Sky map instrument noise

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1 Sky map instrument noise

Let us estimate here the noise in each sky (HealPIX) pixel coming from the intrinsic noise in each detector pixel.

2 NEP

Detector noise is most conveniently quoted in noise-equivalent power (NEP), which is defined as the amount of optical power that must be incident such that the signal-to-noise ratio (S/N) is 1. It is a measure of noise in optical power units. Note that NEP may be referenced at various different points in the instrument, e.g. inside the detector, incident on the detector, incident at the aperture of the telescope. The noise (N) component is intrinsic to the detector, and so is always referenced to inside the detector. Thus, the S/N = 1 condition must always be enforced inside the detector. For NEPs referenced at some other point, the signal (S) component must first be transferred to inside the detector, and then compared. Explicitly,

NEP (strict definition)
The amount of optical signal power incident at the reference point that creates a signal-to-noise ratio of 1 (S/N = 1) as measured inside of the detector.

From Richards[3], we note that the noise in a bolometer is

$$\frac{P_N^2}{B} = 2\int \mathrm{d}\nu h^2\nu^2 2N(n+n^2) = 2\int \mathrm{d}\nu P_\nu h\nu + \int \mathrm{d}\nu P_\nu^2 \frac{c^2}{A\Omega\nu^2} \qquad \left[\frac{\mathrm{W}^2}{\mathrm{Hz}}\right] \tag{1}$$

where the integral is taken over the passband and the first term on the right hand side corresponds to n and the second term corresponds to n^2. B is the detector bandwidth, N is the number of modes ($N = A\Omega/\lambda^2$, from the Antenna Equation), n is the number of photons per mode, the 2 in front of the first integral comes from the conversion between integration time and bandwidth (see below and Appendix A), and the 2 inside the integral comes from the 2 polarization states per photon. The energy per photon is $h\nu$. The term $n+n^2$ is the thermal expectation value for the variation in the number of photons per mode, $\langle(\Delta n)^2\rangle = n + n^2$.

We note that the number of photons per mode is[2]

$$n = \frac{1}{\exp\left(\frac{h\nu}{k_{\mathrm{B}}T}\right) - 1}$$

where T references the signal (sky) temperature, which is the temperature of the CMB[1] in this case, $T = 2.726\,\mathrm{K}$. For the 4 PIPER frequency bands (200, 270, 350, 600 GHz), this gives photon numbers n

$$\begin{aligned} n = 3\times 10^{-2} &\text{ for } \nu = 200\,\mathrm{GHz} \\ 9\times 10^{-3} &\text{ for } \nu = 270\,\mathrm{GHz} \\ 2\times 10^{-3} &\text{ for } \nu = 350\,\mathrm{GHz} \\ 3\times 10^{-5} &\text{ for } \nu = 600\,\mathrm{GHz} \end{aligned}$$

We note that $n \ll 1$ for all frequency bands, so $n^2 \ll n$, and we may ignore the second term in Eq. (1). So we use for the noise power,

$$\frac{P_N^2}{B} = 2\int \mathrm{d}\nu P_\nu h\nu \qquad \left[\frac{\mathrm{W}^2}{\mathrm{Hz}}\right] \tag{2}$$

where P_ν is the power spectral density.

Let us compute the NEP inside the detector using this expression. The NEP is defined as the amount of power required to give a signal to noise ratio of 1 when there is 1 Hz of noise bandwidth. Note that the signal bandwidth is already implicitly integrated out of this expression. So we have,

$$\begin{aligned} \frac{S}{N} = \frac{\mathrm{NEP}}{P_N(1\,\mathrm{Hz})} &= 1 \\ \mathrm{NEP} &= P_N(1\,\mathrm{Hz}) \\ \mathrm{NEP}^2 &= (1\,\mathrm{Hz})\cdot\frac{P_N^2}{B} \end{aligned}$$

Note that in this scenario, NEP has units of W, which strictly matches the definition given above. However, it is conventional to fold the 1 Hz of noise bandwidth back into the definition of NEP to remind what the scaling with bandwidth is and so that a factor of 1 Hz does not need to be carried around,

$$\mathrm{NEP}^2_{\mathrm{conventional}} = \frac{\mathrm{NEP}^2}{1\,\mathrm{Hz}} = \frac{P_N^2}{B}$$

where the explicit definition of conventional NEP is

NEP (conventional)
: The amount of optical signal power incident at the reference point that creates a signal-to-noise ratio of 1 (S/N = 1) as measured inside of the detector, all divided by $1\,\mathrm{Hz}^{1/2}$.

Henceforth, we will use only the conventional NEP and drop the subscript.

Let us now consider a system with detector absorption efficiency[1] η and optical efficiency τ, where $0 < \eta, \varepsilon, \tau \leq 1$. These parameters make no difference to the NEP referenced inside the detector.

We will compute the NEP referenced to power incident on the detector. The signal power incident to the detector will produce an amount of power inside the detector that is reduced by the factor η. So the NEP condition is then

$$\frac{\eta \cdot \mathrm{NEP}_D}{P_N} = 1$$

and we see that the NEP at the detector input is

$$\mathrm{NEP}_D^2 = \frac{1}{\eta^2} \frac{P_N^2}{B} = \frac{2}{\eta^2} \int \mathrm{d}\nu P_\nu h\nu \tag{3}$$

Similarly, the NEP referenced to power incident on the primary of the telescope is determined by noting that the power at the primary must pass through the optical system, with losses according to the optical efficiency τ, and then be absorbed by the detector. So our NEP condition is

$$\frac{\eta\tau \cdot \mathrm{NEP}_P}{P_N} = 1$$

and the NEP at the telescope input is

$$\mathrm{NEP}_P^2 = \frac{1}{\eta^2\tau^2} \frac{P_N^2}{B} = \frac{2}{\eta^2\tau^2} \int \mathrm{d}\nu P_\nu h\nu \tag{4}$$

We will assume that the NEP inside the detector is known, as it is typically measured independently.

As a final note, most of the transformations in this section do not tell us how to calculate the noise power of the bolometer, which is an intrinsic aspect of the bolometer itself and should be computed from the perspective of heat inside the detector. It is a property of the detector that is independent of the optical loading, since optical power is converted to heat, and is interchangeable with all other sources of heat in the detector (such as electrical). Rather, what these transformations tell us is how to translate the intrinsic noise in the detector (encoded as NEP inside the detector) to reference points outside of the detector. Since the power is transported as photons coming out of the detector (in the time-reversed sense), we must understand the properties of the photons and how they transport power in order to understand how the detector NEP translates to noise at other points. That is the purpose of this section.

However, in addition to intrinsic detector noise, there is also intrinsic noise in the signal from the sky in what is called photon noise. Photon noise can be modeled by using $P_\nu = A\Omega\eta\tau B_\nu(T)$,

[1] Alternatively called detector absorptivity.

$$\mathrm{NEP}^2_{\mathrm{photon}} = \left(\frac{2}{1\,\mathrm{Hz}}\right) A\Omega\,\eta\tau \left(\frac{2h^2}{c^2}\right) \int_{\Delta\nu} \mathrm{d}\nu\; \frac{\nu^4}{\exp\left(\frac{h\nu}{k_{\mathrm{B}}T}\right) - 1} \qquad \left[\frac{\mathrm{W}^2}{\mathrm{Hz}}\right] \tag{5}$$

$$\mathrm{NEP}^2_{\mathrm{photon}} = \left(\frac{2}{1\,\mathrm{Hz}}\right) A\Omega\,\eta\tau \frac{2h^2}{c^2} \left(\frac{k_{\mathrm{B}}T}{h}\right)^5 \int_{x_1}^{x_2} \mathrm{d}x\; \frac{x^4}{e^x - 1} \qquad \left[\frac{\mathrm{W}^2}{\mathrm{Hz}}\right] \tag{6}$$

where $x = \frac{h\nu}{k_{\mathrm{B}}T}$. See below for details of the transformation.

Since it originates from the sky, it must be treated slightly differently. For a single detector, there is no difference, but for many detectors, the optical system can correlate pixels. This is impossible for noise intrinsic to the detector, since each detector pixel is a unique independent device. Readout noise could correlate different pixels to each other, but we do not consider this case here.

3 NET, NEQ, NEU, NEV

The first quantity we will examine is the noise-equivalent temperature (NET). This is the same as the NEP, except in temperature difference units $(\mathrm{K_{CMB}})^2$. It is defined as the change in thermodynamic temperature of a blackbody at the reference point that would generate an amount of optical power in the detector such that the S/N is 1. Note that this definition implicitly includes a fair amount of information about the instrument, such as the etendue, the optical efficiency, and the frequency band. This information is required since we must know how a change in the temperature of the sky's photons propagate to the detector, and the path of propagation is through the instrument. Also note that we must assume a base sky temperature since the spectral distribution of the sky is temperature-dependent.

The NET is defined relative to the NEP by

$$\mathrm{NET} = \frac{\mathrm{NEP}}{\left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)} \qquad \left[\mathrm{K_{CMB}}/\sqrt{\mathrm{Hz}}\right] \tag{7}$$

where $\frac{\mathrm{d}P}{\mathrm{d}T}$ is the change in optical power incident on the detector due to a change in sky temperature. This is straight-forward to compute if we first find the power incident on the detector, which is

$$P(T) = \int_{\Delta\nu} \mathrm{d}\nu\; \eta(\nu)\tau(\nu)P_\nu(T) = A\Omega \int_{\Delta\nu} \mathrm{d}\nu\; \eta(\nu)\tau(\nu)B_\nu(T) \qquad [\mathrm{W}] \tag{8}$$

where we have implicitly defined $P_\nu = \eta\tau A\Omega B_\nu(T)$ as the spectral power density, $\Delta\nu$ is the passband and $B_\nu(T)$ is the Planck distribution, which we note already includes both polarization modes,

$$B_\nu(T) = \frac{2h}{c^2} \frac{\nu^3}{\exp\left(\frac{h\nu}{k_{\mathrm{B}}T}\right)}.$$

Then $\frac{\mathrm{d}P}{\mathrm{d}T}$ is given by

[2] Also known as CMB temperature units. See `docs/foregrounds` for a discussion of these units versus regular thermodynamic temperature units.

$$\frac{\mathrm{d}P}{\mathrm{d}T} = A\Omega \int_{\Delta\nu} \mathrm{d}\nu \; \eta(\nu)\tau(\nu)\frac{\mathrm{d}B_\nu(T)}{\mathrm{d}T}$$

The derivative of the Planck distribution can be shown to be

$$\frac{\mathrm{d}B_\nu(T)}{\mathrm{d}T} = \frac{2h^2}{c^2 k_\mathrm{B} T^2} \frac{\nu^4 \exp\left(\frac{h\nu}{k_\mathrm{B}T}\right)}{\left[\exp\left(\frac{h\nu}{k_\mathrm{B}T}\right) - 1\right]^2}$$

so

$$\frac{\mathrm{d}P}{\mathrm{d}T} = A\Omega \int_{\Delta\nu} \mathrm{d}\nu \; \eta(\nu)\frac{2h^2}{c^2 k_\mathrm{B} T^2} \frac{\nu^4 \exp\left(\frac{h\nu}{k_\mathrm{B}T}\right)}{\left[\exp\left(\frac{h\nu}{k_\mathrm{B}T}\right) - 1\right]^2} \qquad \left[\frac{\mathrm{W}}{\mathrm{K_{CMB}}}\right] \tag{9}$$

which may be put in the more numerically convenient form by using the substitution $x = \frac{h}{k_\mathrm{B}T}\nu$,

$$\frac{\mathrm{d}P}{\mathrm{d}T} = A\Omega \frac{2k_\mathrm{B}}{c^2}\left(\frac{k_\mathrm{B}T}{h}\right)^3 \int_{x_1}^{x_2} \mathrm{d}x \; \eta(x)\tau(x)\frac{x^4 e^x}{(e^x - 1)^2} \qquad \left[\frac{\mathrm{W}}{\mathrm{K_{CMB}}}\right] \tag{10}$$

The integral does not have a nice algebraic solution even if $\eta(x)\tau(x)$ is constant, but is amenable to quadrature for realistic bandpass functions $\eta(x)$. We note, as discussed above, that the etendue ($A\Omega$), the band-pass, the efficiency, and the sky temperature are involved in this conversion factor.

The units of P is W, so $\left[\frac{\mathrm{d}P}{\mathrm{d}T}\right] = \frac{\mathrm{W}}{\mathrm{K_{CMB}}}$, and

$$[\mathrm{NET}] = \left[\frac{\mathrm{NEP}}{\left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)}\right] = \frac{\mathrm{W}}{\sqrt{\mathrm{Hz}}}\frac{\mathrm{K}}{\mathrm{W}} = \frac{\mathrm{K}}{\sqrt{\mathrm{Hz}}}$$

However the conventional unit for NET is $\mathrm{K_{CMB}}\sqrt{\mathrm{s}}$, for which the procedure to get the noise figure in K is to divide by the square root of the integration time, i.e. more integration time results in less noise. This is conceptually intuitive, since one would expect that the measurement from each period of time would be independent, so the number of independent samples would go like $f_\mathrm{sample} \cdot T_\mathrm{integration}$, and the noise would go down by the square root of the number of independent samples.

We note that formally the units $\frac{\mathrm{K_{CMB}}}{\sqrt{\mathrm{Hz}}}$ and $\mathrm{K_{CMB}}\sqrt{\mathrm{s}}$ are equivalent. However, the conversion between the two is a bit more subtle. We wish to convert between units of bandwidth to units of integration time, but due to the Nyquist sampling theorem, we must integrate for 2 seconds to get 1 Hz of bandwidth (see Appendix A). So the correct conversion factor is

$$1 = \frac{2\,\mathrm{s}}{1\,\mathrm{Hz}^{-1}}$$

Then the NET may be written in conventional units as

$$\mathrm{NET} = \left(\sqrt{2}\frac{\sqrt{\mathrm{s}}}{\mathrm{Hz}^{-1/2}}\right)\frac{\mathrm{NEP}}{\left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)} \qquad \left[\mathrm{K_{CMB}}\sqrt{\mathrm{s}}\right] \tag{11}$$

Next let us examine the noise-equivalent Q-parameter (NEQ). Again this is simply a unit transformation, this time from temperature intensity units ($\mathrm{K_{CMB}}$) to polarization intensity units (also in $\mathrm{K_{CMB}}$). NEQ is a noise-equivalent measure of the amount of noise in the measurement of the Stokes Q parameter. There is also a noise-equivalent U-parameter (NEU), but it is usually identical to NEQ, so typically only NEQ is listed. The conventional units of NEQ are the same as NET, $\mathrm{K_{CMB}}\sqrt{\mathrm{s}}$. There is also a noise-equivalent V-parameter (NEV), which has the same units and a similar interpretation. It is rarely listed, since the V-parameter is rarely of interest.

To convert from NET to NEQ, we note that in order to measure the temperature intensity, we need only to measure 1 number. However, to measure the polarization, we must measure 3 numbers (Q, U, and V, or equivalently, E_x, E_y, and ϕ). Because of this, we must split our observation time to measuring 3 different things, so for a single parameter, we get only a fraction of the observation time. If the observation time is split between Q, U, and V according to the ratios f_Q, f_U, and f_V (where $0 \leq f_X \leq 1$ and $f_Q + f_U + f_V = 1$), then we will reduce the observation time for Q according to $T_{\mathrm{integration}} \to f_Q T_{\mathrm{integration}}$ (and similarly for U and V). Then since the number of independent measurements is linear in the integration time, $N_{\mathrm{obs}} = f_{\mathrm{sample}} T_{\mathrm{integration}}$, we get only a fraction of the number of independent samples, $N_{\mathrm{obs}} \to f_Q N_{\mathrm{obs}}$. This change results in a $1/\sqrt{f_Q}$ increase in noise, since noise goes like $1/\sqrt{N_{\mathrm{obs}}}$. Thus, the generic conversions from NET to NEQ, NEU, and NEV are

$$\begin{aligned}
\mathrm{NEQ} &= \left(\sqrt{\frac{2}{f_Q}}\frac{\sqrt{\mathrm{s}}}{\mathrm{Hz}^{-1/2}}\right)\frac{\mathrm{NEP}}{\left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)} && \left[\mathrm{K_{CMB}}\sqrt{\mathrm{s}}\right] \\
\mathrm{NEU} &= \left(\sqrt{\frac{2}{f_U}}\frac{\sqrt{\mathrm{s}}}{\mathrm{Hz}^{-1/2}}\right)\frac{\mathrm{NEP}}{\left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)} && \left[\mathrm{K_{CMB}}\sqrt{\mathrm{s}}\right] \\
\mathrm{NEV} &= \left(\sqrt{\frac{2}{f_V}}\frac{\sqrt{\mathrm{s}}}{\mathrm{Hz}^{-1/2}}\right)\frac{\mathrm{NEP}}{\left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)} && \left[\mathrm{K_{CMB}}\sqrt{\mathrm{s}}\right]
\end{aligned}$$

For Piper, the VPM modulation strategy results in a demodulation such that $\sqrt{f_Q} = \sqrt{f_U} = \frac{1}{2} \cdot \frac{4}{3}\sqrt{f_V}$.[3] Combining this with the normalization condition, $f_Q + f_U + f_V = 1$, we see that

$$f_Q = f_U = \frac{4}{17} \simeq 0.2353 \qquad \text{and} \qquad f_V = \frac{9}{17} \simeq 0.5294. \tag{12}$$

which gives us the Piper-specific conversions,

$$\mathrm{NEQ} = \mathrm{NEU} = \left(\frac{\sqrt{34}}{2}\frac{\sqrt{\mathrm{s}}}{\mathrm{Hz}^{-1/2}}\right)\frac{\mathrm{NEP}}{\left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)} = \left(\frac{2.915\sqrt{\mathrm{s}}}{\mathrm{Hz}^{-1/2}}\right)\frac{\mathrm{NEP}}{\left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)} \qquad \left[\mathrm{K_{CMB}}\sqrt{\mathrm{s}}\right] \tag{13}$$

$$\mathrm{NEV} = \left(\frac{\sqrt{34}}{3}\frac{\sqrt{\mathrm{s}}}{\mathrm{Hz}^{-1/2}}\right)\frac{\mathrm{NEP}}{\left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)} = \left(\frac{1.943\sqrt{\mathrm{s}}}{\mathrm{Hz}^{-1/2}}\right)\frac{\mathrm{NEP}}{\left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)} \qquad \left[\mathrm{K_{CMB}}\sqrt{\mathrm{s}}\right] \tag{14}$$

[3] See Piper proposal, if available to you. Particularly, each telescope has 0.8 sensitivity to local Q and 0.6 sensitivity to V. Since sensitivity goes like $\sqrt{f}$, this gives us $\sqrt{f_Q} = \frac{0.8}{0.6}\sqrt{f_V} = \frac{4}{3}\sqrt{f_V}$. But since both telescopes measure V, and instrument Q and U are measured by only one telescope each, the V weight is doubled.

Lastly, note that we have not specified the reference point for the NEP in any of these calculations. All of these conversions are independent of the efficiencies of the telescope and detector, so moving the reference point of the NEP will change the reference point of the derived quantity (e.g. NEQ or NEU) in the same way. Thus, the derived quantities (NEQ, NEU, NEV) have reference points, and the reference point of the derived quantity is the same as the NEP that was used to construct it. Transforming a derived quantity to move the reference point is done in the same way that transforming the NEP is done.

4 Map Sensitivity

Now that we have in hand the NEQ, which describes the noise in a given detector pixel and how it depends on integration time, we turn to estimating the amount of noise in a given sky pixel. We have already accounted for the noise properties of the detector (encoded in the NEP), so the frame of the detectors is no longer convenient. We are really interested in the noise in each sky pixel, not in the noise in each detector pixel, so we must project the detector noise back onto the sky. Then we can accumulate the integration time in each pixel and get the noise in each sky pixel.

Let us write the NEQ at the sky. We will assume that there is perfect transfer from the sky to the telescope primary, i.e. there are no atmospheric effects or CMB secondaries. These would factor in at the transfer from the primary to the sky and would require some other efficiency parameter in addition to η and τ.

$$\mathrm{NEQ}_P^2 = \left(\frac{2}{f_Q}\frac{\mathrm{s}}{\mathrm{Hz}^{-1}}\right)\frac{\mathrm{NEP}_P^2}{\left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)^2} = \left(\frac{2}{f_Q}\frac{\mathrm{s}}{\mathrm{Hz}^{-1}}\right)\frac{1}{\eta^2\tau^2}\frac{\mathrm{NEP}^2}{\left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)^2}.$$

Over the course of an experiment, the detector spends some period of time looking at each unit of solid angle on the sky. Supposing we have some kind of idealized experiment that uniformly sampled a region of the sky with an overlap factor f_s, then the integration time per unit solid angle is

$$t = \frac{f_s T_e}{\Omega_e} = \frac{f_s T_e}{4\pi f_e} \qquad \left[\frac{\mathrm{s}}{\mathrm{sr}}\right] \tag{15}$$

To get a sense of the meaning of f_s, we can imagine two different experiments with the same etendue $A\Omega$. The first experiment has a small beam Ω_1, and so in the experiment period T_e can only cover the experimental region Ω_e by raster scanning. Each beam spot gets an integration time of only $T_e/(\Omega_e/\Omega_1)$. The second experiment has a large beam $\Omega_2 > \Omega_1$, and so can cover the experimental region more quickly, and so cover it more times in the given experimental period. In particular, experiment 2 has an integration time of $T_e/(\Omega_e/\Omega_2)$. If experiment 2 followed a similar raster scan strategy, it would complete $f_s = \frac{T_e/(\Omega_e/\Omega_2)}{T_e/(\Omega_e/\Omega_1)} = \Omega_2/\Omega_1$ scans in the time experiment 2 took to complete a single scan.

Then as we noted before, the integration period and the number of independent samples are related, so we should divide NEQ_P by $\sqrt{t}$ to get the map sensitivity m,

$$m^2 = \frac{\mathrm{NEQ}_P^2}{t} = \frac{1}{t}\left(\frac{2}{f_Q}\frac{\mathrm{s}}{\mathrm{Hz}^{-1}}\right)\frac{1}{\eta^2\tau^2}\frac{\mathrm{NEP}^2}{\left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)^2} = \frac{4\pi f_e}{f_s T_e}\left(\frac{2}{f_Q}\frac{\mathrm{s}}{\mathrm{Hz}^{-1}}\right)\frac{1}{\eta^2\tau^2}\frac{\mathrm{NEP}^2}{\left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)^2} \tag{16}$$

$$m = \frac{1}{\sqrt{t}}\left(\sqrt{\frac{2}{f_Q}\frac{\sqrt{\mathrm{s}}}{\mathrm{Hz}^{-1/2}}}\right)\frac{1}{\eta\tau}\frac{\mathrm{NEP}}{\left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)} \qquad \left[\mathrm{K_{CMB}}\sqrt{\mathrm{sr}}\right] \tag{17}$$

We note that m has units of $[m] = \mathrm{K_{CMB}}\sqrt{\mathrm{sr}}$. It represents the amount of noise one would get in a region that subtends a particular solid angle. Larger regions require more integration time, and so their noise is reduced. Put another way, 1 second of integration time can measure a region of a particular size (in solid angle) to some fixed amount of precision. To measure a larger region (in solid angle), one must put together many of these fixed size regions. Since each subregion is uncorrelated, this involves combining the fluctuations of many random variables, which reduces the total variance of the whole region.

As a final note, the map sensitivity can be trivially extended to include non-uniform experiments. Simply let the integration time per solid angle depend on direction, $t = t(\mathbf{n})$, then

$$m^2(\mathbf{n}) = \frac{\mathrm{NEQ}_P^2}{t(\mathbf{n})} = \frac{1}{t(\mathbf{n})}\left(\frac{2}{f_Q}\frac{\mathrm{s}}{\mathrm{Hz}^{-1}}\right)\frac{1}{\eta^2\tau^2}\frac{\mathrm{NEP}^2}{\left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)^2}. \tag{18}$$

This expression contains the full directional dependence for our purposes, because the NEQ depends solely on the properties telescope and detector. For a space-based telescope, this is valid. For a balloon- or ground-based experiment, it would be wise to include the effects of the atmosphere. Since the atmosphere is most conveniently represented in the coordinate system of the experiment and not in the CMB coordinate system, it is preferable to include atmospheric effects in the integration time and to construct an effective integration time, $t(\mathbf{n}) \to \tilde{t}(\mathbf{n})$. Using the effective integration time $\tilde{t}$, the above equation would then encode all of the directional dependencies.

5 Multiple Detectors

Suppose instead of a single detector we have an array of detectors. We would like to understand how to estimate the sensitivity of the full array from the properties of the single pixel and the experiment properties. We recall that the map sensitivity is computed from the NEQ via the integration time per solid angle,

$$m^2 = \frac{\mathrm{NEQ}_P^2}{t} \qquad\qquad t = \frac{f_e T_e}{\Omega_e}$$

Having many detectors factors into the t, though they can factor into any of T_e or Ω_e. In all cases, changing from 1 detector to N detectors changes the integration time from $T_e \xrightarrow{N} NT_e$, since each of the N detectors is integrating. Additionally, N detectors can map out the sky N times as quickly, so as long as the regions never overlap, $\Omega_e \xrightarrow{N} N\Omega_e$. In this case, the integration time per steradian scales like

$$\begin{aligned} &t \xrightarrow{N} \frac{f_e N T_e}{N\Omega_e} = \frac{f_e T_e}{\Omega_e} \\ &t \xrightarrow{N} t \qquad \text{for detectors tiled normal to the travel direction} \end{aligned}$$

and the experiment map sensitivity is not improved, but the experiment covers a larger area in the same period of time. This scenario is relevant for detectors that are tiled in a direction normal to the direction of travel along the sky.

For the scenario where the $N-1$ additional detectors are integrating a region that the experiment has already covered, then the behavior is different. This scenario is relevant for detectors that are tiled in a direction parallel to the direction of travel along the sky. In this case, the additional detectors are not measuring new solid angles, so the experimental area does not scale with number of detectors, $\Omega_e \xrightarrow{N} \Omega_e$. However, the integration time always scales up, $T_e \xrightarrow{N} NT_e$. Thus, in this case, the integration time per steradian scales like

$$\begin{aligned} t &\xrightarrow{N} \frac{f_e N T_e}{\Omega_e} = N\frac{f_e T_e}{\Omega_e} \\ t &\xrightarrow{N} Nt \qquad \text{for detectors tiled parallel to the travel direction} \end{aligned}$$

Note that the scaling, and thus the integration time per solid angle, depends on the scan strategy.

Next we consider the case where the beams of some adjacent detectors overlap on the sky. The effects of this depends on where the noise originates from. When the beams do not overlap, there is no need to distinguish the source of noise, and noise from any origin may be treated similarly. We will first consider noise that originates from the detector (e.g. phonon noise). Following that we will consider noise that originates from the sky (e.g. photon noise).

Intrinsic Detector Noise

For noise the originates in the detector, the noise is independent of the signal. The noise in each pixel is independent from every other pixel. So in this case, if there is overlap in the beams, then the same signal is measured in more than one pixel. Since we are ignoring photon noise (natural variations in the signal), the signal is always equal to its mean value. In this case, the overlaps are essentially multiple measurements of the same signal, each measurement with uncorrelated noise. Thus, this scenario is no different from the overlapping region being visited twice at two different time periods, a scenario that was discussed above. In this case, the integration time scales with the number of detectors, regardless of the beam overlap.

$$t \xrightarrow{N} Nt$$

Sky Noise

For noise that originates in the sky, the noise and signal are correlated. We again model the signal as the mean, and now the noise is the variance. For regions of the sky that are not overlapping, each instance of noise is a realization of the 0-mean Gaussian with variance equal to the sky variance. For regions that are overlapping, both detector pixels will sample the same region and thus the noise realization will be identical. Thus we have only 1 sample of the noise in the overlap region and we cannot integrate down the noise, even though we have multiple measurements, because the measurements are correlated. For the regions without overlap, everything works as described in the above general case. So we have

$$\begin{aligned} t &\xrightarrow{N} t \quad \text{for overlapping regions} \\ t &\xrightarrow{N} Nt \quad \text{for non-overlapping regions.} \end{aligned}$$

A convenient way of approximating this is by replacing the actual number of pixels with an effective number of pixels. We simply take the width of the beam of the full array and divide by the beam width of a single pixel to get the effective number of independent pixels along that direction N_{eff}. We note that this is only necessary for detectors tiled parallel to the direction of the scan, since for perpendicularly tiled detectors the integration time per solid angle does not scale with number of detectors, so the transformation $N \to N_{\rm eff}$ makes no difference.

6 Pixel Noise

Let us use our map sensitivity to estimate the noise in each pixel of the sky, which depends on our pixelization scheme. Equal-area pixelization schemes are the most tractable, including HealPIX. Each pixel has a fixed angular size, Ω_p. The map sensitivity tells us how much noise is in a region of some angular size, so we simply divide the map sensitivity by the pixel size to get the pixel noise N_p,

$$N_p = \frac{m}{\sqrt{\Omega_p}} = \frac{1}{\sqrt{\Omega_p t}} \left(\sqrt{\frac{2}{f_Q}} \frac{\sqrt{\rm s}}{\rm Hz^{-1/2}} \right) \frac{1}{\eta\tau} \frac{\rm NEP}{\left(\frac{{\rm d}P}{{\rm d}T}\right)} \qquad \left[{\rm K_{CMB}}\sqrt{\rm pixel}\right] \tag{19}$$

where we have included the unitless "pixel" to indicate the scaling of the pixel noise with the number of pixels. Note that the units justify this, since the units of Ω_p may be written $[\Omega_p] = \text{sr/pixel}$. To get the noise in a single pixel, simply divide by 1 pixel. To get the average noise level in a region $N_{\rm pix}$ pixels large, simply divide by $\sqrt{N_{\rm pix}}$.

7 Map Sensitivity in Angular Units

Frequently features are measured in units of radians (or degrees, arcminutes, etc.) rather than steradians. Thus, we would like to know what the map sensitivity for such features are. We suppose that a feature described by some angular size θ is actually a spherical cap with angular diameter θ. Note that the opening angle for a spherical cap with angular diameter θ is $\theta/2$, i.e. suppose the spherical cap is centered on the z-axis, then the polar angle between the z-axis and the edge of the spherical cap is $\theta/2$.

The solid angle of this spherical cap is

$$\begin{aligned} \Omega(\theta) &= \int {\rm d}\Omega = \int_0^{\theta/2} \sin\theta' \ {\rm d}\theta' \int {\rm d}\phi \\ \Omega(\theta) &= 2\pi \left[1 - \cos\left({}^{\theta}/_{2}\right)\right] \end{aligned} \tag{20}$$

We note that in the small angle limit,

$$\Omega(\theta) \simeq \frac{\pi\theta^2}{4}$$

as expected, since it reduces to a flat circle in this limit.

To get the NEQ in radians, we simply use this expression for Ω_e in Eq. 16,

$$m = \sqrt{\frac{2\pi\left[1-\cos\left(\theta/2\right)\right]}{f_s T_e}}\left(\sqrt{\frac{2}{f_Q}}\frac{\sqrt{\mathrm{s}}}{\mathrm{Hz}^{-1/2}}\right)\frac{1}{\eta\tau}\frac{\mathrm{NEP}}{\left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)} \qquad [\mathrm{K_{CMB}}\cdot\mathrm{rad}] \tag{21}$$

$$m \simeq \frac{\sqrt{\pi}\theta}{2\sqrt{f_s T_e}}\left(\sqrt{\frac{2}{f_Q}}\frac{\sqrt{\mathrm{s}}}{\mathrm{Hz}^{-1/2}}\right)\frac{1}{\eta\tau}\frac{\mathrm{NEP}}{\left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)} \qquad [\mathrm{K_{CMB}}\cdot\mathrm{rad}] \tag{22}$$

8 PIPER-like Experiment

Let us estimate some of these parameters for a PIPER-like experiment[4]. A single pixel of PIPER's detectors has a phonon NEP of

$$\mathrm{NEP} = 4\times10^{-18}\,\frac{\mathrm{W}}{\sqrt{\mathrm{Hz}}}$$

and efficiencies η and τ,

Frequency	200 GHz	270 GHz	350 GHz	600 GHz
Bandwidth $\delta\nu/\nu$	0.3	0.3	0.16	0.1
η	0.9	0.9	0.7	0.5
τ	0.55	0.52	0.5	0.42
Single-pixel Area $\left[\mathrm{m}^2\right]$	1.3×10^{-6} [a]	1.3×10^{-6}	1.3×10^{-6}	1.3×10^{-6}
f/N [b]	1.6	1.6	1.6	1.6
Ω [c] [sr]	0.2424	0.2424	0.2424	0.2424
$A\Omega$ $\left[\mathrm{m}^2\ \mathrm{sr}\right]$	3.1×10^{-7}	3.1×10^{-7}	3.1×10^{-7}	3.1×10^{-7}
NEP [d] (phonon) $\left[\frac{\mathrm{W}}{\sqrt{\mathrm{Hz}}}\right]$	4×10^{-18}	4×10^{-18}	4×10^{-18}	4×10^{-18}

[a] $1135\,\mu\mathrm{m}\times1135\,\mu\mathrm{m}$
[b] At the detector.
[c] At the detector.
[d] At the detector.

A single pixel has area

$$A = 1135\,\mu\mathrm{m}\times1135\,\mu\mathrm{m} = 1.288\times10^{-6}\,\mathrm{m}^2$$

and the $f/$number at the detectors is $f/1.6$, which corresponds to

$$\Omega = \frac{4\pi}{(4f/\mathrm{N})^2} = 0.2424\,\mathrm{sr}$$

for a throughput of

$$A\Omega = 3.122\times10^{-7}\,\mathrm{m}^2\ \mathrm{sr}$$

[4] The following values are all taken from the 2014 PIPER proposal.

A Relationship between Integration Time and Bandwidth

Suppose we integrate a signal $x(t)$ for a period of time T. This is equivalent to filtering the signal with a rect filter,

$$h(t) = \mathrm{rect}(t/T) = u(t + {}^{T}\!/_{2}) - u(t - {}^{T}\!/_{2}), \tag{23}$$

where $u(t)$ is the Heaviside step function. In harmonic space, this filter is

$$H(f) = F\{h(t)\} = T\,\mathrm{sinc}\,(fT) = T\frac{\sin(\pi f T)}{\pi f T} \tag{24}$$

where we define our fourier transform as

$$F\{x(t)\} = \int_{-\infty}^{\infty} \mathrm{d}f\; x(t)\exp(-2\pi i f t)$$

The bandwidth of a signal is conventionally defined as the distance in frequency space between the first positive and first negative node. We note that the sinc function is symmetric and has its first node at $f = 1/T$, so the bandwidth is $B = 2/T$. Thus the relationship between integration time and bandwidth is

$$\begin{aligned} T\,\text{seconds integration time} &\longleftrightarrow \frac{2}{T}\,\text{Hz bandwidth} \\ 1\,\text{second integration time} &\longleftrightarrow 2\,\text{Hz bandwidth} \end{aligned}$$

and so the conversion between seconds of integration time and Hz of bandwidth is

$$1 = \frac{1\,\mathrm{s}}{\frac{1}{2}\,\mathrm{Hz}^{-1}} = \frac{2\,\mathrm{s}}{1\,\mathrm{Hz}^{-1}} \tag{25}$$

Note that for linear time-invariant (LTI) systems, whether we do the integration in the time domain all at once or over a series of bursts, the result must be the same. We could simply time-shift all of the individual bursts to be right next to each other, thereby recreating the full-width rect.

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