Normalizing Flows

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Abstract

Notes on normalizing flows.

1 Planar Flow

$$f(\mathbf{z}) = \mathbf{z} + \mathbf{u}h(\mathbf{w}^\top \mathbf{z} + b)$$

$$\frac{\partial f(\mathbf{z})}{\partial \mathbf{z}} = \mathbf{I} + \mathbf{u}h'(\mathbf{w}^\top \mathbf{z} + b)\mathbf{w}^\top$$

Now, using the matrix determinant lemma

$$\begin{aligned}\det \frac{\partial f(\mathbf{z})}{\partial \mathbf{z}} &= (1 + h'(\mathbf{w}^\top \mathbf{z} + b)\mathbf{w}^\top \mathbf{I}^{-1}\mathbf{u}) \det(\mathbf{I}) \\ &= (1 + h'(\mathbf{w}^\top \mathbf{z} + b)\mathbf{w}^\top \mathbf{u})\end{aligned}$$

2 Radial Flow

$$f(\mathbf{z}) = \mathbf{z} + \beta h(\alpha, r)(\mathbf{z} - \mathbf{z}_0)$$

where $h(\alpha, r) = (\alpha + r)^{-1}$ and $r = ||\mathbf{z} - \mathbf{z}_0||$.

$$\begin{aligned}\frac{\partial f(\mathbf{z})}{\partial \mathbf{z}} &= \mathbf{I} + \beta \left((\mathbf{z} - \mathbf{z}_0)h'(\alpha, r)\frac{\partial r}{\partial \mathbf{z}} + h(\alpha, r)\mathbf{I} \right) \\ &= (1 + \beta h(\alpha, r))\mathbf{I} + \beta h'(\alpha, r)(\mathbf{z} - \mathbf{z}_0)\frac{(\mathbf{z} - \mathbf{z}_0)^\top}{||\mathbf{z} - \mathbf{z}_0||}\end{aligned}$$

Again, using the matrix determinant lemma

$$\begin{aligned}\det \frac{\partial f(\mathbf{z})}{\partial \mathbf{z}} &= \left(1 + \beta h'(\alpha, r)\frac{(\mathbf{z} - \mathbf{z}_0)^\top}{||\mathbf{z} - \mathbf{z}_0||}\frac{\mathbf{I}}{(1 + \beta h(\alpha, r))}(\mathbf{z} - \mathbf{z}_0) \right) \det((1 + \beta h(\alpha, r))\mathbf{I}) \\ &= \left(\frac{1 + \beta h(\alpha, r) + \beta h'(\alpha, r)||\mathbf{z} - \mathbf{z}_0||}{(1 + \beta h(\alpha, r))} \right)(1 + \beta h(\alpha, r))^d \\ &= (1 + \beta h(\alpha, r) + \beta h'(\alpha, r)r)\,(1 + \beta h(\alpha, r))^{d-1}\end{aligned}$$