2.2 Collapse of a Spherical Cavitation Bubble

This project is self-contained, although knowledge of the Part IB Fluid Dynamics course is an advantage in interpreting the physical significance of the results.

1 Introduction

The aim of this project is to examine the influence of surface tension on the time of collapse of a spherical cavitation bubble, and to determine the corresponding changes in the large (and often damaging) fluid pressures nearby.

For computational purposes, it is convenient to use dimensionless variables: units are chosen so that the initial radius of the bubble, the fluid density and the pressure at large distance are all unity. (This can be achieved by scaling the usual r, t and ρ variables.)

The bubble is represented by an empty spherical cavity of radius $R(t)$, where t is time, in an unbounded fluid which is initially at rest. It may be shown that the fluid pressure $p(r,t)$ at time t is given by

$$p(r,t) = 1 + \frac{1}{r}\frac{d}{dt}\left(R^2\frac{dR}{dt}\right) - \frac{R^4}{2r^4}\left(\frac{dR}{dt}\right)^2. \tag{1}$$

An evolution equation for $R(t)$ is obtained using the boundary condition

$$p(R,t) = -2\lambda/R, \tag{2}$$

where λ in these units is the (constant) surface tension at the surface ($r = R$) of the bubble. (In general, λ is surface tension scaled by pressure at infinity and initial bubble radius, and it measures the relative importance of surface tension and inertia in the system – an inverse Weber number.)

2 Analytic solutions for $\frac{dR}{dt}$ and $p(r,t)$

Question 1 Show that the equation for $R(t)$ admits a first integral

$$\left(\frac{dR}{dt}\right)^2 = \frac{2}{3}\left(\frac{1}{R^3} - 1\right) + 2\lambda\left(\frac{1}{R^3} - \frac{1}{R}\right). \tag{3}$$

Plot on the same graph dR/dt against R for $\lambda = 0.0$, 0.1, 1.0, 10.0, and 100.0. (Note that you may find it helpful to use logarithmic scales for both dR/dt and R.) Comment on your graph.

Question 2 Show that $p(r,t)$ can be expressed in terms of r and $R(t)$, that is it can be written $p(r,t) \equiv p(r,R)$. Show that $p(r,R)$ has a particularly simple form when expressed in terms of α and β, where

$$\alpha = \tfrac{1}{4}(1-4R^3) + \tfrac{3}{4}\lambda(1-3R^2) \qquad \text{and} \qquad \beta = 1 - R^3 + 3\lambda(1-R^2). \tag{4}$$

Show that the largest pressure *in the fluid* at time t is

$$p_{max} = \begin{cases} 1, & \alpha < 0 \\ 1 + R^{-3}(\alpha^4/\beta)^{1/3}, & \alpha > 0 \end{cases} \tag{5}$$

Plot on the same graph $p(r, R)$ against r for a range of values [about four will do] of R for the case $\lambda = 0$. For each value of R, what is p_{max} and what is the value of r which corresponds to p_{max}?

Plot similar graphs for the cases $\lambda = 0.2$ and $\lambda = 9.0$. You may also find it helpful to plot graphs of $p(r, R)$ against r normalised so that $p_{max} - p_{min} = 1$.

Comment on your graphs.

3 No surface tension case

Question 3 Take the case without surface tension $\lambda = 0$ and write a program to solve equation (3) for $R(t)$. You will find it worthwhile to observe the following points:

- Equation (3) is singular as $R \to 0$; to remove this singularity, define a new dependent variable, x, say, such that
$$x = R^{5/2}.$$
Show that (3) then becomes
$$\dot{x} = -\tfrac{5}{2}\left[\tfrac{2}{3}(1 - x^{6/5}) + 2\lambda(1 - x^{4/5})\right]^{1/2}, \tag{6}$$
with the condition $x = 1$ at $t = 0$. Justify mathematically and/or physically why the negative root has been chosen. Would it have made sense to choose the positive root?
- Equation (6) is non-analytic at $x = 0$. Explain with reasons why as $x \to 0$ higher-order numerical methods (such as Runge–Kutta) do not have their usual benefits over low order schemes such as the Euler method. For your numerical calculations use the Euler method.
- Equation (6) has the trivial solution $x = 1$ for all t. To avoid this, either use an alternative numerical scheme for the first step, or find a series solution for small time and start from a suitable non-zero value of time.

By making the substitution $x = \sin^{5/3}\theta$ and performing a numerical integration, or otherwise, solve (6) for $\lambda = 0$ to obtain a value for the time t_c for collapse.

Using your program, determine the time t_c for collapse. Plot R as a function of t, and show also the behaviour of $p_{max}(t)$ (choosing appropriate scales to show the variation in $p_{max}(t)$). Justify carefully the numerical accuracy of your results.

4 With surface tension case

Question 4 Repeat the computation for a representative set of [positive] values of λ. Interpret your results both physically and mathematically. Comment on the limits $\lambda \ll 1$ and $\lambda \gg 1$.

Question 5 What are likely to be the physical limitations of this model?

So far we have considered dimensionless variables. Now make a rough estimate of the actual initial radius of bubbles that might be generated by a boat propellor. Assuming that the bubbles are empty, use your graphs of dR/dt from Question 1 to estimate a radius below which the physical limitations of this model are likely to be important.

How might you change the model to allow for bubbles in a liquid that are not empty but contain small amounts of vapour or other gas?

Historical Footnote. Calculation of the collapse of a spherical cavitation bubble (in a slightly compressible fluid) was one of the earlier uses of computers in fluid dynamics (see C. Hunter's Ph.D. thesis 1960 in the DAMTP library).

