STA355A2

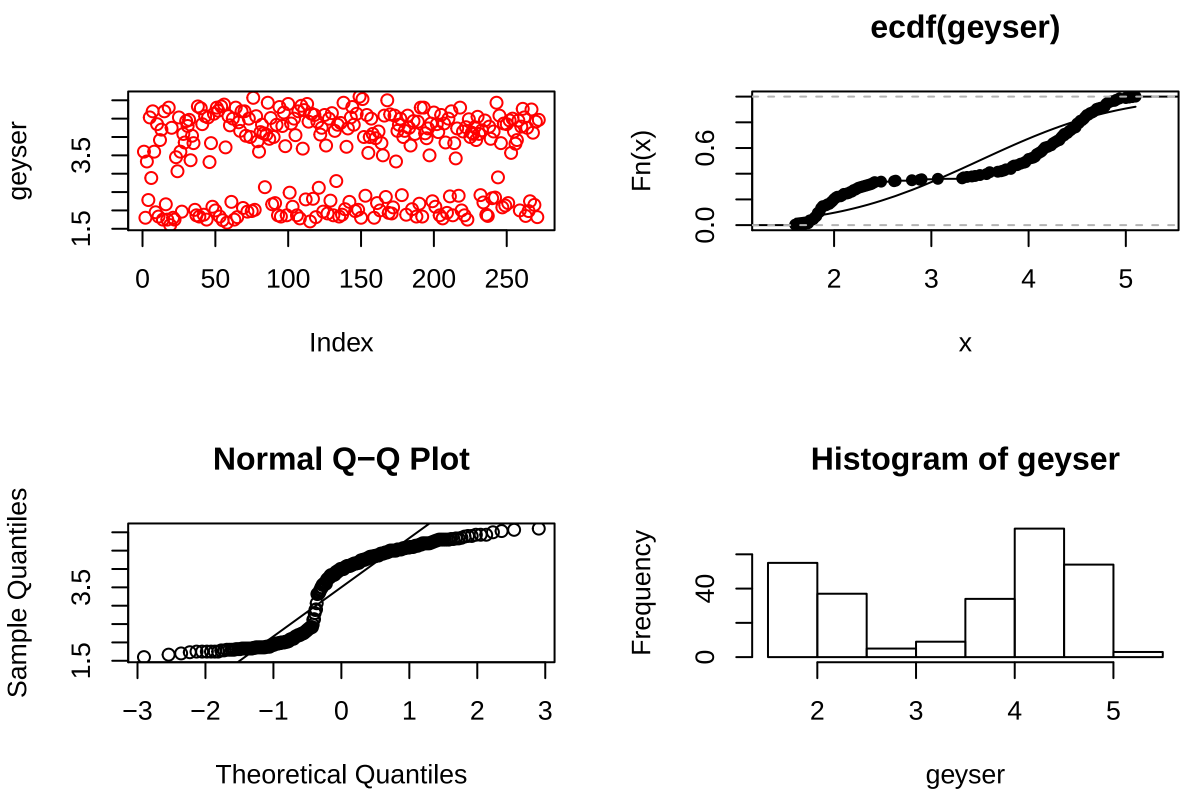
zx

a)

```
geyser = scan("/home/yiche/Desktop/github/UofT_course/STA355/geyser.txt");
sort_geyser = sort(geyser)
par(mfrow = c(2,2))
plot(geyser, col="red")
plot(ecdf(geyser))
lines(sort_geyser, pnorm(sort_geyser, mean(geyser), sqrt(var(geyser))))
x.norm = rnorm(1000, mean(geyser), var(geyser))
qqnorm(geyser); qqline(x.norm);
shapiro.test(geyser)
```

```
##
##  Shapiro-Wilk normality test
##
## data:  geyser
## W = 0.84592, p-value = 9.036e-16
```

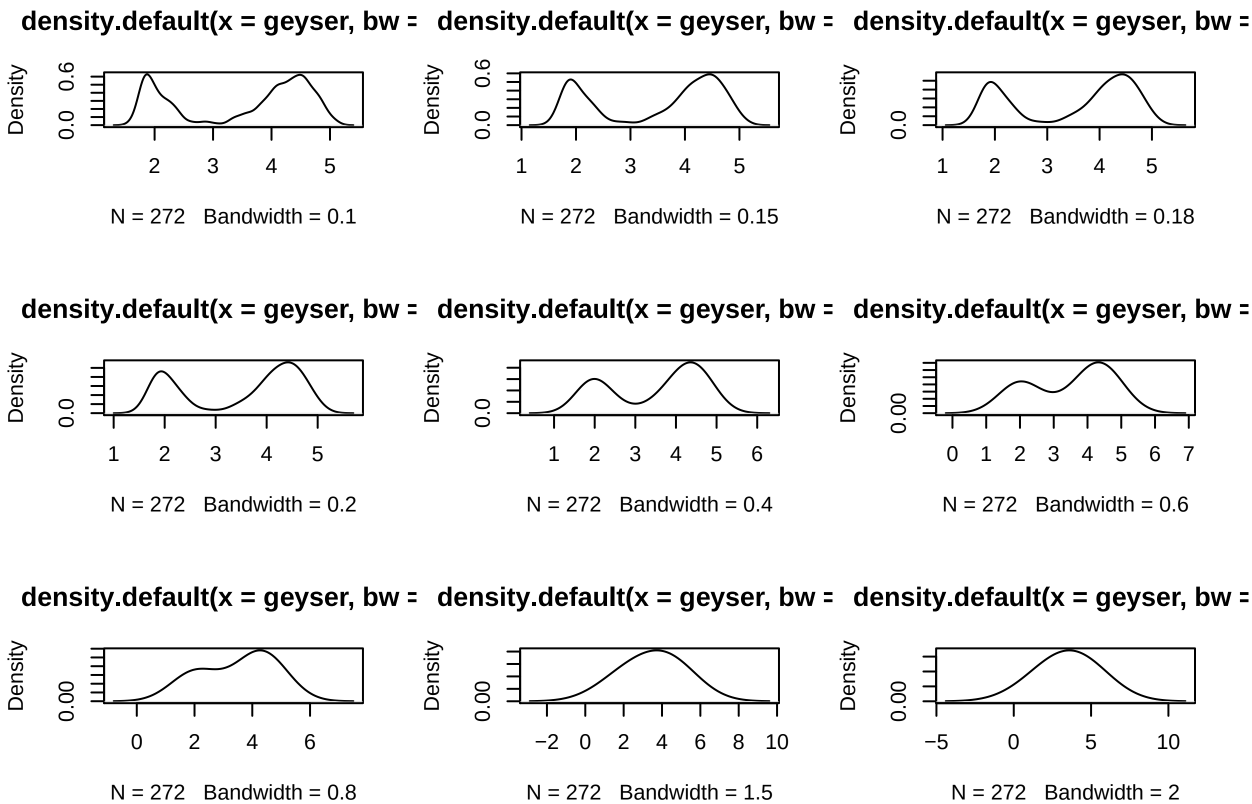
```
hist(geyser)
```



From the histgram we can see data does not follow unimodal.
By comparing the qqplot draw from normal distribution and qqplot from data, these data does not follow normal distribution. Further we use shapiro test, since p value is close to 0, data are unlikely to follow normal distribution.

b)

```
bw = c(0.1, 0.15, 0.18, 0.2, 0.4, 0.6, 0.8, 1.5, 2)
plot_density = function(bw) {
  for (i in bw){
    plot(density(geyser, bw = i))
  }
}
par(mfrow = c(3,3))
plot_density(bw)
```



By tring variety of bandwidth, we can see that as bandwidth increase, the modes are less clear, eventurally when bandwidth larger than 1 in this case, there will exist only one mode. As the bandwidth decrease, more modes and cruvatures occurs.

```
library("mixtools")
```

```
## mixtools package, version 1.1.0, Released 2017-03-10
## This package is based upon work supported by the National Science Foundation under Grant No. SES-0518
```

```
mix_estimation = function() {
  counter1 = 1; exp1 = 0
  counter2 = 1; exp2 = 0
  for(i in geyser) {
    u = runif(1, 2.5,3.5)
    if(i < 3) {
      exp1[counter1] = i
      counter1 = counter1 + 1
    } else {
      exp2[counter2] = i
      counter2 = counter2 + 1
    }
  }
  theta1= (counter1-1)/length(geyser)
  theta2 = (counter2-1)/length(geyser)
  theta1; theta2
  exp1.mean = round(mean(exp1),3); exp2.mean = round(mean(exp2),3);
  exp1.mean; exp2.mean
  std1 = 0; std2 = 0
  for (i in 1:counter1 - 1) {
    std1[i] = (exp1[i] - exp1.mean)^2
  }
  for (i in 1:counter2 - 1) {
    std2[i] = (exp2[i] - exp2.mean)^2
  }
  var1 = round(sum(std1)/(counter1-1),3); var1
  var2 = round(sum(std2)/(counter2-1),3); var2
  result = list(theta = theta1,
                mu1 = exp1.mean, mu2 = exp2.mean,
                sigma1 = var1, sigma2 = var2)
  result
}
mix_estimation()
```

```
## $theta
## [1] 0.3566176
##
## $mu1
## [1] 2.038
##
## $mu2
## [1] 4.291
##
## $sigma1
## [1] 0.07
##
## $sigma2
## [1] 0.168
```

From the histgram we can see there are two peak and one pit in the graph, we use the pit as edge of two

normal distribution, for any number less than 3 it is belongs to first normal distribution, for any number larger than 3 it is belongs to second normal distribution.
We calculate the mean and variance for the first and second distribution respectively and we obtains

$$\mu_1 = 2.04, \mu_2 = 4.29, \sigma_1 = 0.07, \sigma_2 = 0.168$$

. And we estimate theta as the portion of whole data belongs to first normal distribution and second normal distribution, we get

$$\theta_1 = 0.36, \theta_2 = 0.64$$

We can see the mean and theta are approximately matches to the frequency graph for geyser data.
In order to check the accurcy of our approximation, we use EM algorithm the obtains the result.

```r
#use em to verify.
library("mixtools")
myEM = normalmixEM(geyser, mu = c(2, 4),lambda = c(0.5, 0.5), sigma = c(1, 1))
```

```
## number of iterations= 21
```

```r
myEM$mu; myEM$lambda; (myEM$sigma)^2
```

```
## [1] 2.018609 4.273344
```

```
## [1] 0.3484051 0.6515949
```

```
## [1] 0.05551835 0.19102298
```

We can see the approximation use EM are quite close to our approximation, the difference might cause by insufficient sample. Therefore our approximation are reasonable.

a)

$$g(t) = t - L_F(t)$$
$$g'(t) = 1 - L'_F(t)$$

is maximized when

$$g'(t) = 0$$

therefore:

$$L'_F(t) = 1$$
$$L'_F(t) = \frac{F^{-1}(t)}{\mu(F)}$$

Therefore it is maximized when

$$\mu(F) = F^{-1}(t)$$

b)
We subsitute LF(t) in part(a) we get

$$P(F) = t - \frac{1}{\mu(F)\int_0^t F^{-1}(s)ds}$$

Let

$$t = F(\mu(F))$$

then

$$P(F) = F(\mu(F)) - \frac{1}{\mu(F)\int_0^{F(\mu(F))} F^{-1}(s)ds}$$

use change of variables:

$$x = F^{-1}(s), s = F(x), ds = f(x)dx$$

$$= \int_0^{\mu(F)} f(x)dx - \frac{1}{\mu(F)} \int_0^{\mu(F))} xf(x)dx$$

$$= \frac{1}{\mu(F)} \int_0^{\mu(F)} (\mu(F) - x)f(x)dx$$

Since

$$E_F[|x - \mu(F)|] = \int_0^{\inf} |x - \mu(F)|f(x)fx$$

$$= \int_0^{\inf} |x - \mu(F)|f(x)fx$$

$$= \int_0^{\mu(F)} (\mu(F) - x)f(x)dx + \int_{\mu(F)}^{\inf} (x - \mu(F))f(x)dx$$

while u(F) is mean, two sides are equal, thus

$$\int_0^{\mu(F)} (\mu(F) - x)f(x)fx = \int_{\mu(F)}^{0} (\mu(F) - x)f(x)fx$$

We have

$$P(F) = \frac{1}{2\mu(F)} \int_0^{\inf} |\mu(F) - x|f(x)dx$$

c)

```r
library("stats4")
income = scan("/home/yiche/Desktop/github/UofT_course/STA355/incomes.txt");
n = length(income); income.log = log(income)
```

```r
library("exptest")
pietra.exp.test(income, nrepl = 1)
```

```
##
##  Test for exponentiality based on the Pietra statistic
##
## data:  income
## Pn = 0.34196, p-value < 2.2e-16
```

The unbiased estimator using jackknife is

$$\bar{X}_{-i} = \frac{1}{n-1}\sum_{j\neq i} X_j$$

```r
jackknife = function(x) {
  sigma = NULL
  estimator1 = mean(abs(x - mean(x)))/(2*mean(x))
  for (i in 1:length(x)) {
    xi = x[-i]
    estimator2 = mean(abs(xi - mean(xi)))/(2*mean(xi))
    sigma = c(sigma, n*estimator1 - (n-1)*estimator2)
  }
  se = sqrt(var(sigma)/n)
  result = list(std_err = se)
  result
}
jackknife(income)
```

```
## $std_err
## [1] 0.02150457
```

Therefore the estimate of P(x) is 0.34196 and its stardard error is 0.0215.

d)
We knoe that the normal distribution is:

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Loglikelihood of normal distribution is:

$$-\frac{n}{2} loog(2\pi) - \frac{n}{2} log(\sigma^2) - \frac{\sum(x_i - \mu)^2}{2\sigma^2}$$

$$\frac{\partial}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{\sum(x_i - \mu)^2}{2\sigma^4}$$

Use MLE:

$$-\frac{n}{2\sigma^2} + \frac{\sum(x_i - \mu)^2}{2\sigma^4} = 0$$

$$\sigma^2 = \frac{\sum(x_i - \mu)^2}{n}$$

$$\frac{\partial}{\partial (\sigma^2)^2} = \frac{n}{2\sigma^4} - \frac{\sum(x_i - \mu)^2}{2\sigma^6}$$

After subsistution of mle variance we obtain:

$$\frac{\partial}{\partial (\sigma^2)^2} = -\frac{n}{2\sigma^4}$$

We can get estimated std err using fisher information:

$$se(\sigma^2) = \frac{\partial}{\partial \sigma^4} = (\frac{n}{2\sigma^4})^{-\frac{1}{2}}$$

```
D(expression(2*pnorm(sigma/2)-1),"sigma")
```

```
## 2 * (dnorm(sigma/2) * (1/2))
```

$$se(P(\sigma^2)) = |P'(\sigma^2)| se(\sigma^2)$$

Where

$$P'(\sigma^2) = dnorm(\frac{\sigma^2}{2})$$

Then we have

$$se(P(\sigma^2)) = |dnorm(\frac{\sigma^2}{2})| * (\frac{n}{2\sigma^4})^{-\frac{1}{2}}$$

We use income data to calculate std error:

```
mean_logincome = mean(income.log); mean(income.log)
```

```
## [1] 10.44447
```

```
mle_var = sum((income.log - mean_logincome)^2)/n; mle_var
```

```
## [1] 0.7492891
```

```
se_P = abs(dnorm(mle_var/2)*(n/(2*(mle_var)^2))^(-0.5)); se_P
```

```
## [1] 0.02786641
```