Probability
Lab Assignment
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(10) 1. Consider the experiment of flipping one fair coin whose two sides we will call heads (H) and tails (T).

(a) What is the sample space Ω?

The sample space is : {H,T}

(b) What is the probability of each outcome in Ω?

The probability of each outcome (if the coin is fair) is:
Heads = 1/2
Tails = 1/2

(c) Take a coin (say, a US quarter) out of your pocket, and flip it 10 times. How many times does it came up H, and how many does times it came up T? Do this entire thing again a few (at least 3) more times.

For the first batch of 10, heads came up 6 times and tails came up 4 times. The following are the next 3 batches.

1) 4 heads, 6 tails; 2) 4 heads, 6 tails; 3) 5 heads, 5 tails

(d) Read the abstract[1] of the paper "Dynamical Bias in the Coin Toss" by Persi Diaconis, Susan Holmes, and Richard Montgomery[2] In a few sentences discuss what this might mean for the distinction between theoretical models and actual data in real life.

Theoretical models do not take into account all the potential variables for differences between what is conceptually envisioned and what actually occurs. The discrepancy for a flipping a coin appears to be from the means of a human hand flipping the coin. This difference changes the theoretical value slightly.

[1]Read the rest if you like as well!
[2]https://statweb.stanford.edu/ susan/papers/headswithJ.pdf

(10) 2. Consider the experiment of flipping two fair coins, one with our left hand (call this the "first coin") and one with our right hand (call this the "second coin").

(a) What is the sample space Ω?

The sample space is: {HH, HT, TH, TT}

(b) Let A be the event that "the first coin came up H and the second coin came up T." Compute $P(A)$.

P(A) = P(H) * P(T) = 1/2 * 1/2 = 1/4

(c) Let B be the event that "exactly one H came up." Compute $P(B)$.

P(B) = P(H) * P(T) + P(T) * P(H) = 1/2 * 1/2 + 1/2 * 1/2 = 1/2

(10) 3. Suppose instead we flip seven fair coins.

(a) Let C be the event that "the first coin came up H and all other coins came up T'." Compute $P(C)$.

P(C) = P(H) * P(T)* P(T)* P(T)* P(T)* P(T)* P(T) = 1/2 * 1/2* 1/2* 1/2* 1/2* 1/2* 1/2= 1/128

(b) Let D be the event that "exactly one H came up." Compute $P(D)$.

P(D) = 7 * 1/128= 7/128

(30) 4. Consider the experiment of flipping three fair coins. Let X be the discrete random variable that counts how many heads came up in the experiment.

(a) What are the possible values that X can take?

X:{0,1,2,3}

(b) For each of these values x_i, what is $P(X = x_i)$?

P(X=0) = P(T)*P(T)*P(T) = 1/8
P(X=1) = P(H)*P(T)*P(T) + P(T)*P(H)*P(T) + P(T)*P(T)*P(H) = 3/8
P(X=2) = P(H)*P(H)*P(T) + P(H)*P(T)*P(H) + P(T)*P(H)*P(H)= 3/8
P(X=3) = P(H)*P(H)*P(H) = 1/8

(c) What is the expected value $E[X]$?

E[X] = 0 *1/8 + 1*3/8 + 2*3/8 + 3 *1/8 = 1.5

(d) What is the variance $Var[X]$? What is the standard deviation?

Var[X]=E[(x-E[x])^2]= (0-1.5)^2*1/8 + (1-1.5)^2*3/8 + (2-1.5)^2*3/8 + (3-1.5)^2 1/8
= .28125 + .09375 + .09375 + .28125= .75

Standard Deviation = sqrt(Var[x]) = .866

(e) What is the probability that a specific run of the experiment will result in $X = E[X]$?

The probability of having X = E[X] after one specific run is zero because the outputs are discrete, not continuous, meaning that you cannot get a fraction of a head as a result.

(f) Take three coins out of your pocket, and run this experiment yourself a large number of times. Each time you run it, note the value of X. Visualize your results in the form of a histogram.

# of occurrances
4
3
2
1
0
1
2
3
# of heads

(20) 5. Consider a fair six-sided die, with faces numbered 1 through 6. Let's play the following game which will produce a discrete random variable X. We roll this fair die four times. If we never see a 1, then $X = 0$. Otherwise, let X be the roll on which we *first* see a 1. For example, if we roll $(2,1,3,1)$, then $X = 2$. If we roll $(6,4,2,1)$, then $X = 4$, and if we roll $(3,6,2,2)$, then $X = 0$.

(a) Play this game a large number of times and sketch (to scale) a histogram of the observed X-values.

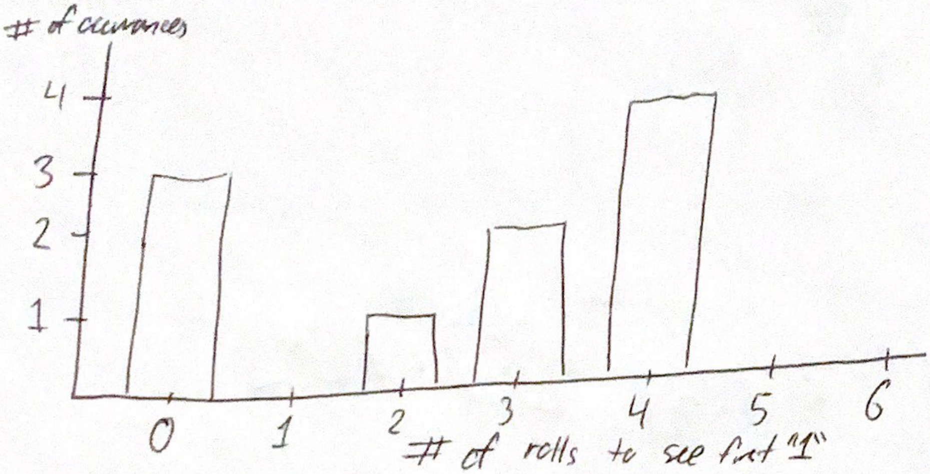
(b) Compute the theoretical expected value $E[X]$.

3 (2, 4, 1, 4) 0 (4, 6, 6, 2) 0 (3, 4, 2, 6)

2 (2, 1, 1, 5) 4 (3, 2, 5, 1) 4 (3, 4, 6, 1)

4 (6, 6, 5, 1) 0 (6, 5, 4, 2)

4 (2, 3, 5, 1) 3 (4, 4, 1, 5)



$$P(X=0) = \frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6} = P(x\neq 1)\cdot P(x\neq 1)\cdot P(x\neq 1)\cdot P(x\neq 1) \approx .482$$

$$P(X=1) = \frac{1}{6} = P(x=1) \approx .167$$

$$P(X=2) = \frac{5}{6}\cdot\frac{1}{6} = P(x\neq 1)\cdot P(x=1) \approx .139$$

$$P(X=3) = \frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6} = P(x\neq 1)\cdot P(x\neq 1)\cdot P(x=1) \approx .116$$

$$P(X=4) = \frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6} \approx .096$$

$$\sum P(X=1\ldots 4) = 1$$

$$E[X] = 0\cdot P(x=0) + 1\cdot P(X=1) + 2\cdot P(x=2) + 3\cdot P(X=3) + 4\cdot P(X=4)$$

$$= 0 + .167 + .278 + .348 + .384 = \boxed{1.177}$$

(30) 6. Now let's make the last problem a little bit harder (warning: you will need to use your Calculus skills, specifically series and sequences, a little bit here!). The game is as follows. We roll a fair six-sided die over and over again, until we see our first 1. Then we stop. Let Y be the roll on which we see this 1.

Compute the expected value $E[Y]$.

$Y: \{ 1, 2, \dots \infty \}$ all $\oplus$ real integers

$P(Y=1) = \frac{1}{6} \approx .167$

$P(Y=2) = \frac{5}{6} \cdot \frac{1}{6} \approx .139$

$P(Y=3) = \frac{5}{6}\frac{5}{6}\frac{1}{6} \approx .116$

$\vdots$

$P(Y=N) = \left(\frac{5}{6}\right)^{N-1} \frac{1}{6}$

$$E[Y] = \sum_{i=1}^{N} y_i \cdot P(Y = y_i)$$

$$= \frac{1}{6} \sum_{i=1}^{\infty} i \left(\frac{5}{6}\right)^{i-1}$$

$$\frac{d}{dx}\left(\sum_{i=0}^{\infty} x^i\right) = \left(\frac{1}{1-x}\right)\frac{d}{dx}$$

$$\sum_{i=1}^{\infty} i x^{i-1} = \frac{1}{(1-x)^2}$$

$$\therefore \frac{1}{6} \sum_{i=1}^{\infty} i \left(\frac{5}{6}\right)^{i-1} = \frac{1}{6} \cdot \frac{1}{(1-5/6)^2} = \boxed{6} = E[Y]$$

(30) 7. Now you're going to play a game called Solo Pig[3]. The rules are as follows. You start with zero points. At each turn of the game, you have one of two options, Stop or Roll. If you Stop, the game ends and you keep the points you have. If you Roll, then one of two things happens. If you roll a 1, the game ends and you finish with zero points. If you roll something other than a 1, then you add the number of dots rolled to your score, and you get another turn. The game continues until you roll a 1 or you choose Stop. For example, a series of rolls $(5,3,3,4,1)$ would give me zero points, but a series of rolls $(5,3,3,4)$ followed by a Stop choice would leave me with 15 total points.

Play this game a few times yourselves, experimenting with different strategies. Write down the results. Then, using your answer to the previous question, come up with what you think is the optimal strategy for Solo Pig[4]

After rolling several times, it can be seen that there is an average number of rolls before the chance of rolling a 1 is too high. This is very similar to the problem before because as you increase the number of rolls, you decrease your chances of success. I believe that the optimal strategy would be to continue rolling the die until the combined probability is reduced to just above 1 in 2 odds. Since the goal is to roll anything but a 1, the probability is 5/6 per roll. After 3 rolls, the probability of not getting a 1 is about 57.9%. After 4 rolls however, the chance drops below to 48.2%. This was a pattern that was starting to be seen after some experimenting.

[3]There's a more common, and more complicated dice game called Pig, which I encourage you to look up.
[4]That is, at what total score is it a good idea to Stop rather than to Roll?

(10) 8. Provide two different possible sets of 5 observations that have a mean equal to 20, yet a mean of 20 is a potentially misleading summary statistic.

One possible set of 5 observations could be the amount of points that are possible to be earned per round
Points:{10, 15, 20, 25, 30}

Another possible set is another game where the points are widely out of proportional, so the points that can be earned are
Points:{100, 90, 80, 70, -240}

9. Suppose there are 40 people in the classroom across the hall, and you know that the mean age of the people in that room is 20 years old and half are wearing blue shirts while half are wearing white shirts. What can you say about a "typical" occupant of that classroom? What can you say about the population of people in that classroom as a whole?

Knowing that the half the people are wearing blue and the other half are wearing white, only gives information that any one sample would be wearing one or the other. This information is independent of the age of an occupant since a set has not been drawn from the ages of the people in the classroom. Since only the mean of the people is 20 years old, it is unknown if the majority are within a few years of 20 or if half the group is 30 while the other half is 10. From the vague bits of information, there are no significant characteristics that can be assessed about a typical occupant other than that they will be wearing either a blue or white shirt and will be older than 0 years old (since humans only use positive integers when assessing how many years old).