Sky map instrument noise

Justin Lazear

March 16, 2015

1 Sky map instrument noise

Let us estimate here the noise in each sky (healpix) pixel coming from the intrinsic noise in each detector pixel.

1.1 NEP

Detector noise is most conveniently quoted in noise-equivalent power (NEP), which is defined as the amount of optical power that must be incident such that the signal-to-noise ratio (S/N) is 1. It is a measure of noise in optical power units. Note that NEP may be referenced at various different points in the instrument, e.g. inside the detector, incident on the detector, incident at the aperture of the telescope. The noise (N) component is intrinsic to the detector, and so is always referenced to inside the detector. Thus, the S/N = 1 condition must always be enforced inside the detector. For NEPs referenced at some other point, the signal (S) component must first be transferred to inside the detector, and then compared. Explicitly,

NEP (strict definition)
: The amount of optical signal power incident at the reference point that creates a signal-to-noise ratio of 1 (S/N = 1) as measured inside of the detector.

From Richards[3], we note that the noise in a bolometer is

$$\frac{P_N^2}{B} = 2\int \mathrm{d}\nu h^2\nu^2 2N(n+n^2) = 2\int \mathrm{d}\nu P_\nu h\nu + \int \mathrm{d}\nu P_\nu^2 \frac{c^2}{A\Omega\nu^2} \tag{1}$$

where the first term on the right hand side corresponds to n and the second term corresponds to n^2. B is the detector bandwidth, N is the number of modes ($N = A\Omega/\lambda^2$, from the Antenna Equation), n is the number of photons per mode, the 2 in front of the first integral comes from the conversion between integration time and bandwidth (see below and Appendix A), and the 2 inside the integral comes from the 2 polarization states per photon. The energy per photon is $h\nu$. The term $n + n^2$ is the thermal expectation value for the variation in the number of photons per mode, $\langle(\Delta n)^2\rangle = n + n^2$.

We note that the number of photons per mode is[2]

$$n = \frac{1}{\exp\left(\frac{h\nu}{k_{\mathrm{B}}T}\right) - 1}$$

where T references the signal (sky) temperature, which is the temperature of the CMB[1] in this case, $T = 2.726\,\mathrm{K}$. For the 4 PIPER frequency bands (200, 270, 350, 600 GHz), this gives photon numbers n

$$\begin{aligned} n &= 3 \times 10^{-2} \text{ for } \nu = 200\,\mathrm{GHz} \\ & 9 \times 10^{-3} \text{ for } \nu = 270\,\mathrm{GHz} \\ & 2 \times 10^{-3} \text{ for } \nu = 350\,\mathrm{GHz} \\ & 3 \times 10^{-5} \text{ for } \nu = 600\,\mathrm{GHz} \end{aligned}$$

We note that $n \ll 1$ for all frequency bands, so $n^2 \ll n$, and we may ignore the second term in Eq. (1). So we use for the noise power,

$$\frac{P_N^2}{B} = 2 \int \mathrm{d}\nu P_\nu h\nu \tag{2}$$

where P_ν is the power spectral density.

Let us compute the NEP inside the detector using this expression. The NEP is defined as the amount of power required to give a signal to noise ratio of 1 when there is 1 Hz of noise bandwidth. Note that the signal bandwidth is already implicitly integrated out of this expression. So we have,

$$\begin{aligned} \frac{S}{N} = \frac{\mathrm{NEP}}{P_N(1\,\mathrm{Hz})} &= 1 \\ \mathrm{NEP} &= P_N(1\,\mathrm{Hz}) \\ \mathrm{NEP}^2 &= (1\,\mathrm{Hz}) \cdot \frac{P_N^2}{B} \end{aligned}$$

Note that in this scenario, NEP has units of W, which strictly matches the definition given above. However, it is conventional to fold the 1 Hz of noise bandwidth back into the definition of NEP to remind what the scaling with bandwidth is and so that a factor of 1 Hz does not need to be carried around,

$$\mathrm{NEP}^2_{\text{conventional}} = \frac{\mathrm{NEP}^2}{1\,\mathrm{Hz}} = \frac{P_N^2}{B}$$

where the explicit definition of conventional NEP is

NEP (conventional)
: The amount of optical signal power incident at the reference point that creates a signal-to-noise ratio of 1 (S/N = 1) as measured inside of the detector, all divided by $1\,\mathrm{Hz}^{1/2}$.

Henceforth, we will use only the conventional NEP and drop the subscript.

Let us now consider a system with detector absorption efficiency[1] η and optical efficiency τ, where $0 < \eta, \varepsilon, \tau \leq 1$. These parameters make no difference to the NEP referenced inside the detector.

We will compute the NEP referenced to power incident on the detector. The signal power incident to the detector will produce an amount of power inside the detector that is reduced by the factor η. So the NEP condition is then

$$\frac{\eta \cdot \mathrm{NEP}_D}{P_N} = 1$$

and we see that the NEP at the detector input is

$$\mathrm{NEP}_D^2 = \frac{1}{\eta^2} \frac{P_N^2}{B} = \frac{2}{\eta^2} \int \mathrm{d}\nu P_\nu h\nu \tag{3}$$

Similarly, the NEP referenced to power incident on the primary of the telescope is determined by noting that the power at the primary must pass through the optical system, with losses according to the optical efficiency τ, and then be absorbed by the detector. So our NEP condition is

$$\frac{\eta\tau \cdot \mathrm{NEP}_P}{P_N} = 1$$

and the NEP at the telescope input is

$$\mathrm{NEP}_P^2 = \frac{1}{\eta^2\tau^2} \frac{P_N^2}{B} = \frac{2}{\eta^2\tau^2} \int \mathrm{d}\nu P_\nu h\nu \tag{4}$$

We will assume that the NEP inside the detector is known, as it is typically measured independently.

As a final note, most of the transformations in this section do not tell us how to calculate the noise power of the bolometer, which is an intrinsic aspect of the bolometer itself and should be computed from the perspective of heat inside the detector. It is a property of the detector that is independent of the optical loading, since optical power is converted to heat, and is interchangeable with all other sources of heat in the detector (such as electrical). Rather, what these transformations tell us is how to translate the intrinsic noise in the detector (encoded as NEP inside the detector) to reference points outside of the detector. Since the power is transported as photons coming out of the detector (in the time-reversed sense), we must understand the properties of the photons and how they transport power in order to understand how the detector NEP translates to noise at other points. That is the purpose of this section.

1.2 NET, NEQ, NEU, NEV

The first quantity we will examine is the noise-equivalent temperature (NET). This is the same as the NEP, except in temperature difference units ($\mathrm{K}_{\mathrm{CMB}}$)[2]. It is defined as the change in thermodynamic temperature

[1] Alternatively called detector absorptivity.

[2] Also known as CMB temperature units. See `docs/foregrounds` for a discussion of these units versus regular thermodynamic temperature units.

of a blackbody at the reference point that would generate an amount of optical power in the detector such that the S/N is 1. Note that this definition implicitly includes a fair amount of information about the instrument, such as the etendue, the optical efficiency, and the frequency band. This information is required since we must know how a change in the temperature of the sky's photons propagate to the detector, and the path of propagation is through the instrument. Also note that we must assume a base sky temperature since the spectral distribution of the sky is temperature-dependent.

The NET is defined relative to the NEP by

$$\mathrm{NET} = \frac{\mathrm{NEP}}{\left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)} \qquad \text{in } \mathrm{K_{CMB}}/\sqrt{\mathrm{Hz}} \tag{5}$$

where $\frac{\mathrm{d}P}{\mathrm{d}T}$ is the change in optical power incident on the detector due to a change in sky temperature. This is straight-forward to compute if we first find the power incident on the detector, which is

$$P(T) = \int_{\Delta\nu} \mathrm{d}\nu\ \eta(\nu)\tau(\nu)P_\nu(T) = A\Omega \int_{\Delta\nu} \mathrm{d}\nu\ \eta(\nu)\tau(\nu)B_\nu(T) \tag{6}$$

where $\Delta\nu$ is the passband and $B_\nu(T)$ is the Planck distribution, which we note already includes both polarization modes,

$$B_\nu(T) = \frac{2h}{c^2} \frac{\nu^3}{\exp\left(\frac{h\nu}{k_\mathrm{B}T}\right)}.$$

Then $\frac{\mathrm{d}P}{\mathrm{d}T}$ is given by

$$\frac{\mathrm{d}P}{\mathrm{d}T} = A\Omega \int_{\Delta\nu} \mathrm{d}\nu\ \eta(\nu)\tau(\nu)\frac{\mathrm{d}B_\nu(T)}{\mathrm{d}T}$$

The derivative of the Planck distribution can be shown to be

$$\frac{\mathrm{d}B_\nu(T)}{\mathrm{d}T} = \frac{2h^2}{c^2 k_\mathrm{B} T^2} \frac{\nu^4 \exp\left(\frac{h\nu}{k_\mathrm{B}T}\right)}{\left[\exp\left(\frac{h\nu}{k_\mathrm{B}T}\right) - 1\right]^2}$$

so

$$\frac{\mathrm{d}P}{\mathrm{d}T} = A\Omega \int_{\Delta\nu} \mathrm{d}\nu\ \eta(\nu) \frac{2h^2}{c^2 k_\mathrm{B} T^2} \frac{\nu^4 \exp\left(\frac{h\nu}{k_\mathrm{B}T}\right)}{\left[\exp\left(\frac{h\nu}{k_\mathrm{B}T}\right) - 1\right]^2} \tag{7}$$

which may be put in the more numerically convenient form by using the substitution $x = \frac{h}{k_\mathrm{B}T}\nu$,

$$\frac{\mathrm{d}P}{\mathrm{d}T} = A\Omega \frac{2k_\mathrm{B}}{c^2} \left(\frac{k_\mathrm{B}T}{h}\right)^3 \int_{x_1}^{x_2} \mathrm{d}x\ \eta(x)\tau(x) \frac{x^4 e^x}{(e^x - 1)^2} \tag{8}$$

The integral does not have a nice algebraic solution even if $\eta(x)\tau(x)$ is constant, but is amenable to quadrature for realistic bandpass functions $\eta(x)$. We note, as discussed above, that the etendue ($A\Omega$), the band-pass, the efficiency, and the sky temperature are involved in this conversion factor.

The units of P is W, so $\left[\frac{\mathrm{d}P}{\mathrm{d}T}\right] = \frac{\mathrm{W}}{\mathrm{K_{CMB}}}$, and

$$[\mathrm{NET}] = \left[\frac{\mathrm{NEP}}{\left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)}\right] = \frac{\mathrm{W}}{\sqrt{\mathrm{Hz}}}\frac{\mathrm{K}}{\mathrm{W}} = \frac{\mathrm{K}}{\sqrt{\mathrm{Hz}}}$$

However the conventional unit for NET is $\mathrm{K_{CMB}}\sqrt{\mathrm{s}}$, for which the procedure to get the noise figure in K is to divide by the square root of the integration time, i.e. more integration time results in less noise. This is conceptually intuitive, since one would expect that the measurement from each period of time would be independent, so the number of independent samples would go like $f_{\mathrm{sample}} \cdot T_{\mathrm{integration}}$, and the noise would go down by the square root of the number of independent samples.

We note that formally the units $\frac{\mathrm{K_{CMB}}}{\sqrt{\mathrm{Hz}}}$ and $\mathrm{K_{CMB}}\sqrt{\mathrm{s}}$ are equivalent. However, the conversion between the two is a bit more subtle. We wish to convert between units of bandwidth to units of integration time, but due to the Nyquist sampling theorem, we must integrate for 2 seconds to get 1 Hz of bandwidth (see Appendix A). So the correct conversion factor is

$$1 = \frac{2\,\mathrm{s}}{1\,\mathrm{Hz}^{-1}}$$

Then the NET may be written in conventional units as

$$\mathrm{NET} = \left(\sqrt{2}\frac{\sqrt{\mathrm{s}}}{\mathrm{Hz}^{-1/2}}\right)\frac{\mathrm{NEP}}{\left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)} \qquad \text{in } \mathrm{K_{CMB}}\sqrt{\mathrm{s}} \tag{9}$$

Next let us examine the noise-equivalent Q-parameter (NEQ). Again this is simply a unit transformation, this time from temperature intensity units ($\mathrm{K_{CMB}}$) to polarization intensity units (also in $\mathrm{K_{CMB}}$). NEQ is a noise-equivalent measure of the amount of noise in the measurement of the Stokes Q parameter. There is also a noise-equivalent U-parameter (NEU), but it is usually identical to NEQ, so typically only NEQ is listed. The conventional units of NEQ are the same as NET, $\mathrm{K_{CMB}}\sqrt{\mathrm{s}}$. There is also a noise-equivalent V-parameter (NEV), which has the same units and a similar interpretation. It is rarely listed, since the V-parameter is rarely of interest.

To convert from NET to NEQ, we note that in order to measure the temperature intensity, we need only to measure 1 number. However, to measure the polarization, we must measure 3 numbers (Q, U, and V, or equivalently, E_x, E_y, and ϕ). Because of this, we must split our observation time to measuring 3 different things, so for a single parameter, we get only a fraction of the observation time. If the observation time is split between Q, U, and V according to the ratios f_Q, f_U, and f_V (where $0 \leq f_X \leq 1$ and $f_Q + f_U + f_V = 1$), then we will reduce the observation time for Q according to $T_{\mathrm{integration}} \to f_Q T_{\mathrm{integration}}$ (and similarly for U and V). Then since the number of independent measurements is linear in the integration time, $N_{\mathrm{obs}} = f_{\mathrm{sample}} T_{\mathrm{integration}}$, we get only a fraction of the number of independent samples, $N_{\mathrm{obs}} \to f_Q N_{\mathrm{obs}}$. This change results in a $1/\sqrt{f_Q}$ increase in noise, since noise goes like $1/\sqrt{N_{\mathrm{obs}}}$. Thus, the generic conversions from NET to NEQ, NEU, and NEV are

$$\text{NEQ} = \left(\sqrt{\frac{2}{f_Q}}\frac{\sqrt{\text{s}}}{\text{Hz}^{-1/2}}\right)\frac{\text{NEP}}{\left(\frac{\text{d}P}{\text{d}T}\right)} \qquad \text{in } \text{K}_{\text{CMB}}\sqrt{\text{s}}$$

$$\text{NEU} = \left(\sqrt{\frac{2}{f_U}}\frac{\sqrt{\text{s}}}{\text{Hz}^{-1/2}}\right)\frac{\text{NEP}}{\left(\frac{\text{d}P}{\text{d}T}\right)} \qquad \text{in } \text{K}_{\text{CMB}}\sqrt{\text{s}}$$

$$\text{NEV} = \left(\sqrt{\frac{2}{f_V}}\frac{\sqrt{\text{s}}}{\text{Hz}^{-1/2}}\right)\frac{\text{NEP}}{\left(\frac{\text{d}P}{\text{d}T}\right)} \qquad \text{in } \text{K}_{\text{CMB}}\sqrt{\text{s}}$$

For PIPER, the VPM modulation strategy results in a demodulation such that $\sqrt{f_Q} = \sqrt{f_U} = \frac{4}{3}\sqrt{f_V}$.[3] Combining this with the normalization condition, $f_Q + f_U + f_V = 1$, we see that

$$f_Q = f_U = \frac{16}{41} \simeq 0.3902 \qquad \text{and} \qquad f_V = \frac{9}{41} \simeq 0.2195. \tag{10}$$

which gives us the PIPER-specific conversions,

$$\text{NEQ} = \text{NEU} = \left(\frac{\sqrt{82}}{4}\frac{\sqrt{\text{s}}}{\text{Hz}^{-1/2}}\right)\frac{\text{NEP}}{\left(\frac{\text{d}P}{\text{d}T}\right)} = \left(\frac{2.264\sqrt{\text{s}}}{\text{Hz}^{-1/2}}\right)\frac{\text{NEP}}{\left(\frac{\text{d}P}{\text{d}T}\right)} \qquad \text{in } \text{K}_{\text{CMB}}\sqrt{\text{s}} \tag{11}$$

$$\text{NEV} = \left(\frac{\sqrt{82}}{3}\frac{\sqrt{\text{s}}}{\text{Hz}^{-1/2}}\right)\frac{\text{NEP}}{\left(\frac{\text{d}P}{\text{d}T}\right)} = \left(\frac{3.018\sqrt{\text{s}}}{\text{Hz}^{-1/2}}\right)\frac{\text{NEP}}{\left(\frac{\text{d}P}{\text{d}T}\right)} \qquad \text{in } \text{K}_{\text{CMB}}\sqrt{\text{s}} \tag{12}$$

Lastly, note that we have not specified the reference point for the NEP in any of these calculations. All of these conversions are independent of the efficiencies of the telescope and detector, so moving the reference point of the NEP will change the reference point of the derived quantity (e.g. NEQ or NEU) in the same way. Thus, the derived quantities (NEQ, NEU, NEV) have reference points, and the reference point of the derived quantity is the same as the NEP that was used to construct it. Transforming a derived quantity to move the reference point is done in the same way that transforming the NEP is done.

1.3 Sensitivity

Now that we have in hand the NEQ, which describes the noise in a given detector pixel and how it depends on integration time, we turn to estimating the amount of noise in a given sky pixel. We have already accounted for the noise properties of the detector (encoded in the NEP), so the frame of the detectors is no longer convenient. We are really interested in the noise in each sky pixel, not in the noise in each detector pixel, so we must project the detector noise back onto the sky. Then we can accumulate the integration time in each pixel and get the noise in each sky pixel.

Let us write the NEQ at the telescope input,

[3] See PIPER proposal, if available to you...

$$
\begin{aligned}
\mathrm{NEQ}_P^2 &= \left(\frac{2}{f_Q}\frac{\mathrm{s}}{\mathrm{Hz}^{-1}}\right)\frac{\mathrm{NEP}_P^2}{\left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)^2} = \left(\frac{2}{f_Q}\frac{\mathrm{s}}{\mathrm{Hz}^{-1}}\right)\frac{1}{\eta^2\tau^2}\frac{\mathrm{NEP}^2}{\left(\frac{\mathrm{d}P}{\mathrm{d}T}\right)^2} \\
&= \left(\frac{2}{f_Q}\frac{\mathrm{s}}{\mathrm{Hz}^{-1}}\right)\frac{1}{\eta^2\tau^2}\frac{2\int \mathrm{d}\nu P_\nu h\nu}{\left[A\Omega\frac{2k_\mathrm{B}}{c^2}\left(\frac{k_\mathrm{B}T}{h}\right)^3\eta\tau\int\mathrm{d}x\frac{x^4e^x}{e^x-1}\right]^2} \\
&= \left(\frac{2}{f_Q}\frac{\mathrm{s}}{\mathrm{Hz}^{-1}}\right)\frac{1}{\eta^2\tau^2}\frac{4A\Omega\left(\frac{k_\mathrm{B}T}{h}\right)^5\eta\tau\int\frac{x^4}{e^x-1}\,\mathrm{d}x}{(A\Omega)^2\frac{4k_\mathrm{B}^2}{c^4}\left(\frac{k_\mathrm{B}T}{h}\right)^6\eta^2\tau^2\left[\int\mathrm{d}x\frac{x^4e^x}{e^x-1}\right]^2} \\
\mathrm{NEQ}_P^2 &= \left(\frac{2}{f_Q}\frac{\mathrm{s}}{\mathrm{Hz}^{-1}}\right)\frac{1}{\eta^3\tau^3}\frac{I_1}{I_2^2}\frac{h^3c^2}{k_\mathrm{B}^3T}\frac{1}{A\Omega}
\end{aligned} \tag{13}
$$

where $I_1 = \int \mathrm{d}x\frac{x^4}{e^x-1}$ and $I_2 = \int \mathrm{d}x\frac{x^4e^x}{e^x-1}$.

A Relationship between Integration Time and Bandwidth

Suppose we integrate a signal $x(t)$ for a period of time T. This is equivalent to filtering the signal with a rect filter,

$$h(t) = \mathrm{rect}(t/T) = u(t - {}^{T}\!/_{2}) - u(t + {}^{T}\!/_{2}), \tag{14}$$

where $u(t)$ is the Heaviside step function. In harmonic space, this filter is

$$H(f) = F\{h(t)\} = T\,\mathrm{sinc}\,(fT) = T\frac{\sin(\pi fT)}{\pi fT} \tag{15}$$

where we define our fourier transform as

$$F\{x(t)\} = \int_{-\infty}^{\infty} \mathrm{d}f\ x(t)\exp(-2\pi ift)$$

The bandwidth of a signal is conventionally defined as the distance in frequency space between the first positive and first negative node. We note that the sinc function is symmetric and has its first node at $f = 1/T$, so the bandwidth is $B = 2/T$. Thus the relationship between integration time and bandwidth is

$$\begin{aligned} T\,\text{seconds integration time} &\longleftrightarrow \frac{2}{T}\,\text{Hz bandwidth} \\ 1\,\text{second integration time} &\longleftrightarrow 2\,\text{Hz bandwidth} \end{aligned}$$

and so the conversion between seconds of integration time and Hz of bandwidth is

$$1 = \frac{1\,\mathrm{s}}{\frac{1}{2}\,\mathrm{Hz}^{-1}} = \frac{2\,\mathrm{s}}{1\,\mathrm{Hz}^{-1}} \tag{16}$$

Note that for linear time-invariant (LTI) systems, whether we do the integration in the time domain all at once or over a series of bursts, the result must be the same. We could simply time-shift all of the individual bursts to be right next to each other, thereby recreating the full-width rect.

References

[1] D. J. Fixsen. The Temperature of the Cosmic Microwave Background. *The Astrophysical Journal*, 707:916–920, December 2009.

[2] Charles Kittel and Herbert Kroemer. *Thermal Physics*. W. H. Freeman, second edition edition, January 1980.

[3] P. L. Richards. Bolometers for infrared and millimeter waves. *Journal of Applied Physics*, 76(1):1–24, July 1994.