his budget constraint is allowed. One can borrowing for his balance $(x_i - e_i)$ from bank at an exogenous market borrowing rate $r$. However, for convenience, lending is unprofitable, since we assume market lending rate is $0$. So cost of bid is

$$c(x_i, e_i) = \begin{cases} x_i & \text{if } x_i \leq e_i \\ x_i + r(x_i - e_i) & \text{if } x_i > e_i \end{cases} \tag{1}$$

Prize is given to only one bidder with the highest bid. (Ties are broken randomly). The winner pays his bid and others pay nothing.

In the second stage, the winner know the realization of prize's value $v$, and decides whether to default. If defaulting, all his wealth will be confiscated by bank. The timeline of this game is figure out below (Figure 1):

**FIGURE 1:** Timeline without Cap




In the economy, bank plays a role as capital market which loose the liquidity constraint to allow more captical to invest in the auction. Definitely, participation of bank will make this auction more competitive in the one hand, since it allows every bidder to decide his bid using his private information and ability sufficiently regardless of his budget. However, in the other hand, bidders with "shallow pocket", even with nothing, would join the auction and submit a surprisingly high bid, for his little default cost. To see more deeply whether and when the second case will happen, the following two lemma do a favor.

**Lemma 1.** If revelation of $v$ is $\overline{v}$, then the winner will never default.

*Proof.* Since $\{x_i \mid c(x_i, e_i) > \overline{v}\}$ is dominated by $x_i = 0$,

$$\text{payoff } |_{v=\overline{v}} = \begin{cases} \overline{v} - c(x_i, e_i) \geq 0 & \text{not default} \\ -e_i \leq 0 & \text{default} \end{cases}$$

That means deciding not to default weakly dominates to default. □

**Lemma 2.** If revelation of $v$ is $0$, then the winner will default only if he has claimed a debt.

*Proof.*

$$\text{payoff } |_{v=0} = \begin{cases} -c(x_i, e_i) & \text{not default} \\ -e_i & \text{default} \end{cases}$$

When the winner has claimed a debt, $x_i > e_i$, thus $c(x_i, e_i) = x_i + r(x_i - e_i) > x_i > e_i$. This implies the winner will choose to default. At the other case, he will not claim to default if $x_i \leq e_i$. □

According to the above lemmas, we could see clearly that default only occurs at the winner who claims a debt (i.e.$x_i > e_i$) when $0$ value happens. We could ask furtherly, who indeed will declare a debt, a high budget one or a low budget one. Intuitively, high-budget bidder does not needs fund, because it is costly. While low-budget bidder would prefer it to increasing his winning probability with little default cost. Next lemma will select out a set of bidder who will never bid more than budget.

**Lemma 3.** For bidder whose budget is no less than $(1 - \theta)\overline{v}$, his bid $x$ will always no larger than $e$.

*Proof.* For bidder with $e \geq (1 - \theta)\overline{v}$, when selecting any $x > e$, we get

$$\begin{aligned} V(x, e) &= (1 - \theta)(\overline{v} - x - r(x - e)) - \theta e \\ &= (1 - \theta)\overline{v} - (1 - \theta)x - \theta e - (x - e)(1 - \theta)r \\ &< (1 - \theta)\overline{v} - e - (x - e)(1 - \theta)r \\ &< 0 \end{aligned}$$

while $V(0, e) = 0$. This indicates that any $x > e$ is dominated by $x = 0$. □

This lemma told someone who bears with a sufficiently large budget (more specificly, whose budget is no less than the expected value of prize), will never bid more than budget.

Combining lemma 2 and 3, we could see default could only occurs at bidder whose budget is less than $(1-\theta)\overline{v}$ (i.e. the expected value of prize).

Actually, the symmetric Bayesian Nash equilibrium of this auction game is a complicated set which contains many non-monotone and discontinuous bidding strategies. Charles Z.Zheng (2001) restricts his attention to monotonic and continuous function set, and figures out an unique solution. He point out when the borrowing rate is below the threshold $\frac{\theta}{1-\theta}$, the winner is the most budget-constrainted bidder and is most likely to declare bankruptcy while the bidding strategy is strictly decrearsing on $[\underline{e}, (1-\theta)\overline{v})$ and weakly decreasing on $[(1-\theta)\overline{v}, \overline{e}]$.

**Lemma 4.** (*Charles Z.Zheng, 2001*). In the case of $r \in [0, \frac{\theta}{1-\theta})$, there exists an unique continuous symmetric equilibrium of the auction game given by

$$\widetilde{\beta}(e) = \begin{cases} E_{e^L_{-i}}[\frac{\overline{v}+r' \min(e^L_{-i},(1-\theta)\overline{v})}{1+r} \mid e^L_{-i} > e] & \text{if } \overline{e} \leq e < (1-\theta)\overline{v} \\ (1-\theta)\overline{v} & \text{otherwise} \end{cases} \tag{2}$$

where $r' = r - \frac{\theta}{1-\theta}$ and $e^L_{-i}$ denotes the lowest budget among a bidders' rivals.

If $r \in [0, \frac{\theta}{1-\theta})$, bankruptcy rate is always stays at a fixed relatively high level.

**Lemma 5.** In the case of $r \in [0, \frac{\theta}{1-\theta})$, bankruptcy rate is $\theta[1-(1-F((1-\theta)\overline{v}))^n]$.

*Proof.*

$$\begin{aligned} \text{Prob}(e^L < (1-\theta)\overline{v}) &= 1 - \text{Prob}(e^L \geq (1-\theta)\overline{v}) \\ &= 1 - \text{Prob}(e_i \geq (1-\theta)\overline{v}, \forall i \in N) \\ &= 1 - (1-F((1-\theta)\overline{v}))^n \end{aligned}$$

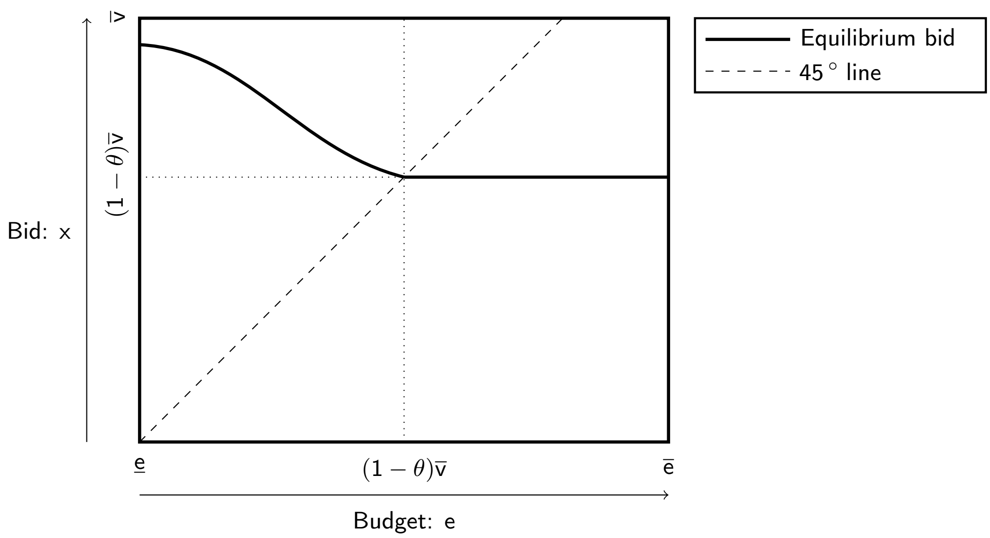
Then we obtain

$$\begin{aligned} \text{Bankruptcy rate} &= \theta \cdot \text{Prob}(e^L < (1-\theta)\overline{v}) \\ &= \theta[1-(1-F((1-\theta)\overline{v}))^n] \end{aligned}$$

□

This result of "high bids and broke winners" is really shocking, one for the bad allocation of the prize in which the poorest wins, another for strange bidding strategy by which any bidder's bid is no less than the expected value of prize. This fact shows in Figure 2.

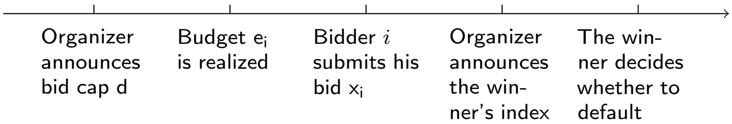
**FIGURE 2:** Equilibrium Bid without Cap



All these strange phenomenons result from the incomplete financial market. Finicial institution cannot figure out what is going on in the auction in detail which leads to asymmetic information dilemma. The exogenous borrowing rate is dependent on the whole economy which beyond our thinking, so the rate cannot precisely measure the risk of this auction. If borrowing rate in outside market is relative low, specificly, below the threshold, a large amount of speculators swarm into the auction and give surprisingly high bid of the prize which is much more than its real value. They use leverages which lower down their bankruptcy cost and their probability of winning. However, their crazy strategies will lead a large part of the limited captical to a small and risky market, and make it to be a "GOLD RUSH". When risk occurs, the winner can not afford the payment of contract he signed with bank, he will take a series of bankruptcy processes, in which exceeding amount will be paid by bank, not him. At this moment, bubble breaks, accompanied by plenty amount of investors' losses of wealth. This may generate a permanent shock to our economy, such as causing a lot of fear and fright, hurting expectations ,even confidence.

Next, we assume the organizer of this auction care both expected revenue and bankrutpcy rate a lot. And the most intuitional method for him to control the bankruptcy rate is adding a bid cap (denoted by $d \geq 0$) to set an upper bound of the bid. Intuitively, this method will restrict the abnormal behaviors of low-budget bidders, and lower down the bankruptcy rate. Our extensional auction proceeds under following timeline (Figure 3):

**FIGURE 3:** Timeline with Cap




As it is shown above, we assume the organizer announces the bid cap $d$ before budget $e_i$ being realized. This implies bid cap $d$ is a common knowledge.

## 2 Characterization of equilibria

In this part, we are going to explore what equilibria looks like and how bankruptcy rate changes with respect to different bid cap.

We first consider the case with redundant cap where bid cap is too large to have actrual constraint on any bidders. A first-price auction we has discussed in last section arises.

**Proposition 1.** Let $r \in [0, \frac{\theta}{1-\theta})$ and $d > E_{e^L_{-i}}[\frac{\overline{v}+r' \min(e^L_{-i},(1-\theta)\overline{v})}{1+r}]$, there exists an unique continuous symmetric equilibrium of the auction game given by

$$\beta(e,d) = \widetilde{\beta}(e) = \begin{cases} E_{e^L_{-i}}[\frac{\overline{v}+r' \min(e^L_{-i},(1-\theta)\overline{v})}{1+r} \mid e^L_{-i} > e] & \text{if } \overline{e} \leq e < (1-\theta)\overline{v} \\ (1-\theta)\overline{v} & \text{otherwise} \end{cases} \tag{3}$$

where $r' = r - \frac{\theta}{1-\theta}$ and $e^L_{-i}$ denotes the lowest budget among a bidders' rivals. And the bankruptcy rate is $\theta[1-(1-F((1-\theta)\overline{v}))^n]$.

*Proof.*

$$\begin{aligned} \max_{e\in[\underline{e},\overline{e}]} \widetilde{\beta}(e) &= \widetilde{\beta}(\underline{e}) \\ &= E_{e^L_{-i}}[\frac{\overline{v}+r' \min(e^L_{-i},(1-\theta)\overline{v})}{1+r}] \end{aligned}$$

Thus, if $d > E_{e^L_{-i}}[\frac{\overline{v}+r' \min(e^L_{-i},(1-\theta)\overline{v})}{1+r}]$, then the bid cap is ineffective. According to lemma 4, the symmetric and continuous equilibrium is unique, and

$$\beta(e,d) = \widetilde{\beta}(e)$$

Since every bidder submits same bid as the case without cap, the bankruptcy rate will stay at last value, that is

$$\text{Bankruptcy rate}(d) = \theta[1-(1-F((1-\theta)\overline{v}))^n]$$

□

**FIGURE 4:** Equilibrium Bid with Cap: d $> \widetilde{\beta}(\underline{e})$

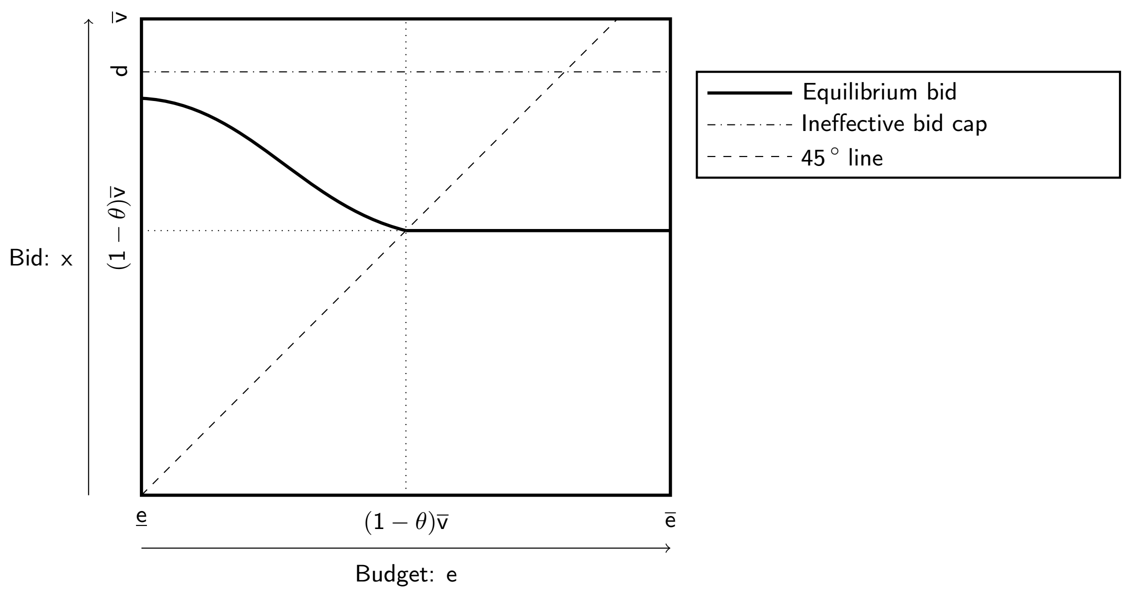



Figure 4 illustrates these facts.

We then consider the case with effective cap.

**Proposition 2.** Let $r \in [0, \frac{\theta}{1-\theta})$ and $(1-\theta)\overline{v} < d \leq E_{e^L_{-i}}[\frac{\overline{v}+r'\min(e^L_{-i},(1-\theta)\overline{v})}{1+r}]$. Then the bid cap is effective, and there exists a symmetric monotone equilibrium where bidding strategy is given by

$$\beta(e_i, d) = \begin{cases} d & \text{if } \underline{e} \leq e_i < \widetilde{e} \\ \widetilde{\beta}(e_i) & \text{if } \widetilde{e} \leq e_i < (1-\theta)\overline{v} \\ (1-\theta)\overline{v} & \text{if } (1-\theta)\overline{v} \leq e_i \leq \overline{e} \end{cases} \tag{4}$$

and the bankruptcy rate is given by

$$\text{Bankruptcy rate}(d) = \theta[1-(1-F((1-\theta)\overline{v}))^n] \tag{5}$$

where the critical value $\widetilde{e} = \widetilde{e}(d)$ is strictly monotonic decreasing, and defined by

$$d = \frac{nF(\widetilde{e})(1-F(\widetilde{(e)}))^{n-1}}{1-(1-F(\widetilde{e}))^n}[\beta\widetilde{e} - \frac{\overline{v}+r'\widetilde{e}}{1+r}] + \frac{\overline{v}+r'\widetilde{e}}{1+r} \tag{6}$$

where $r' = r - \frac{\theta}{1-\theta}$

*Proof.* See the Appendix. □

Proposition 2 indicates that bid cap can lower down the bid which is bigger than the upper bound, however, it will also incentive bidders whose former bids are a little lower than cap to increase their bids to the upper bound $d$. What disappoints us is that bidders who claimed for debts before, will still need to be funded now. This fact tells us the bankruptcy rate will stay unchanged, which means this bid cap is useless for our purpose.

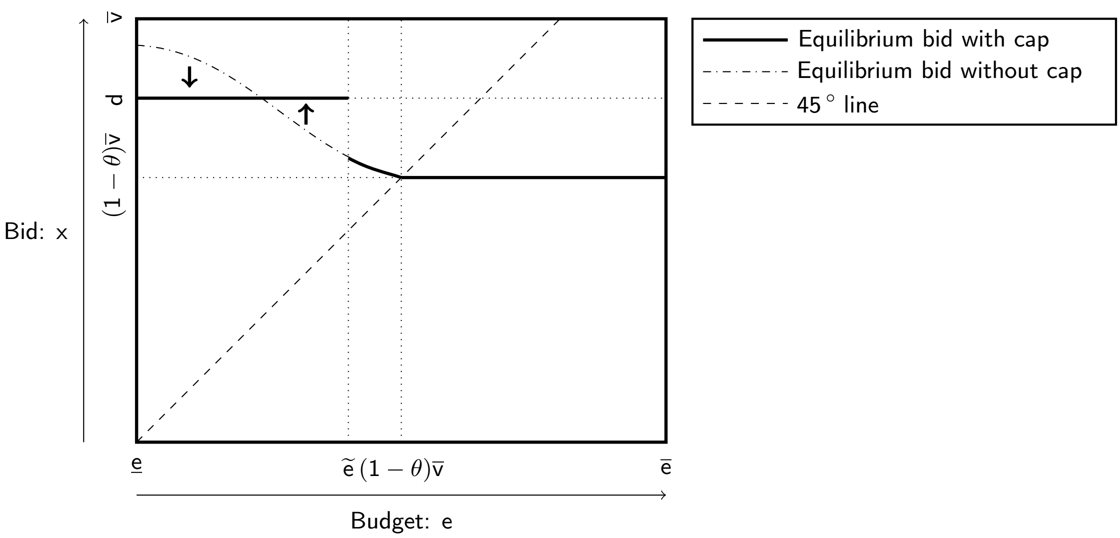
Figure 5 illustrates these facts.

Next, we assume our auction organizer set a further smaller bid cap which is auctually lower than the expected value of the prize. This relatively low cap will both influences behaviors of bidders who claimed debt before and affact choice of bidders whose budgets are larger than the expected value of the prize.

**Proposition 3.** Let $r \in [0, \frac{\theta}{1-\theta})$ and $0 < d \leq (1-\theta)\overline{v}$. Then the bid cap is effective, and there exists a symmetric monotone equilibrium where bidding strategy is given by

$$\beta(e_i, d) = d \ \ \forall e_i \in [\underline{e}, \overline{e}] \tag{7}$$

**FIGURE 5:** Equilibrium Bid with Cap: $d > (1-\theta)\bar{v}$




and the bankruptcy rate is given by

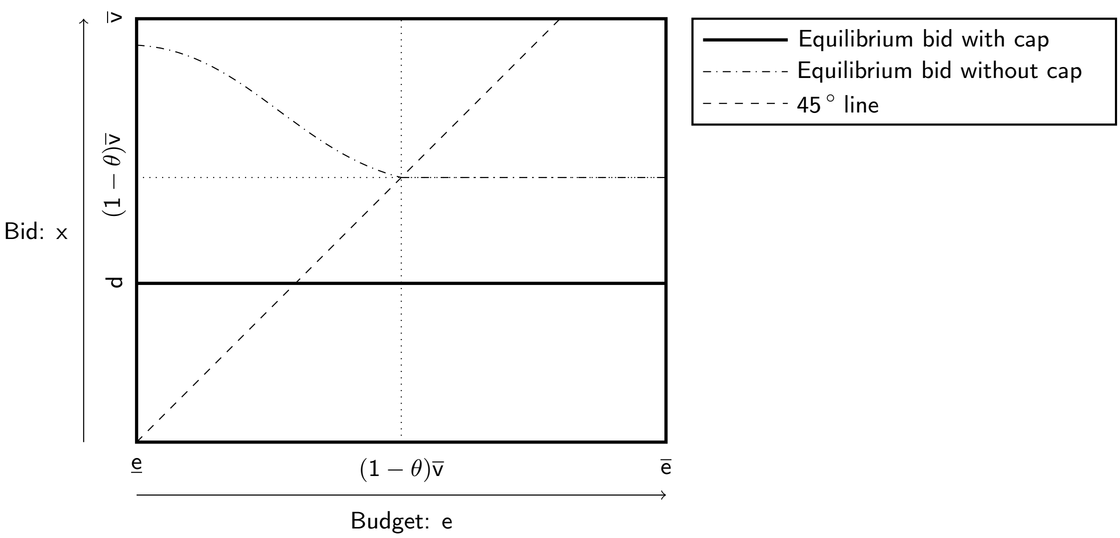
$$\text{Bankruptcy rate}(d) = \theta F(d) \tag{8}$$

which is strictly increasing with respect to bid cap $d$.

*Proof.* See the Apendix. □

Proposition 3 states that if we set cap smaller than the expected value of the prize, bankruptcy rate will be lowered down along with the decease of bid cap. This result is satisfying for us, and if we gradually let the cap smaller and smaller , finally set it equal to the lowest budget in budget's support, the bankruptcy rate will be 0. Unfortunately, as bid cap decreases, the competitive of the game will lower down, difference among bidders will gradually disappear, and the degree of information disclosure will decrease. Finally, all these bad news of game structure will indeed be followed by an reduction of expected revenue of auction organizer. This results in a trade off against revenue and bankruptcy rate. Figure 6 illustrates these facts.

**FIGURE 6:** Equilibrium Bid with Cap: $0 \leq d \leq (1-\theta)\bar{v}$




## 3 Anlysis of the change of bankruptcy rate with respect to bid cap

According to proposition 1-3, we could obtain

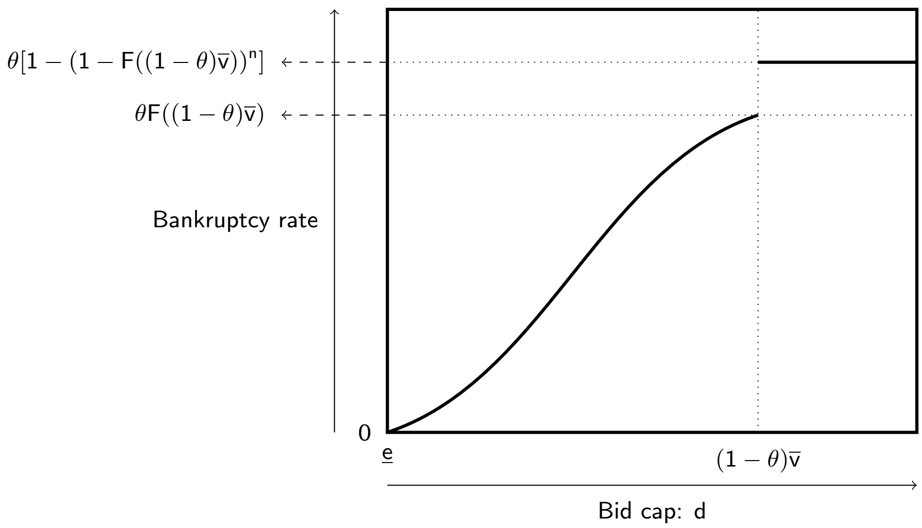
$$\text{Bankruptcy rate}(d) = \begin{cases} \theta[1-(1-F((1-\theta)\overline{v}))^n] & \text{if } d > (1-\theta)\overline{v} \\ \theta F(d) & \text{if } 0 < d \leq (1-\theta)\overline{v} \end{cases} \tag{9}$$

This is a weakly increasing function with a discontinuous point $d = (1-\theta)\overline{v}$. To see it in directer, we calculate two directions' derivative as following

$$\begin{aligned} \lim_{d \to (1-\theta)\overline{v}^+} \text{Bankrutpcy rate}(d) &= \theta[1-(1-F((1-\theta)\overline{v}))^n] \\ &> \theta[1-(1-F((1-\theta)\overline{v}))] \\ &= \theta F((1-\theta)\overline{v}) \\ &= \lim_{d \to (1-\theta)\overline{v}^-} \text{Bankrutpcy rate}(d) \end{aligned}$$

At point $d = (1-\theta)\overline{v}$, function's right derivative is larger than left derivative, thus there is a jump. This fact shows in Figure 7.

**FIGURE 7:** Bankruptcy rate w.r.t bid cap




# Appendices

## A Proofs of propositions

***Proof of proposition 2.*** In this case, $\widetilde{\beta}(\cdot)$ is never a symmetric equilibrium bidding strategy, since $\widetilde{\beta}(\underline{e})$ is larger than cap $d$, which is forbidden. We claim equation (4) and (6) is the best response for bidder $i$ against others' strategies.

*Relation between $d$ and $\widetilde{e}$.*

$$\lim_{\widetilde{e} \to \underline{e}} d(\widetilde{e}) = [\lim_{\widetilde{e} \to \underline{e}} \frac{nF(\widetilde{e})}{1-(1-F(\widetilde{e}))^n}][\beta\underline{e} - \frac{\overline{v}+r'\underline{e}}{1+r}] + \frac{\overline{v}+r'\underline{e}}{1+r}$$

And

$$\begin{aligned} \lim_{\widetilde{e} \to \underline{e}} \frac{nF(\widetilde{e})}{1-(1-F(\widetilde{e}))^n} &= lim_{y \to 1} \frac{n(1-y)}{1-y^n} \\ &= lim_{y \to 1} \frac{-n}{-ny^{n-1}} \\ &= 1 \end{aligned}$$

Thus

$$\lim_{\widetilde{e}\to\underline{e}} d(\widetilde{e}) = \widetilde{\beta}(\underline{e})$$

Because $\lim_{\widetilde{e}\to(1-\theta)\overline{v}} \frac{\overline{v}+r'e}{1+r} = (1-\theta)\overline{v} = \widetilde{\beta}((1-\theta)\overline{v})$, we can get

$$\lim_{\widetilde{e}\to(1-\theta)\overline{v}} d(\widetilde{e}) = (1-\theta)\overline{v}$$

What is more, d is strictly decreasing with $\widetilde{e}$, for

$$d'(\widetilde{e}) = (1-(1-F(\widetilde{e}))^n - nF(\widetilde{e})(1-F(\widetilde{e}))^{n-1})[\frac{nf(\widetilde{e})(1-F(\widetilde{e}))^{n-1}}{(1-(1-F(\widetilde{e}))^n)^2}\cdot(\widetilde{\beta}(\widetilde{e}) - \frac{\overline{v}+r'\widetilde{e}}{1+r}) + \frac{1}{1-(1-F(\widetilde{e}))^n}\cdot\frac{r'}{1+r}]$$
$$< 0$$

Remark:

$$\begin{aligned}1-(1-F(\widetilde{e}))^n - nF(\widetilde{e})(1-F(\widetilde{e}))^{n-1} &= F(\widetilde{e})[\frac{1-(1-F(\widetilde{e}))^n}{F(e)} - n(1-F(\widetilde{e}))^{n-1}] \\ &= F(\widetilde{e})[1+(1-F(\widetilde{e}))+\cdots+(1-F(\widetilde{e}))^{n-1} - n(1-F(\widetilde{e}))^{n-1}] \\ &< 0\end{aligned}$$

$$\widetilde{\beta}(\widetilde{e}) - \frac{\overline{v}+r'\widetilde{e}}{1+r} < 0$$

$$\frac{r'}{1+r} < 0$$

∎

*Proof of $\beta(\cdot)$ is the best reponse for $e \geq (1-\theta)\overline{v}$.* According to Lemma 3, we know that $x < e$, for all $e \geq (1-\theta)\overline{v}$. Then we can obtain that if $x < \beta(e,d) = (1-\theta)\overline{v}$, then

$$EV(x,e) = V(x,e)\cdot \text{Prob}_\beta(\text{win} \mid x) = 0$$

and if $(1-\theta)\overline{v} = \beta(e,d) < x \leq e$, then

$$\begin{aligned}EV(x,e) &= V(x,e)\cdot \text{Prob}_\beta(\text{win} \mid x) \\ &= [(1-\theta)(\overline{v}-x) - \theta x]\cdot \text{Prob}_\beta(\text{win} \mid x) \\ &< 0\end{aligned}$$

In conclusion, we can claim that $\beta(\cdot)$ is the best reponse for $e \geq (1-\theta)\overline{v}$.

*Proof of $\beta(\cdot)$ is the best reponse for $\widetilde{e} \leq e < (1-\theta)\overline{v}$.* $\forall e_i \in [\widetilde{e}, (1-\theta)\overline{v}]$, $e_i \in (\beta(\widetilde{e}), d)$ will never be the best response, which is dominated by $\beta(\widetilde{e})$. Since $\beta([\underline{e}, (1-\theta)\overline{v}]) = [0,d] \setminus (\beta(\widetilde{e}), d)$, we could regard choice of bidder $i$ as selecting $\hat{e}$ and submit $\beta(\hat{e})$. The expected payoff generated by his chioce $\hat{e}$ is

$$EV(e_i,\hat{e}_i) = \begin{cases} [(1-\theta)(\overline{v} - r(\beta(\hat{e}) - e_i) - \beta(\hat{e})) - \theta e_i]\cdot(1-F(\hat{e}))^{n-1} & \text{if } \hat{e}_i \geq \widetilde{e} \\ [(1-\theta)(\overline{v} - r(d-e_i) - d) - \theta e_i]\cdot\frac{1}{nF(\widetilde{e})}\cdot(1-(1-F(\widetilde{e}))^n) & \text{if } \hat{e}_i < \widetilde{e}\end{cases}$$

For $\hat{e}_i \geq \widetilde{e}$:

$$D_2EV(e_i,\hat{e}_i) = (1-\theta)(n-1)f(\hat{e})(1-F(\hat{e}))^{n-2}r'(\hat{e}-e)$$

Thus, we could obtain

$$D_2EV(e_i,\hat{e}_i) = \begin{cases} >0 & \text{if } \widetilde{e} \leq \hat{e}_i < e_i \\ =0 & \text{if } \hat{e}_i = e_i \\ <0 & \text{if } e_i < \hat{e}_i \leq (1-\theta)\overline{v}\end{cases}$$

So bidder $i$'s optimal choice $\hat{e}_i \in [\widetilde{e}, (1-\theta)\overline{v}]$ is $e_i$.
Next, we compare expected payoff when $x_i = d$ with $x_i = \beta(e_i)$.

$$\begin{aligned}\Delta(e_i) &= EV(e_i, x_i = \beta(e_i)) - EV(e_i, x_i = d) \\ &= [(1-\theta)(\overline{v} - r(\beta(e_i) - e_i) - \beta(e_i)) - \theta e_i](1 - F(e))^{n-1} \\ &\quad - [(1-\theta)(\overline{v} - r(d - e_i) - d) - \theta e_i]\frac{1}{nF(\widetilde{e})}(1 - (1 - F(\widetilde{e}))^n)\end{aligned}$$

$\Delta(\widetilde{e})$ is equal to 0, and if $e_i > \widetilde{e}$

$$\begin{aligned}\Delta'(e_i) &= [(1-\theta)r'][(1 - F(e_i))^{n-1} - \frac{1 - (1 - F(\widetilde{e}))^n}{nF\widetilde{e}}] \\ &> [(1-\theta)r'][(1 - F(\widetilde{e}))^{n-1} - \frac{1 - (1 - F(\widetilde{e}))^n}{nF\widetilde{e}}] \\ &= -[(1-\theta)r']\frac{1}{n}[1 + (1 - F(\widetilde{e})) + \cdots (1 - F(\widetilde{e}))^{n-1} - n(1 - F(\widetilde{e}))^{n-1}] \\ &> 0\end{aligned}$$

which means $\Delta(e_i) > 0$. That is to say, $\beta(\cdot)$ is the best response for bidder i with budget in $[\widetilde{e}, (1-\theta)\overline{v}]$, and at critical point $\widetilde{e}$, bidder is exactly indifferent between submitting $\widetilde{\beta}(\widetilde{e})$ and $d$.

For $\underline{e} \leq e_i < \widetilde{e}$, it can also be verified that $x = d$ is the best response using same method. ∎ □

***Proof of proposition 3.***

*Proof of $\beta(\cdot)$ is the best reponse when $0 < d \leq (1-\theta)\overline{v}$.* Assume that $0 < d \leq (1-\theta)\overline{v}$, we are going to verify that $\beta(e) = d \; \forall e \in [\underline{e}, \overline{e}]$ is the best response for all bidders.

Suppose bidder $i$ submits $x_i < d$, against all rivals follow $\beta(e)$, then his $\text{Prob}_\beta(\text{win} \mid x_i) = 0$. This implies that his expected payoff is equal to 0. If we can verify for all bidders his expected payoff is no less than 0 when he submits $x_i = d$, then we have proved this proposition.

We devided it into two stages, first is $e \geq d$, another is $e < d$.

For bidder $i$ with $e \geq d$:

$$\begin{aligned}EV(e, d) &= V(e, d) \cdot \text{Prob}_\beta(\text{win} \mid x = d) \\ &= [(1-\theta)\overline{v} - d] \cdot \frac{1}{n} \\ &\geq 0\end{aligned}$$

For bidder $i$ with $e < d$:

$$\begin{aligned}EV(e, d) &= V(e, d) \cdot \text{Prob}_\beta(\text{win} \mid x = d) \\ &= [(1-\theta)(\overline{v} - d - r(d - e)) - \theta e] \cdot \frac{1}{n} \\ &> [(1-\theta)\overline{v} - (1-\theta)^2\overline{v}(1 + r) + (1-\theta)re - \theta e] \cdot \frac{1}{n} \\ &= [-(1-\theta)r'((1-\theta)\overline{v} - e)] \cdot \frac{1}{n} \\ &> 0\end{aligned}$$

∎

*Calculation of bankruptcy rate.* Assume that $0 < d \leq (1-\theta)\overline{v}$, we are going to calculate the bankruptcy rate with respect to cap.

$$
\begin{aligned}
\text{Bankrutpcy rate}(d) &= [\sum_{m=0}^{n} C_n^m F(d)^m (1-F(d))^{n-m} \frac{m}{n}]\theta \\
&= [\sum_{m=0}^{n} C_{n-1}^{m-1} F(d)^m (1-F(d))^{n-m}]\theta \\
&= [\sum_{m=1}^{n} C_{n-1}^{m-1} F(d)^m (1-F(d))^{n-m}]\theta \\
&= F(d)[\sum_{m-1=0}^{n-1} C_{n-1}^{m-1} F(d)^{m-1} (1-F(d))^{n-m}]\theta \\
&= F(d)\theta
\end{aligned}
$$

$\blacksquare$ $\square$

# B Nomenclature

| | |
|---|---|
| $e$ | Budget |
| $F$ (resp. $f$) | Distribution (resp. density) function of Budget |
| $[\underline{e}, \overline{e}]$ | The support of F |
| $x$ | Effort(bid) |
| $v$ | Prize valuation |
| $\overline{v}$ | The highest prize valuation |
| $\theta$ | Probability of $v = 0$ |
| $n$ | Numbers of bidders |
| $N$ | The set of bidders |
| $i$ | Index of bidders |
| $d$ | Bid cap |
| $\beta$ | Symmetric equilibrium bidding strategy with bid cap |
| $ER$ | Expected revenue for organizer with bid cap |
| $V$ | Expected payoff for a bidder conditional on his winning |
| $EV$ | Expected payoff for a bidder with bid cap |
| $\widetilde{\beta}$ | Symmetric equilibrium bidding strategy without bid cap |
| $\widetilde{ER}$ | Expected revenue for organizer without bid cap |
| $\widetilde{EV}$ | Expected payoff for a bidder without bid cap |
| $D$ | Derivative |
| Prob(win) | Probability of winning |
| $\widetilde{e}$ | Citical budget where bidder is indifferent between submitting $d$ and $\widetilde{\beta}(\widetilde{e})$ |