Numerical Explorations of FLRW Cosmologies

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1 Introduction

To attain a picture of the universe as a whole, we must look at it on the largest possible scales. Since the universe is almost certainly much larger than the observable universe, or indeed the universe we will ever observe [1, 2, 3, 4], some guesswork is necessary [4]. The most popular—assumption used to study the universe as a whole is the *Copernican principle* [5], the notion that, on the larges scales, the universe looks the same everywhere and in every direction [2, 4].

In modern language, we assume that the universe is homogeneous and isotropic. That is, we assume that the energy-momentum tensor that feeds Einstein's equations is the same everywhere and in space and has no preferred direction [2, 3, 4]. Obviously, this isn't the only approach one can take. Roughly speaking one can generate a spacetime with an arbitrary amount of symmetry between no symmetry and maximal symmetry. The homogeneous isotropic universe is almost as symmetric as one can get. One can add time-like symmetry to attain de Sitter, anti de Sitter, or Minkowski spacetimes, but that's as symmetric as you can get [2, 3, 4]. Relaxing the symmetry conditions on the energy-momentum tensor results in, for example, the mixmaster universe [3]. However, observational evidence, most strongly the cosmic microwave background, suggests that the universe is homogeneous and isotropic to a very high degree [6].

In this work, we derive the equations governing the evolution of a homogeneous isotropic universe and numerically explore how such a universe could evolve. In section 2 we briefly derive the so-called Friedmann equations, the equations governing such spacetimes. In section 3, we briefly discuss the numerical tools we use to evolve these equations. In section 4, we present our results. Finally, in section 5, we offer some concluding remarks.

2 The Friedmann-Lemaitre-Robertson-Walker Metric

First, we derive the metric of a homogenous, isotropic spacetime. Then we derive the analytic formulae for its evolution. Because we derived the Friedmann equations using tetrads in class [7], we decided to work in coordinates for another perspective.

2.1 Deriving the Metric

The following discussion borrows from [2], [3], and [4]. However, we mostly follow [2]. It is possible to rigorously and precisely define the notions of homogeneous and isotropic [4]. However, one can prove that homogenous and isotropic spacetimes are foliated by maximally symmetric spacelike hypersurfaces [2]. Intuitively, at least, it's not hard to convince oneself that homogeneity and isotropy are sufficient to enforce maximal symmetry. There are three classes of such hypersurfaces all of constant scalar curvature: spherical spaces, hyperbolic spaces, and flat spaces. Spherical spaces have constant positive curvature. Hyperbolic spaces have constant negative curvature, and flat spaces have zero curvature [2].

Let us consider a homogenous, isotropic, four-dimensional Lorentzian spacetime. We can write the metric as [2]

$$ds^2 = -dt^2 + a^2(t)d\sigma^2, \tag{1}$$

where t is the time coordinate, $a(t)$ is a scalar function known as the *scale factor*, and $d\sigma^2$ is the metric of the maximally symmetric spacelike hypersurface. Up to some overall scaling (which we can hide in $a(t)$, we have [4]

$$d\sigma^2 = \begin{cases} d\chi^2 + \sin^2(\chi)(d\theta^2 + \sin^2(\theta)d\phi^2) & \text{for positive curvature} \\ d\chi^2 + \chi^2(d\theta^2 + \sin^2 d\phi^2) & \text{for no curvature} \\ d\chi^2 + \sinh^2(\chi)(d\theta^2 + \sin^2(\theta)d\phi^2) & \text{for negative curvature} \end{cases}, \tag{2}$$

where χ, θ, and ϕ are the usual spherical coordinates. However, if we choose r such that

$$d\chi = \frac{dr}{\sqrt{1 - kr^2}}, \tag{3}$$

where

$$k \in \{-1, 0, 1\} \tag{4}$$

is the normalized scalar curvature of the hypersurface, we can renormalize our equation to put it in a nicer form [2, 3],

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\right], \tag{5}$$

where

$$d\Omega = d\theta^2 + \sin^2(\theta)d\phi^2 \tag{6}$$

is the angular piece of the metric. This is known as the *Friedmann-Lemaitre-Roberston-Walker* (FRLW) metric.

2.2 Evolution and the Friedmann Equation

The following discussion draws primarily from the lecture notes [7], specifically lecture 16. If we calculate the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \tag{7}$$

(using, e.g., a tool like Maple's differential geometry package [8]) in the frame induced by our choice of coordinates, we find that it is diagonal [7]:[1]

$$G^t{}_t = -3\frac{k + \dot{a}^2}{a^2} \tag{8}$$

$$G^i{}_i = -\frac{k + 2a\ddot{a} + \dot{a}^2}{a^2} \;\forall\; i \in \{1, 2, 3\}, \tag{9}$$

$$G^\mu{}_\nu = 0 \;\forall\; \mu \neq \nu \tag{10}$$

where we have suppressed the t-dependence of a and where $\dot{a}$ is a time-derivative of a in the usual way.

If we feed the Einstein tensor into Einstein's equation in geometric units,

$$G_{\mu\nu} = 8\pi\tilde{T}_{\mu\nu} - \Lambda g_{\mu\nu}, \tag{11}$$

[1] We write the tensor as a linear operator with one raised index because this produces the nicest form of the equations.

since $g_{\mu\nu}$ is diagonal, we see that $T^{\mu}_{\ \nu}$ must be diagonal too. Indeed, with one index raised, it looks like the energy-momentum tensor for a perfect fluid in its rest frame:

$$\begin{aligned} T^{\mu}_{\ \nu} &= \tilde{T}^{\mu}_{\ \nu} - \frac{\Lambda}{8\pi}\mathcal{I} \\ &= \begin{bmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix} - \frac{\Lambda}{8\pi}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \end{aligned} \tag{12}$$

where $\mathcal{I}$ is the identity operator, ρ is the density of the fluid, and p is the pressure of the fluid. $\tilde{T}^{0}_{\ 0}$ is negative because we've raised one index. Obviously, this is just a mathematical coincidence, and we've even manipulated our definition of pressure to hide our choice of coordinates. However, one can make weak physical arguments to justify it [2, 3].

The time-time component of Einstein's equation (11) then gives the first Friedmann equation [2, 3, 7]:

$$\frac{\dot{a}^2 + k}{a^2} = \frac{8\pi\rho + \Lambda}{3}. \tag{13}$$

Any space-space diagonal component gives the second Friedmann equation [2, 3, 7]:

$$-\frac{k + 2a\ddot{a} + \dot{a}^2}{a^2} = 8\pi p + \Lambda. \tag{14}$$

To use these equations to describe an FLRW universe, we need one more piece of information: the relationship between density and pressure, called an equation of state [2, 3, 7]. We parametrize our ignorance of the equation of state as

$$p = \omega(\rho)\rho, \tag{15}$$

where $\omega(\rho)$ is some scalar function.

In the simple case when ω is a constant, we get several different regimes. In the case of $\omega \geq 0$, as for a radiation-dominated or matter-dominated universe [3, 2, 7],[2] the scale factor increases monotonically and at a decelerating rate [3, 2, 7]. In the case of $\omega = -1$, one gets a dark-energy dominated universe with accelerating expansion [3, 2, 7]. And if ω is simply very close to -1, one gets an inflationary universe [3, 2, 7]. As a test, will numerically explore the cases when ω is constant and we have a known regime. We will also numerically explore the cases when ω is a more sophisticated object.

3 Numerical Approach

In any numerical problem, there are two steps. The first is two frame the problem in a way compatible with numerical methods. The second step is to actually design and implement a method. To limit our problem domain, from here on out, we assume that

$$k = 0. \tag{16}$$

3.1 Framing the Problem

We will use a numerical algorithm to solve initial value problems. These algorithms are usually formulated as first order ODE solvers. The ode is written in the form

$$\frac{d\vec{y}}{dt} = \vec{f}(y, t) \text{ with } \vec{y}(0) = \vec{y}_0, \tag{17}$$

[2] If $\omega = 0$, we have a matter-dominated universe where the density scales as one over the scale factor cubed, in other words, one over volume. If $\omega = 1/3$, we have a radiation-dominated universe, where the density scales as one over the scale factor to the fourth. The extra scaling of the scale factor is due to cosmological redshift [2, 7].

where $\vec{y}$ is a vector containing all the functions of t one wishes to solve for. Let's see if we can't write the Friedmann equations in such a form.

Our first step is to reduce the second-order ODE system to a first-order system. We define a new variable, $b(t)$ such that

$$\dot{a}(t) = b(t). \tag{18}$$

Our next step is to remove the variable ρ from our system. To do so, we assume that the density $\rho(t)$ can be written as a function of the scale factor a and its derivatives. Fortunately, we can solve equation (13) for ρ [7] to find:

$$\rho(a, b) = \frac{3b - \Lambda a^2}{8\pi a^2}, \tag{19}$$

where we have substituted equation (18) and used equation (16). Finally, we solve equation (14) for $\dot{b} = \ddot{a}$:

$$\dot{b} = -\frac{b^2 + 8\pi\omega\rho a^2 + \Lambda a^2}{2a}. \tag{20}$$

So, putting it all together, if we define the two-vector

$$\vec{y}(t) = \begin{bmatrix} a(t) \\ b(t) \end{bmatrix}, \tag{21}$$

then we can write the equations governing the evolution of the universe as

$$\frac{d}{dt}\vec{y}(t) = \vec{f}(y) = \begin{bmatrix} b \\ -\dfrac{b^2 + 8\pi\omega(\rho(a,b))\rho(a,b)a^2 + \Lambda a^2}{2a} \end{bmatrix}, \tag{22}$$

where $\rho(a, b)$ is given in equation (19) and $\omega(\rho(a, b))$ is defined by the $\omega(\rho)$ that we choose.

3.2 Initial Data

Even with evolution equations, we're still missing some critical information. A first-order ODE system needs one initial value for each unknown function.

Obviously, equation (22) breaks down when the scale factor is zero. Therefore we must be careful to avoid starting a simulation too close to the big bang singularity. For this reason, we will always assume that $a(0) = 1$. However, we still need to choose an initial value for b. This depends on the energy density in the initial universe. We will play around with this value as we explore our numerical solutions. However, since observations tell us the universe is expanding, we will usually assume that $b(0) \geq 0$.

3.3 Numerical Method

For a given $\omega(\rho)$ and set of initial data, we solve the ODE system using a "Runge-Kutta" algorithm. Before we define Runge-Kutta, let's first describe a simpler, similar, method. The definition of a derivative is

$$\frac{d}{dt}\vec{y}(t) = \lim_{h\to 0}\left[\frac{\vec{y}(t+h) - \vec{y}(t)}{h}\right]. \tag{23}$$

Or, alternatively, if h is sufficiently small,

$$\vec{y}(t+h) = \vec{y}(t) + h\frac{d\vec{y}}{dt}(t). \tag{24}$$

If we know $\vec{y}(t_0)$ and $\frac{d}{dt}\vec{y}(t_0)$, then we can use equation (24) to solve for $\vec{y}(t+h)$. Then, let $t_1 = t_0 + h$ and use equation (24) to solve for $\vec{y}(t_1 + h)$. In this way, we can solve for $\vec{y}(t)$ for all $t > t_0$. This method is called the "forward Euler" method [9].

Runge-Kutta methods are more sophisticated. One can use a Taylor series expansion to define a more accurate iterative scheme that relies on higher-order derivatives, not just first derivatives. However, since we only have first derivative information, we simulate higher-order derivatives by evaluating the first derivative at a number of different values of t. Then, of course, the higher-order derivatives are finite differences of these evaluations [10, 9].

We use a fourth-order Runge-Kutta method—which means the method effectively incorporates the first four derivatives of a function—with adaptive step sizes: the 4(5) Runge-Kutta-Fehlberg method [11]. We use our own implementation, written in C++. The Runge-Kutta solver by itself can be found here: [12].[3] In the context of FLRW metrics, our implementation can be found here: [13]. Both pieces of code are open-sourced, and the reader should feel free to download and explore either program.

4 Results

5 Conclusion

References

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[3] It is good coding practice to make one's code as flexible as possible. By separating the initial-value solver from a specific application, we make it possible to reuse our code.