Numerical Explorations of FLRW Cosmologies

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1 Introduction

To attain a picture of the universe as a whole, we must look at it on the largest possible scales. Since the universe is almost certainly much larger than the observable universe, or indeed the universe we will ever observe [1, 2, 3, 4], some guesswork is necessary [4]. The most popular—assumption used to study the universe as a whole is the *Copernican principle* [5], the notion that, on the larges scales, the universe looks the same everywhere and in every direction [2, 4].

In modern language, we assume that the universe is homogeneous and isotropic. That is, we assume that the energy-momentum tensor that feeds Einstein's equations is the same everywhere and in space and has no preferred direction [2, 3, 4]. Obviously, this isn't the only approach one can take. Roughly speaking one can generate a spacetime with an arbitrary amount of symmetry between no symmetry and maximal symmetry. The homogeneous isotropic universe is almost as symmetric as one can get. One can add time-like symmetry to attain de Sitter, anti de Sitter, or Minkowski spacetimes, but that's as symmetric as you can get [2, 3, 4]. Relaxing the symmetry conditions on the energy-momentum tensor results in, for example, the mixmaster universe [3]. However, observational evidence, most strongly the cosmic microwave background, suggests that the universe is homogeneous and isotropic to a very high degree [6].

In this work, we derive the equations governing the evolution of a homogeneous isotropic universe and numerically explore how such a universe could evolve. In section 2 we briefly derive the so-called Friedmann equations, the equations governing such spacetimes. In section 3, we briefly discuss the numerical tools we use to evolve these equations. In section 4, we present our results. Finally, in section 5, we offer some concluding remarks.

2 The Friedmann-Lemaitre-Robertson-Walker Metric

First, we derive the metric of a homogenous, isotropic spacetime. Then we derive the analytic formulae for its evolution. Because we derived the Friedmann equations using tetrads in class [7], we decided to work in coordinates for another perspective.

2.1 Deriving the Metric

The following discussion borrows from [2], [3], and [4]. However, we mostly follow [2]. It is possible to rigorously and precisely define the notions of homogeneous and isotropic [4]. However, one can prove that homogenous and isotropic spacetimes are foliated by maximally symmetric spacelike hypersurfaces [2]. Intuitively, at least, it's not hard to convince oneself that homogeneity and isotropy are sufficient to enforce maximal symmetry. There are three classes of such hypersurfaces all of constant scalar curvature: spherical spaces, hyperbolic spaces, and flat spaces. Spherical spaces have constant positive curvature. Hyperbolic spaces have constant negative curvature, and flat spaces have zero curvature [2].

Let us consider a homogenous, isotropic, four-dimensional Lorentzian spacetime. We can write the metric as [2]

$$ds^2 = -dt^2 + a^2(t)d\sigma^2, \tag{1}$$

where t is the time coordinate, $a(t)$ is a scalar function known as the *scale factor*, and $d\sigma^2$ is the metric of the maximally symmetric spacelike hypersurface. Up to some overall scaling (which we can hide in $a(t)$, we have [4]

$$d\sigma^2 = \begin{cases} d\chi^2 + \sin^2(\chi)(d\theta^2 + \sin^2(\theta)d\phi^2) & \text{for positive curvature} \\ d\chi^2 + \chi^2(d\theta^2 + \sin^2 d\phi^2) & \text{for no curvature} \\ d\chi^2 + \sinh^2(\chi)(d\theta^2 + \sin^2(\theta)d\phi^2) & \text{for negative curvature} \end{cases}, \tag{2}$$

where χ, θ, and ϕ are the usual spherical coordinates. However, if we choose r such that

$$d\chi = \frac{dr}{\sqrt{1-kr^2}}, \tag{3}$$

where

$$k \in \{-1, 0, 1\} \tag{4}$$

is the normalized scalar curvature of the hypersurface, we can renormalize our equation to put it in a nicer form [2, 3],

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right], \tag{5}$$

where

$$d\Omega = d\theta^2 + \sin^2(\theta)d\phi^2 \tag{6}$$

is the angular piece of the metric. This is known as the *Friedmann-Lemaitre-Roberston-Walker* (FRLW) metric.

2.2 Evolution and the Friedmann Equation

The following discussion draws primarily from the lecture notes [7], specifically lecture 16. If we calculate the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \tag{7}$$

(using, e.g., a tool like Maple's differential geometry package [8]) in the frame induced by our choice of coordinates, we find that it is diagonal [7]:[1]

$$G^t{}_t = -3\frac{k + \dot{a}^2}{a^2} \tag{8}$$

$$G^i{}_i = -\frac{k + 2a\ddot{a} + \dot{a}^2}{a^2} \ \forall\ i \in \{1, 2, 3\}, \tag{9}$$

$$G^\mu{}_\nu = 0 \ \forall\ \mu \neq \nu \tag{10}$$

where we have suppressed the t-dependence of a and where $\dot{a}$ is a time-derivative of a in the usual way.

If we feed the Einstein tensor into Einstein's equation in geometric units,

$$G_{\mu\nu} = 8\pi\tilde{T}_{\mu\nu} - \Lambda g_{\mu\nu}, \tag{11}$$

[1] We write the tensor as a linear operator with one raised index because this produces the nicest form of the equations.

since $g_{\mu\nu}$ is diagonal, we see that $T^{\mu}_{\ \nu}$ must be diagonal too. Indeed, with one index raised, it looks like the energy-momentum tensor for a perfect fluid in its rest frame:

$$\begin{aligned}
T^{\mu}_{\ \nu} &= \tilde{T}^{\mu}_{\ \nu} - \frac{\Lambda}{8\pi}\mathcal{I} \\
&= \begin{bmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix} - \frac{\Lambda}{8\pi}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
\end{aligned} \tag{12}$$

where $\mathcal{I}$ is the identity operator, ρ is the density of the fluid, and p is the pressure of the fluid. $\tilde{T}^{0}_{\ 0}$ is negative because we've raised one index. Obviously, this is just a mathematical coincidence, and we've even manipulated our definition of pressure to hide our choice of coordinates. However, one can make weak physical arguments to justify it [2, 3].

The time-time component of Einstein's equation (11) then gives the first Friedmann equation [2, 3, 7]:

$$\frac{\dot{a}^2 + k}{a^2} = \frac{8\pi\rho + \Lambda}{3}. \tag{13}$$

Any space-space diagonal component gives a second equation [2, 3, 7]:

$$-\frac{k + 2a\ddot{a} + \dot{a}^2}{a^2} = 8\pi p + \Lambda. \tag{14}$$

Now we take

$$-\frac{1}{2}a\left[(\text{equation (13)}) + \frac{1}{3}(\text{equation (14)})\right], \tag{15}$$

which yields

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p) + \frac{\Lambda}{3}, \tag{16}$$

which is the second Friedmann equation [7]. To use these equations to describe an FLRW universe, we need one more piece of information: the relationship between density and pressure, called an equation of state [2, 3, 7]. We parametrize our ignorance of the equation of state as

$$p = \omega(\rho)\rho, \tag{17}$$

where $\omega(\rho)$ is some scalar function.

Finally, we hide the cosmological constant as a type of matter, which we call dark matter. This corresponds to resetting

$$\rho \to \rho - \frac{\Lambda}{8\pi}. \tag{18}$$

In this case, the Friedmann equations are

$$\frac{\dot{a}^2 + k}{a^2} = \frac{8\pi}{3}\rho \tag{19}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p). \tag{20}$$

Then a cosmological constant corresponds to an energy density that does not dilute with time... and one with negative pressure. Unless otherwise stated, we will make this choice.

In the simple case when ω is a constant, we get several different regimes. In the case of $\omega \geq 0$, as for a radiation-dominated or matter-dominated universe [3, 2, 7],[2] the scale factor increases monotonically

[2]If $\omega = 0$, we have a matter-dominated universe where the density scales as one over the scale factor cubed, in other words, one over volume. If $\omega = 1/3$, we have a radiation-dominated universe, where the density scales as one over the scale factor to the fourth. The extra scaling of the scale factor is due to cosmological redshift [2, 7].

and at a decelerating rate [3, 2, 7]. In the case of $\omega = -1$, one gets a dark-energy dominated universe with accelerating expansion [3, 2, 7]. And if ω is simply very close to -1, one gets an inflationary universe [3, 2, 7]. As a test, will numerically explore the cases when ω is constant and we have a known regime. We will also numerically explore the cases when ω is a more sophisticated object.

3 Numerical Approach

In any numerical problem, there are two steps. The first is two frame the problem in a way compatible with numerical methods. The second step is to actually design and implement a method. To limit our problem domain, from here on out, we assume that

$$k = 0. \tag{21}$$

3.1 Framing the Problem

We will use a numerical algorithm to solve initial value problems. These algorithms are usually formulated as first order ODE solvers. The ode is written in the form

$$\frac{d\vec{y}}{dt} = \vec{f}(y,t) \text{ with } \vec{y}(0) = \vec{y}_0, \tag{22}$$

where $\vec{y}$ is a vector containing all the functions of t one wishes to solve for. Let's see if we can't write the Friedmann equations in such a form.

Our first step is to reduce the second-order ODE system to a first-order system. We will do this by eliminating $\ddot{a}$ in favor of some other variable. If we differentiate equation (19), we get

$$-2\frac{\dot{a}^2}{a^3} + \frac{\ddot{a}}{a^2} = \frac{8\pi}{3}\dot{\rho}. \tag{23}$$

We can substitute $\ddot{a}$ in from (20) to obtain an equation for the time evolution of ρ:

$$\dot{\rho} = -3\left(\frac{\dot{a}}{a}\right)(\rho + p), \tag{24}$$

which expresses conservation of mass energy. As a last step, we use equation (19) and equation (17) to eliminate $\dot{a}$ and p from equation (24) to attain an evolution equation for ρ:

$$\dot{\rho} = \mp 3\left[\rho + \omega(\rho)\rho\right]\sqrt{\frac{8\pi}{3}\rho - k}. \tag{25}$$

So, putting it all together, if we define the two-vector

$$\vec{y}(t) = \begin{bmatrix} a(t) \\ \rho(t) \end{bmatrix}, \tag{26}$$

then we can write the equations governing the evolution of the universe as

$$\frac{d}{dt}\vec{y}(t) = \vec{f}(y) = \pm\begin{bmatrix} \sqrt{8\pi a^2\rho/3} \\ -3\left(\rho + \omega(\rho)\rho\right)\sqrt{\frac{8\pi}{3}\rho - k} \end{bmatrix}. \tag{27}$$

Since our universe is expanding and we're interested in an expanding universe, we choose $\dot{a} > 0$. This gives us

$$\vec{f}(y) = \begin{bmatrix} \sqrt{8\pi a^2\rho/3} \\ -3\left(\rho + \omega(\rho)\rho\right)\sqrt{\frac{8\pi}{3}\rho - k} \end{bmatrix}. \tag{28}$$

3.2 Initial Data

Even with evolution equations, we're still missing some critical information. A first-order ODE system needs one initial value for each unknown function.

Obviously, equation (28) breaks down when the scale factor is zero. Therefore we must be careful to avoid starting a simulation too close to the big bang singularity. For this reason, we will never assume that $a(0) = 0$. If convenient, we will choose $a(0)$ to be close to zero. There is one exception, which is the case of a dark-energy dominated universe, where we will choose $a(0) = 1$, since such a universe is exponentially growing.

We still need to choose a value for $\rho(0)$, however, so long as $\rho(0) > 0$, we should expect it to not matter much. General relativity is invariant under an overall re-scaling of energy [3, 4]. However, to get a better idea of qualitative behavior, we will chose a number of values of $\rho(0)$ in each case.

3.3 Numerical Method

For a given $\omega(\rho)$ and set of initial data, we solve the ODE system using a "Runge-Kutta" algorithm. Before we define Runge-Kutta, let's first describe a simpler, similar, method. The definition of a derivative is

$$\frac{d}{dt}\vec{y}(t) = \lim_{h\to 0}\left[\frac{\vec{y}(t+h)-\vec{y}(t)}{h}\right]. \tag{29}$$

Or, alternatively, if h is sufficiently small,

$$\vec{y}(t+h) = \vec{y}(t) + h\frac{d\vec{y}}{dt}(t). \tag{30}$$

If we know $\vec{y}(t_0)$ and $\frac{d}{dt}\vec{y}(t_0)$, then we can use equation (30) to solve for $\vec{y}(t+h)$. Then, let $t_1 = t_0 + h$ and use equation (30) to solve for $\vec{y}(t_1 + h)$. In this way, we can solve for $\vec{y}(t)$ for all $t > t_0$. This method is called the "forward Euler" method [9].

Runge-Kutta methods are more sophisticated. One can use a Taylor series expansion to define a more accurate iterative scheme that relies on higher-order derivatives, not just first derivatives. However, since we only have first derivative information, we simulate higher-order derivatives by evaluating the first derivative at a number of different values of t. Then, of course, the higher-order derivatives are finite differences of these evaluations [9, 10].

We use a fourth-order Runge-Kutta method—which means the method effectively incorporates the first four derivatives of a function—with adaptive step sizes: the 4(5) Runge-Kutta-Fehlberg method [11]. We use our own implementation, written in C++. The Runge-Kutta solver by itself can be found here: [12].[3] In the context of FLRW metrics, our implementation can be found here: [13]. Both pieces of code are open-sourced, and the reader should feel free to download and explore either program.

4 Results

Before we delve into unknown territory, it's a good idea to explore known physically interesting FLRW solutions. This is what we do now.

4.1 Known Solutions

If the universe is dominated by radiation, as in the case in the early universe, just after reheating [7], we should expect the scale factor to grow as $a(t) = (t/t_0)^{1/2}$ [7]. To account for not quite starting at $a = 0$, we fit our numerical solutions to

$$a(t) = a_0 t^{1/2} + b, \tag{31}$$

[3] It is good coding practice to make one's code as flexible as possible. By separating the initial-value solver from a specific application, we make it possible to reuse our code.

where a_0 and be are constants. Figure 1 shows scale factors for several different choices of $\rho(0)$ and a fit to an analytic scale factor solution using equation (31), which matches the data extremely well.[4]

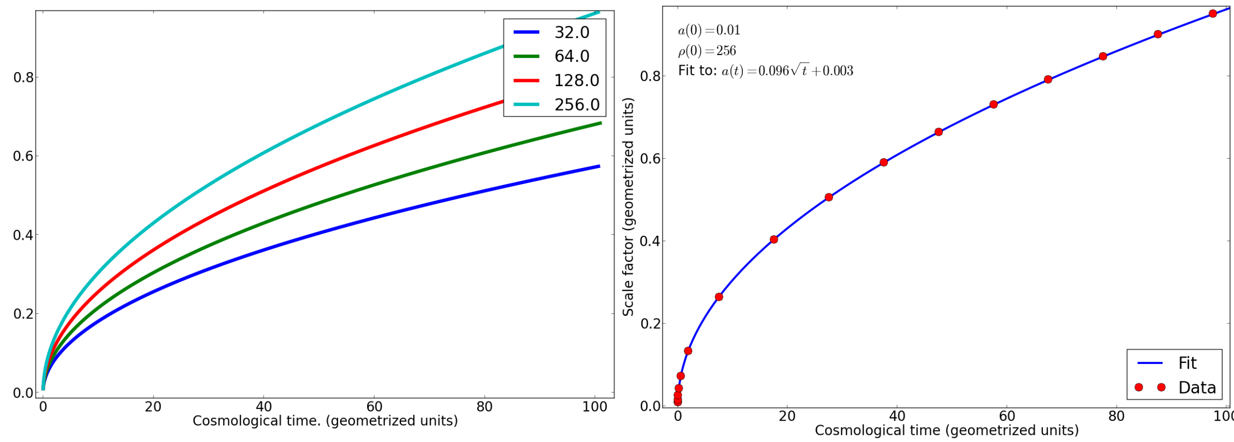

Figure 1: The Radiation-Dominated Universe. Left: Scale factor as a function of time for a number of different values of $\rho(0)$. Right: A fit to equation (31) for $\rho(0) = 256$.

If the universe is dominated by matter, as was the case until extremely recently [7], then we expect the scale factor to scale as $a(t) = (t/t_0)^{2/3}$ [7]. To account for not quite starting at $a = 0$, we fit our numerical solutions to

$$a(t) = a_0 t^{2/3} + b. \tag{32}$$

Figure 2 shows the scale factors and fit.

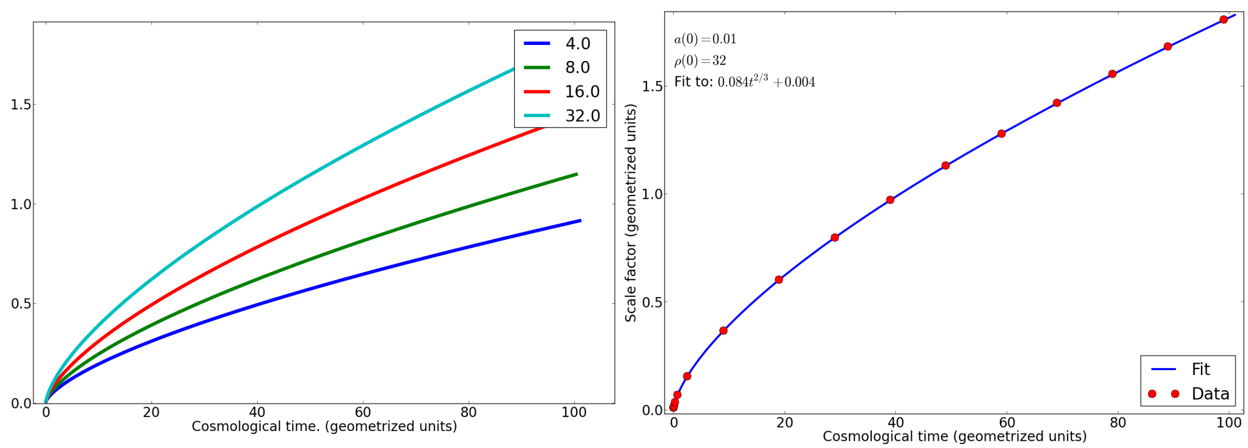

Figure 2: The Matter-Dominated Universe. Left: Scale factor as a function of time for a number of different values of $\rho(0)$. Right: A fit to equation (32) for $\rho(0) = 32$.

If the universe is dominated by dark energy, as will be the case very soon [7], then we expect the scale factor to scale as $a(t) = a_0 e^{t_0 t}$. For convenience, we fit to

$$a(t) = e^{t_0 t + b}, \tag{33}$$

[4]In all plots showing a fit, we've removed most of the data points. There are a minimum of 100 data points generated per simulation. However these are hidden to demonstrate the accuracy of the fit.

where t_0 and a_0 are constants. Figure 3 shows the scale factors and fit.

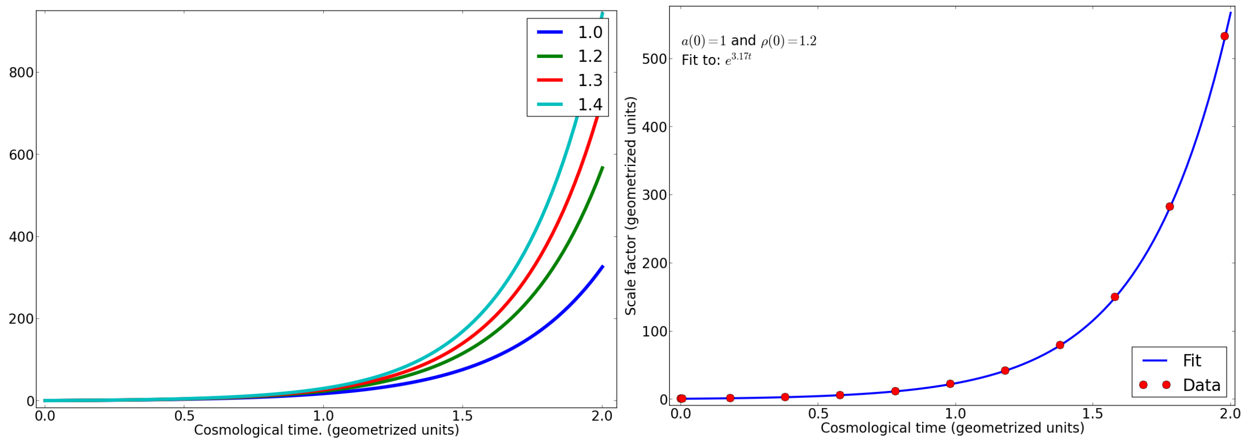

Figure 3: The Dark Energy-Dominated Universe. Left: Scale factor as a function of time for a number of different values of $\rho(0)$. Right: A fit to equation (32) for $\rho(0) = 1.2$.

4.2 A Multi-Regime Universe

Now that we've confirmed appropriate behavior in each of the analytically explored regimes, it's time to move into unknown territory! We will attempt to construct a history of the universe by splicing together the known solutions to the Friedmann equations in a smooth way.

To this end, we let ω vary smoothly with ρ. We want the radiation, matter, and dark energy dominated regimes of the universe to emerge naturally, separated by transition regions. Furthermore, we want $\omega(\rho)$ to be monotonically increasing, since we know that we first had a radiation dominated universe—radiation scales as $a^{1/4}$—then a matter dominated universe—matter scales as $a^{1/3}$—and finally a dark energy dominated universe—dark energy does not scale with a at all [2, 3, 7]. The dark energy regime should take over when ρ is very small and the normal matter and radiation has diluted away [2, 3, 7].

One function that smoothly, asymptotically, and monotonically interpolates between two constant values is the error function,

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \tag{34}$$

$\operatorname{erf}(-\infty) = -1$ and $\operatorname{erf}(+\infty) = +1$. However, the transition region is very sharp and the error function is effectively -1 for $x < -2$ and effectively $+1$ for $x > 2$. We can therefore create a smoothly varying $\omega(\rho)$ with the desired properties by taking a linear combination of error functions:

$$\omega(\rho) = -1 + \left(\frac{1}{2}\right)\left(1 + \frac{1}{3}\right) + \frac{1}{2}\operatorname{erf}\left(\frac{\rho - \rho_{DE}}{\Delta_{DE}}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\operatorname{erf}\left(\frac{\rho - \rho_{RM}}{\Delta_{RM}}\right), \tag{35}$$

where ρ_{DE} is the value of ρ marking the change of regimes from a matter-dominated to a dark energy dominated universe and Δ_{DE} is the width of the transition region. ρ_{RM} and Δ_{RM} fill analogous roles for the transition from a radiation to a matter dominated universe.

By varying ρ_{DE}, ρ_{RM}, Δ_{DE}, and Δ_{RM}, we can select a function where the transition between regimes is very abrupt or where the transition is very gradual. We chose four such functions, as shown in figure 4: an equation of state with abrupt transitions ($\rho_{DE} = 10, \Delta_{DE} = \Delta_{RM} = 1, \rho_{RM} = 20$), one with moderate transitions ($\rho_{DE} = 20, \Delta_{DE} = \Delta_{RM} = 5, \rho_{RM} = 40$), one with gradual transitions ($\rho_{DE} = 25, \Delta_{DE} = \Delta_{RM} = 12, \rho_{RM} = 80$). And, finally, as a check, we chose a function with no well defined epochs at all ($\rho_{DE} = 37, \Delta_{DE} = \Delta_{RM} = 20, \rho_{RM} = 92$).

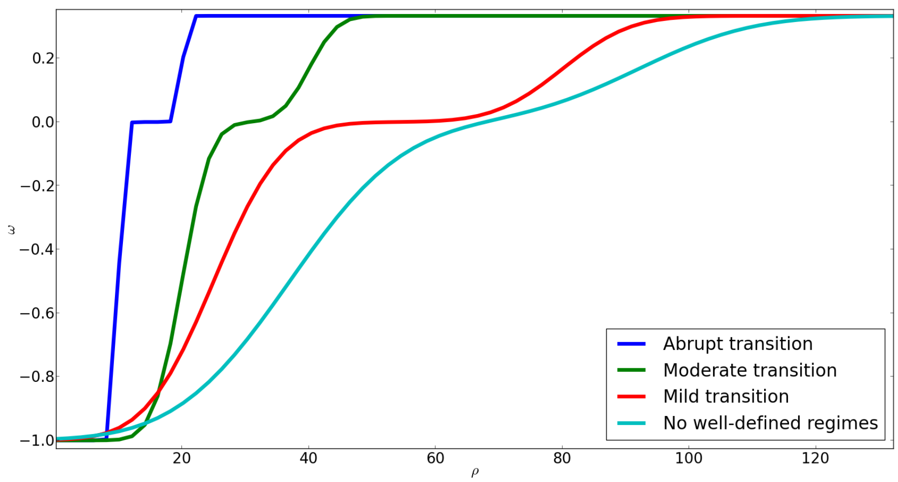

Figure 4: The equation of state variable ω (see equation (17)) as a function of energy density ρ, as defined in equation (35). We choose four possible scenarios: an equation of state with abrupt transitions ($\rho_{DE} = 10, \Delta_{DE} = \Delta_{RM} = 1, \rho_{RM} = 20$), one with moderate transitions ($\rho_{DE} = 20, \Delta_{DE} = \Delta_{RM} = 5, \rho_{RM} = 40$), one with gradual transitions ($\rho_{DE} = 25, \Delta_{DE} = \Delta_{RM} = 12, \rho_{RM} = 80$), and one with almost no transitions at all ($\rho_{DE} = 37, \Delta_{DE} = \Delta_{RM} = 20, \rho_{RM} = 92$)

The evolution of $a(t)$ for these four choices of $\omega(\rho)$ is shown in figure 5, where we also show the evolution of ρ and p on the same plots. (We have rescaled $a(t)$ so that it fits nicely with the other variables.) In these plots, we can see hints of each of the three analytically predicted epochs of growth. Initially $\ddot{a} < 0$, but it transitions to $\ddot{a} > 0$.

Surprisingly, so long as we allow $\omega(\rho)$ to vary from $+1/3$ to 0 to -1, $a(t)$ seems remarkably resilient to the details such as the abruptness of the transition or the precise location of the transition region. Indeed, when we plotted all four scale factor functions on the same scale, they overlapped completely.

We postulate that his unexpected resilience is due to the a^{-2} term in the formula for $\dot{\rho}$. Although different fixed values of ω drive $a(t)$ at different rates, $a(t)$ in turn drives $\rho(t)$ into a regime that balances out this change. This stability is a boon to cosmologists, since it means that, for the purpose of predictions a piecewise function composed of the analytic solutions is sufficient—we don't need to look for a more detailed or accurate equation of state.

4.3 $\omega \approx -1.2$

The Pan-STARRS1 survey [14] predicts that $\omega \approx -1.2$, which is a very strange type of matter indeed. When $\omega = -1$, it emerges from the cosmological constant term and the energy density ρ stays constant as the scale factor changes. If $\omega < -1$, then the energy density must *increase* with the scale factor! Since we expect energy density to *dilute* with increased scale factor [2, 7], this is decidedly odd!

We can test this hypothesis with our FLRW simulation code. And, indeed, the energy density slowly grows with scale factor, as shown in figure 6. The result is that the scale factor grows with time more quickly than an exponential. We fit $\ln(a(t))$ to a polynomial and found a cubic to fit well. Figure 6 shows the results of one such fit.

5 Conclusion

In summary, given that the universe is foliated by homogeneous and isotropic hypersurfaces, we have re-derived how to solve for the evolution of space as a function of time [2, 3, 4, 7]. We have also used a fourth-

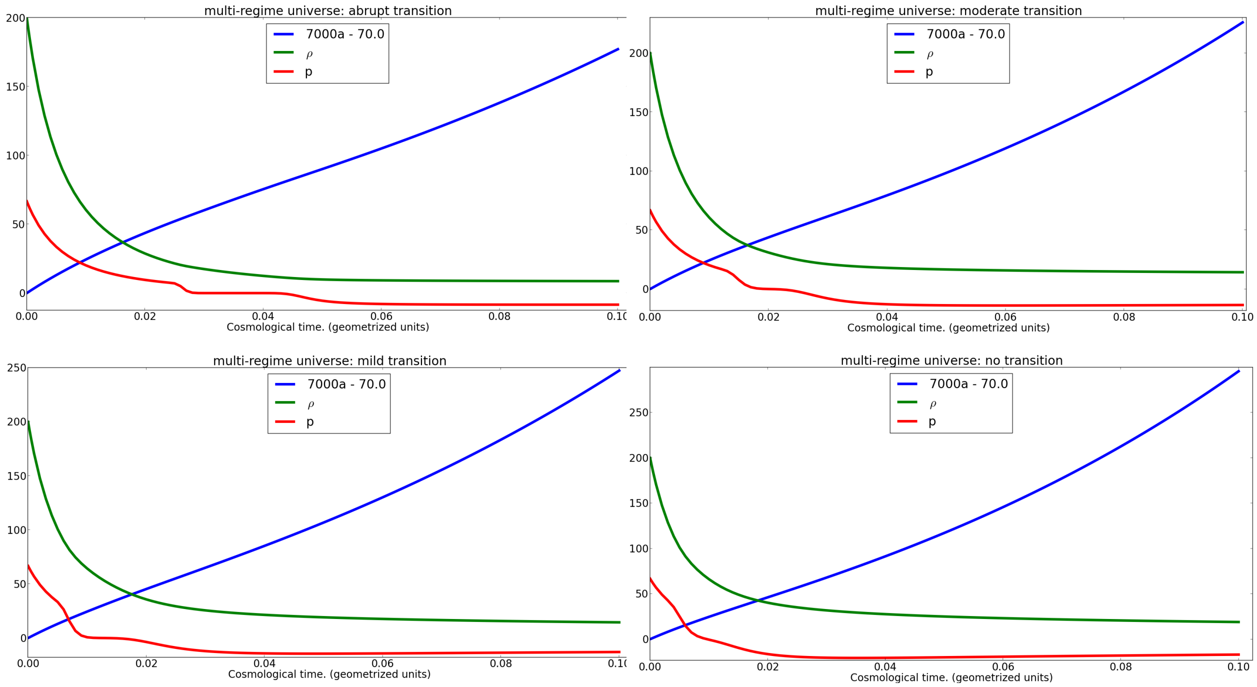

Figure 5: $a(t)$ with a variable $\omega(\rho)$. We plot the scale factor for the four equations of state show in figure 4.

order Runge-Kutta-Fehlberg method [11, 12] to numerically solve for the universe given these evolution equations. We have reproduced the analytically known epochs of evolution and solved for the universe as it transitions between analytically known radiation-dominated, matter-dominated, and dark energy-dominated regimes. Finally, we have solved for a universe dominated by an equation of state where $\omega < -1$ and the energy density increases slowly with scale factor. In this case, we discovered faster than exponential growth, which agrees qualitatively with expectations.

It is a pleasant surprise to find that the evolution of the universe extremely resilient to a choice of equation of state, $p = \omega(\rho)\rho$, so long as $\omega(\rho)$ varies smoothly and monotonically from $\omega = -1$ to $\omega = +1/3$. This means that it is sufficient for cosmologists to study the evolution of the universe epoch by epoch, and not worry too much about the transitional periods.

We would have liked to study the big bounce scenario wherein the scale factor shrinks to some minimum value and then grows monotonically. This is certainly analytically feasible. In the case of a positively curved spacelike foliation dominated by dark energy, for example, we analytically attain de Sitter space, where $a(t) \approx \cosh(t)$. Unfortunately, $\dot{a}$ must equal zero in this scenario, and this condition makes the first-order version of the Friedmann equations singular. A numerical treatment of the big bounce seems difficult if we continue to treat it as an initial value problem. A better approach might be to treat the big bounce scenario as a boundary value problem with initial and final values of $a(t)$ and a negative value for $\dot{a}(t)$. In general, the numerical methods to treat initial value problems are very different from those used to treat boundary value problems [9]. Thus it would be necessary to implement another approach (e.g., finite elements) anew. This would be an interesting future project.

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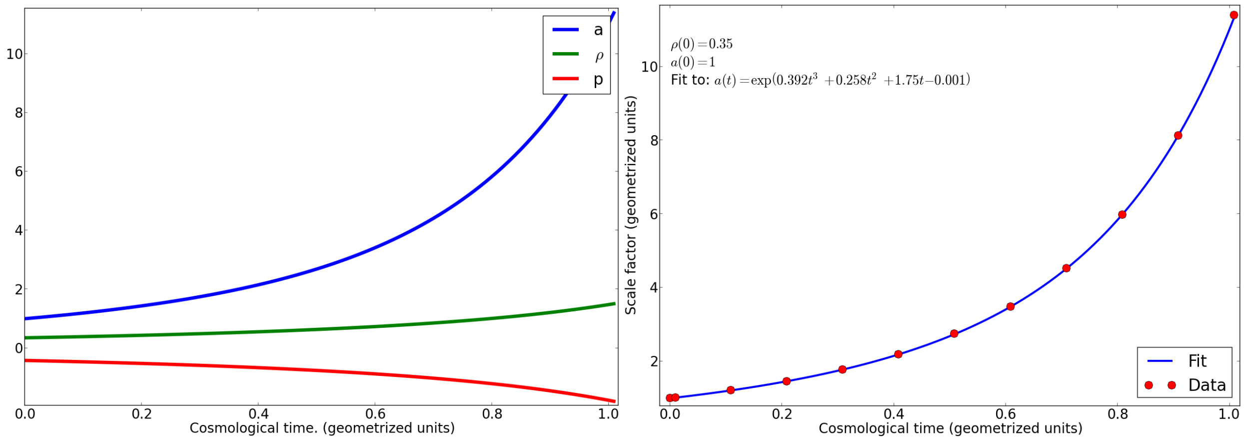

Figure 6: Our future universe? We solve for the universe when $\omega \approx -1.2$. Left: Scale factor, density, and pressure as a function of time. Right: A fit to equation $a(t) = \exp(a_0 t^3 + bt^2 + ct + d)$. In both cases, $\rho(0) = 0.35$.

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