CMP418: Algorithm and Complexity Analysis (3 units)

Lecture 4: Decrease and Conquer

MR. M. YUSUF

Outline

- Decrease and Conquers
  - Insertion Sort and Topological Sorting
- Decrease-by-a Constantan-Factor Algorithm
  - Binary Search
- Variable-Size-Decrease Algorithms
  - Computing a median and Selection Problem
  - Search and Insertion in a Binary Search Tree

What is Decrease and Conquer?

- Decrease and Conquer approach is based on exploiting the relationship between a solution to a ***given instance*** of a problem and a solution to its ***smaller instance***.
  - Top-down: recursive
  - Bottom-up: iterative
- 3 major types:
  - Decrease by a **constant**
  - Decrease by a **constant factor**
  - Decrease by **Variable size**

Decrease and Conquer…

- Decrease by a constant
  - Compute a^n where $a \neq 0$ and n is a nonnegative
  - $a^n = a^{n-1} \times a$
  - Top down: recursive
    - $$f(n) = \begin{cases} f(n-1) \times a & \text{if } n > 0 \\ 1 & \text{if } n = 0 \end{cases}$$
  - Bottom up: iterative
    - Multiply 1 by a, n times

Decrease and Conquer…

- Decrease by a constant factor (usually 2)

$$a^n = \begin{cases} \left(a^{n/2}\right)^2 & \text{If n is even and positive} \\ \left(a^{(n-1)/2}\right)^2 \cdot a & \text{If n is odd} \\ 1 & \text{If n = 0} \end{cases}$$

$$\mathbf{5^{29} = (5^{(29-1)/2})^2 . 5 = (5^{14})^2 = 5^{14}. 5^{14}}$$

Decrease and Conquer…

- Decrease by a constant factor (usually 2)

```
ALGORITHM Exponentiate(a, n)
    if n = 0
        return 1
    tmp <- Exponentiate(a, n >> 1)
    if (n & 1) = 0   // n is even
        return tmp*tmp
    else
        return tmp*tmp*a
```

$$a^n = \begin{cases} \left(a^{n/2}\right)^2 & \text{If n is even and positive} \\ \left(a^{(n-1)/2}\right)^2 \cdot a & \text{If n is odd} \\ 1 & \text{If n = 0} \end{cases}$$

What's the time complexity ? Θ(log n)

Decrease and Conquer: Insertion Sort

```
ALGORITHM InsertionSort(A[0..n-1])
for i <- 1 to n-1 do
    v <- A[i]
    j <- i-1
    while j ≥ 0 and A[j] > v do
        A[j+1] <- A[j]
        j <- j-1
    A[j+1] <- v
```

89 | **45** 68 90 29 34 17

45 89 | **68** 90 29 34 17

45 68 89 | **90** 29 34 17

45 68 89 90 | **29** 34 17

29 45 68 89 90 | **34** 17

29 34 45 68 89 90 | **17**

17 29 34 45 68 89 90

Input size: n
Basic op: A[j] > v

Why not j ≥ 0 ?
C(n) depends on input type ?

Decrease and Conquer: Insertion Sort…

```
ALGORITHM InsertionSort(A[0..n-1])
for i <- 1 to n-1 do
    v <- A[i]
    j <- i-1
    while j ≥ 0 and A[j] > v do
        A[j+1] <- A[j]
        j <- j-1
    A[j+1] <- v
```

What is the worst case scenario ?

A[j] > v executes highest # of times

When does that happen ?

A[j] > A[i] for j = i-1, i-2, …, 0

Worst case input:
An array of strictly decreasing values

What is the best case ?

A[i-1] ≤ A[i] for i = 1, 2, …, n-1

$C_{worst}(n) = \sum_{i=1}^{n-1}\sum_{j=0}^{i-1} 1 = \sum_{i=1}^{n-1} i = \frac{(n-1)n}{2} \in \Theta(n^2)$

$C_{best}(n) = \sum_{i=1}^{n-1} 1 = n-1 \in \Theta(n)$

$C_{avg}(n) \approx \frac{n^2}{4} \in \Theta(n^2)$

For almost sorted files, insertion sort's performance is excellent!

Can you improve it ?

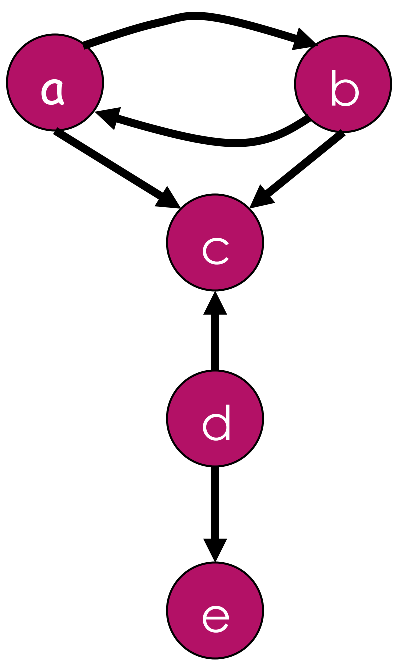
In place? Stable ?

Topological Sorting

- Ordering of the vertices of a directed graph such that for every edge uv, u comes before v in the ordering
- First studied in 1960s in the context of PERT (Project Evaluation and Review Technique) for scheduling in project management.
  - Jobs are vertices, there is an edge from x to y if job x must be completed before job y can be started
  - Then topological sorting gives an order in which to perform the jobs

Directed Graph or Digraph

- Directions for all edges



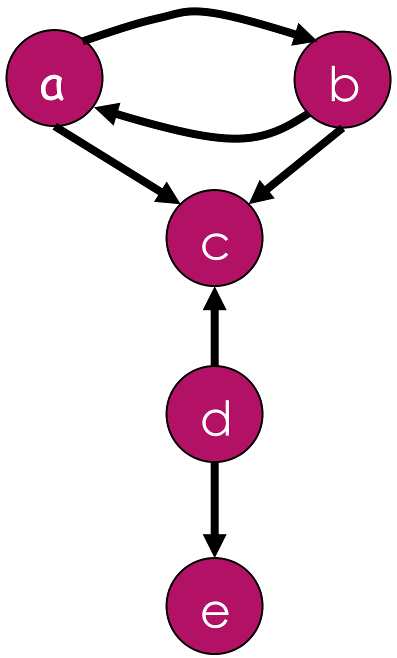
a ⟶ b ⟶ c
b ⟶ a ⟶ c
c ⟶
d ⟶ c ⟶ e
e ⟶

One node for one edge

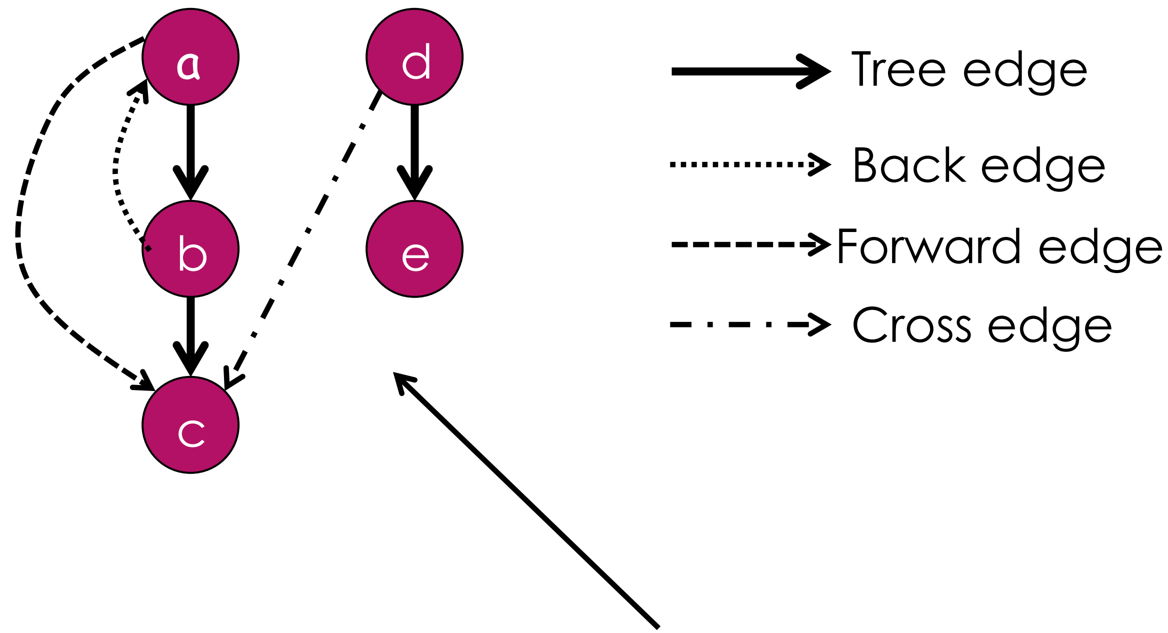
$$
\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}
\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}
$$

Not symmetric

Digraph



a ⟶ b ⟶ c
b ⟶ a ⟶ c
c ⟶
d ⟶ c ⟶ e
e ⟶



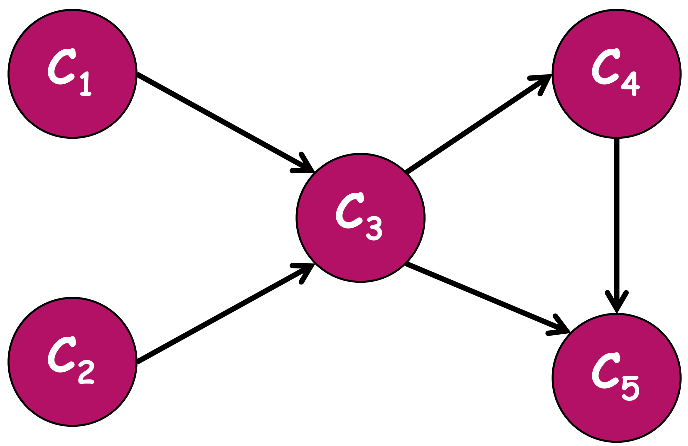
Directed cycle: **a, b, a**

DFS forest

If no directed cycles, the digraph is called "directed acyclic graph" or "DAG"

Topological Sorting Scenario 2

- Set of 5 courses: { C_1, C_2, C_3, C_4, C_5 }
- C_1 and C_2 have no prerequisites
- C_3 requires C_1 and C_2
- C_4 requires C_3
- C_5 requires C_3 and C_4



C_1, C_2, C_3, C_4, C_5 !

Topological Sorting: Usage

- A large project – e.g., in construction, research, or software development – that involves a multitude of interrelated tasks with known prerequisites
  - Schedule to minimize the total completion time
- Instruction scheduling in program compilation, resolving symbol dependencies in linkers, etc.

Outline

- Decrease and Conquers
  - Insertion Sort and Topological Sorting
- **Decrease-by-a Constantan-Factor Algorithm**
  - Binary Search
- Variable-Size-Decrease Algorithms
  - Computing a median and Selection Problem
  - Search and Insertion in a Binary Search Tree

Decrease-by-a-Constant-Factor Algorithms

- Binary Search
  - Highly efficient way to search for a key K in a sorted array A[0..n-1]

Compare K with A's middle element A[m]

If they match, stop.

Else if K < A[m] do the same for the first half (LHS) of A

Else if K > A[m] do the same for the second half (RHS) of A

Decrease-by-a-Constant-Factor Algorithms: Binary Search

K (key)

$\updownarrow$

LHS		**RHS**
$\underbrace{A[0] \dots A[m-1]}$	A[m]	$\underbrace{A[m+1] \dots A[n-1]}$
Search here if K < A[m]		**Search here if K > A[m]**

Let us apply binary search for K = 70 on the following array:

3,	**14,**	**27,**	**31,**	**39,**	**42,**	**55,**	**70,**	**74,**	**81,**	**85,**	**93,**	**98**
0	1	2	3	4	5	6	7	8	9	10	11	12

Binary Search Algorithm

```
ALGORITHM BinarySearch(A[0..n-1], K)
//Input: A[0..n-1] sorted in ascending order and a search key K
//Output: An index of A's element equal to K or -1 if no such element
l <- 0
r <- n-1
while l ≤ r do
    m <- ⌊(l + r)/2⌋
    if K = A[m]
        return m
    else if K < A[m]
        r <- m-1
    else
        l <- m+1
return -1
```

Best-case input: K is the mid position of a sorted element …

Worst-case input: K is absent or some K is at some special position…

$C_{worst}(n) = C_{worst}(\lfloor n/2 \rfloor) + 1$
$C_{worst}(1) = 1$

To simplify, assume $n = 2^k$
Then $C_{worst}(n) \in \Theta(\log n)$

Outline

- Decrease and Conquers
  - Insertion Sort and Topological Sorting
- Algorithms for Generating Combinatorial Objects
  - Generating Permutations and Generating Subsets
- Decrease-by-a Constantan-Factor Algorithm
  - Binary Search
- **Variable-Size-Decrease Algorithms**
  - Computing a median and Selection Problem
  - Search and Insertion in a Binary Search Tree

Variable-Size-Decrease

- Problem size decreases at each iteration in variable amount
- Euclid's algorithm for computing the greatest common divisor of two integers is one example

Computing Median and the Selection Problem

- Find the k-th smallest element in a list of n numbers
- For k = 1 or k = n, we could just scan the list
- More interesting case is for k = $\lceil n/2 \rceil$
  - Find an element that is not larger than one half of the list's elements and not smaller than the other half; this middle element is called the "median"
  - Important problem in statistics

Median Selection Problem

- We shall take advantage of the idea of "partitioning" a given list around some value **p** of, say the list's first element. This element is called the "pivot".

- This method is referred to as Lomuto partitioning & Hoare's partitioning

M. Yusuf

Median Selection Problem…

p

i → i

61	93	56	90	11

s

A[i] **not less than** p

p

i

61	93	56	90	11

s → s

A[i] **less than** p

p

i → i

61	56	93	90	11

s

p

i → i

61	56	93	90	11

s

A[i] **not less than** p

p

i

61	56	93	90	11

s → s

A[i] **less than** p

p

i

61	56	11	90	93

s

11	56	61	90	93

Lomuto Partitioning Algorithm

ALGORITHM LomutoPartition(A[L..R])
//Partition subarray by Lomuto's algorithm using first element as pivot
//Input: A subarray A[L..R] of array A[0..n-1], defined by its
//left and right indices l and r (L ≤ R)
//Output: Partition of A[L..R] and the new position of the pivot

```
p <- A[l]
s <- L
for i <- L+1 to R do
    if A[i] < p
        s <- s+1
        swap(A[s], A[i])
swap(A[l], A[s])
return s
```

Median Selection Problem

- We want to use the Lomuto partitioning to efficiently find out the k-th smallest element of A[0..n-1]
- Let s be the partition's split position
- If s = k-1, pivot p is the k-th smallest
- If s > k-1, the k-th smallest (of entire array) is the k-th smallest of the left part of the partitioned array
- If s < k-1, the k-th smallest (of entire array) is the [(k-1)-(s+1)+1]-th smallest of the right part of the partitioned array

Median Selection Problem: QuickSelect Algorithm

ALGORITHM QuickSelect(A[l..r], k)

//Input: Subarray A[l..r] of array A[0..n-1] of orderable elements and integer k (1 ≤ k ≤ r-l+1)

//Output: The value of the k-th smallest element in A[l..r]

s <- LomutoPartiotion(A[l..r])

if s = k-1

    return A[s]

else if s > l+k-1

    QuickSelect(A[l..s-1], k)

else

    QuickSelect(A[s+1..r], k-1-s)

Median Selection Problem: QuickSelect Algorithm…

$k = \lceil 9/2 \rceil = 5$

s	i							
4	1	10	8	7	12	9	2	15

	s	i						
4	1	10	8	7	12	9	2	15

	s						i	
4	1	10	8	7	12	9	2	15

		s					i	
4	1	2	8	7	12	9	10	15

		s						i
4	1	2	8	7	12	9	10	15

		s						
2	1	4	8	7	12	9	10	15

s = 2 < k-1 = 4, so we proceed with the right part…

Median Selection Problem: QuickSelect Algorithm…

$k = \lceil 9/2 \rceil = 5$

2	1	4	8	7	12	9	10	15

s	i				
8	7	12	9	10	15

	s	i			
8	7	12	9	10	15

	s				i
8	7	12	9	10	15

	s				
7	8	12	9	10	15

Now s (=4) = k-1
(=5-1=4),
we have found the median!

Thank You