Internal Linear Combination Method for Foreground Subtraction

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1 Internal Linear Combination method of foreground subtraction

Consider a measurement of the full CMB sky in k different frequency bands. The product of such a measurement is k temperature maps

$$T_i(p) = \text{Temperature map of frequency band } \nu_i \text{ with pixel index } p \tag{1}$$

where i is the frequency band index with $i = 1, \ldots, k$ and p is the pixel index with $p = 1, \ldots, N$. We will choose to represent the maps in thermodynamic temperature units[1].

Such maps are usually encoded in the HEALpix[3] format. Care must be taken to ensure that all maps that are used have the same number of pixels (typically $N_{\text{side}} = 512$ for WMAP) and that all maps have been smoothed to the same resolution (typically 1°).

The temperature maps can (theoretically) be decomposed into a CMB component and a residual component

$$T_i(p) = T_{\text{CMB}}(p) + R_i(p), \tag{4}$$

where the $T_{\text{CMB}}(p)$ component does not dependent on frequency (since we have chosen thermodynamic temperature units), and the $R_i(p)$ component encodes *all* sources of signal that are not from the CMB.

[1] Thermodynamic temperature units are defined relative to the Planck distribution, and are a measure of power per unit area. The key point to keep in mind that the thermodynamic temperature units are referencing a blackbody (Planck) spectrum, so any additional information required to get to the desired units must be pulled from the properties of the map (e.g. area).

Sky maps are usually given in terms of spectral intensity, i.e. energy per unit area per unit solid angle per unit frequency, with SI unit $\frac{\text{W}}{\text{m}^2\,\text{sr Hz}}$. Commonly, intensity is given in terms of mega-Janskies per steradian, where $1\,\text{Jy} = 10^{-26}\,\frac{\text{W}}{\text{m}^2\,\text{Hz}}$, so $1\,\frac{\text{w}}{\text{m}^2\,\text{sr Hz}} = 10^{20}\,\frac{\text{MJy}}{\text{sr}}$.

Spectral intensity I_ν is related to thermodynamic temperature units T by the Planck distribution,

$$I_\nu = \frac{2h\nu^3}{c^2} \frac{1}{\exp(h\nu/k_B T) - 1} = B_\nu(T). \tag{2}$$

From inspection, we can see that T has units of Kelvin, but encodes the same information as spectral intensity.

The last common unit is antenna temperature T_A, which is the low-frequency limit of the above relation,

$$I_\nu = \frac{2k_B \nu^2}{c^2} T_A, \tag{3}$$

where we note that all we have done is expand the inverse exponential term in the $h\nu/k_B T \ll 1$ limit. The antenna temperature also has units of Kelvin.

We note that the conversions between spectral intensity and the temperatures are frequency-dependent, and do not simply have a numerical value. Additionally, since the CMB spectrum is a blackbody (ignoring spectral distortions), its thermodynamic temperature is constant for all frequencies.

Since the CMB component will be constant in frequency, we expect that we can combine the various maps in different frequencies to construct an estimator of the CMB map $T_{\mathrm{CMB}}(p)$. We construct an estimator using a linear combination of the maps in different frequency bands with to-be-determined weights $\zeta_i(p)$[1, 2, 4],

$$\begin{aligned}\hat{T}(p) &= \sum_{i=1}^{k} \zeta_i(p) T_i(p) &(5)\\ &= \sum_i \zeta_i(p)\left[T_{\mathrm{CMB}}(p) + R_i(p)\right]\\ \hat{T}(p) &= T_{\mathrm{CMB}}(p) \sum_i \zeta_i(p) + \sum_i \zeta_i(p) R_i(p). &(6)\end{aligned}$$

We note that there are $\mathrm{num}\left(\zeta_i(p)\right) = Nk$ unknown parameters in our estimator. In order for the estimator $\hat{T}(p)$ to have unity gain in $T_{\mathrm{CMB}}(p)$, we must have

$$\sum_i \zeta_i(p) = 1 \qquad \forall\ p \tag{7}$$

which provides N constraint equations. This results in

$$\hat{T}(p) = T_{\mathrm{CMB}}(p) + \sum_{i=1}^{k} \zeta_i(p) R_i(p). \tag{8}$$

Further properties of the estimator $\hat{T}(p)$ will be determined by how we choose to constrain the remaining $(N-1)k$ degrees of freedom.

We note some statistical properties of the estimator.

$$\begin{aligned}\left\langle \hat{T} \right\rangle &= \left\langle T_{\mathrm{CMB}} \right\rangle + \sum_i \left\langle \zeta_i R_i \right\rangle\\ \mathrm{var}\,\hat{T} &= \mathrm{var}\left(T_{\mathrm{CMB}}\right) + \sum_i \mathrm{var}\left(\zeta_i R_i\right) + \sum_i \mathrm{cov}\left(T_{\mathrm{CMB}}, \zeta_i R_i\right) + \sum_{i \neq j} \mathrm{cov}\left(\zeta_i R_i, \zeta_j R_j\right)\end{aligned}$$

where the expectation value, variance, and covariance operators are defined over the pixels[2].

We can write the variance of the estimator in a matrix formalism by expanding the definition of the covariance operator to allow for vector[3] arguments,

[2] I.e.,

$$\begin{aligned}\langle x \rangle &= \frac{1}{N} \sum_{p=1}^{N} x(p)\\ \mathrm{var}\,(x) = \mathrm{cov}\,(x, x) &= \left\langle x^2 \right\rangle - \langle x \rangle^2\\ \mathrm{cov}\,(x, y) &= \langle xy \rangle - \langle x \rangle \langle y \rangle\end{aligned}$$

[3] More precisely, the covariance operator is a functional and we are extending it to allow for a vector of discrete input functions. In this formalism, we think of each map as a discrete function with a single argument p and domain $p = 1, \ldots, N$.

$$\mathbf{cov}\left(a,\mathbf{x}\right)=\begin{pmatrix}\operatorname{cov}\left(a,x_1\right)\\ \vdots\\ \operatorname{cov}\left(a,x_n\right)\end{pmatrix},\qquad \mathbf{cov}\left(a,\mathbf{x}^T\right)=\begin{pmatrix}\operatorname{cov}\left(a,x_1\right) & \cdots & \operatorname{cov}\left(a,x_n\right)\end{pmatrix}=\mathbf{cov}\left(a,\mathbf{x}\right)^T$$

$$\operatorname{cov}\left(\mathbf{x},\mathbf{y}^T\right)=\operatorname{cov}\left(\mathbf{x}^T,\mathbf{y}\right)=\operatorname{cov}\left(\mathbf{x},\mathbf{y}^T\right)^T=\begin{pmatrix}\operatorname{cov}\left(x_1,y_1\right) & \cdots & \operatorname{cov}\left(x_1,y_n\right)\\ \vdots & \ddots & \vdots\\ \operatorname{cov}\left(x_n,y_1\right) & \cdots & \operatorname{cov}\left(x_n,y_n\right)\end{pmatrix}.$$

Then we define $\mathbf{v}^T=\left(\begin{smallmatrix}T_{\mathrm{CMB}} & \boldsymbol{\eta}^T\end{smallmatrix}\right)=\left(T_{\mathrm{CMB}},\ \zeta_1 R_1,\ \ldots,\ \zeta_k R_k\right)$, so

$$\operatorname{var}\hat{T}=\mathbf{1}^T\operatorname{cov}\left(\mathbf{v},\mathbf{v}^T\right)\mathbf{1}\tag{9}$$

and we note that $\operatorname{cov}\left(\mathbf{v},\mathbf{v}^T\right)$ has the form

$$\operatorname{cov}\left(\mathbf{v},\mathbf{v}^T\right)=\begin{pmatrix}\operatorname{cov}\left(T_{\mathrm{CMB}},T_{\mathrm{CMB}}\right) & \mathbf{cov}\left(T_{\mathrm{CMB}},\boldsymbol{\eta}^T\right)\\ \mathbf{cov}\left(T_{\mathrm{CMB}},\boldsymbol{\eta}\right) & \operatorname{cov}\left(\boldsymbol{\eta},\boldsymbol{\eta}^T\right)\end{pmatrix}\tag{10}$$

As a final note, all we can say about the estimator at this point is that it is unity gain in T_{CMB} and that $\operatorname{var}\hat{T}\geq 0$ (since covariance matrices are positive semi-definite). $\hat{T}$ may be biased and its variance may not be an accurate estimate of $\operatorname{var}\left(T_{\mathrm{CMB}}\right)$.

2 Constant weighting factors

Suppose that the weight factors $\zeta_i(p)$ were uniform across the entire map, so $\zeta_i(p)=\zeta_i$. Then

$$\operatorname{cov}\left(T_{\mathrm{CMB}},\zeta_i R_i\right)=\zeta_i\operatorname{cov}\left(T_{\mathrm{CMB}},R_i\right)\quad\text{and}\quad\operatorname{cov}\left(\zeta_i R_i,\zeta_j R_j\right)=\zeta_i\zeta_j\operatorname{cov}\left(R_i,R_j\right)\tag{11}$$

so

$$\mathbf{cov}\left(T_{\mathrm{CMB}},\boldsymbol{\eta}\right)=\operatorname{diag}\left(\boldsymbol{\zeta}\right)\mathbf{cov}\left(T_{\mathrm{CMB}},\mathbf{R_i}\right)\quad\text{and}\quad\operatorname{cov}\left(\boldsymbol{\eta},\boldsymbol{\eta}^T\right)=\operatorname{diag}\left(\boldsymbol{\zeta}\right)\operatorname{cov}\left(\mathbf{R},\mathbf{R}^T\right)\operatorname{diag}\left(\boldsymbol{\zeta}\right)\tag{12}$$

and $\operatorname{var}\hat{T}$ may be written

$$\begin{aligned}\operatorname{var}\hat{T}&=\begin{pmatrix}1 & \boldsymbol{\zeta}^T\end{pmatrix}\begin{pmatrix}\operatorname{cov}\left(T_{\mathrm{CMB}},T_{\mathrm{CMB}}\right) & \mathbf{cov}\left(T_{\mathrm{CMB}},\mathbf{R}^T\right)\\ \mathbf{cov}\left(T_{\mathrm{CMB}},\mathbf{R}\right) & \operatorname{cov}\left(\mathbf{R},\mathbf{R}^T\right)\end{pmatrix}\begin{pmatrix}1\\ \boldsymbol{\zeta}\end{pmatrix}\\ \operatorname{var}\hat{T}&=\operatorname{var}\left(T_{\mathrm{CMB}}\right)+2\,\mathbf{cov}\left(T_{\mathrm{CMB}},\mathbf{R}^T\right)\boldsymbol{\zeta}+\boldsymbol{\zeta}^T\operatorname{cov}\left(\mathbf{R},\mathbf{R}^T\right)\boldsymbol{\zeta}\end{aligned}\tag{13}$$

which we note is a convex[4] quadratic form in $\boldsymbol{\zeta}$, which therefore must have a unique minimum.

[4] The positive semi-definiteness of $\operatorname{cov}\left(\mathbf{R},\mathbf{R}^T\right)$ implies that the function is convex.

We may minimize $\operatorname{var}\hat{T}$ by finding the extremum of the RHS of the Eq. 13.

$$\begin{aligned} 0 = \frac{\partial \operatorname{var}\hat{T}}{\partial \boldsymbol{\zeta}} = \left(\frac{\partial \operatorname{var}\hat{T}}{\partial \zeta_1},\ \ldots,\ \frac{\partial \operatorname{var}\hat{T}}{\partial \zeta_n}\right) &= 2\,\mathbf{cov}\left(T_{\mathrm{CMB}}, \mathbf{R}\right) + 2\boldsymbol{\zeta}^T \operatorname{cov}\left(\mathbf{R}, \mathbf{R}^T\right) \\ \boldsymbol{\zeta}^* &= -\operatorname{cov}\left(\mathbf{R}, \mathbf{R}^T\right)^{-1} \mathbf{cov}\left(T_{\mathrm{CMB}}, \mathbf{R}\right) \end{aligned} \tag{14}$$

which matches the result of Hinshaw 2007[4] (Eq. 10).

Does $\boldsymbol{\zeta}^*$ satisfy $\sum_i \zeta_i^* = \mathbf{1}^T\boldsymbol{\zeta}^* = -\mathbf{1}^T \operatorname{cov}\left(\mathbf{R}, \mathbf{R}^T\right)^{-1} \mathbf{cov}\left(T_{\mathrm{CMB}}, \mathbf{R}\right) = 1$?

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