PGCS713:Discrete Structures (3 units)

Mathematical Logic

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Chapter 3 Outline

- Introduction
- Propositional and Logical Operators

- Truth Tables and Propositions Generated by a Set
- Equivalence and Implication
- The Laws of Logic

Introduction

- By the end of this chapter, you should be able to;
  - Recognize valid logical **statements** and **arguments** used by
  - Lawyers in court room
    - Physician examining a patient

- Engineer solving a structural problem
- Scientist in designing circuit of computer

- A Statement is a declarative sentence that can be T(1) or F(0)
  - E.g. sun shines
  - Humans walk
- We are dealing with **Nonimperative** statements not imperative
  - Slap him, fight her etc

Propositional and Logical Operators

- **Proposition:** is a sentence to which ***one*** and ***only one*** of the terms is ***true*** or ***false*** can be meaningfully applied

Capital Proposition denotes statements P, Q, R
Lower Proposition are not statement but use for proofs p, q, r

- Examples of propositions.
- "Four is even", → True/False
- "4 ∈ {1, 3, 5}", → True/False
- "43 > 21" → True/False

- we expect that logical propositions contain connectives like the word **"and"** and **"or".** E.g.
  - School may resume in January ***OR*** not
  - When coming to school come along with hand sanitizer **AND** face mask

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Connectives

1	��	
2	¬��	
3	�� ∧ ��	
4	�� ∨ ��	

5	�� → ��	
6	�� ↔ ��	
7	�� ← ��	
8	�� ⊕ ��	�

9	�� ↑ ��	��

How to translate Proposition into English

- P = You plagiarize R = You write an assignment
- Q = You will get caught S = You will fail

(R ∧ ��) → (Q ∧ ��)

Solution

- If (You write an assignment **and** You plagiarize)
- Then (You will get caught **and** You will fail)

How to translate English into Proposition Logic

- **If** Mr. Yusuf will **not** be online **then** class rep will **not** be online **and** some students will be happy **or** some student will **not** be happy

Solution

- P = “Mr. Yusuf will be online”

- $\neg P$ = “Mr. Yusuf will **not** be online”
- �� = “class rep will be online”
- $\neg Q$ =class rep will **not** be online **and**
- R = “some student will be happy”
- $\neg R$ = “some student will **not** be happy”

$$(\neg P) \rightarrow (\neg Q \wedge (\text{��} \vee \neg R))$$

Exercise 1a to be submitted Sunday 11:59 PM

- Translate the following sentences, given the following statements
  1. P: I finish writing my discrete structure assignment before dinner
  2. Q: I shall swim tonight

3. R: The moon is shining
4. S: The humidity is high

- If the moon is shinning, I shall swim tonight
- Finishing the writing of my discrete structures assignment before dinner is necessary for me to swim tonight.
- High humidity moon light are sufficient for me to swim tonight

Exercise1b to be submitted Sunday 11:59 PM

- Determine which is a statement?

1. In Nazareth, Jesus was born
2. $20x + 30.4y$ is an integer

3. Tell me your age
4. We can not leave without our parents
5. Show me you answers
6. Send you airtime

Propositional and Logical Operators cont.

- **Logical Conjunction:** If **p** and **q** are propositions, their **conjunction**, p and q (denoted $p \wedge q$), is defined by the **truth table. �� ∧ �� = min(p, q)**

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

- **0 and 1** denotes false and true
- order of cases in a truth table is standardized
  - 00, 01, 10, 11 for two binary digits. **How about for three binary digits?**

Propositional and Logical Operators cont.

- **Logical Disjunction:** If **p** and **q** are propositions, their **disjunction**, p or q (denoted p v q), is defined by the **truth table.** **��** v **�� = max(p, q)**

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

- **Logical Negation:** If **p** is a proposition, its **negation**, **not p**, denoted **¬p**, and is defined by the **truth table**

p	$\neg p$
0	1
1	0

- Note: Negation is the only standard operator that acts on a **single proposition**; hence only **two** cases are needed

Propositional and Logical Operators cont.

- **Conditional Statement:** The conditional statement **"If p then q** ", denoted $p \rightarrow q$, is defined by the **truth table.**

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1

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Exercise 1c Draw a truth table for the stmt.

E.g. Assume your instructor told you "If you receive a grade of ***95*** or better in the final examination, then you will receive an ***A*** in this course". Your instructor has made a ***promise*** to you. If you ***fulfill his condition***, you expect the conclusion (getting an A) to be forthcoming. Suppose your graded final has been returned to you. Has your instructor told the truth or is your instructor guilty of a falsehood? If your; 0 0

- ***Case I:*** FES < 95 (***the condition is false***) and Grade **!= A** (***the conclusion is false***). **The instructor told the *truth***.

0 1

- ***Case II:*** FES < 95, yet Grade = A . **The instructor told the truth**.

1 0

- ***Case III:*** FES > 95, but Grade **!= A**. **The instructor lied**.

1 1

- ***Case IV:*** FES > 95, and Grade = A. **The instructor told the truth.**

  - Summary: the only case in which a **conditional proposition = false** is when the **condition = true** and the **conclusion = false**. 23/01/2022 04:20

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Propositional and Logical Operators cont.

- **Contrapositive:** The contrapositive of the proposition $p \rightarrow q$ is the proposition **$\neg q \rightarrow \neg p$.**

- $\neg q \to \neg p$ have the same **logical meaning with** $p \to q$

- **Biconditional Proposition.** If p and q are propositions, the **biconditional** statement ***"p if and only if q",*** denoted **$p \leftrightarrow q$ and p iff q**, is defined by the truth table

p	q	$p \leftrightarrow q$
0	0	1
0	1	0
1	0	0
1	1	1

- Note that ***$p \leftrightarrow q$*** is true when p and q have the ***same truth values***. It is common to abbreviate ***"if and only if" = "iff".***

Propositional and Logical Operators cont.

- All of the following are equivalent to **"If p then q"**:
  - p **implies** q.

- q **follows from** p.
- p, **only if** q.
- q, **if** p.
- p **is sufficient for** q.
- q **is necessary for** p
- All of the following are equivalent to **"p if and only if q"**:
  - p is necessary and sufficient for q.
  - p is equivalent to q.
  - If p, then q, and if q, then p.
  - If p, then q and conversely.

- **Logic** is the study of ***consequence***
- Fundamental of symbology is discussed along with argument

- An **<u>argument</u>** is a set of statements, one of which is called the **<u>conclusion</u>** and the rest of which are called **<u>premises</u>**.
- An **argument** is said to be **valid** if the **conclusion** **<u>MUST</u>** be **true** whenever the **premises** **<u>are all true</u>**.
- An argument is **invalid** if it is **not valid**;
- it is possible for all the premises to be true and the conclusion to be false.

If Edith eats her vegetables, then she can have a cookie.
Edith eats her vegetables.

$\therefore$ Edith gets a cookie.

Florence must eat her vegetables in order to get a cookie.
Florence eats her vegetables.

$\therefore$ Florence gets a cookie.

- Are the arguments bellow valid? First one Yes, the Second?

- Just because Florence ***must*** eat her vegetables, we have not said that doing so would be ***enough.***
  - She might also need to do other tasks (like clean her room, for example)
  - So is the second statement **valid?**

- A proposition is simply a statement.
- Propositional logic studies **the ways statements can interact** with each other.
- Propositional logic does not really care about the **content** of the statements.
- "if the moon is made of cheese then basketballs are round"
  - "if spiders have eight legs then Sam walks with a limp"
  - Are the statement same? ($P \to Q$)

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

p	$\neg p$
0	1
1	0

- Generate truth table for the stmt. $\neg P \vee Q$.

P	Q	$\neg P$	$\neg P \vee Q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

- Analyze the statement, "if you get more doubles than any other player you will lose, **or** that if you lose you must have bought

Tautology

AND

the most properties", using truth tables.

- P = "you get more doubles than any other player"

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- Q = "you will lose",
- R = "you must have bought the most properties"

- A statement which is true on the basis of its logical form alone is said to be tautology.
- **Tautologies** are always true but they don't tell us much

about the world

Logical Equivalence

- Two molecular statements **P** and **Q** are logically equivalent provided **P** is **true** precisely when **Q** is **true**.
  - That is, **P** and **Q** have the same truth value under any assignment of truth values to their atomic parts.
- $\neg P \lor Q$ is identical to the final column in the truth table for $P \rightarrow Q$.

P	Q	$P \to Q$	$\neg P \vee Q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Logical Equivalence

- This says that no matter what **P** and **Q** are, the statements **¬ P** v **Q** and **P** → **Q** either both true or both false.
  - The statements are **logically equivalent.**

Logical Equivalence e.g.

- Are the statements, "**it will not rain or snow**" and "**it will not rain and it will not snow**" logically equivalent? Let

= "**it will not rain or snow**"

P	Q	$\neg(P \vee Q)$	$\neg P \wedge \neg Q$
T	T	F	F

= "**it will not rain and it will not snow**"

- Every row the truth values for the two statements are equal, the two statements are logically equivalent.

De Morgan's Laws and Disjunctions.

- **De Morgan’s Laws**
- $\neg(P \wedge Q)$ is logically equivalent to $\neg P \vee \neg Q$.

- **Disjunctions**
- P → Q is logically equivalent to ¬P v Q.
  - Example: "If a number is a multiple of 4, then it is even" is equivalent to, "a number is not a multiple of 4 or (else) it is even".
- **Double Negation.**
- ¬¬P is logically equivalent to P.
  - Example: "It is not the case that c is not odd" means "c is odd".

Sequence of logically Equivalent Statement

- To verify that two statements are logically equivalent, you can use <u>truth tables</u> or a <u>sequence of logically equivalent replacements.</u>

- E.g. Prove that the statements $\neg(P \to Q)$ and $P \land \neg Q$ are logically equivalent without using truth tables.
  - Start with first part of the statement
    - $\neg(P \to Q) = \neg(\neg P \lor Q)$. (disjunction)
    - $\neg(\neg P \lor Q) = \neg\neg P \land \neg Q$. (DE Morgan's law)
    - $\neg\neg P \land \neg Q = P \land \neg Q$ (double negation)
- This illustrates that the **negation** of an **implication** is **NOT** an implication: it is a **conjunction**!
  - Hence $\neg(P \to Q)$ and $P \land \neg Q$ are logically equivalent

Examples of Verifying Equivalences.

- Are the statements $(P \vee Q) \rightarrow R$ and $(P \rightarrow R) \vee (Q \rightarrow R)$ logically equivalent?

$(P \rightarrow R) \vee (Q \rightarrow R)$ is **true**,
$(P \vee Q) \rightarrow R$ is **false**. On two cases

Statements are **not logically equivalent**

However it implies that we can deduce $(P \rightarrow R) \vee (Q \rightarrow R)$ from $(P \vee Q) \rightarrow R$ but reverse not the case

- So a statement is said to be equivalent if they can be deduced from each other

Examples of Deduction rule,

- From **<u>logical examples</u>** from slide 17
- If Edith eats her vegetables, then she can have a cookie. Edith ate her vegetables. Therefore Edith gets a cookie. Let
  - **P** denote <u>"Edith eats her vegetables"</u> and
  - **Q** denote <u>"Edith can have a cookie"</u>.
  - The **premises are P → Q and P**, **Q is conclusion**
  - Which rows of the truth table correspond to both = true?
  - So if P → Q and P are both true,
  - we see that Q must be true as well.

famous rule called **Modus Ponens.**

Examples of Deduction rule,

and **– P** → **Q** are both True, then **Q** is also True

so the argument form is valid
(valid deduction rule)

Examples of Deduction rule,

is a valid deduction rule. Instead

we say Valid

invalid

invalid

Prove it

► Decide weather

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Beyond Propositions

- Not every statement can be analyzed using logical connectives alone
  - E.g. All primes greater than 2 are odd.

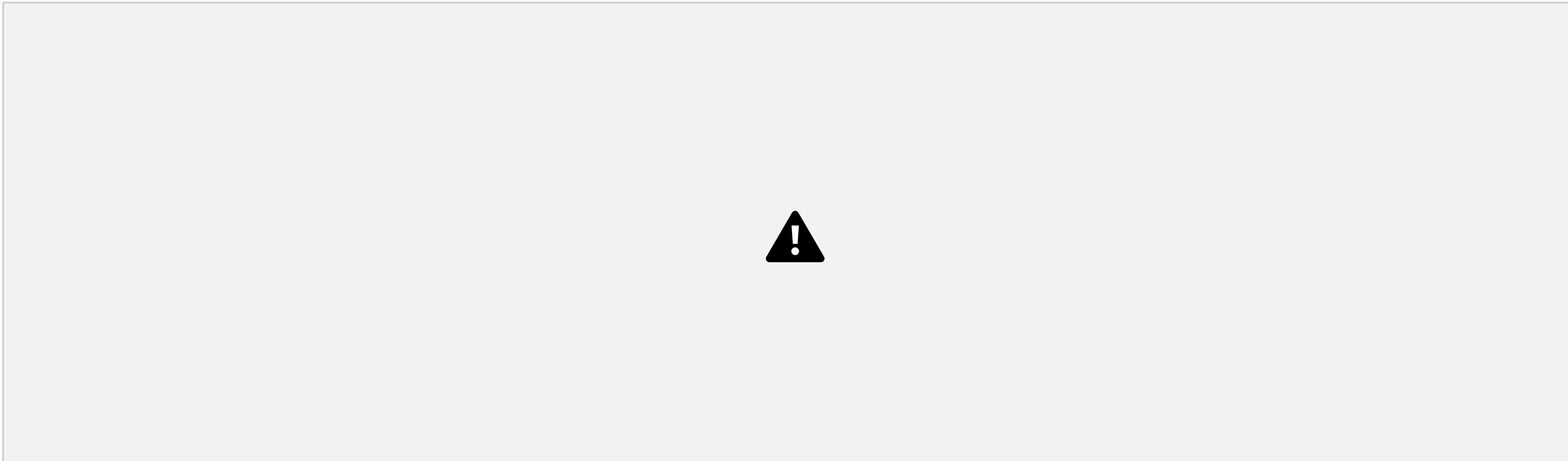
- **Quantifiers** are needed to represent the stmt. **Symbolically**
- $\forall x\, ((P(x) \wedge x > 2) \rightarrow O(x))$.
- $P(x)$ to denote **"x is prime"**
- $O(x)$ to denote **"x is odd"**
- **P** and **O** are **predicates (!proposition)**

- **Predicate logic** allows us to analyze statements at a **higher resolution**, **digging down** into the individual **propositions** P, Q, etc.

Basic Logical Laws - Equivalences

Basic Logical Laws - Common Implications and

Equivalences



Proofs

������

1. Given

(�� ∨ 0) ∧ (�� ∨ 1)

�� ∧ (�� ∨ 1)

����������������� �������

�� ∧ 1

�������������������

� ������� ��

����������������

3. Given

¬¬�� ∨ ((�� ∨ 0) ∧ ¬¬��)

2. Given

¬(¬�� ∧ ¬��)

¬¬�� ∨ ¬¬�� DeMogans Law �� ∨

�� Double Negation

�� ∨ ((�� ∨ 0) ∧ ¬¬��) Double Negation �� ∨ ((�� ∨ 0) ∧ ��) Double Negation �� ∨ (�� ∧ ��) Identity Law �� Absorption Law

Thank You

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