Discrete Mathematics

December 13, 2021

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Instructions: Research about *The Pigeonhole Principle* and develop examples of its applications.

About The Pigeonhole Principle

The Pigeonhole principle says:

> If n pigeonholes are occupied by n+1 pigeons, at least one pigeonhole is occupied by more than one pigeon.

In fact, from a mathematical perspective, let p be the pigeons and h the pigeonholes, then at least one pigeonhole contains at least $\frac{p}{h}$ pigeons.

Examples with applications

1. If m is an odd positive integer, then prove that there exists a positive integer n such that m divides $2^n - 1$.

   Let m be an odd positive integer, we consider the finite sequence of powers of 2:

   $$2^1, 2^2, ..., 2^m.$$

   Since m is odd, it does not divide to any of the above powers. So the possible remainders of $2^i \div m$ (with $1 \leq i \leq m$) are:

   $$1,\ 2,\ ...,\ m-1$$

   Since there are $m-1$ possible remainders (pigeonholes) and m powers of 2 (pigeons), it follows that there exist at least two positive integers p and q (say $p < q \leq m$) such that 2^p and 2^q have the same remainders in the division by m. Thus, it turns out that m divides $2^q - 2^p$. Let us note that

   $$2^q - 2^p = 2^p(2^{q-p} - 1)$$

   Since p and q are positive integers, with $p < q$, then $q - p$ is a positive integer (say n) and since m does not divide 2^p, then m divides $2^n - 1$.

   $\therefore$ There is a positive integer n such that m divides $2^n - 1$.

2. Show that if any eight positive integers are chosen, two of them will have the same remainder when divided by 7.

   The remainders that result from dividing any positive integer by 7 are: $\{0, 1, 2, 3, 4, 5, 6\}$; that is, seven possibilities, which can be considered as the pigeonholes, under the analogy of the pigeonhole principle, as shown:

   $$\boxed{0}\ \boxed{1}\ \boxed{2}\ \boxed{3}\ \boxed{4}\ \boxed{5}\ \boxed{6}$$

   Then we take 8 integers (inhabitants) that when arranged in the houses, by principle of the pigeonhole, at least in one pigeonhole, there will be more than one inhabitant, as shown: