Catalog of Friedmann Universes

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Abstract

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$$H_0 \int dt = \int \frac{da}{\sqrt{\Omega_{r,0}/a^2 + \Omega_{m,0}/a + a^2\Omega_{\Lambda,0} + 1 - \Omega_0}} \tag{0.1}$$

1 | Single Component Flat Universe Models

1.1 Matter Dominated Universe

$$\Omega_m = 1 \implies \Omega_r = \Omega_\Lambda = \Omega_k = 0$$

$$H_0 \int dt = \int_0^a \frac{da'}{\sqrt{\Omega_{m,0}/a'}} = \int_0^a \sqrt{a'}da' \implies a(t) = \left(\frac{3}{2}H_0 t\right)^{2/3} \tag{1.1}$$

by integrating RHS 0 to 1, one can find the age of such a universe as

$$t_0 = \frac{2}{3}H_0 = 9.32 \text{ Gyr} \tag{1.2}$$

1.2 Radiation Dominated Universe

$$\Omega_r = 1 \implies \Omega_m = \Omega_\Lambda = \Omega_k = 0$$

$$H_0 \int dt = \int_0^a \frac{da'}{\sqrt{\Omega_{r,0}/a'^2}} = \int_0^a a' da' \implies a(t) = (2H_0 t)^{1/2} \tag{1.3}$$

by integrating RHS 0 to 1, one can find the age of such a universe as

$$t_0 = \frac{1}{2}H_0 = 6.99 \text{ Gyr} \tag{1.4}$$

1.3 Λ Dominated Universe (de Sitter)

$$\Omega_\Lambda = 1 \implies \Omega_m = \Omega_r = \Omega_k = 0$$

$$H_0 \int dt = \int_0^a \frac{da'}{\sqrt{a'^2\Omega_{\Lambda,0}}} = \int_0^a \frac{da'}{a'} \implies a(t) = e^{H_0 t} \tag{1.5}$$

1.4 Empty Universe

$$\Omega_k = 1 \implies \Omega_m = \Omega_r = \Omega_\Lambda = 0$$

$$H_0 \int dt = \int_0^a da' \implies a(t) = H_0 t \tag{1.6}$$

by integrating RHS 0 to 1, one can find the age of such a universe as

$$t_0 = \frac{1}{H_0} = 13.98 \text{ Gyr} \tag{1.7}$$

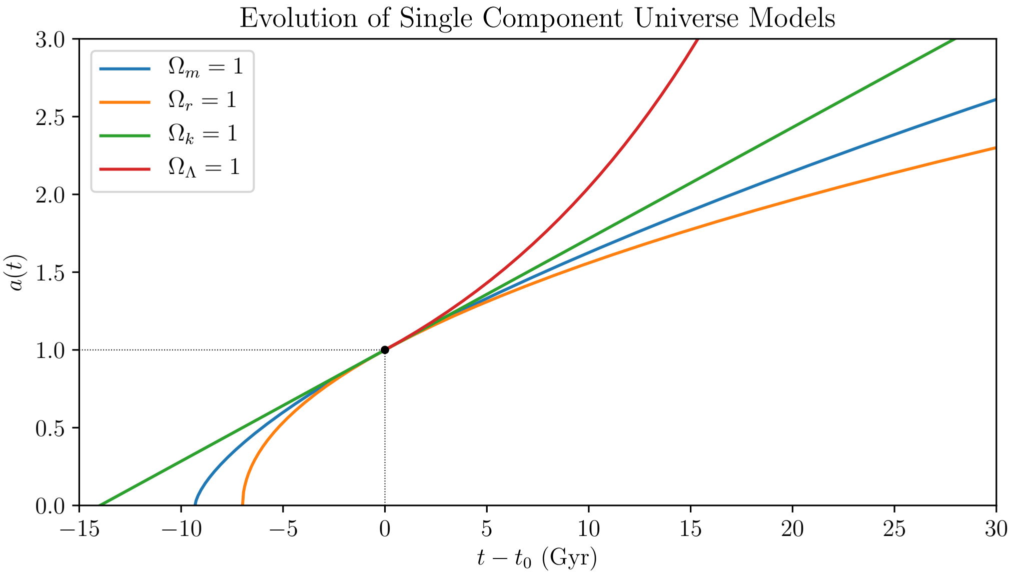

Figure 1.1. Evolution of Various Single Component Universe Models

{fig:s

2 | Multiple - Component Universe Models

2.1 Matter - Curvature Models

For matter + curvature case, the equation

$$H_0 \int dt = \int \frac{da}{\sqrt{\Omega_{r,0}/a^2 + \Omega_{m,0}/a + a^2\Omega_{\Lambda,0} + 1 - \Omega_0}}$$

becomes

$$H_0 \int dt = \int \frac{da}{\sqrt{\Omega_{m,0}/a + 1 - \Omega_0}}$$

Since the matter (and DM) is the only component let us denote $\Omega_{m,0}$ as Ω_0.

The solution differs for $\Omega_0 > 1$ and $\Omega_0 < 1$ cases, an elegant solution is provided in AstroBaki.

2.1.1 Closed ($\Omega_0 > 1$) Models

Evolution of the scale factor is given in parametric form,

$$\begin{aligned} a(\theta) &= \frac{1}{2}\frac{\Omega_0}{\Omega_0 - 1}(1 - \cos\theta) \\ t(\theta) &= \frac{1}{2H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}(\theta - \sin\theta) \end{aligned} \tag{2.1}$$

{eq:Ma

where $\theta \to (0, 2\pi)$, and the age of such a universe can be found by

$$t_0 = \frac{1}{H_0}\left[\frac{\Omega_0}{2(\Omega_0 - 1)^{3/2}} \arccos\left(\frac{2 - \Omega_0}{\Omega_0}\right) - \frac{1}{\Omega_0 - 1}\right] \tag{2.2}$$

Therefore, time elapses between the Big Bang at $\theta = 0$ and the Big Crunch at $\theta = 2\pi$ is

$$t_{\text{crunch}} = \frac{\pi}{H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} \tag{2.3}$$

Another useful parameter is the scale factor at the time of maximum expansion, here maximum also implies that $H(t) = 0$,

$$\begin{aligned} \frac{H^2(t)}{H_0^2} &= \frac{\Omega_0}{a^3} + \frac{1 - \Omega_0}{a^2} \\ 0 &= \frac{\Omega_0}{a_{\text{max}}^3} + \frac{1 - \Omega_0}{a_{\text{max}}^2} \end{aligned} \tag{2.4}$$

Thus,

$$a_{\text{max}} = \frac{\Omega_0}{\Omega_0 - 1} \tag{2.5}$$

a_{max} and the corresponding time to this maximum expansion are shown in Figure 2.1.

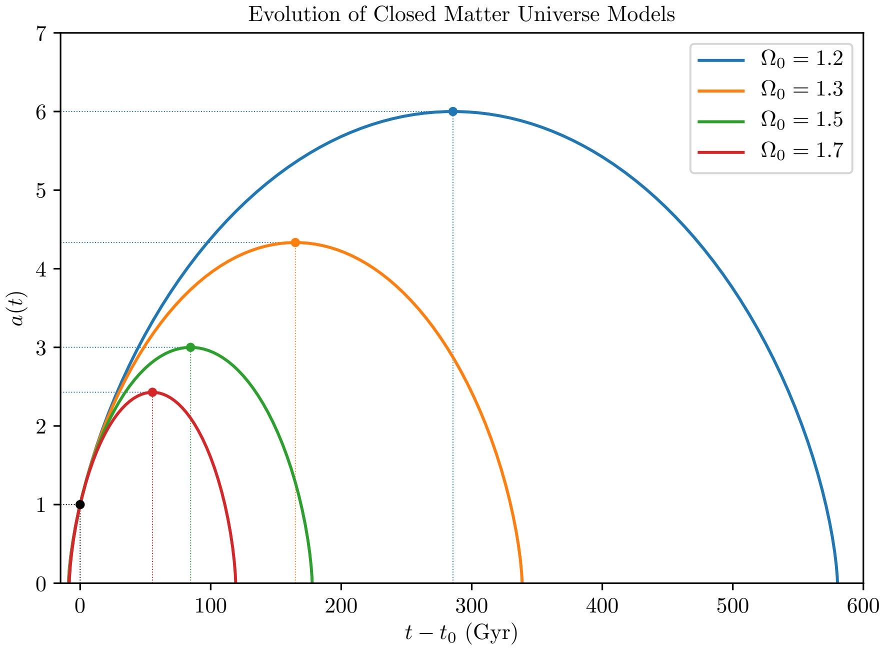

Figure 2.1. Evolution of Closed Matter + Curvature Universe Models {fig:M

2.1.2 Open ($\Omega_0 < 1$) Models

Evolution of the scale factor is given in parametric form,

$$
\begin{aligned}
a(\eta) &= \frac{1}{2}\frac{\Omega_0}{1-\Omega_0}(\cosh\eta - 1) \\
t(\eta) &= \frac{1}{2H_0}\frac{\Omega_0}{(1-\Omega_0)^{3/2}}(\sinh\eta - \eta)
\end{aligned}
\tag{2.6}
$$

where $\eta \;\rightarrow (0, \infty)$, and the age of such a universe can be found by

$$
t_0 = \frac{1}{H_0}\left[\frac{1}{1-\Omega_0} - \frac{\Omega_0}{2(1-\Omega_0)^{3/2}}\,\text{arccosh}\left(\frac{2-\Omega_0}{\Omega_0}\right)\right]
\tag{2.7}
$$

Plot of various open universe models is given in Figure 2.2, and in order to see the things together examples of open, flat, and closed universe models are shown in Figure 2.3.

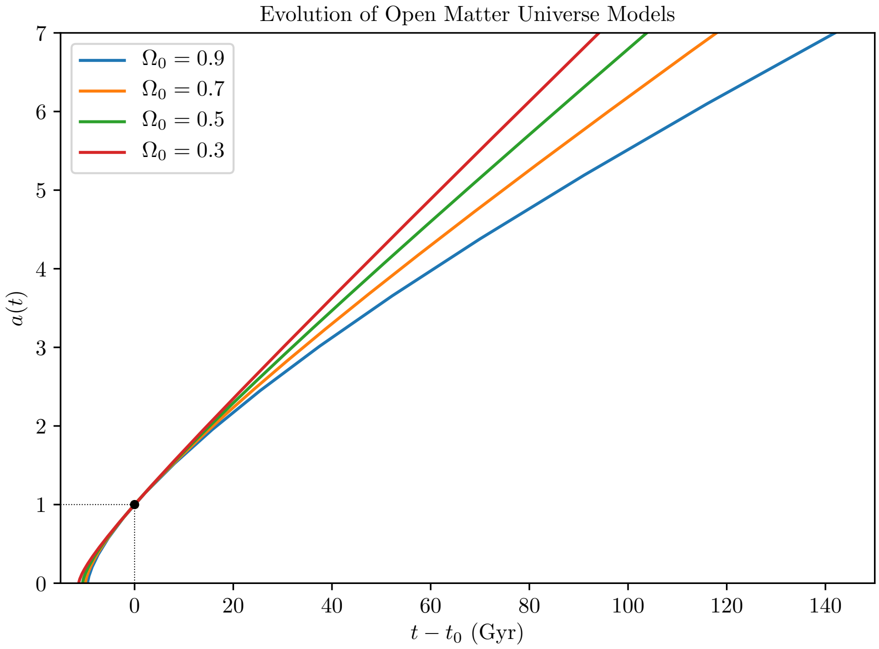

Figure 2.2. Evolution of Open Matter + Curvature Universe Models {fig:M

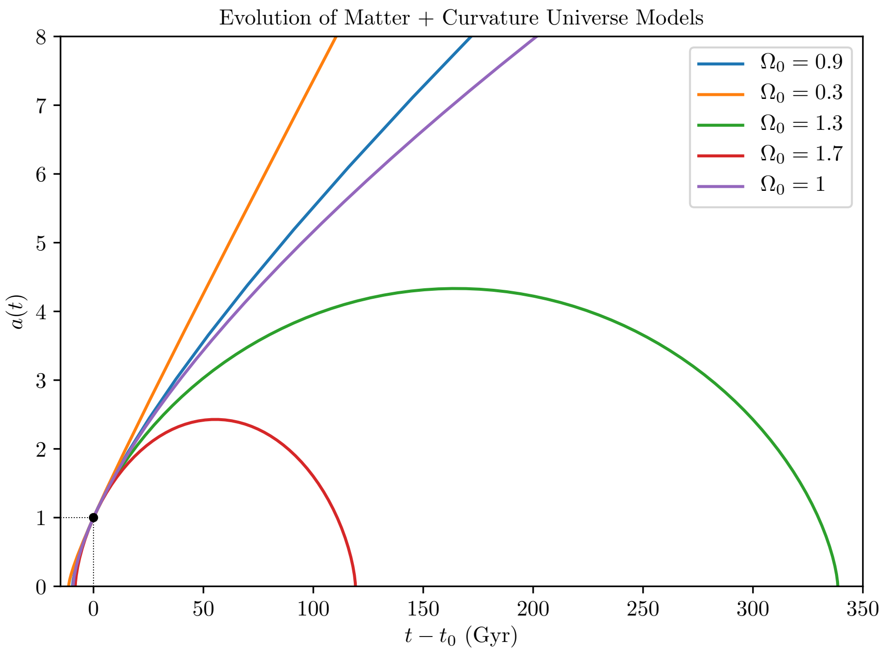

Figure 2.3. Evolution of Various Matter + Curvature Universe Models {fig:M

2.2 Matter - Cosmological Constant (Λ) Models

Consider a spatially flat universe which contains matter and cosmological constant[1], therefore Friedmann equations reduces to

$$\frac{H^2}{H_0^2} = \frac{\Omega_{m,0}}{a^3} + \Omega_\Lambda \tag{2.8}$$

but, the requirement for a flat universe brings a constraint on $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$

$$\Omega_{\Lambda,0} = 1 - \Omega_{m,0}$$

thus,

$$\frac{H^2}{H_0^2} = \frac{\Omega_{m,0}}{a^3} + (1 - \Omega_{m,0}) \tag{2.9}$$

2.2.1 Negative Cosmological Constant

In a flat universe with $\Omega_{\Lambda,0} < 0$, the negative cosmological constant provides an attractive force, NOT the repulsive force of a positive cosmological constant, such a universe will cease to expand at a maximum scale factor. To find maximum scale factor, let us just use the fact that $H\left(t_{a_{\max}}\right) = 0$ in Friedmann equation

$$0 = \frac{\Omega_{m,0}}{a_{\max}^3} + (1 - \Omega_0)$$

Therefore,

$$\boxed{a_{\max} = \left(\frac{\Omega_{m,0}}{\Omega_{m,0} - 1}\right)^{1/3}} \tag{2.10}$$

and will collapse back to $a = 0$ at a cosmic time

$$t_{\text{crunch}} = \frac{2\pi}{3H_0}\frac{1}{\sqrt{\Omega_{m,0} - 1}} \tag{2.11}$$

Note that for a given H_0, the larger $\Omega_{m,0}$, the shorter lifetime of the universe.

Friedmann equation can be integrated to yield the analytic solution

$$H_0 t = \frac{2}{3\sqrt{\Omega_{m,0} - 1}} \arcsin\left[\left(\frac{a}{a_{\max}}\right)^{3/2}\right] \tag{2.12}$$ {eq:ma

which can be inverted to $a(t)$ as

$$a(t) = \left[a_{\max}^{3/2} \sin\left(\frac{3}{2} H_0 t \sqrt{\Omega_{m,0} - 1}\right)\right]^{2/3} \tag{2.13}$$

[1] Such a universe is of particular interest to us, since it appears to be a close approximation to our own universe at the present day.

and the present age of such a universe is

$$t_0 = \frac{2}{3H_0\sqrt{\Omega_{m,0}-1}} \arcsin\left(\sqrt{\frac{\Omega_{m,0}-1}{\Omega_{m,0}}}\right) \tag{2.14}$$

2.2.2 Positive Cosmological Constant

Although a negative cosmological constant is permitted by the laws of physics, it appears that we live in a universe where the cosmological constant is non - negative! A flat universe with $\Omega_{\Lambda,0} > 0$ will continue to expand forever it is expanding at $t = t_0$, which is an another example of Big Chill universe.

In a flat universe with $\Omega_{m,0} < 1$ and $\Omega_{\Lambda,0} > 0$, the density contributions of matter and cosmological constant are equal at the scale factor (matter - cosmological constant equality)

$$a_{m\Lambda} = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3} = \left(\frac{\Omega_{m,0}}{1-\Omega_{m,0}}\right)^{1/3} \tag{2.15}$$

For a flat, $\Omega_{\Lambda,0} > 0$ universe, the Friedmann equation can be integrated to yield the analytic solution

$$H_0 t = \frac{2}{3\sqrt{1-\Omega_{m,0}}} \ln\left[(a/a_{m\lambda})^{3/2} + \sqrt{1+(a/a_{m\lambda})^3}\right] \tag{2.16}$$

Equation can be inverted analytically to obtain $a(t)$, or can be inverted numerically when plotting. The present age of such a universe is

$$t_0 = \frac{2}{3H_0\sqrt{1-\Omega_{m,0}}} \ln\left(\frac{1+\sqrt{1-\Omega_{m,0}}}{\sqrt{\Omega_{m,0}}}\right) \tag{2.17}$$

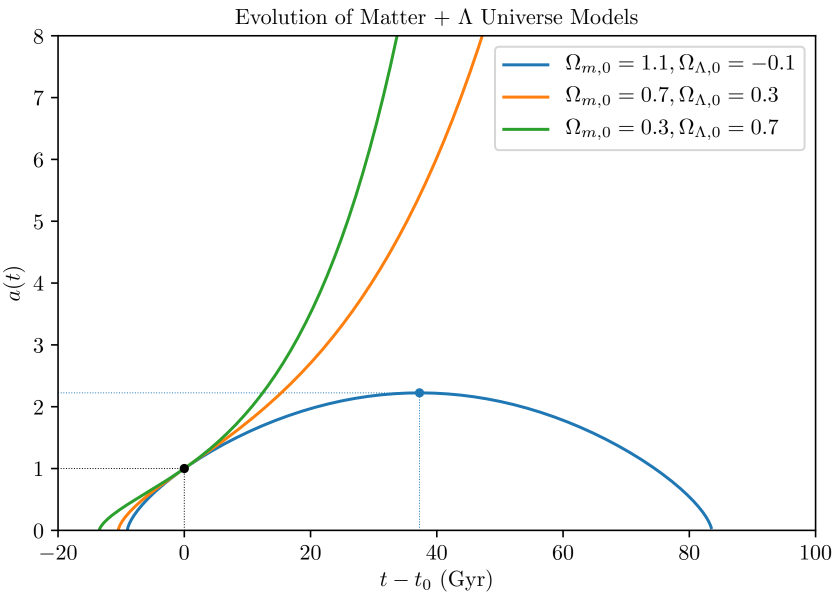

Figure 2.4. Evolution of Various Matter + Λ Universe Models {fig:M

2.3 Matter - Curvature - Cosmological Constant (Λ) Models

If a universe containing both matter and cosmological constant is curved ($k = 0$), then a wide range of behaviors is possible for $a(t)$ which are given in Table 2.1.

Model	$\Omega_{m,0}$	$\Omega_{r,0}$	$\Omega_{\Lambda,0}$
Loitering	0.3	0	1.71
Λ Collapse	1	0	-0.3
Big Bounce	0.3	0	1.8
Λ - CDM	0.27	8×10^{-5}	0.73

Table 2.1. Examples of various Matter - Curvature - Λ Models {tab:m

Plots of these models are given in Figure 2.5. For such models Friedmann equation reduces to

$$\frac{H^2}{H_0^2} = \frac{\Omega_{m,0}}{a^3} + \frac{1 - \Omega_{m,0} - \Omega_{\Lambda,0}}{a^2} + \Omega_{\Lambda,0} \tag{2.18}$$

According to the above equation, for some choices of $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$, the value of H^2 will be positive for small values of a (where matter dominates) and for large values of a (where cosmological constant dominates), but will be negative for intermediate values of a (where the curvature term dominates).

Since negative values of H^2 are unphysical, this means that these universes have a forbidden range of scale factors. Suppose such a universe starts out with $a \gg 1$ and $H < 0$. As the universe contracts, however, the negative curvature term becomes dominant, causing the contraction to stop at a minimum scale factor $a = a_{\text{min}}$, and then expand outward again in a **Big Bounce**. Thus, it is possible to have a universe which expands outward at late times, but which never had an initial Big Bang, with $a = 0$ at $t = 0$.

Another possibility is a **loitering** (also called Lemaitre) universe. Such a universe starts in a matter-dominated state, expanding outward with $a \propto t^{2/3}$. Then, it enters a stage (called the loitering stage) in which a is very nearly constant for a long period of time. During this time it is almost Einstein's static universe. After the loitering stage, the cosmological constant takes over, and the universe starts to expand exponentially.

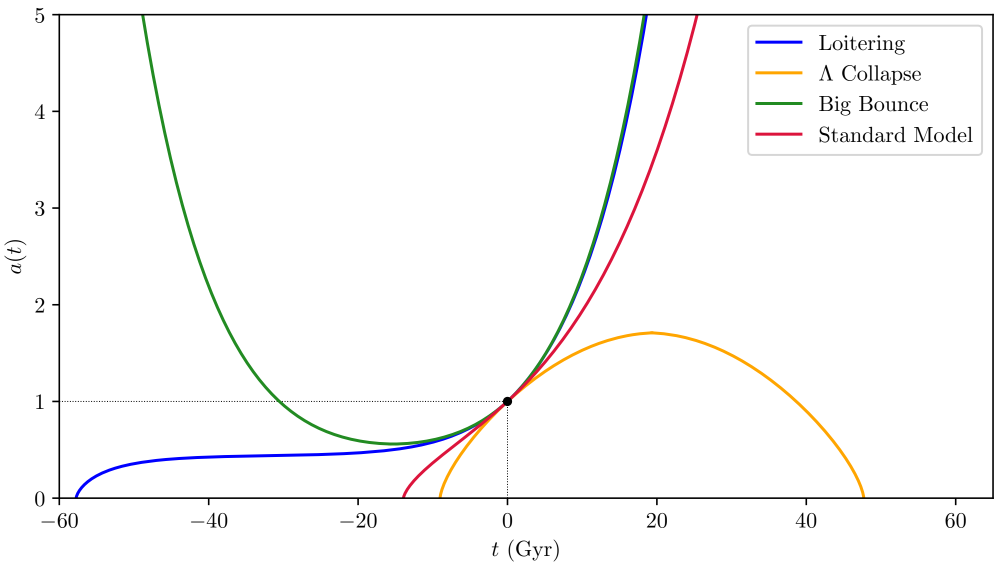

Figure 2.5. Evolution of Various Matter - Curvature - Λ Universe Models {fig:M

2.4 Standard Model of the Universe (Λ - CDM)

$$\boxed{H_0 \int dt = \int \frac{da}{\sqrt{\Omega_{r,0}/a^2 + \Omega_{m,0}/a + a^2\Omega_{\Lambda,0} + 1 - \Omega_0}}} \tag{2.19}$$

Ingredient	Amount
Photons	$\Omega_{\gamma,0} = 5.0 \times 10^{-5}$
Neutrinos	$\Omega_{\nu,0} = 5.0 \times 10^{-5}$
Total Radiation	$\Omega_{r,0} = 8.4 \times 10^{-5}$
Baryonic Matter	$\Omega_{bary,0} = 0.04$
Non-Baryonic Dark Matter	$\Omega_{dm,0} = 0.26$
Cosmological Constant	$\Omega_{\Lambda,0} \approx 0.70$

Table 2.2. Properties of the Standard Model {tab:p

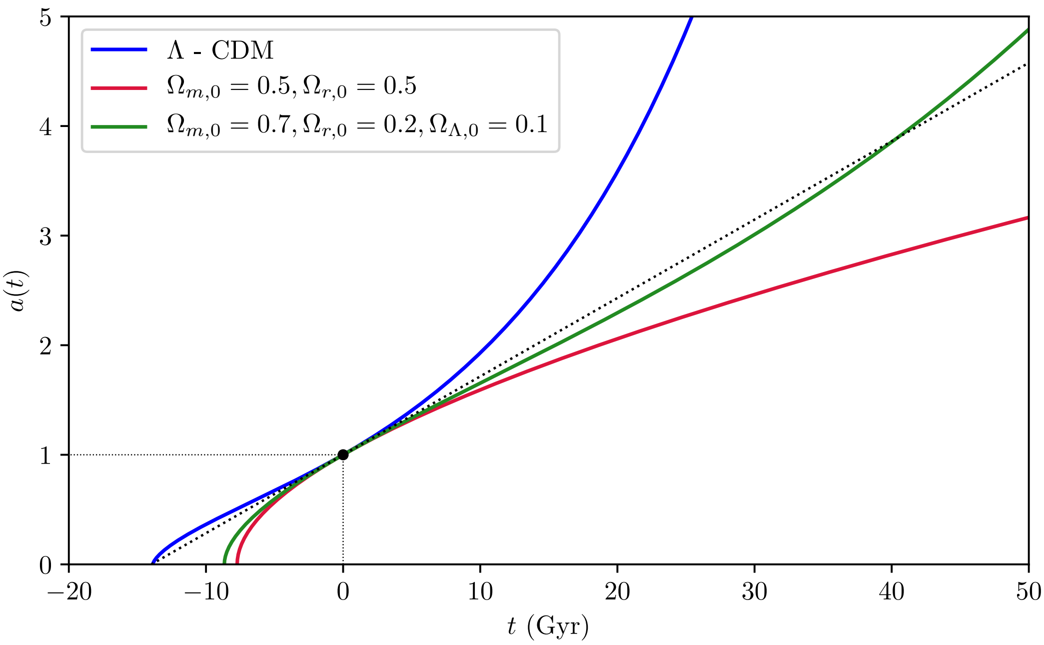

Figure 2.6. Evolution of Various Matter - Radiation - Λ Universe Models {fig:M

A | Python Codes

In this section, we provide example scripts for analytical (Program 1) and numerical(Program 2) universe models. Complete codes, Jupyter notebooks and animated version can be found here.

Listing 1. Matter + Lambda Universe Analytic Solution | code:Matter_Lambda

```
#  *  coding: utf 8  *
Created on Wed Aug 01 21:12:02 2018@author:  Kemal

from __future__ import division
import numpy as np
import matplotlib.pyplot as plt
import matplotlib.animation as animation
import matplotlib as mpl

plt.rc('text',fontsize = 11 ,usetex=True)  #In order to use LaTeX
plt.rc('font', family='serif') #In order to use Serif (mathced font with LaTeX)
#mpl.style.use('seaborn')

# Some unit conversion factors:
H0 = 70                # Hubble constant H = 70 km/s/Mpc
Mpc = 3.085677581e19   # km
km =  1.0
Gyr = 3.1536e16        # second

H_0 = (H0 * km * Gyr) /  Mpc    #Here H_0 is in the units of 1/Gyr

# A flat universe with matter + cosmological constant

def Matter_Negative_Lambda(Omega_0,t):

    a_max = (Omega_0 / (Omega_0 - 1))**(1./3)
    t_crunch = (2*np.pi / (3*H_0)) * (1/(np.sqrt(Omega_0 - 1)))
    a = (np.sqrt(Omega_0 / (Omega_0 - 1))*np.sin((3./2)*H_0*t*np.sqrt(Omega_0 -1 )))**(2./3)
    age = (2. / (3*H_0*np.sqrt(Omega_0 - 1))) * np.arcsin(np.sqrt((Omega_0 - 1)/(Omega_0)))
    return [a,a_max,t_crunch,age]



def Matter_Positive_Lambda(Omega_0,t):
    a_ml = (Omega_0 / (1 - Omega_0))**(1./3)
    age = (2. / (3*H_0*np.sqrt(1 - Omega_0 ))) * np.log( (np.sqrt(1 - Omega_0) + 1)/np.sqrt(Omega_0) )
    return [age, a_ml]

def Matter_Positive_Lambda_inverter(Omega_0,a):
    a_ml = (Omega_0 / (1 - Omega_0))**(1./3)
    t = ( (2 / (3*np.sqrt(1-Omega_0))) * np.log(  (a/a_ml)**(3/2) + np.sqrt( 1 + (a/a_ml)**(3))))/H_0
    return t


```

```python
t= np.linspace(0,100,1000)

ML1 = Matter_Negative_Lambda(1.1,t)
a1 = ML1[0]
a_max1 = ML1[1]
t_crunch = ML1[2]
age1 = ML1[3]

fig1 = plt.figure()
ax1 = fig1.add_subplot(111)

ax1.set_xlabel('$t t_0$ (Gyr)')
ax1.set_ylabel('$a(t)$')
ax1.set_title('Evolution of Matter + $\Lambda$ Universe Models',fontsize = 11)

ax1.set_xlim(-20,100)
ax1.set_ylim(0,8)

ax1.plot(t-age1,a1,label = '$\Omega_{m,0} = 1.1,  \Omega_{\Lambda,0} =  0.1$')
ax1.scatter((t_crunch/2)-age1, a_max1,s=13)
plt.hlines(a_max1,-20,(t_crunch/2)-age1,color = 'C0',linestyle = ':',linewidth = 0.5)
plt.vlines((t_crunch/2)-age1,0,a_max1,color = 'C0',linestyle = ':',linewidth = 0.5)

#Positive Lambda Universe
a = np.linspace(0,30,1000)
t_invert = Matter_Positive_Lambda_inverter(0.7,a)
mat_lam2 = Matter_Positive_Lambda(0.7,t)
age_ml2 =mat_lam2[0]
a_ml2 = mat_lam2[1]


ax1.plot(t_invert - age_ml2,a,label = '$\Omega_{m,0} = 0.7,  \Omega_{\Lambda,0} = 0.3$')


ax1.scatter(0,1,color = 'black',zorder = 10,s = 13)
plt.hlines(1,-20,0,color = 'black',linestyle = ':',linewidth = 0.5)
plt.vlines(0,0,1,color = 'black',linestyle = ':',linewidth = 0.5)

t_invert3 = Matter_Positive_Lambda_inverter(0.3,a)
mat_lam3 = Matter_Positive_Lambda(0.3,t)
age_ml3 =mat_lam3[0]
ax1.plot(t_invert3 - age_ml3,a,label = '$\Omega_{m,0} = 0.3,  \Omega_{\Lambda,0} = 0.7$')

plt.legend()
plt.show()


'''
fig2 = plt.figure()
ax2 = fig2.add_subplot(111)
```

```

ax2.set_xlabel('$t $ (Gyr)')
ax2.set_ylabel('$a(t)$')
ax2.set_title('Evolution of Matter + $\Lambda$ Universe Models',fontsize = 11)

ax2.set_xlim( 15,120)
ax2.set_ylim(0,8)



ax2.plot(t_invert   age_ml2,a,label = '$\Omega_{m,0} = 0.7,  \Omega_{\Lambda,0} = 0.3$')
'''
```

Listing 2. Matter + Radiation + Lambda Numerical Solution code:Standard_Model

```
# * coding: utf 8 *
Created on Thu Aug 02 22:01:46 2018@author: Kemal

from __future__ import division
import numpy as np
import matplotlib.pyplot as plt
import matplotlib.animation as animation
import matplotlib as mpl

plt.rc('text',fontsize = 11 ,usetex=True)  #In order to use LaTeX
plt.rc('font', family='serif') #In order to use Serif (mathced font with LaTeX)
#mpl.style.use('seaborn')

# Some unit conversion factors:
H0 = 70                # Hubble constant H = 70 km/s/Mpc
Mpc = 3.085677581e19   # km
km =  1.0
Gyr = 3.1536e16        # second

H_0 = (H0 * km * Gyr) /  Mpc    #Here H_0 is in the units of 1/Gyr

eps = 1e-4
def Friedmann(a,m,r,l):
    Omega_0 = m + r + l
    t = (1/ np.sqrt( (r/(a*a)) + (m/a) + (l * (a*a)) + (1-Omega_0)  ) ) /H_0
    #t_0 = (1/( (m/a) + (lam * (a*a)))**(0.5))/gyr # Matter   Lambda Flat Universe Approximation

    return t
def EmptyUniverse(t):
    a_e = H_0*t
    age_e = 1/H_0
    return a_e,age_e

def Trapezoidal(a,b,m,r,l):
    n = 10000                 # Step number
    deltaX = (b-a)/n         # Step size

    AGE = 0

    x = np.zeros(n)
    y = np.zeros(n)
    z = np.zeros(n)

    for i in range(n):
        x[i] = a + i*deltaX
        y[i] = Friedmann(x[i],m,r,l)
        z[i] = (deltaX/2) * (2*np.sum(y) - y[0] - y[n-1])

        if (x[i] == 1 or 1-eps <= x[i] <= 1+eps):
```

```
            AGE = z[i]

    #result = (deltaX/2) * (2*np.sum(y)   y[0]   y[n 1])

    print 'Age of the universe with m = %5.3f, r = %5.3f, lambda = %5.3f is %5.3f Gyr' %(m,r,l,AGE)

    return x,z,AGE

a,t,age = Trapezoidal(1e-10,10,0.27,8e-5,0.73)
a1,t1,age1 = Trapezoidal(1e-10,10,0.5,0.5,0)
a2,t2,age2 = Trapezoidal(1e-10,10,0.7,0.2,0.1)
a3,t3,age3 = Trapezoidal(1e-10,10,0.3,0.7,-0.1)

t_i = 0
t_f = 90
steps = 1000
t_e = np.linspace(t_i,t_f,steps)

a_e,age_e = EmptyUniverse(t_e)

fig1 = plt.figure()
ax1 = fig1.add_subplot(111)

ax1.plot(t-age, a, color='blue', label = '$\Lambda$   CDM')
ax1.plot(t1-age1, a1, color='crimson', label = '$\Omega_{m,0} = 0.5,\Omega_{r,0} = 0.5$')
ax1.plot(t2-age2, a2, color='forestgreen', label = '$\Omega_{m,0} = 0.7,\Omega_{r,0} = 0.2, \Omega_{\Lambda,0} = 0.1 $')
ax1.plot(t_e-age_e,a_e,color = 'black',linestyle = ':',linewidth = 1)

ax1.set_xlim(-20,50)
ax1.set_ylim(0,5)

ax1.set_xlabel('$t  t_0$ (Gyr)')
ax1.set_ylabel('$a(t)$')
ax1.legend()

ax1.scatter(0,1,color = 'black',zorder = 10,s = 13)
plt.hlines(1,-60,0,color = 'black',linestyle = ':',linewidth = 0.5)
plt.vlines(0,0,1,color = 'black',linestyle = ':',linewidth = 0.5)
```