FIG. 1. Georg Friedrich Bernhard Riemann

1. Analytic Sets

Definition 1.1. Let G be a domain in $\mathbb{C}^n$, a close set $A \subseteq G$ is called **analytic set**, if and only if for any $z \in A$, there exists an open set $V \subseteq G$ containing z and finite many holomorphic functions $\{f_1, f_2, , , f_q\}$,s.t. $Z(f_1, f_2, , , f_q) = A \cap$V.

Proposition 1.2. *Let A be an analytic set in domain G, if A has a inner point,then $A = G$, else $G \backslash A$ is dense and connected.*

Proof. (The proof is not complete, also considerting the case that A has countable infinite discrete points.)
Case 1: Let G be a ball in $\mathbb{C}^n$ and $\{f_1, , , , f_q\}$be a set of finite many holomorphic functions on defined on G s.t. $A = Z(f_1, , , f_q)$.
If A has at least one inner point, then $\{f_1, , f_q\} = \{0\}$, therefore $A = G$.
If else, for any $z_1, z_2 \in G \backslash A$, let L be the unique complex line connecting z_1 and z_2, then $\{f_1, , f_q\}|_L$ can be viewed as a set of holomorphic funcitons of one variable, without loss of genelarity, assume that $L \nsubseteq Z(f_1)$, then $L \cap Z(f_1)$is discrete and therefore, ww can find a path in $L \subseteq G = B$ connecting z_1 and z_2, and avoiding A.(It a fact in algebraic topology.)
Case 2: Let G be a general domain in $\mathbb{C}^n$,and assume that A has no inner point.

FIG. 2. Augustin Louis Cauchy
柯西男爵抱有坚定的保皇信念和极端的宗教观点. 在他工作的这个活跃期间正值波旁王朝复辟，而在 1830 年的七月革命之后，柯西与皇族家庭一起移居到了意大利. 但到了 1838 年他返回了祖国并重新在一所天主教会学院里教数学，到了 1848 年他成了巴黎大学文理学院 (Sorbonne) 的教授 (但他拒绝宣普效忠于政府)

For any path γ in G connecting z_1 and z_2, we can find finite open balls covering the path, then we can find an another path homotopic to γ avoiding A. □

2. Analytic Continuation

If A in analytic in domain G. we call A is proper iff $A \neq G$.

Theorem 2.1 (**Riemann Continuation Theorem**). *Let A analytic proper in domain $G \subseteq \mathbb{C}^n$, and f holomorphic in $A \nsubseteq G$, and f is locally bounded in any point $z \in A$, the f can be holomorphicly continuated to G.*

Proof. Case 1: n=1. It's the known case in complex analysis of one variable.
Case 2: n>1.For any $z_0 \in A$we can find a complex L through z_0,and **locally** $L \cap A = \{z_0\}$.(why?).
Through complex linear transformation and moving orign, L satisfies the

FIG. 3. Kodaira Kunihiko（小平邦彦）

system$\{z_2 = z_2 = ... = z_n = 0\}$ Let r_1,$r^{'}$ be suffiently small positive numbers s.t.

$$P = \{z = (z_1, z^{'}) \in \mathbb{C} \times \mathbb{C}^{n-1} | z^{'} = (z_2, , , z_n), |z_1| < r_1, |z^{'}| < r^{'}\} \subseteq G.$$

and $A \cap \{z \in \mathbb{C}^n | \ |z_1| = r_1, |z^{'}| < r^{'}\} = \emptyset$, moreover for any $0 \leq c \leq r_1$, $A \cap \{z \in \mathbb{C}^n | \ |z_1| = r_1, |z^{'}| < c\}$can be viewed as the intersection $\mathbb{D} \cap A$. Then through the one dimensional case, we can continuate f on $G \backslash A$ holomorphicly to P and therefore to G. $\square$

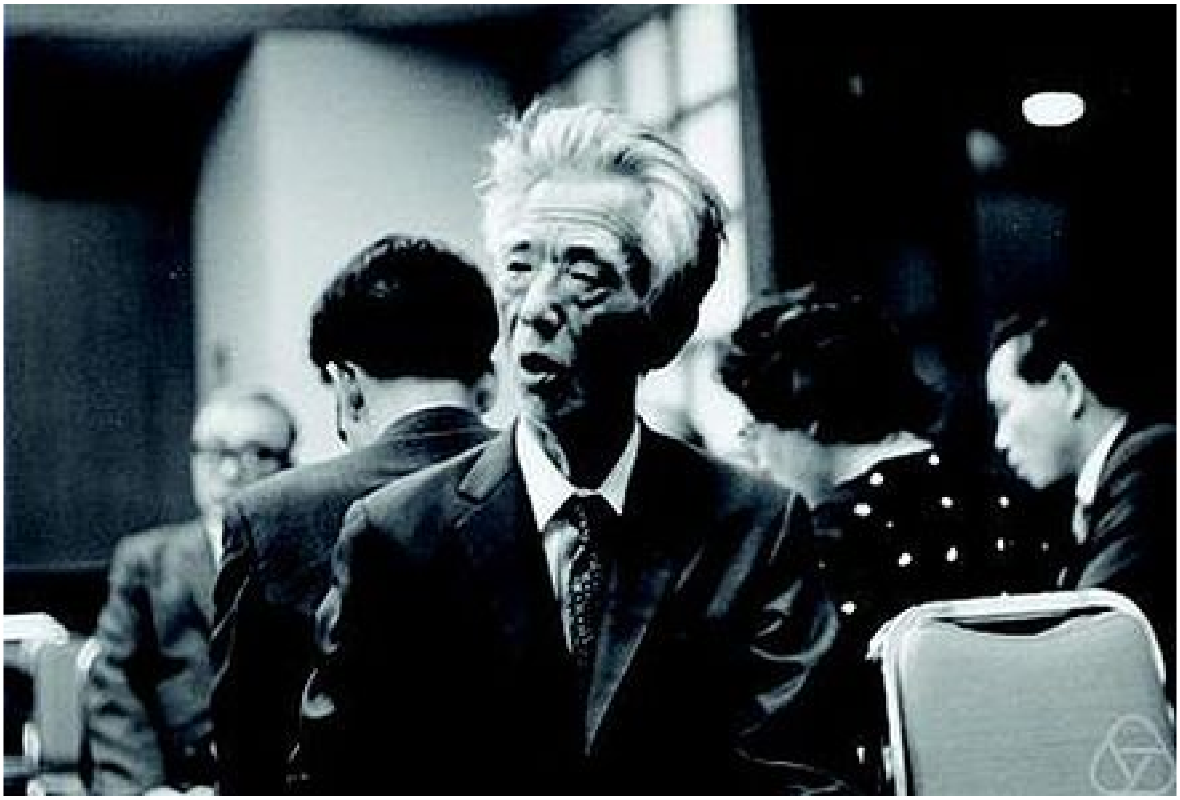

FIG. 4. Kiyoshi Oka（冈洁）
1973 年摄于京都

3. REGULAR POINTS OF ANALYTIC SETS

Definition 3.1. Let G be a domain in $G \subseteq \mathbb{C}^n$, A is locally defined at $z_0 \in A$ by $Z(f_1, f_2, , , f_q)$where $f_i \in \mathscr{O}(G)$, then $rank_{z_0}(f_1, f_2, , , f_q) := rank(J(f_1, f_2, , , f_q)(z_0)) = rank(\dfrac{\partial f_i}{\partial z_j}(z_0)) \leq q$. If such $\{f_1, f_2, , , f_q\}$ exists and $rank_{z_0}(f_1, f_2, , , f_q = 0)$ we call A is regular q codimensional at z_0. $z_0$is called a regular point of A. All regular points called the regular set of A(written as A_{reg}), the set of other points is called singluar set.(A_{sing})

Theorem 3.2 (**local parameterization**). *Let A analytic in domain*$G \subseteq \mathbb{C}^n, z_0 \in A$, *then A is q codimensional regular at* ***iff*** *at* z_0 *G is locally biholomorphic to an open set* $W \subseteq \mathbb{C}^n$ *centered at the orign, and the biholomorphic map satisfies:* $F(z_0) = 0$*and* $F(U \cap A) = \{w = (w_1, , , w_n) | \ w_{n-q+1} = ... = w_n = 0\}$

Proof. The proof is similiar to the proof of implict function theory and omitted here. □

It is obvious that A_{reg} is open in A.

Theorem 3.3. *Let G be a domain of $\mathbb{C}^n$, and $f \in \mathscr{O}(G)$ and f is not constant, then $Z(f)$ has at least one regular point.*

Proof. Case 1: n=1, obvious.
Case 2: n>1. Assume that $A = Z(F) = \emptyset$, then $df = 0$ on A.
For any $z_0 \in A$, we can always find $n_0 \in \mathbb{N}_+$, and a multiplied index v_0, and $\lambda \in \{1,2,,,n\}$, s.t $|v_0| = n_0$ and $(D^{v_0} f)(z_0) \neq 0$, moreover $D^v f)(z) = 0$ for any $z \in A$, $|v| \leq n$ we define $M = \{z \in G| \ D^{v_0} f(z) = 0\}$, then $A \subseteq M$ and M is regular at z_0. Through a local parameterization at z_0, we have $z_0 = 0$ and $M = \{z = (z_1,,,z_n)| \ z_1 = 0\}$, then $f(\zeta, \vec{0}) \neq 0$ for $|\zeta| \neq 0$and small enough. Using **Argument principle**, we notice that viewing z_0 as variable, for $|\vec{\zeta}| < r$ small enough. $f(\eta, \vec{\zeta})$has zero points in ball $|\vec{\zeta}| < r$,and therefore the points fall in A. Consequently, we have $A = M$. This leads to the contradiction. ☐