EGMO Solutions

Kempu33334

October 20, 2025

Contents

1 Angle Chasing

Problem 1.5

Solve the first part of Example 1.1.

Let the intersections of the diagonals of $WXYZ$ be A. Then, we have that

$$\angle AZY = 180 - (90 + \angle AYZ) = 40^\circ.$$

Hence,

$$\angle Z = \angle AZW + \angle AZY = \boxed{70^\circ}.$$

Problem 1.6

Let ABC be a triangle inscribed in a circle ω. Show that $\overline{AC} \perp \overline{CB}$ if and only if $\overline{AB}$ is a diameter of ω.

Let the center of ω be O. We start with the if direction. Given that $\overline{AB}$ is a diameter of ω, we know that

$$\angle AOB = 180^\circ.$$

Thus, because of the Inscribed Angle Theorem, we know that $\angle ACB$ must be equal to 90° as desired.

Now for the only if direction. If $\overline{AC} \perp \overline{CB}$, then we know that

$$\angle ACB = 90^\circ.$$

By the Inscribed Angle Theorem, this implies that

$$\angle AOB = 180^\circ$$

which means that $\overline{AB}$ is a diameter of ω as desired. $\square$

Problem 1.7

Let O and H denote the circumcenter and orthocenter of an acute $\triangle ABC$, respectively. Show that $\angle BAH = \angle CAO$.

By the Inscribed Angle Theorem,

$$2\angle ABC = \angle AOC.$$

Negating both sides and adding 180, we get that

$$\begin{aligned} 180 - 2\angle ABC &= 2(90 - \angle ABC) \\ &= 2\angle BAH \end{aligned}$$

and

$$180 - \angle AOC = 2\angle CAO$$

since $\triangle AOC$ is isosceles ($OA = OC$). Finally, this means that $\angle BAH = \angle CAO$ as desired. $\square$

Problem 1.10

Show that a trapezoid is cyclic if and only if it is isosceles.

Let the aforementioned trapezoid be $ABCD$ where $\overline{BC} \parallel \overline{AD}$. We start with the if direction. Since $ABCD$ is isosceles, we know that

$$\angle BAD = \angle CDA = 180 - \angle BCD$$

so the opposite angles add up to $180°$, implying that $ABCD$ is cyclic.

Now, we do the only if direction. Since $ABCD$ is cyclic, we know that

$$\angle BAD = 180 - \angle BCD = \angle CDA$$

which means that $ABCD$ is isosceles as desired. □

Problem 1.11

Quadrilateral $ABCD$ has $\angle ABC = \angle ADC = 90°$. Show that $ABCD$ is cyclic, and that $(ABCD)$ (that is, the circumcircle of $ABCD$) has diameter $\overline{AC}$.

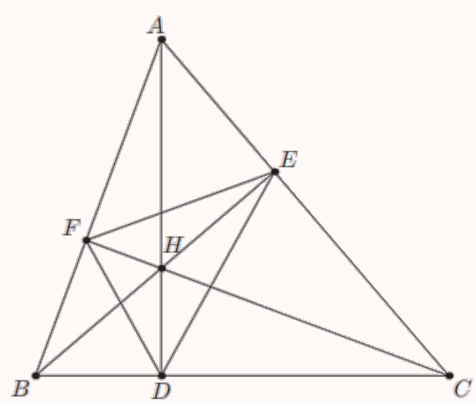
Since

$$\angle ABC + \angle ADC = 180°$$

we know that $ABCD$ is cyclic. In addition, by Problem 1.6, we know that $(ABCD)$ has diameter $\overline{AC}$ as desired. □

Problem 1.16

In the figure below, show that $\triangle AEF$, $\triangle BFD$, and $\triangle CDE$ are each similar to $\triangle ABC$.



We will show the necessary result for one of them ($\triangle AEF$), as it easily generalizes. Clearly, we know that $\triangle ABC$ and $\triangle AEF$ share $\angle A$. In addition,

$$\begin{aligned}\angle B &= 90 - \angle BAD \\ &= \angle AHF \\ &= \angle AEF.\end{aligned}$$

Thus, we know that two of the angles are equal, so

$$\triangle ABC \sim \triangle AEF$$

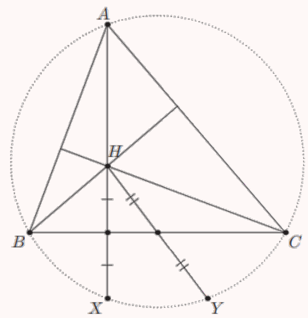
as required. □

Problem 1.17

Let H be the orthocenter of $\triangle ABC$, as in the diagram below. Let X be the reflection of H over $\overline{BC}$ and Y the reflection over the midpoint of $\overline{BC}$.

(a) Show that X lies on (ABC).

(b) Show that $\overline{AY}$ is a diameter of (ABC).



We start with part (a). Let the foot of the altitude from A be D, from B be E, and the foot from C be F. Then, we can write the following equations:

$$\begin{aligned}\angle A + \angle BXC &= \angle A + \angle BHC \\ &= \angle A + \angle FHE \\ &= 180\end{aligned}$$

so $ABXC$ is cyclic, implying that X lies on (ABC) as desired.

We proceed with part (b). Define M to be the midpoint of $\overline{BC}$. Then, we know that

$$\triangle HDM \sim \triangle HXY.$$

This means that

$$\angle HXY = \angle AXY = 90.$$

Next, we show that Y lies on (ABC), by showing that $AXYC$ is cyclic. We can say that:

$$\begin{aligned}\angle BCY + \angle C &= \angle HBC + \angle C \\ &= 90\end{aligned}$$

where $\angle BCY = \angle HBC$ since $\triangle HBM \cong YCM$. This, in conjuction with the fact that $\angle AXY = 90$ means that $AXYC$ is cyclic. So, Y lies on (ABC) as desired. Finally, this means that $\overline{AY}$ is the diameter of (AXY) or equivalently (ABC), as required. $\square$

Remark

The complex number proof can be found as Problem 6.14.

Problem 1.28

We claimed that $\measuredangle FKD + \measuredangle DKE + \measuredangle EKF = 0$ in the above proof. Verify this.

Let P be a point that is collinear with D and K. Then, we know that

$$\measuredangle DKE + \measuredangle EKP = 0$$

and

$$\measuredangle PKF + \measuredangle FKD = 0.$$

Adding these gives the desired conclusion. $\square$

Remark

Throughout this text, we will use $\measuredangle$ to denote directed angles.

Problem 1.29

Show that for any distinct points A, B, C, D we have $\measuredangle ABC + \measuredangle BCD + \measuredangle CDA + \measuredangle DAB = 0$.

We know that:

$$\begin{aligned} \measuredangle BAD + \measuredangle ADB + \measuredangle DBA &= 0 \\ \measuredangle BDC + \measuredangle DCB + \measuredangle CBD &= 0. \end{aligned}$$

Adding these gives that

$$\measuredangle BAD + (\measuredangle ADB + \measuredangle BDC) + (\measuredangle DBA + \measuredangle CBD) + \measuredangle DCB = 0$$

and simplifying, we have that

$$\measuredangle BAD + \measuredangle ADC + \measuredangle CBA + \measuredangle DCB = 0.$$

Finally, negating the entire equation gives the desired conclusion. $\square$

Problem 1.30

Points A, B, C lie on a circle with center O. Show that $\measuredangle OAC = 90^\circ - \measuredangle CBA$.

Let A' be the reflection of A across O. Then, we know that since AA' is a diameter,

$$\measuredangle ACA' = 90^\circ.$$

This implies the following:

$$\begin{aligned} \measuredangle ABC &= \measuredangle AA'C \\ &= -\measuredangle A'CA - \measuredangle CAA' \\ &= 90 + \measuredangle A'AC \\ &= 90 + \measuredangle OAC. \end{aligned}$$

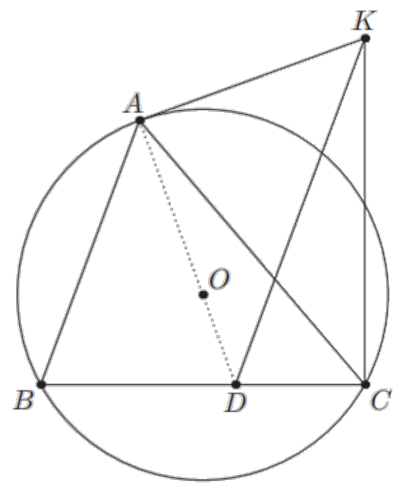
This implies that

$$\measuredangle OAC = 90 + \measuredangle ABC$$

as desired. $\square$

Problem 1.33

Let ABC be a triangle and let ray AO meet $\overline{BC}$ at D. Point K is selected so that $\overline{KA}$ is tangent to (ABC) and $\angle KCB = 90°$. Prove that $\overline{KD} \parallel \overline{AB}$.



Since

$$\angle KAD = \angle KCD = 90°$$

we know that quadrilateral $KADC$ is cyclic. Because of this, we can write that:

$$\begin{aligned}\angle KDB &= 180 - \angle KDC \\ &= 180 - \angle KAC \\ &= 180 - \angle B\end{aligned}$$

which implies that $\overline{KD} \parallel \overline{AB}$, as required. □

Problem 1.34

In scalene triangle ABC, let K be the intersection of the angle bisector of $\angle A$ and the perpendicular bisector of $\overline{BC}$. Prove that the points A, B, C, K are concyclic.

Let K' be the intersection of the angle bisector of $\angle A$ and (ABC) not at A. We wish to show that $K' = K$. By the Incenter-Excenter Lemma, we know that

$$K'B = K'C$$

implying that K' lies on the perpendicular bisector of $\overline{BC}$. Hence, K' is the same point as K, and since $ABK'C$ is cyclic, we have the required result. □

Problem 1.36

Let $ABCDE$ be a convex pentagon such that $BCDE$ is a square with center O and $\angle A = 90°$. Prove that $\overline{AO}$ bisects $\angle BAE$.

Clearly, $ABOE$ is cyclic since $\angle BOE = 90°$. This means that

$$\angle BAO = \angle BEO = 45°.$$

So, $\overline{AO}$ bisects $\angle BAE$ as desired. $\square$

Problem 1.37 (BAMO 1999/2)

Let $O = (0,0)$, $A = (0,a)$, and $B = (0,b)$, where $0 < a < b$ are reals. Let Γ be a circle with diameter $\overline{AB}$ and let P be any other point on Γ. Line $\overline{PA}$ meets the x-axis again at Q. Prove that $\angle BQP = \angle BOP$.

We wish to show that quadrilateral $BPOQ$ is cyclic. We can write the following:

$$\begin{aligned} \measuredangle PQO &= \measuredangle AQO \\ &= 90 - \measuredangle OAQ \\ &= 90 - \measuredangle BAP \\ &= \measuredangle PBA \\ &= \measuredangle PBO \end{aligned}$$

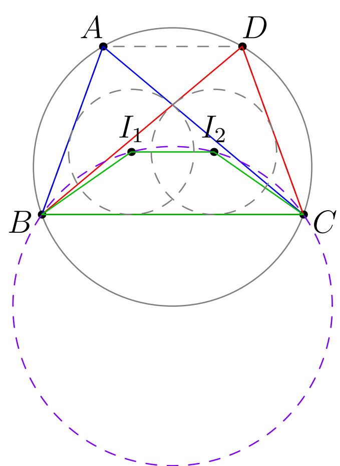
so $BPOQ$ is indeed cyclic. This implies that

$$\measuredangle BQP = \measuredangle BOP$$

as desired. $\square$

Problem 1.38

In cyclic quadrilateral $ABCD$, let I_1 and I_2 denote the incenters of ABC and DBC, respectively. Prove that I_1I_2BC is cyclic.



Firstly, since $ABCD$ is cyclic, we know that

$$\angle BAC = \angle BDC.$$

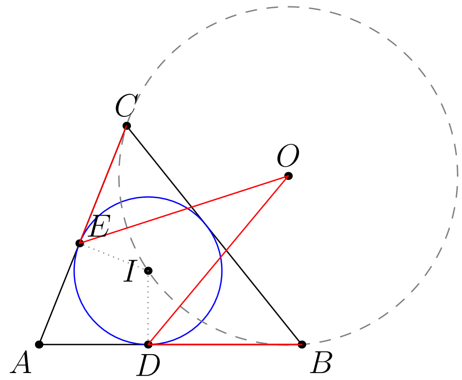
Dividing by two and adding 90, we get that $90 + \frac{\angle BAC}{2} = 90 + \frac{\angle BDC}{2}$, which is equivalent to

$$\angle BI_1C = \angle BI_2C$$

so BI_1I_2C is cyclic, as desired. $\square$

Problem 1.39 (CGMO 2012/5)

Let ABC be a triangle. The incircle of $\triangle ABC$ is tangent to $\overline{AB}$ and $\overline{AC}$ at D and E respectively. Let O denote the circumcenter of $\triangle BCI$. Prove that $\angle ODB = \angle OEC$.



We know that $AEID$ is cyclic since

$$\angle AEI + \angle ADI = 180^\circ.$$

In addition, we know that $\overline{AI}$ bisects $\angle EID$, and since A, I, and O are collinear (by the Incenter-Excenter Lemma), we know that $\angle EIO = \angle DIO$. This implies that

$$\triangle EIO \sim \triangle DIO$$

since $DI = EI$, which means that

$$\angle IEO = \angle IDO.$$

Finally, this implies that $\angle ADO = \angle AEO$, which result in the desired claim after negating and adding 180. $\square$

Problem 1.40 (Canada 1991/3)

Let P be a point inside circle ω. Consider the set of chords of ω that contain P. Prove that their midpoints all lie on a circle.

We claim that all the midpoints lie on the circle with diameter $\overline{OP}$.

Let M be any one of these midpoints. We must show that $\angle OMP = 90^\circ$ for any M. Let A be one endpoint of the chord, and B, the other. Then, we know that

$$\triangle OMA \sim \triangle OMB$$

since all the corresponding sides are equal ($OA = OB$ as well). Thus, this means that

$$\angle OMA = \angle OMB = 90^\circ$$

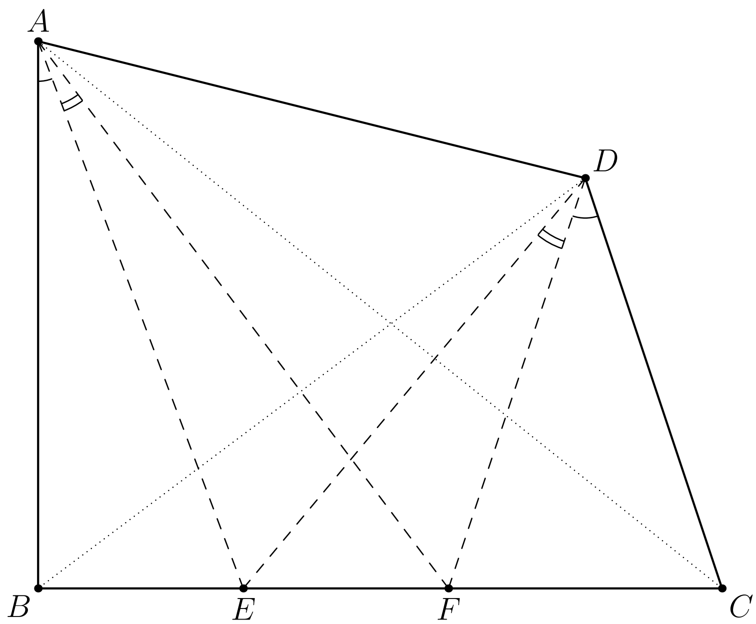
as desired. $\square$

Remark

Note that there are a few exceptions. If $O = M$, then obviously M lies on the circle with diameter $\overline{OP}$. Similarly, if $P = M$, then M obviously lies on that circle.

Problem 1.41 (Russian Olympiad 1996)

Points E and F are on side $\overline{BC}$ of convex quadrilateral $ABCD$ (with E closer than F to B). It is known that $\angle BAE = \angle CDF$ and $\angle EAF = \angle FDE$. Prove that $\angle FAC = \angle EDB$.



Clearly, quadrilateral $ADEF$ is cyclic. Thus, we can write the following:

$$\begin{aligned}\angle D &= \angle ADF + \angle FDC \\ &= 180 - \angle AEF + \angle FDC \\ &= \angle AEB + \angle FDC \\ &= 180 - \angle BAE - \angle B + \angle FDC \\ &= 180 - \angle B.\end{aligned}$$

Thus, $ABCD$ is cyclic. This implies that

$$\angle BAC = \angle CDB$$

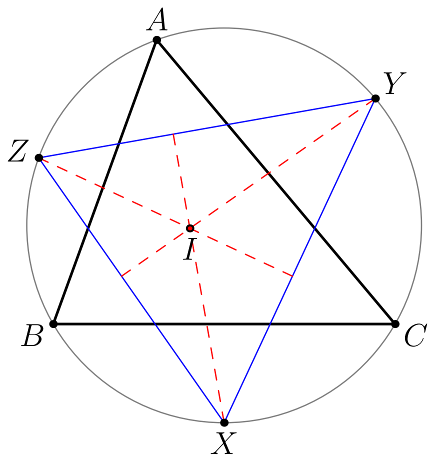
which implies that

$$\angle FAC = \angle EDB$$

due to the congruent angles. □

Problem 1.42

Let ABC be an acute triangle inscribed in circle Ω. Let X be the midpoint of the arc $\widehat{BC}$ not containing A and define Y, Z similarly. Show that the orthocenter of $\triangle XYZ$ is the incenter I of $\triangle ABC$.



We will show that $\overline{XI} \perp \overline{YZ}$ since the result easily generalizes to the other sides. By the Incenter-Excenter Lemma, we know that A, I, and X are collinear, and similarly for the other points. This means that:

$$\begin{aligned} \angle ZXI &= \angle ZXA \\ &= \angle ZCA \\ &= \frac{\angle C}{2}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \angle Z &= \frac{m\,\widehat{XY}}{2} \\ &= 90 - \frac{m\,\widehat{AB}}{4} \\ &= 90 - \frac{\angle C}{2}. \end{aligned}$$

Thus, we get that $\overline{IX} \perp \overline{YZ}$ as required. $\square$

Problem 1.43 (JMO 2011/5)

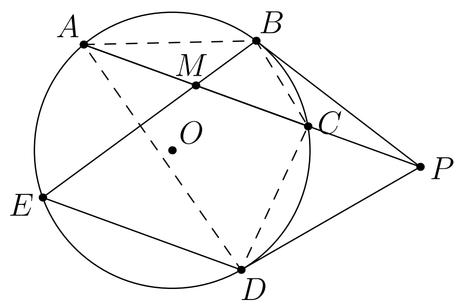
Points A, B, C, D, E lie on a circle ω and point P lies outside the circle. The given points are such that:

(i) lines PB and PD are tangent to ω

(ii) P, A, C are collinear

(iii) $\overline{DE} \parallel \overline{AC}$

Prove that $\overline{BE}$ bisects $\overline{AC}$.



Let O be the center of ω. Since $\overline{DE} \parallel \overline{AC}$, we have that

$$m\,\widehat{AE} = m\,\widehat{CD}.$$

Then, we know that:

$$\begin{aligned}\angle BMP = \angle BMC \\ &= \frac{m\,\widehat{AE}}{2} + \frac{m\,\widehat{BC}}{2} \\ &= \frac{m\,\widehat{CD}}{2} + \frac{m\,\widehat{BC}}{2} \\ &= \frac{m\,\widehat{BD}}{2} \\ &= \angle BOP.\end{aligned}$$

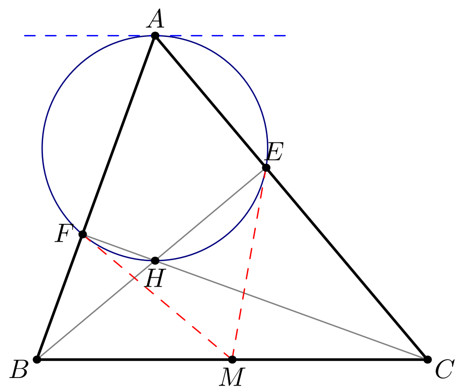
Thus, quadrilateral $POMB$ is cyclic. Thus, we have that

$$\angle OMC = \angle OMP = \angle OBP = 90^\circ$$

so M is the midpoint of $\overline{AC}$, as required. $\square$

Problem 1.44 (Three Tangents)

Let ABC be an acute triangle. Let $\overline{BE}$ and $\overline{CF}$ be altitudes of $\triangle ABC$, and denote by M the midpoint of $\overline{BC}$. Prove that $\overline{ME}$, $\overline{MF}$, and the line through A parallel to $\overline{BC}$ are all tangents to (AEF).



We will show that $\overline{ME}$ is tangent, and $\overline{MF}$ can be derived easily using a similar method. We know that

$$\angle HAE = 90 - \angle C.$$

In addition, since $\triangle BEC$ is a right triangle, and $\overline{EM}$ is the median, we know that $\triangle BME$ is isosceles, and so

$$\angle MEB = \angle MBE = 90 - \angle C.$$

So

$$\angle HAE = \angle MEB = \angle MEH$$

implying that $\overline{ME}$ is tangent to (AEF). The proof for $\overline{MF}$ is similar, so we omit it. Now let P be a point on $\overline{BC}$ on the side of B. Then, we have that

$$\angle FAP = \angle B$$

due to parallel lines. In addition, we can say that

$$\angle AHF = 90 - \angle HAF = \angle B.$$

So, we have that

$$\angle FAP = \angle AHF$$

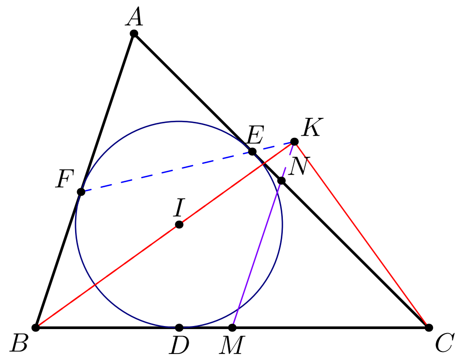
implying that $\overline{AP}$ is tangent to (AEF), as desired. $\square$

Problem 1.45 (Right Angles on Incircle Chord)

The incircle of $\triangle ABC$ is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Let M and N be the midpoints of $\overline{BC}$ and $\overline{AC}$, respectively. Ray BI meets line EF at K.

(i) Show that $\overline{BK} \perp \overline{CK}$.

(ii) Show K lies on line MN.



We start with the first part. Clearly, we know that $\triangle BMK$ is isosceles. This implies that

$$\angle BKM = \frac{\angle B}{2}.$$

In addition, this also implies that $\triangle KMC$ is isosceles. So, we know that

$$\angle MKC = 90 - \frac{\angle KMC}{2} = \frac{\angle BMK}{2} = 90 - \frac{\angle B}{2}.$$

Thus, we know that $\angle BKC$ is right, as required.

We proceed with the second part. All we must show is that

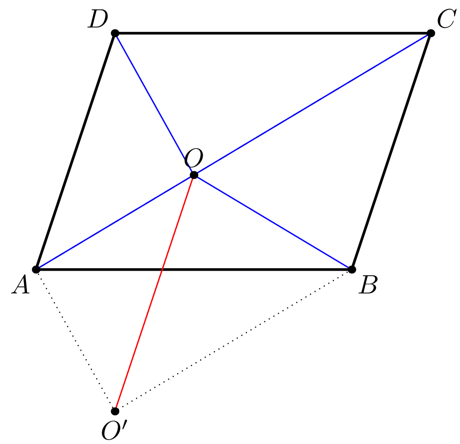
$$\angle KMC = \angle NMC = \angle B.$$

We already established that $\triangle KMC$ is isosceles. Hence, we know that

$$\angle KMC = 180 - 2\angle MKC = \angle B$$

as required. $\square$

Problem 1.46 (Canada 1997/4)
The point O is situated inside the parallelogram $ABCD$ such that $\angle AOB + \angle COD = 180°$. Prove that $\angle OBC = \angle ODC$.



Let O' be defined as the translation of O using the vector $\overrightarrow{DA}$ (or equivalently $\overrightarrow{CB}$). Now, we can write that

$$\triangle DOA \cong \triangle O'AO$$
$$\triangle COB \cong \triangle O'BO$$

because of the parallelograms created. Because of this, we can write that

$$\begin{aligned}\angle AO'B &= \angle AO'O + \angle BO'O \\ &= \angle ADO + \angle BCO \\ &= 180 - \angle ODC - \angle OCD \\ &= \angle DOC \\ &= 180 - \angle AOB\end{aligned}$$

so quadrilateral $AO'BO$ is cyclic. Finally, we can say that:

$$\begin{aligned}\angle OBA &= \angle OO'A \\ &= \angle ODA\end{aligned}$$

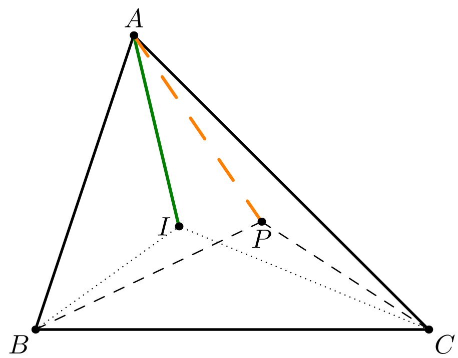
and since $\angle B = \angle D$ this implies that

$$\angle OBC = \angle OBA$$

as required. □

Problem 1.47 (IMO 2006/1)

Let ABC be a triangle with incenter I. A point P in the interior of the triangle satisfies $\angle PBA + \angle PCA = \angle PBC + \angle PCB$. Show that $AP \geq AI$ and that equality holds if and only if $P = I$.



We can rewrite the condition as

$$\angle PBA + \angle PCA = \angle B - \angle PBA + \angle C - \angle PCA$$

which simplifies to

$$\angle PBA + \angle PCA = \frac{\angle B}{2} + \frac{\angle C}{2}.$$

We can rearrange this to

$$\angle PBA - \frac{\angle B}{2} = \frac{\angle C}{2} - \angle PCA.$$

This implies that P must lie on (BIC).

It is well known that the minimum distance from any point to a circle can be found by finding the distance from that point to intersection of the segment with endpoints of the center of the circle and that point. In addition, by the Incenter-Excenter Lemma, we know that A, I and the center of (BIC) are collinear, so the minimum distance from A to (BIC) is AI. Hence, we must have

$$AP \geq AI$$

for any P, with equality that holds if and only if $P = I$, as required. □

Problem 1.48 (Simson Line)

Let ABC be a triangle and P be any point on (ABC). Let X, Y, Z be the feet of the perpendiculars from P onto lines BC, CA, and AB. Prove that points X, Y, Z are collinear.

It suffices to show that

$$\measuredangle PYX = \measuredangle PYZ.$$

We can establish that $PYXC$ is cyclic since

$$\measuredangle PYC = \measuredangle PXC = 90^\circ.$$

Similarly, we know that $PZAY$ is cyclic since

$$\measuredangle PYA = \measuredangle PZA = 90^\circ.$$

Then, we can say that

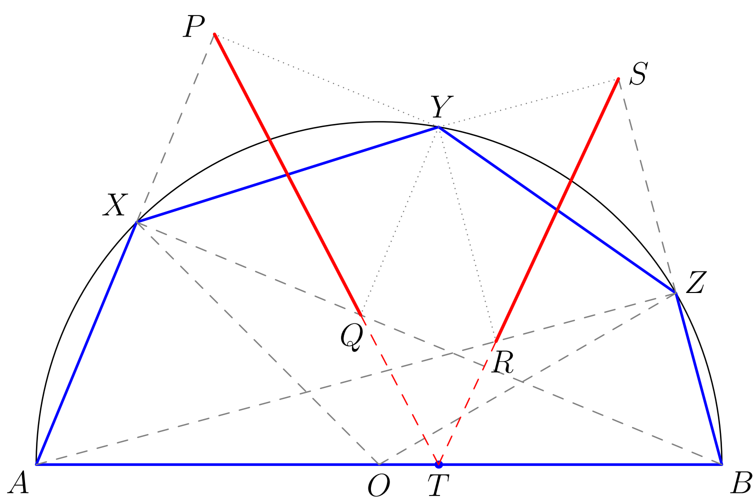
$$\begin{aligned}\measuredangle PYX &= \measuredangle PCX \\ &= \measuredangle PCB \\ &= \measuredangle PAB \\ &= \measuredangle PAZ \\ &= \measuredangle PYZ\end{aligned}$$

where the third line follows because $PABC$ is cyclic. This is the desired conclusion. □

Remark
A proof by complex numbers can be found as part (a) of Problem 6.22.

Problem 1.49 (USAMO 2010/1)

Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter $\overline{AB}$. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment $\overline{AB}$.



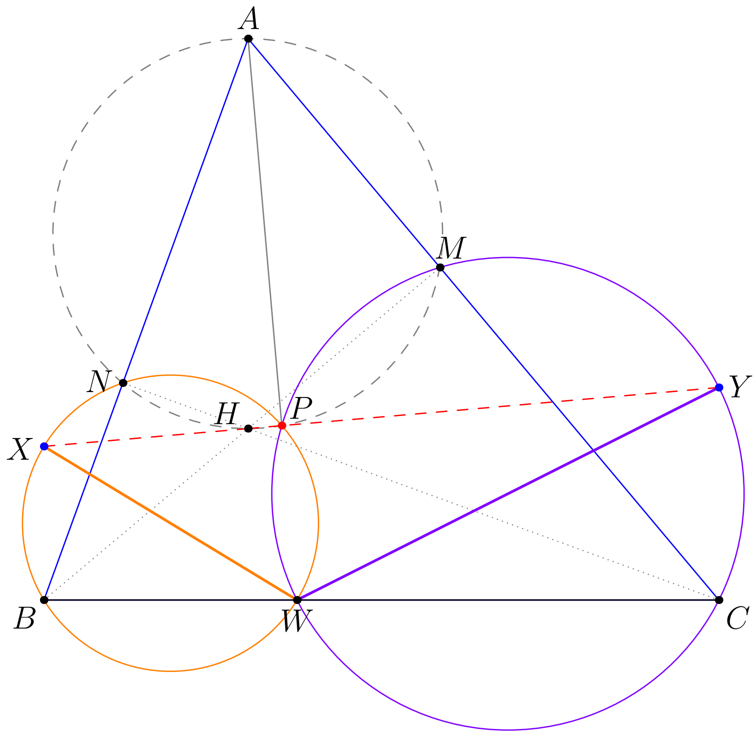
Let T be the intersection between lines PQ and RS. Notice that we can write $\angle QTR$ as

$$\begin{aligned}\angle QTR &= \angle PQY - \angle QYT + \angle SRY - \angle RYT \\ &= \angle PQY + \angle SRY - \angle QYR \\ &= \angle PXY + \angle SZY - \angle QYR \\ &= 180 - \angle AXY + 180 - \angle YZB - \angle ZAB - \angle XBA \\ &= \angle ABY + \angle YAB - \angle ZAB - \angle XBA \\ &= \angle XBY + \angle YAZ \\ &= \frac{\angle XOZ}{2}\end{aligned}$$

as required. □

Problem 1.50 (IMO 2013/4)

Let ABC be an acute triangle with orthocenter H, and let W be a point on the side $\overline{BC}$, between B and C. The points M and N are the feet of the altitudes drawn from B and C, respectively. ω_1 is the circumcircle of triangle BWN and X is a point such that $\overline{WX}$ is a diameter of ω_1. Similarly, ω_2 is the circumcircle of triangle CWM and Y is a point such that $\overline{WY}$ is a diameter of ω_2. Show that the points X, Y, and H are collinear.



Let P be the second intersection between the two circumcircles (other than W). We will show that X, H, and P are collinear, and the same result can easily be replicated for Y. First, note that by the Miquel Point, $ANPM$ is cyclic, and because

$$\angle ANH = \angle AMH = 90^\circ$$

$ANHM$ is also cyclic. This implies that pentagon $ANHPM$ is cyclic. Next, we prove that A, P, and W are collinear. This is equivalent to show that

$$\angle APM + \angle WPM = 180^\circ.$$

We can do this as follows:

$$\begin{aligned} \angle APM &= \angle AHM \\ &= 90 - \angle HAM \\ &= \angle C \\ &= 180 - \angle WPM \end{aligned}$$

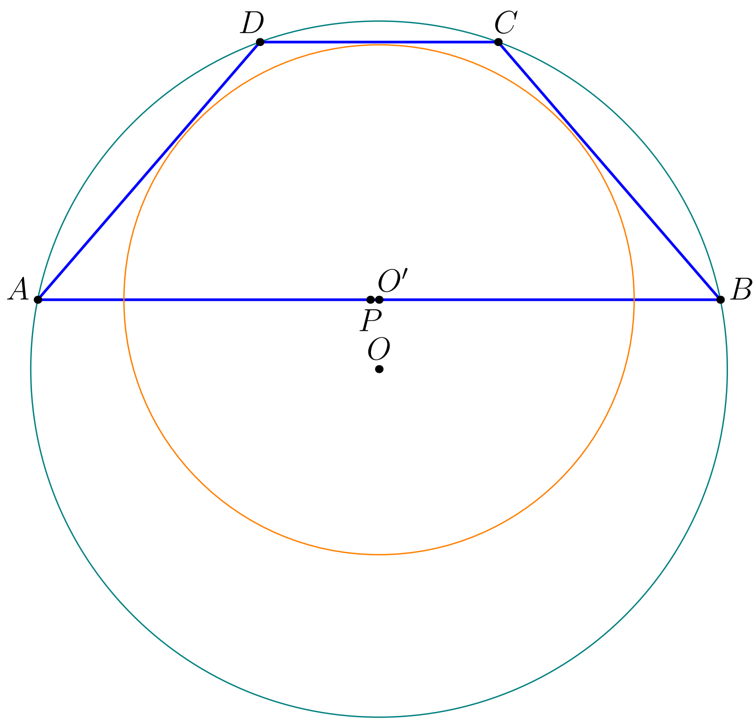
as required. Now, we can say that

$$\angle APH = \angle XPW = 90^\circ$$

so X, H, and P are collinear. We repeat this with Y to get a similar result, which implies that X, H, and Y are collinear, as required. $\square$

Problem 1.51 (IMO 1985/1)

A circle has center on the side $\overline{AB}$ of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + BC = AB$.



Let P be the point on $\overline{AB}$ such that

$$AD = AP.$$

We wish to show that

$$BC = BP.$$

We can first notice that $DPO'C$ is cyclic since

$$\begin{aligned} \angle DPO' &= 180 - \angle DPA \\ &= 90 + \frac{\angle DAP}{2} \\ &= 90 + \frac{180 - \angle C}{2} \\ &= 180 - \angle DCO'. \end{aligned}$$

Next, we will show that $\triangle PBC$ is isosceles. We can say that:

$$\begin{aligned}
\angle CPB &= \angle CPO' \\
&= \angle CDO' \\
&= \frac{\angle D}{2} \\
&= 90 - \frac{\angle B}{2}
\end{aligned}$$

so $\triangle PBC$ is indeed isosceles, implying that

$$BP = BC$$

as desired. $\square$

2 Circles

Problem 2.5

Prove the following (Theorem 2.3): Consider a circle ω and an arbitrary point P.

(a) The quantity $\text{Pow}_\omega(P)$ is positive, zero, or negative according to whether P is outside, on, or inside ω, respectively.

(b) If is a line through P intersecting ω at two distinct points X and Y, then

$$PX \cdot PY = |\text{Pow}_\omega(P)|.$$

(c) If P is outside ω and $\overline{PA}$ is a tangent to ω at a point A on ω, then

$$PA^2 = \text{Pow}_\omega(P).$$

For completeness, we define Pow.

Definition

Let O be the center of ω. We define $\text{Pow}_\omega(P)$ as the quantity $OP^2 - r^2$ if r is the radius of ω.

We start with part (a). If P is inside ω, then $OP < r$, so $\text{Pow}_\omega(P)$ is negative. If P is on ω, then $OP = r$, so $\text{Pow}_\omega(P)$ is zero. Similarly, if P is outside ω, then $OP > r$, so $\text{Pow}_\omega(P)$ is positive.

We continue with part (b). Let the diameter through P intersect ω at A and B. Then, we know that

$$\triangle APX \sim \triangle YPB.$$

Then, we know that

$$\frac{AP}{YP} = \frac{PX}{PB}.$$

So, we know that

$$PX \cdot PY = PA \cdot PB = (OP + r)(OP - r) = |\text{Pow}_\omega(P)|$$

as required.

We finish with part (c). We already know that

$$PA \cdot PB = PX \cdot PY.$$

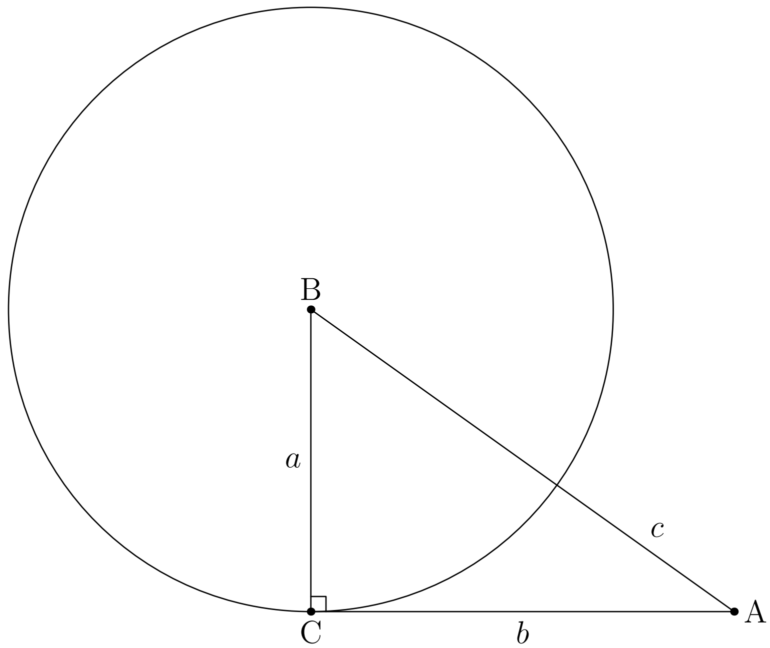
Now, move the line creating A and B closer and closer to the circumference. Consider the limiting case, when $A = B$. In this case, we can write that

$$PA^2 = PX \cdot PY = \text{Pow}_\omega(P)$$

as required. $\square$

Problem 2.6

Let ABC be a right triangle with $\angle ACB = 90^\circ$. Give a proof of the Pythagorean theorem.



Note that by Power of a Point, we have that

$$b^2 = c(c + 2a).$$

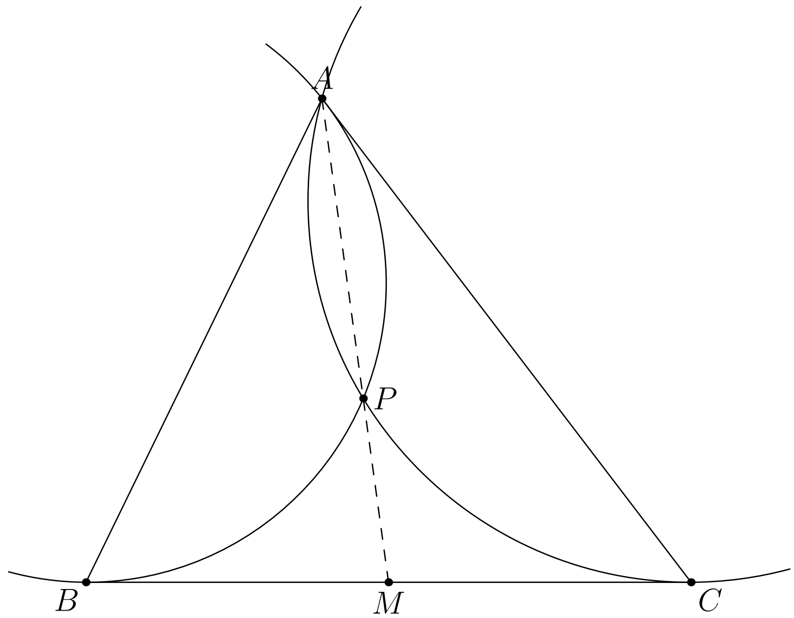
Solving this equation yields the following:

$$b^2 = c^2 + 2ac$$
$$a^2 + b^2 = (a + c)^2$$

which is equivalent to the Pythagorean Theorem in the given diagram.

Problem 2.11

Let ABC be a triangle and consider a point P in its interior. Suppose that $\overline{BC}$ is tangent to the circumcircles of triangles ABP and ACP. Prove that ray AP bisects $\overline{BC}$.



We know that the midpoint of $\overline{BC}$ (which we call M) lies on the radical axis of (ABP) and (ACP) because

$$\text{Pow}_{(ABP)}(M) = \text{Pow}_{(ACP)}(M)$$

which is due to the fact that the length of the tangents to those circles are equal. $\square$

Problem 2.12

Show that the orthocenter of a triangle exists using radical axes. That is, if $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are altitudes of a triangle ABC, show that the altitudes are concurrent.

Consider the circles with diameters $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$. These circles' common chords (considered pairwise) are the aforementioned altitudes. Thus, since the radical center exists, so does the orthocenter. $\square$

Problem 2.18

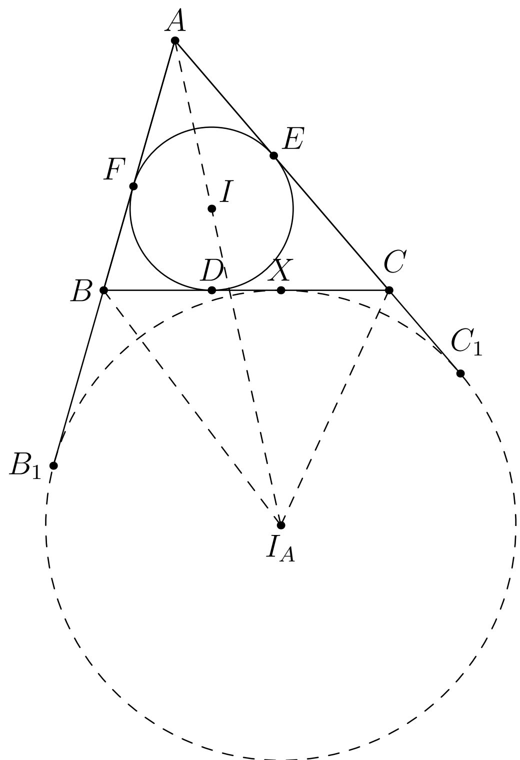
Let the external angle bisectors of B and C in a triangle ABC intersect at I_A. Show that I_A is the center of a circle tangent to $\overline{BC}$, the extension of $\overline{AB}$ through B, and the extension of $\overline{AC}$ through C. Furthermore, show that I_A lies on ray AI.

Clearly, the circle tangent to $\overline{BC}$, the extension of $\overline{AB}$ through B, and the extension of $\overline{AC}$ through C exists. Now, since this circle is tangent to $\overline{BC}$ and the the extension of $\overline{AB}$ through B, we know that the center lies on the external angle bisector of B. Similarly, the center also lies on the external angle bisector of C, as desired. Finally, by the Incenter-Excenter Lemma, A, I, and I_A are collinear. $\square$

Problem 2.19

Prove that the A-exradius has length

$$r_a = \frac{s}{s-a} r.$$



Clearly,

$$\triangle AFI \sim \triangle AB_1I_A.$$

Thus, we have that

$$\frac{AB_1}{B_1I_A} = \frac{AF}{FI} = \frac{s-a}{r}.$$

So, we know that

$$\frac{s-a}{r} = \frac{AB_1}{B_1I_A} = \frac{s}{r_a}$$

as required. □

Problem 2.20

Let ABC be a triangle. Suppose its incircle and A-excircle are tangent to $\overline{BC}$ at X and D, respectively. Show that $BX = CD$ and $BD = CX$.

Refer to the diagram in Problem 2.19.

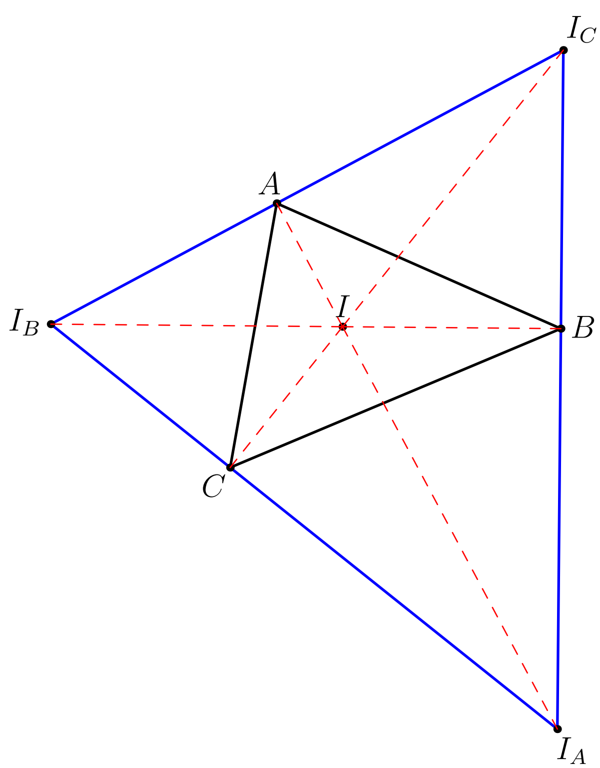
We will show the former, and that will imply the latter. We know that:

$$\begin{aligned} BX &= BB_1 \\ &= AB_1 - AB \\ &= s - c \\ &= DC \end{aligned}$$

as required. □

Problem 2.24

Let ABC be a triangle with I_A, I_B, and I_C as excenters. Prove that triangle $I_AI_BI_C$ has orthocenter I and that triangle ABC is its orthic triangle.



We already know that any vertex is collinear with the incenter and its corresponding excenter (by the Incenter-Excenter Lemma). Thus, all we must prove is that the segment from any vertex to the corresponding excenter is perpendicular to the segment formed by the other two excenters. If we prove that

$$\overline{I_BB} \perp \overline{I_AI_C}$$

then we can easily generalize this to the other sides.

$$\begin{aligned}\angle I_B B I_A &= \angle I_B BC + \angle CBI_A \\ &= \angle IBC + \frac{180 - \angle B}{2} \\ &= \frac{\angle B}{2} + \frac{180 - \angle B}{2} \\ &= 90\end{aligned}$$

as claimed. $\square$

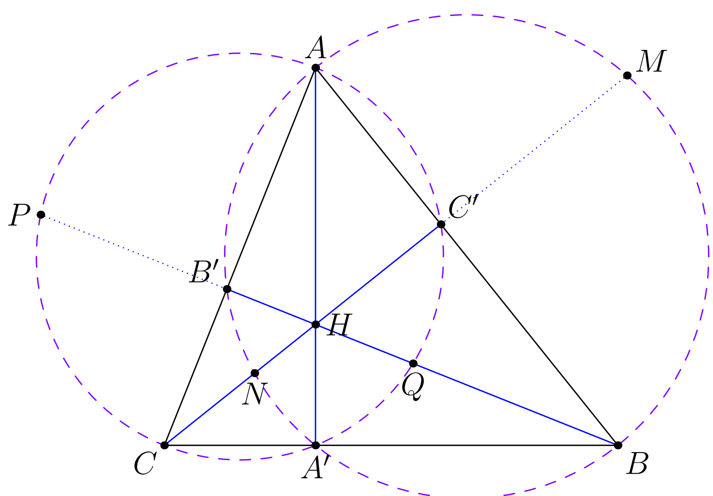
Problem 2.25 (Pitot's Theorem)

Let $ABCD$ be a quadrilateral. If a circle can be inscribed in it, prove that $AB + CD = BC + DA$.

We leave many details to the reader, as this proof is relatively elementary. The circle tangent to $ABCD$ will divide up the sides into two parts, each of which will have a corresponding congruent side on the side adjacent to it. Adding up the opposite sides, will then result in equality.

Problem 2.26 (USAMO 1990/5)

An acute-angled triangle ABC is given in the plane. The circle with diameter $\overline{AB}$ intersects altitude $\overline{CC'}$ and its extension at points M and N, and the circle with diameter $\overline{AC}$ intersects altitude $\overline{BB'}$ and its extensions at P and Q. Prove that the points M, N, P, Q lie on a common circle.



Clearly, lines MN and PQ intersect on the radical axis of the two circles ($\overline{AA'}$), since they intersect at H. Thus, we know that M, N, P, and Q lie on a common circle by the Radical Center of Intersecting Circles.

We can write this out if necessary: Let ω_1 be the circle with diameter $\overline{AB}$, and ω_2 the circle with diameter $\overline{AC}$. Then,

$$HP \cdot HQ = \text{Pow}_{\omega_2}(H) = \text{Pow}_{\omega_1}(H) = HM \cdot HN$$

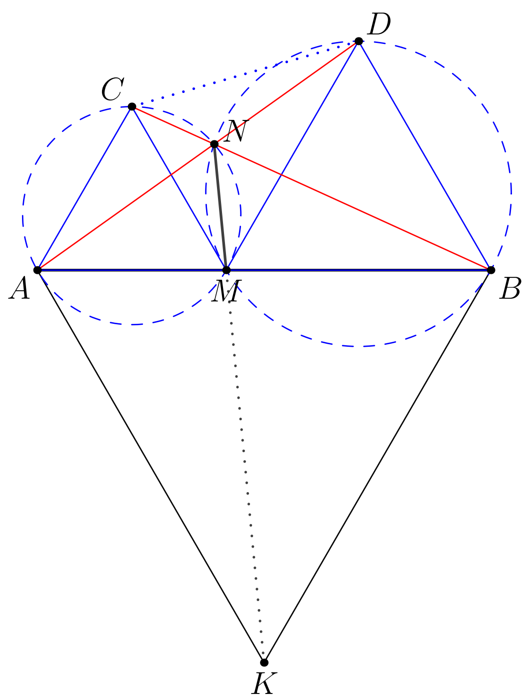
so M, N, P, and Q are indeed concyclic by the Converse of Power of a Point. $\square$

Problem 2.27 (BAMO 2012/4)

Given a segment $\overline{AB}$ in the plane, choose on it a point M different from A and B. Two equilateral triangles AMC and BMD in the plane are constructed on the same side of segment $\overline{AB}$. The circumcircles of the two triangles intersect in point M and another point N.

(a) Prove that $\overline{AD}$ and $\overline{BC}$ pass through point N.

(b) Prove that no matter where one chooses the point M along segment $\overline{AB}$, all lines MN will pass through some fixed point K in the plane.



We start with part (a). Clearly, quadrilaterals $ACNM$ and $BDNM$ are cyclic. Now, we just need to show that N lies on both $\overline{AD}$ and $\overline{BC}$. We can do this as follows:

$$\begin{aligned}\measuredangle AND &= \measuredangle ANM + \measuredangle MND \\ &= \measuredangle ACM + \measuredangle MBD \\ &= 0\end{aligned}$$

so N lies on $\overline{AD}$. We can do the same for $\overline{BC}$ and get that N lies on $\overline{BC}$ as well, as required.

We proceed with part (b). We wish to show that this point K is the point on the opposite side of $\overline{AB}$ such that $\triangle AKB$ is equilateral. Hence, we can show that line NM always passes through K. Now, since $\overline{NM}$ is the radical axis of the two circumcircles, it suffices to show that the tangents to the two circles are always equal in length.

We start by showing that $\overline{KA}$ is tangent to (ACM). Let the center of (ACM) be O_1. Then, we know that:

$$\begin{aligned}\measuredangle O_1AK &= \measuredangle O_1AM + \measuredangle MAK \\ &= 90.\end{aligned}$$

We can show something similar for the other circle as well. Thus, $\overline{AK}$ is tangent to (ACM) and

$\overline{BK}$ is tangent to (BDM). Finally, since

$$AK = BK$$

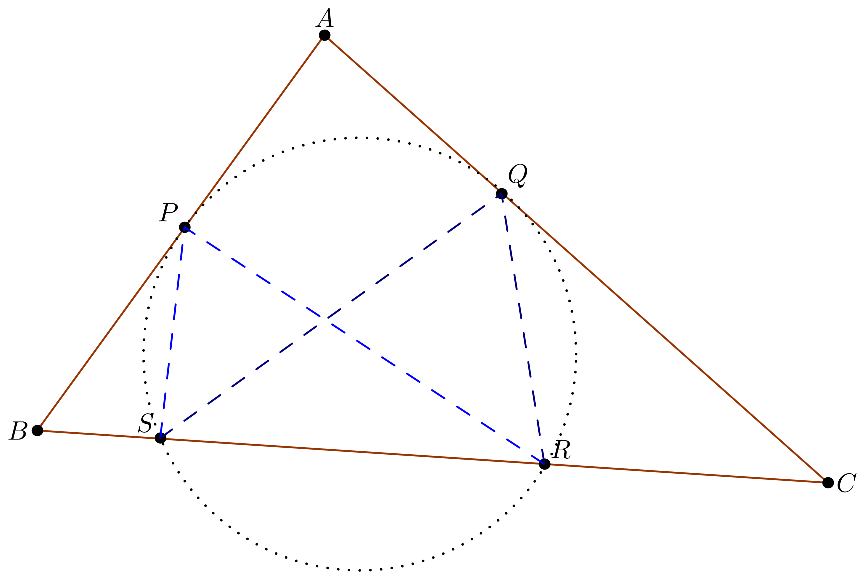
we know that

$$\text{Pow}_{(ACM)}(K) = \text{Pow}_{(BDM)}(K).$$

So, K lies on line NM, as desired. $\square$

Problem 2.28 (JMO 2012/1)

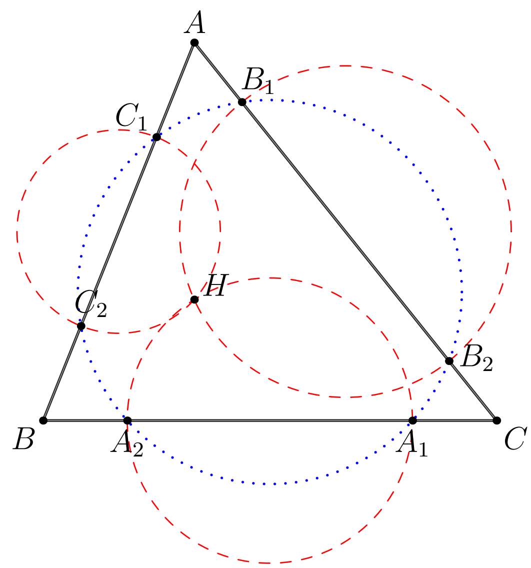
Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic.



Assume for the sake of contradiction that the circumcircles (PRS) and (QRS) are distinct. Then, the given condition tells us that $\overline{AC}$ is tangent to (QRS) at Q and $\overline{AB}$ is tangent to (PRS) at P. In addition, we know that A must lie on the radical axis of (PRS) and (QRS). However, we know that the radical axis is line BC, so if the two circles are distinct, then $\triangle ABC$ is degenerate, which cannot happen. Hence, (PRS) and (QRS) are the same circle, implying that P, Q, R, and S are cyclic. $\square$

Problem 2.29 (IMO 2008/1)

Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of $\overline{BC}$ and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1, and C_2. Prove that six points A_1, A_2, B_1, B_2, C_1, and C_2 are concyclic.



If we show that B_1, B_2, C_1, and C_2 are cyclic, then we can apply the same reasoning to get that any other pair of pairs (A_i, B_i, C_i for $i = 1, 2$) are cyclic as well, which would imply that they are all concyclic. Thus, we will show that the circle through B_1, B_2, C_1, and C_2 exists. Now, let N be the other intersection of (C_1C_2H) and (B_1B_2H). Then, we know that $\overline{NH}$ is perpendicular to the segment formed by the centers of these two circles, since it is the radical axis. However, this implies that line

$$NH \perp \overline{BC}$$

since the segment formed by the centers of these two circles is parallel to $\overline{BC}$. Finally, because

$$AH \perp \overline{BC}$$

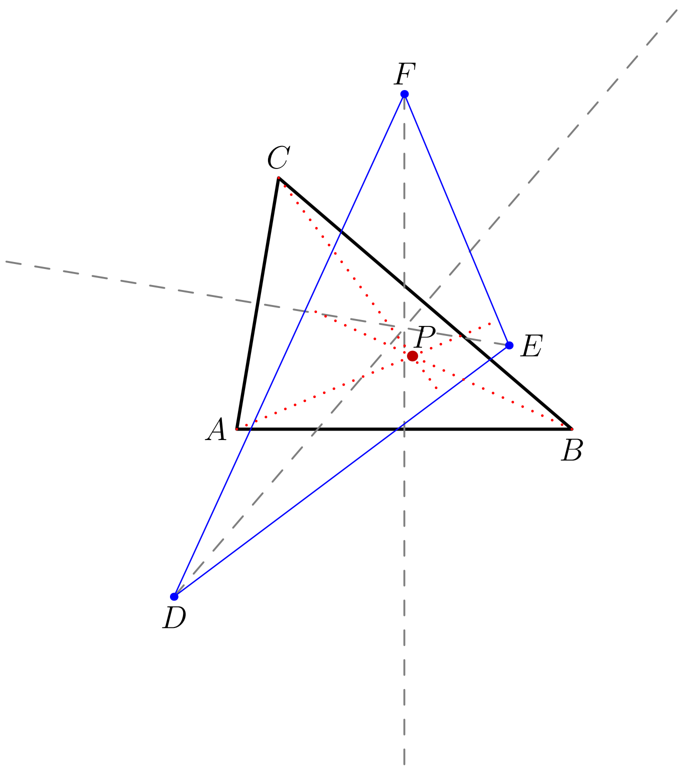
we know that A lies on the radical axis. Thus, since lines C_1C_2 and B_1B_2 concur at A, we know that they are concyclic, as desired. □

Remark

The center of the circle through A_1, A_2, B_1, B_2, C_1, and C_2 is actually O, the circumcenter of $\triangle ABC$.

Problem 2.30 (USAMO 1997/2)

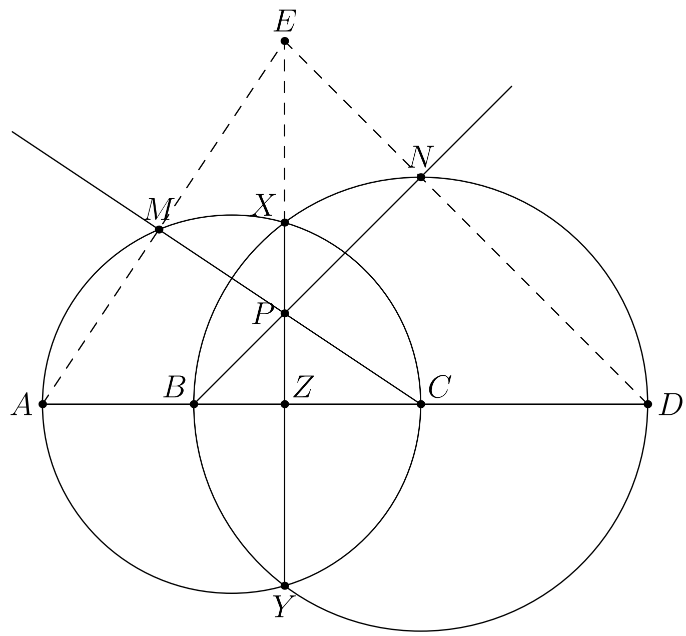
Let ABC be a triangle. Take points D, E, F on the perpendicular bisectors of $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ respectively. Show that the lines through A, B, C perpendicular to $\overline{EF}$, $\overline{FD}$, $\overline{DE}$ respectively are concurrent.



Clearly, since D is on the perpendicular bisector of $\overline{BC}$ and similarly for the others, we can define a circle Γ_D centered at D through B and C, and likewise for the other two points. Then, since the lines in question mentioned in the problem are perpendicular to the sides of $\triangle DEF$, and go through vertices of $\triangle ABC$, we know that they are the radical axes of the circles. Thus, the point in question (labeled P in the diagram) is just the radical center of Γ_D, Γ_E, and Γ_F, which we know to exist. $\square$

Problem 2.31 (IMO 1995/1)

Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters $\overline{AC}$ and $\overline{BD}$ intersect at X and Y. The line XY meets $\overline{BC}$ at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter $\overline{AC}$ at C and M, and the line BP intersects the circle with diameter $\overline{BD}$ at B and N. Prove that the lines AM, DN, XY are concurrent.



We wish to show that the intersection of lines AM and DN, denoted E lies on the radical axis of the two circles in the diagram. Note that E has the property that

$$\text{Pow}_{(AMC)}(E) = EM \cdot EA$$

and

$$\text{Pow}_{(BND)}(E) = EN \cdot ED.$$

We wish to show that these quantities are equal. Now, we know that $\triangle EMP \sim \triangle EZA$ due to the shared angle and the fact that they are both right. Then, we know that

$$\frac{EM}{EP} = \frac{EZ}{EA}.$$

In addition, we know that $\triangle ENP \sim \triangle EZD$ for similar reasons, so

$$\frac{EN}{EP} = \frac{EZ}{ED}.$$

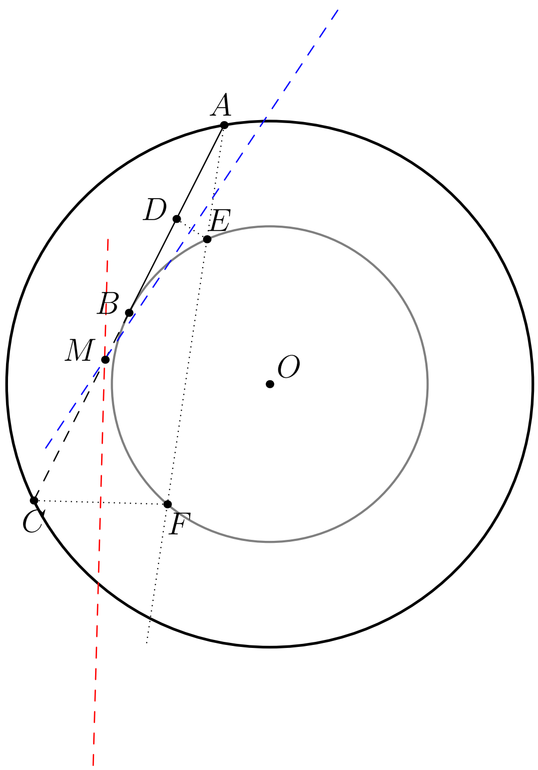
Thus:

$$\begin{aligned}
\text{Pow}_{(AMC)}(E) &= EM \cdot EA \\
&= EZ \cdot EP \\
&= EN \cdot ED \\
&= \text{Pow}_{(BND)}(E)
\end{aligned}$$

as required. So, E lies on the radical axis of the two circles, and hence lies on line XY. $\square$

Problem 2.32 (USAMO 1998/2)

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be concentric circles, with $\mathcal{C}_2$ in the interior of $\mathcal{C}_1$. From a point A on $\mathcal{C}_1$ one draws the tangent $\overline{AB}$ to $\mathcal{C}_2$ ($B \in \mathcal{C}_2$). Let C be the second point of intersection of ray AB and $\mathcal{C}_1$, and let D be the midpoint of $\overline{AB}$. A line passing through A intersects $\mathcal{C}_2$ at E and F in such a way that the perpendicular bisectors of $\overline{DE}$ and $\overline{CF}$ intersect at a point M on AB. Find, with proof, the ratio $\frac{AM}{MC}$.



We claim that $\frac{AM}{MC} = \frac{5}{3}$.

We wish to show that quadilateral $CDEF$ is cyclic. We can write that

$$AE \cdot AF = AB^2 = AD \cdot AC$$

by Power of a Point. Thus, we know that

$$\triangle ADE \sim \triangle AFC.$$

This then implies that $\measuredangle ADE = \measuredangle EFC$, so we know that

$$\measuredangle EFC = \measuredangle ADE = \measuredangle EFC$$

implying that quadrilateral $CDEF$ is indeed cyclic. Now, we know that

$$MC = MF$$

and

$$MD = ME$$

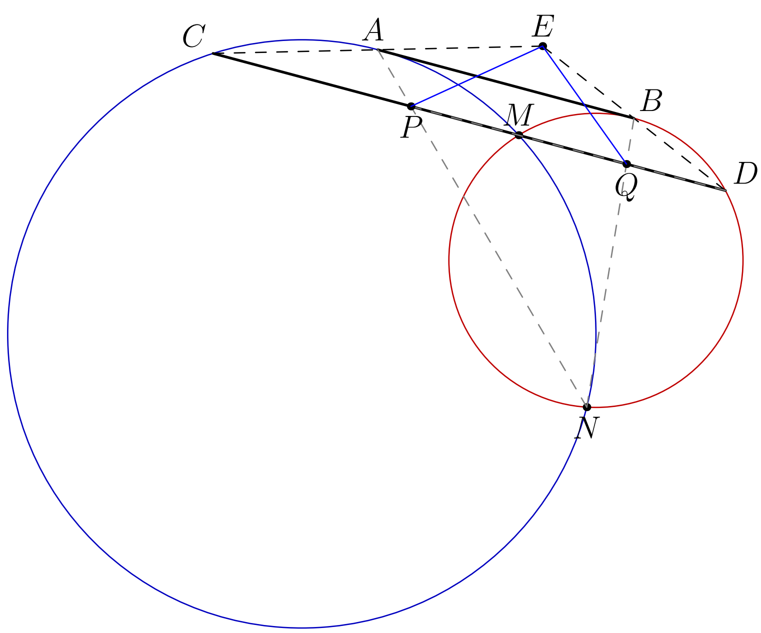
because of the perpendicular bisectors, and so M is the center of $(CDEF)$. This means that we can write:

$$\begin{aligned}\frac{AM}{MC} &= \frac{AM}{MD} \\ &= \frac{MD + AD}{MD} \\ &= 1 + \frac{AB}{2MD} \\ &= 1 + \frac{AB}{AC - AD} \\ &= 1 + \frac{AB}{2AB - \frac{AB}{2}} \\ &= \boxed{\frac{5}{3}}\end{aligned}$$

as required. □

Problem 2.33 (IMO 2000/1)

Two circles G_1 and G_2 intersect at two points M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through the point M, with C on G_1 and D on G_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.



Because of $\overline{AB}$ is tangent to the two circles, we know that line MN bisects $\overline{AB}$, and since

$$\triangle NPQ \sim \triangle NAB$$

we know that

$$MP = MQ.$$

Now, we know that

$$\triangle AEB \cong \triangle AMB$$

because of the tangency at A and B. Thus, we know that they are just reflections of each other across $\overline{AB}$. This means that $\overline{EM} \perp \overline{AB}$, and so

$$\overline{EM} \perp \overline{CD}$$

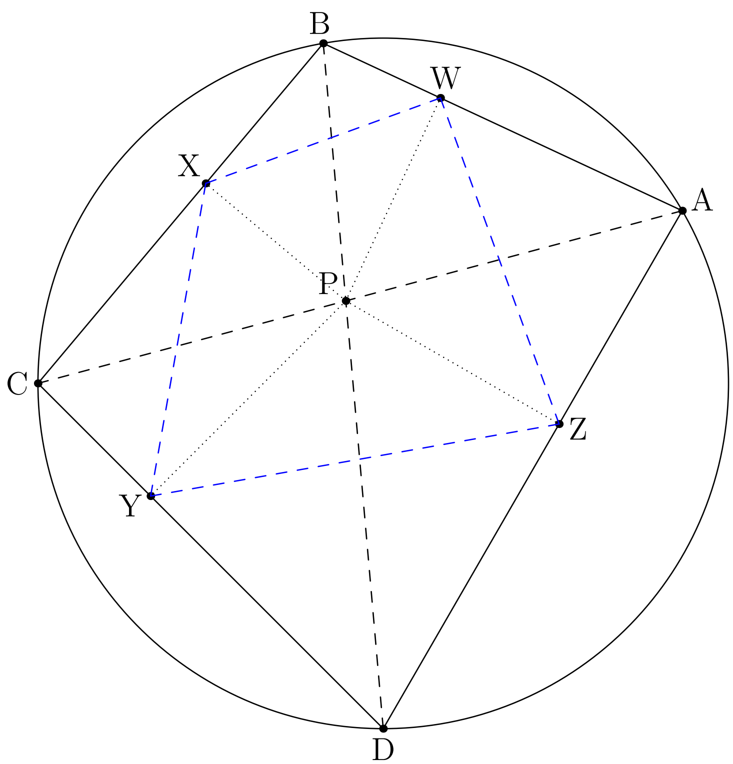
implying that

$$\overline{EM} \perp \overline{PQ}.$$

This, in conjunction with the fact that $MP = MQ$ implies that $\triangle EPQ$ is isosceles, so $EP = EQ$, as required. $\square$

Problem 2.34 (Canada 1990/3)

Let $ABCD$ be a cyclic quadrilateral whose diagonals meet at P. Let W, X, Y, Z be the feet of P onto $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$, respectively. Show that $WX + YZ = XY + WZ$.



By the Converse of Pitot's Theorem, we just have to show that there exists a point which is equidistant to all the sides of quadrilateral $WXYZ$.

Claim

The point P is equidistant to all the sides of $WXYZ$.

We can prove this by showing that P lies on the angle bisectors of quadrilateral $WXYZ$. We will show that for one angle bisector, and the result easily follows by replicating it for the others.

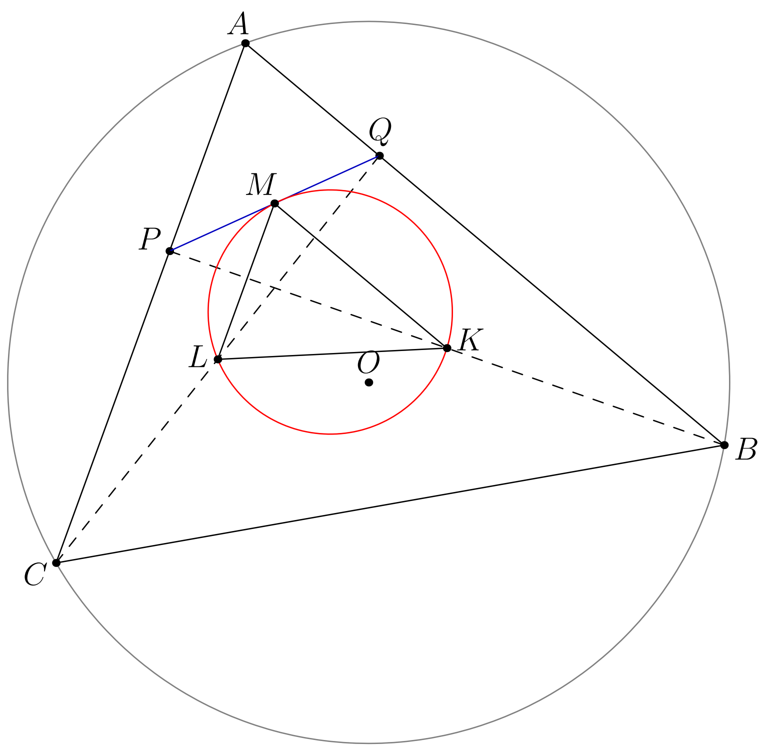
We can say that:

$$\begin{aligned}\angle PYX &= \angle PCX \\ &= \angle ACB \\ &= \angle ADB \\ &= \angle PDZ \\ &= \angle PYZ\end{aligned}$$

so P does indeed lie on the angle bisector of $\angle Y$, as desired. $\square$

Problem 2.35 (IMO 2009/2, synthetic)

Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides $\overline{CA}$ and $\overline{AB}$, respectively. Let K, L, and M be the midpoints of the segments $\overline{BP}$, $\overline{CQ}$, and $\overline{PQ}$, respectively, and let be the circle passing through K, L, and M. Suppose that the line PQ is tangent to the circle. Prove that $OP = OQ$.



Claim

We claim that

$$AP \cdot PC = AQ \cdot QB.$$

Proof. First, we know that

$$\triangle CQP \sim \triangle LQM$$

and

$$\triangle BPQ \sim \triangle KPM$$

for obvious reasons (note that they are both in the ratio $2:1$). Next, we can say that

$$\angle APQ = 180 - \angle CPQ = 180 - \angle LMQ = \angle K$$

and similarly,

$$\angle AQP = 180 - \angle BQP = 180 - \angle KMP = \angle L.$$

Thus, we know that

$$\triangle APQ \sim \triangle MKL.$$

Finally, we can write that

$$\frac{AP}{AQ} = \frac{MK}{ML} = \frac{QB}{PC}$$

so the claim is indeed true, as required. $\square$

Thus, we know that

$$\text{Pow}_{(ABC)}(P) = \text{Pow}_{(ABC)}(Q)$$

which implies that

$$OP^2 - R^2 = OQ^2 - R^2$$

which gives the necessary result. $\square$

Remark
A solution by complex numbers can be found as Problem 6.41.

Problem 2.36

Let $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ be the altitudes of a scalene triangle ABC with circumcenter O. Prove that (AOD), (BOE), and (COF) intersect at point X other than O.

Claim
The circles (AOD), (BOE), and (COF) are coaxial.

Proof. We already know that the circles share the point O. If we show that there exists another point (call it H) such that

$$\text{Pow}_{(AOD)}(H) = \text{Pow}_{(BOE)}(H) = \text{Pow}_{(COF)}(H)$$

then the point line OH is the radical axis of all the circles, and thus they will be coaxial. We claim that the orthocenter satisfies the conditions of point H.

Consider any two of those circles. The chords of those circles that do not contain O will intersect at H, and since the quadrilateral formed by those four points is cyclic, we know that H lies on the radical axis of those two circles. Hence, H lies on the radical axis of *all* the three circles, as required. $\square$

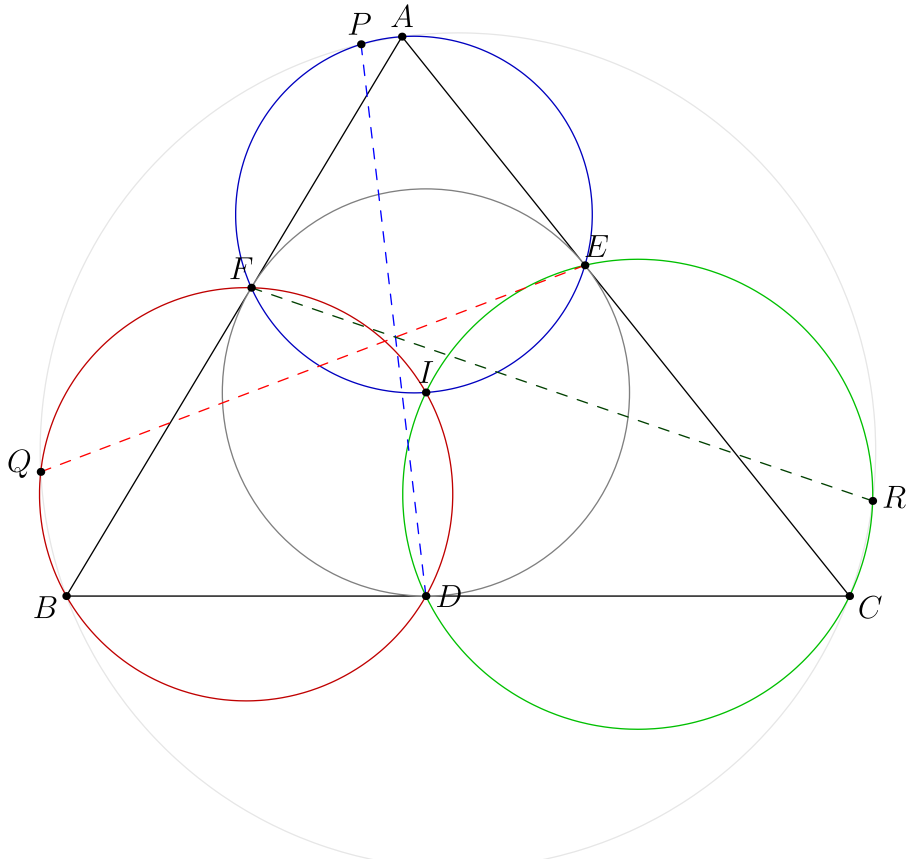
Since the circles are coaxial, they intersect at the same two points, and so there exists another point X where all the three circumcircles intersect, as desired. $\square$

Problem 2.37 (Canada 2007/5)

Let the incircle of triangle ABC touch sides BC, CA and AB at D, E and F, respectively. Let ω, ω_1, ω_2 and ω_3 denote the circumcircles of triangles ABC, AEF, BDF and CDE respectively. Let ω and ω_1 intersect at A and P, ω and ω_2 intersect at B and Q, ω and ω_3 intersect at C and R.

(a) Prove that ω_1, ω_2 and ω_3 intersect in a common point.

(b) Show that lines PD, QE and RF are concurrent.



We start with part (a). We claim that the common point is I; the incenter of $\triangle ABC$. Obviously, quadrilaterals $BFID$, $AEIF$ and $CDIE$ are cyclic due to the tangency, hence I must lie on all of their circumcircles, as desired.

We continue with part (b). We claim that the lines all intersect at point H. We will first show that quadrilateral $PEDQ$ is cyclic. Let the lines QD and PE intersect at K. Then, we know that

$$KD \cdot KQ = \text{Pow}_{(BQFID)}(K) = KI \cdot KF = \text{Pow}_{(APFIE)}(K) = KE \cdot KP.$$

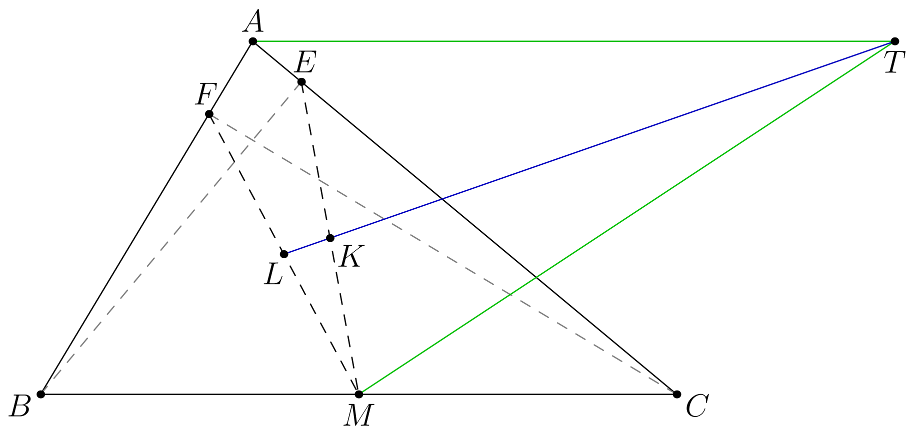
Thus, K lies on the radical axis of the two circles, so $PEDQ$ is cyclic, as desired. Similarly, we get that $PRDF$, and $QFER$ are cyclic. Thus, since the radical center exists, we know that there exists a point where all the lines intersect. Thus, H exists, as required. $\square$

Remark

The point H is actually the orthocenter of $\triangle ABC$.

Problem 2.38 (Iran TST 2011/1)

In acute triangle ABC, $\angle B$ is greater than $\angle C$. Let M be the midpoint of $\overline{BC}$ and let E and F be the feet of the altitudes from B and C, respectively. Let K and L be the midpoints of $\overline{ME}$ and $\overline{MF}$, respectively, and let T be on line KL such that $\overline{TA} \parallel \overline{BC}$. Prove that $TA = TM$.



Consider the circle (AEF). By the Three Tangents Lemma (proved in Problem 1.44 above), we know that lines AT, MF, and ME are tangent to (AEF) at A, F and E respectively. In addition, define ω to be a circle with radius zero centered at M. Clearly, we know that line LK is the radical axis of ω and (AEF), and since T lies on this radical axis, we know that

$$\text{Pow}_{(AEF)}(T) = \text{Pow}_{\omega}(T)$$

implying that

$$TA = TM$$

as required. $\square$

3 Lengths and Ratios

Problem 3.2 (Angle Bisector Theorem)

Let ABC be a triangle and D a point on $\overline{BC}$ so that $\overline{AD}$ is the internal angle bisector of $\angle BAC$. Show that
$$\frac{AB}{AC} = \frac{DB}{DC}.$$

By the Law of Sines, we know that
$$\frac{BD}{\sin(\angle BAD)} = \frac{AB}{\sin(\angle BDA)}$$
and
$$\frac{CD}{\sin(\angle CAD)} = \frac{AC}{\sin(\angle CDA)}.$$
Dividing the first equation by the second, we get that
$$\frac{BD\sin(\angle CAD)}{CD\sin(\angle BAD)} = \frac{BD}{CD} = \frac{AB\sin(\angle CDA)}{AC\sin(\angle BDA)} = \frac{AB\sin(\angle CDA)}{AC\sin(\angle CDA)} = \frac{AB}{AC}$$
as required. $\square$

Problem 3.5

Show the trigonometric form of Ceva holds.

We state Ceva's Theorem for completeness.

Theorem (Ceva's Theorem)

Let $\overline{AX}$, $\overline{BY}$, and $\overline{CZ}$ be cevians of $\triangle ABC$. They concur if and only if
$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

By the Law of Sines, we know that
$$\frac{BX}{\sin(\angle BAX)} = \frac{AX}{\sin(\angle B)}$$
and
$$\frac{CX}{\sin(\angle CAX)} = \frac{AX}{\sin(\angle C)}.$$
Taking the first equation and dividing it by the second, we get that
$$\frac{BX\sin(\angle CAX)}{CX\sin(\angle BAX)} = \frac{\sin(\angle C)}{\sin(\angle B)}.$$
If we take this for all the sides and multiply them together, we get that
$$\begin{aligned}\frac{CX\sin(\angle BAX)}{BX\sin(\angle CAX)} \cdot \frac{CY\sin(\angle CBY)}{AY\sin(\angle ABY)} \cdot \frac{AZ\sin(\angle ACZ)}{BZ\sin(\angle BCZ)} &= \frac{\sin(\angle BAX)\sin(\angle CBY)\sin(\angle ACZ)}{\sin(\angle CAX)\sin(\angle ABY)\sin(\angle BCZ)} \\ &= \frac{\sin(\angle B)}{\sin(\angle C)} \cdot \frac{\sin(\angle C)}{\sin(\angle A)} \cdot \frac{\sin(\angle A)}{\sin(\angle B)} \\ &= 1\end{aligned}$$

as required. $\square$

Problem 3.6

Let $\overline{AM}$, $\overline{BE}$, and $\overline{CF}$ be concurrent cevians of a triangle ABC. Show that $\overline{EF} \parallel \overline{BC}$ if and only if $BM = MC$.

We start by showing the if direction. Since $BM = MC$, by Ceva's Theorem, we know that

$$\frac{AF}{FB} \cdot \frac{CE}{EA} = 1$$

so

$$\frac{AF}{AE} = \frac{FB}{CE} = \frac{AF + FB}{AE + CE} = \frac{AB}{AC}.$$

Thus, we know that

$$\triangle FAE \sim \triangle BAC.$$

Hence, $\angle AFE = \angle ABC$, so $\overline{EF} \parallel \overline{BC}$, as desired.

Now we show the only if direction. Since $\overline{EF} \parallel \overline{BC}$, we know that

$$\triangle FAE \sim \triangle BAC.$$

Hence,

$$\frac{AF}{AE} = \frac{FB}{CE}.$$

Thus, by Ceva's Theorem,

$$\frac{BM}{MC} = 1$$

and so $BM = MC$, as required. $\square$

Problem 3.12

Prove, by taking a negative homothety that the centroid of a triangle divides the median into a $2 : 1$ ratio.

Let there exist a triangle $\triangle ABC$, which has medians $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$. Then, take a homothety centered at G with scale factor -2. Then, we know that $\triangle DEF$ maps to $\triangle ABC$, so we know that

$$\frac{GD}{GA} = \frac{GE}{GB} = \frac{GF}{GC} = \frac{1}{2}$$

as desired. $\square$

Problem 3.13 (Euler Line)

In triangle ABC, prove that O, G, H (with their usual meanings) are collinear and that G divides OH in a $2 : 1$ ratio.

We begin with the following claim:

Claim

Let there exist a triangle ABC. The orthocenter of the medial triangle of $\triangle ABC$ is the circumcenter of $\triangle ABC$.

Proof. The perpendicular bisectors of $\triangle ABC$ are clearly the altitudes of its medial triangle. Hence, the intersection of the perpendicular bisectors of $\triangle ABC$, is the same as the intersection of the altitudes of the medial triangle. However, these are clearly the definition of the circumcenter and orthocenter respectively, so we have the desired conclusion. $\square$

Thus, taking a homothety centered at G with scale factor $-\frac{1}{2}$, $\triangle ABC$ maps to its medial triangle. In addition, H will map to the orthocenter of the medial triangle, which is the same as the circumcenter (O) of $\triangle ABC$ by the claim. Thus, O, G, and H are collinear, and $\frac{HG}{GO} = 2$, as required. $\square$

Problem 3.16 (Gergonne point)

Let ABC be a triangle with contact triangle DEF. Prove that $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ concur. The point of concurrency is the Gergonne point of triangle ABC.

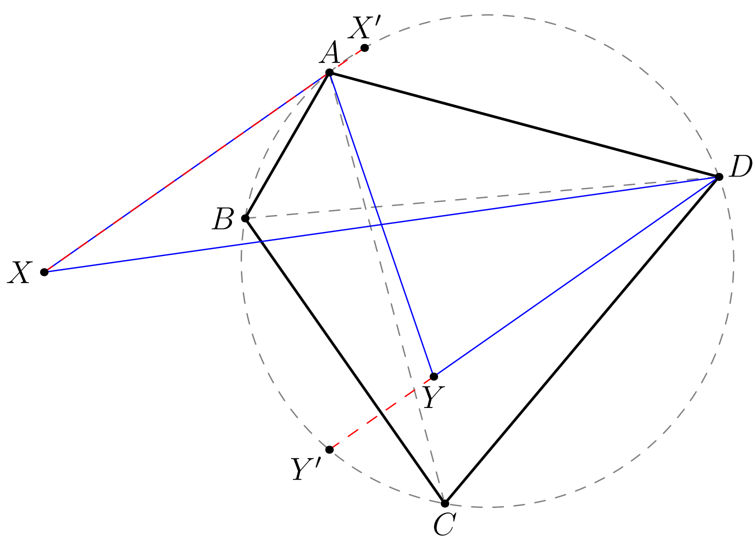
We can write that

$$\frac{AF}{FC} \cdot \frac{CD}{DB} \cdot \frac{BE}{EA} = \frac{AF}{FC} \cdot \frac{FC}{BD} \cdot \frac{BD}{AF} = 1.$$

Thus, by Ceva's Theorem, we know that they are concurrent, so we are done. $\square$

Problem 3.17

In cyclic quadrilateral $ABCD$, points X and Y are the orthocenters of $\triangle ABC$ and $\triangle BCD$. Show that $AXYD$ is a parallelogram.



Let X' be the reflection of X across line BC, and analogously for Y.

Claim

Quadrilateral $Y'AX'D$ is an isosceles trapezoid.

Proof. Since a trapezoid is cyclic if and only if it's isosceles, we just have to show that $Y'AX'D$ is a trapezoid. However, we know that $\overline{AX'} \parallel \overline{Y'D}$ because they are both perpendicular to $\overline{BC}$. Hence, we have the desired claim. $\square$

Now, by the claim, we know that

$$AD = X'Y' = XY.$$

In addition, we know that $XA \parallel YD$, so $AXYD$ is indeed a parallelogram, as desired. $\square$

Problem 3.18

Let $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ be concurrent cevians in a triangle, meeting at P. Prove that

$$\frac{PD}{AD} + \frac{PE}{BE} + \frac{PF}{CF} = 1.$$

We can write that:

$$\frac{PD}{AD} = \frac{[PBD]}{[PBA]} = \frac{[PCD]}{[PCA]} = \frac{[PBC]}{[ABC]}$$

where the last equality is by the identity $\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$. In fact, we can write similar statements for the other ratios as well. Hence, we get that

$$\frac{PD}{AD} + \frac{PE}{BE} + \frac{PF}{CF} = \frac{[PBC]}{[ABC]} + \frac{[PAC]}{[ABC]} + \frac{[PAB]}{[ABC]} = 1$$

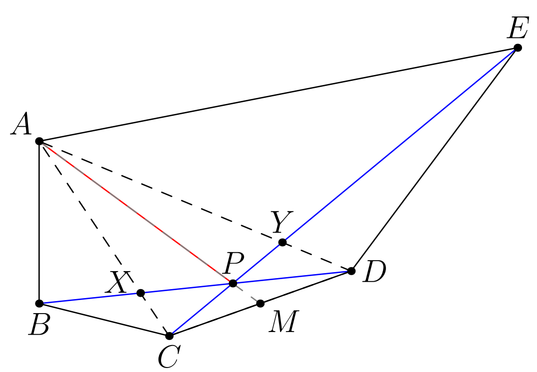
as required. $\square$

Problem 3.19 (ISL 2006/G3)

Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \qquad \text{and} \qquad \angle ABC = \angle ACD = \angle ADE.$$

The diagonals BD and CE meet at P. Prove that the line AP bisects the side CD.



Let M be the point at which ray AP intersects $\overline{CD}$. By Ceva's Theorem, we know that

$$\frac{AX}{XC} \cdot \frac{CM}{MD} \cdot \frac{DY}{YA} = 1.$$

However, because

$$ABCD \sim ACDE$$

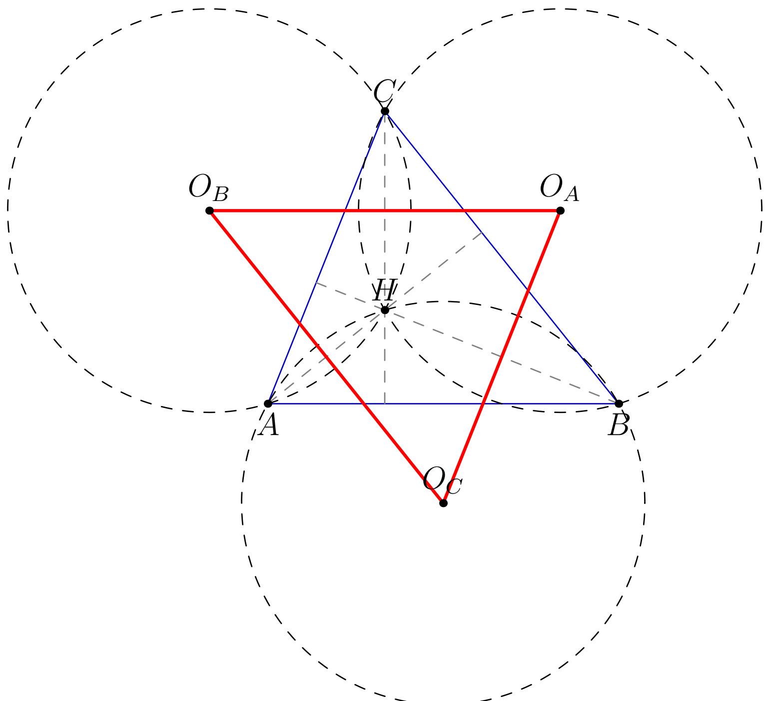
we know that

$$\frac{AX}{XC} = \frac{AY}{YD}.$$

Hence, we know that
$$\frac{CM}{MD} = 1$$
so M is indeed the midpoint of $\overline{CD}$, as required.

Problem 3.20 (BAMO 2013/3)

Let H be the orthocenter of an acute triangle ABC. Consider the circumcenters of triangles ABH, BCH, and CAH. Prove that they are the vertices of a triangle that is congruent to ABC.



Define points O_A, O_B, and O_C as described in the figure; the centers of the given circumcircles. Then, we have the following claim:

Claim

The circles (AHC), (AHB), and (BHC) are congruent.

Proof. We wish to show that they have equal radii. However, let H_B be the reflection of H across $\overline{AC}$. Then, we know that
$$(ABC) = (AH_BC) \cong (AHC).$$
Hence, all the required circumcircles are congruent to (ABC), proving the claim. $\square$

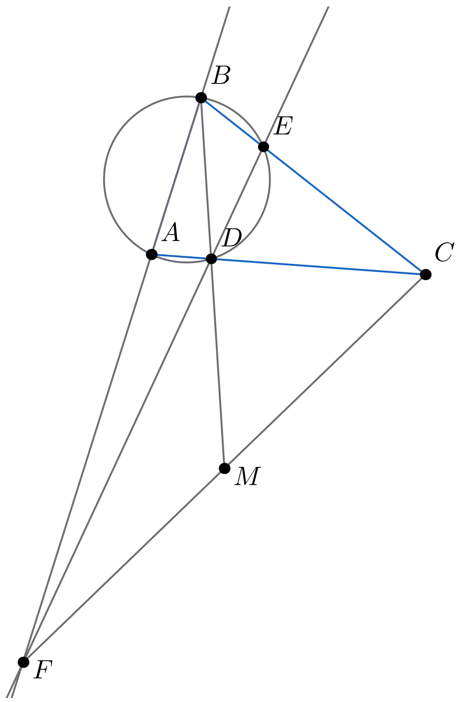
Now, take a homothety h with scale factor $\frac{1}{2}$ centered at H. Then, let $B' = h(B)$ and $C' = h(C)$. Then, we know that C' is the midpoint of $\overline{O_BO_A}$, and B' is the midpoint of $\overline{O_AO_C}$ because of the congruent circles. Hence, let g be a homothety centered at O_A with scale factor 2. Then, we know that $g(C')$ sends C' to O_B, and $g(B') = O_B$. Hence, we know that
$$O_BO_C = 2(\tfrac{1}{2}BC) = BC.$$

We get similar statements for the other sides, implying that

$$\triangle ABC \cong \triangle O_A O_B O_C.$$

Remark
Triangle $O_A O_B O_C$ is infact a rotation of $\triangle ABC$ around N_9.

Problem 3.21 (USAMO 2003/4)
Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Lines AB and DE intersect at F, while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.



We will start with the if direction. Since $\frac{MB}{MC} = \frac{MC}{MD}$, and they share an angle, we know that

$$\triangle MBC \sim \triangle MCD.$$

Now, we can write that

$$\measuredangle AEB = \measuredangle ADB = \measuredangle MDC = \measuredangle MCB = \measuredangle C.$$

Hence, we know that $AE \parallel FC$. This means that

$$\triangle BAE \sim \triangle BFC \implies \frac{BA}{BE} = \frac{BF}{BC} = \frac{AF}{EC}$$

due to the property that $\frac{a}{b} = \frac{c}{d} = \frac{c-a}{d-b}$. This means that

$$\frac{BA}{AF} \cdot \frac{FM}{MC} \cdot \frac{CE}{EB} = \frac{FM}{MC} = 1$$

by Ceva's Theorem. This is the necessary conclusion.

We now prove the only if direction. Since $MF = MC$, we know that

$$\frac{BA}{AF} \cdot \frac{CE}{EB} = 1$$

by Ceva. This implies that $AE \parallel FC$. Hence, we know that

$$\measuredangle C = \measuredangle MCB = \measuredangle AEB = \measuredangle ADB = \measuredangle MDC.$$

Hence, we know that $\triangle MBC \sim \triangle MCD$, implying that $\frac{MB}{MC} = \frac{MC}{MD}$, as required. $\square$

Problem 3.22 (Monge's Theorem)

Consider disjoint circles ω_1, ω_2, ω_3 in the plane, no two congruent. For each pair of circles, we construct the intersection of their common external tangents. Prove that these three intersections are collinear.

By Menelaus' Theorem, it suffices to show that the produt of the ratios of the circles centers to the point at which they intersect is -1. However, by similar triangles, this is equivalent to the product of the ratio of the radii, so we are done. $\square$

Problem 3.23 (Cevian Nest)

Let $\overline{AX}$, $\overline{BY}$, $\overline{CZ}$ be concurrent cevians of $\triangle ABC$. Let $\overline{XD}$, $\overline{YE}$, $\overline{ZF}$ be concurrent cevians in triangle XYZ. Prove that rays AD, BE, CF concur.

By Ceva's Theorem, we know that

$$\frac{ZD}{YD} \cdot \frac{YF}{FX} \cdot \frac{XE}{EZ} = 1.$$

In addition, by the Law of Sines, we can write that

$$\begin{aligned}
\frac{\sin(\angle ZAD)}{\sin(\angle YAD)} &= \frac{ZD}{YD} \\
\frac{\sin(\angle YCF)}{\sin(\angle XCF)} &= \frac{YF}{XF} \\
\frac{\sin(\angle XBE)}{\sin(\angle ZBE)} &= \frac{XE}{ZE}.
\end{aligned}$$

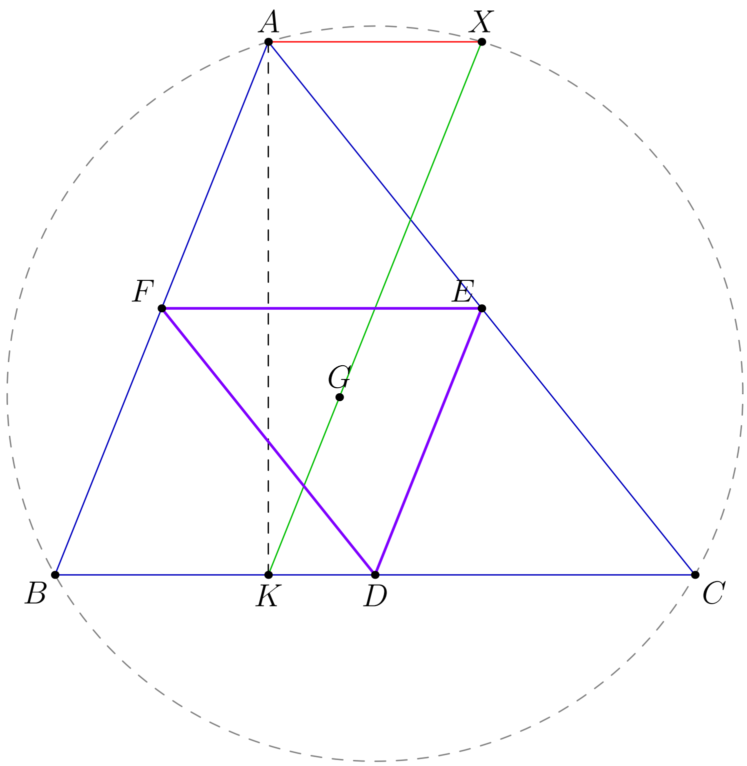
Multiplying these equations gives that:

$$\frac{\sin(\angle ZAD)}{\sin(\angle YAD)} \cdot \frac{\sin(\angle YCF)}{\sin(\angle XCF)} \cdot \frac{\sin(\angle XBE)}{\sin(\angle ZBE)} = \frac{ZD}{YD} \cdot \frac{YF}{FX} \cdot \frac{XE}{EZ} = 1.$$

So, by Trig Ceva, we know that the rays AD, BE, and CF concur, as required. $\square$

Problem 3.24

Let ABC be an acute triangle and suppose X is a point on (ABC) with $\overline{AX} \parallel \overline{BC}$ and $X \neq A$. Denote by G the centroid of triangle ABC, and by K the foot of the altitude from A to $\overline{BC}$. Prove that K, G, X are collinear.



Let h be the homothety centered at G with scale factor $-\frac{1}{2}$. This homothety will send $\triangle ABC$ to its medial triangle DEF. Note that G is both the centroid of $\triangle ABC$, and $\triangle DEF$. All that we need to show that is that K satisfies the same properties with respect to $\triangle DEF$ as X does with respect to $\triangle ABC$.

Claim

We claim that $\overline{DK} \parallel \overline{EF}$.

Proof. Clear, from the fact that X and D both lie on line BC and $BC \parallel \overline{EF}$. ☐

Claim

We claim that quadrilateral $DEFK$ is cyclic.

Proof. Clearly, we have that

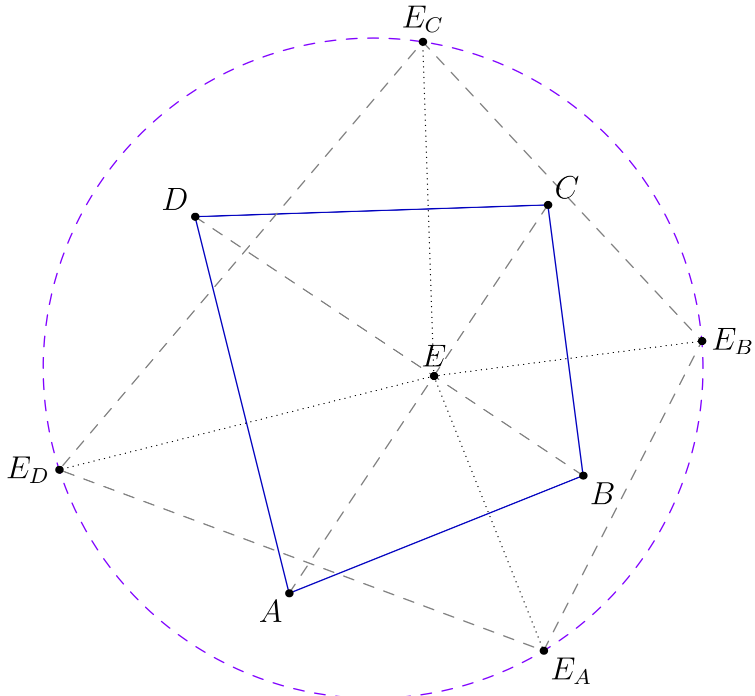
$$\measuredangle FKD = 90 + \measuredangle FKA = 90 + \measuredangle KAF = 90 + \measuredangle KAB = \measuredangle KBA = \measuredangle DBF = \measuredangle FED$$

as required. ☐

These two claims in conjunction, show that $h(X) = K$, so K, G, and X are collinear, as desired.

Problem 3.25 (USAMO 1993/2)

Let $ABCD$ be a quadrilateral whose diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular and intersect at E. Prove that the reflections of E across $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ are concyclic.



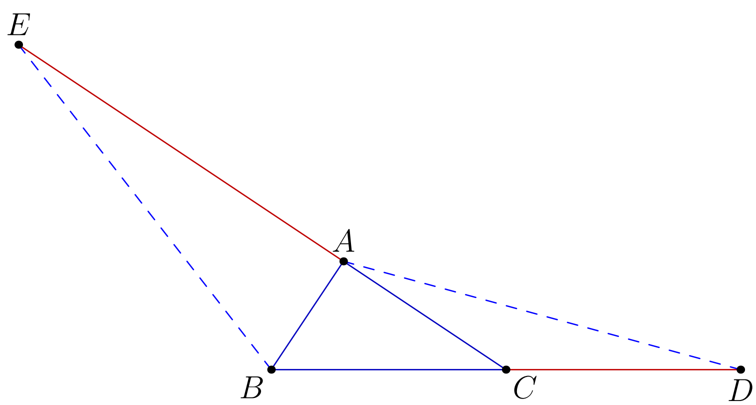
Define E_A, E_B, E_C, and E_D as shown in the diagram. Take a homothety h with scale factor $\frac{1}{2}$, centered at E, and let $h(E_A) = E'_A$ and similarly for the other points. Then, we know that:

$$\begin{aligned}
\angle E'_D E'_A E'_B + \angle E'_D E'_C E'_B &= \angle E'_D E'_A E + \angle E E'_A E'_B + \angle E_D E'_C E + \angle E E'_C E'_B \\
&= \angle E'_D AE + \angle E'_B BE + \angle E'_D DE + \angle E'_B CE \\
&= (\angle E'_D AE + \angle E'_D DE + \angle DEA) + (\angle E'_B BE + \angle E'_B CE + \angle CEB) \\
&\quad - \angle DEA - \angle CEB \\
&= 180 + 180 - 90 - 90 \\
&= 180
\end{aligned}$$

where the second line is due to the fact that the quadrilateral $EE'_D AE'_A$ is cyclic, as well as the other analogous quadrilaterals. This means that $E'_A E'_B E'_C E'_D$ is cyclic, so $E_A E_B E_C E_D$ is also cyclic, as desired. $\square$

Problem 3.26 (EGMO 2013/1, synthetic)

The side BC of the triangle ABC is extended beyond C to D so that $CD = BC$. The side CA is extended beyond A to E so that $AE = 2CA$. Prove that if $AD = BE$ then the triangle ABC is right-angled.



Notice that in $\triangle EBD$, $\overline{EC}$ is a median and the centroid of $\triangle EBD$ is A. Hence, line DA bisects $\overline{BE}$. Now, let M be the midpoint of $\overline{BE}$. Then, we know that

$$AM = \tfrac{1}{2}AD = \tfrac{1}{2}BE = BM = EM.$$

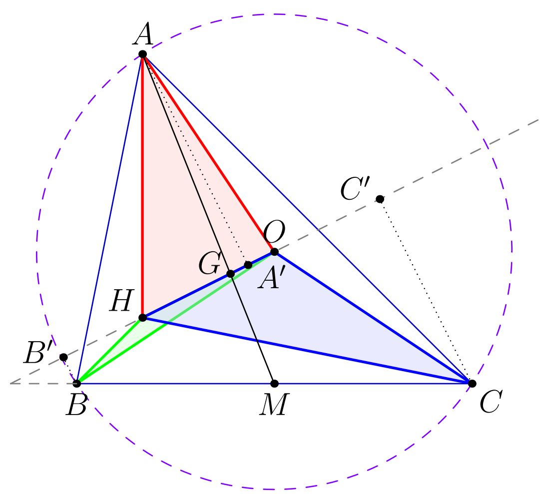
Hence, $\triangle BAE$ is right, implying that $\triangle ABC$ is also right, as required. $\square$

Remark

For a computational method, look to Problem 5.19 and for a barycentric bash, Problem 7.34.

Problem 3.27 (APMO 2004/4)

Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Prove that the area of one of the triangles AOH, BOH, and COH is equal to the sum of the areas of the other two.



Let G be the centroid of $\triangle ABC$, A' be the foot of the altitude from A to line HG, and similarly for B', and C'. In addition, let M be the midpoint of $\overline{BC}$. Then, WLOG let line HO intersect line

BC outside $\overline{BC}$. This means that we can write that

$$[BHO] + [CHO] = 2[MHO]$$

because M is the midpoint of that segment, hence the altitude to line HO will be the average of BB' and CC'. This means that

$$[BHO] + [CHO] = 2[MHO] = [AHO]$$

as required. □

Problem 3.28 (ISL 2001/G1)

Let A_1 be the center of the square inscribed in acute triangle ABC with two vertices of the square on side $\overline{BC}$. Thus one of the two remaining vertices of the square is on side $\overline{AB}$ and the other is on $\overline{AC}$. Points B_1 and C_1 are defined in a similar way for inscribed squares with two vertices on sides $\overline{AC}$ and $\overline{AB}$, respectively. Prove that lines AA_1, BB_1, CC_1 are concurrent.

For each of the A_1, B_1, and C_1, take a homothety centered at its respective point (for example for A_1, it would be A) sending one of the sides of its square to the side of the triangle opposite its respective point. Call the new centers A_2, B_2, and C_2 respectively. Then, we need to show that

$$\frac{\sin(\angle BAA_2)}{\sin(\angle CAA_2)} \cdot \frac{\sin(\angle ACC_2)}{\sin(\angle BCC_2)} \cdot \frac{\sin(\angle CBB_2)}{\sin(\angle ABB_2)} = 1.$$

However, by the Law of Sines, we know that

$$\frac{\sin(\angle BAA_2)}{BA_2} = \frac{\sin(\angle ABA_2)}{AA_2}$$

and

$$\frac{CA_2}{\sin(\angle CAA_2)} = \frac{AA_2}{\sin(\angle ACA_2)}.$$

Multiplying the two gives that

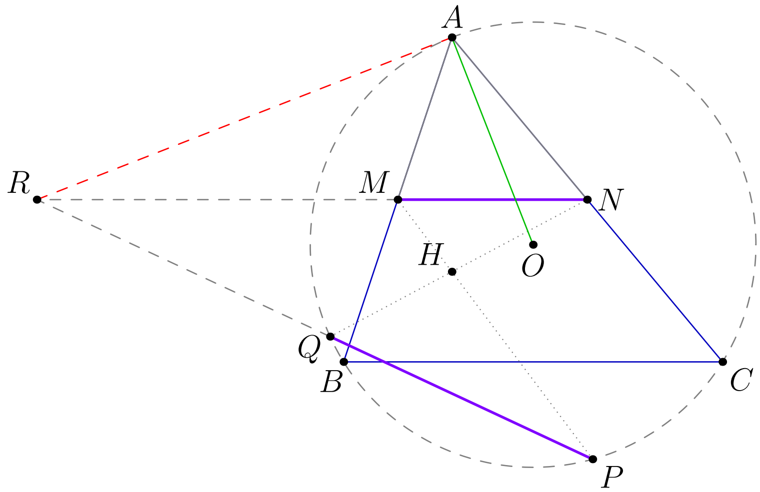
$$\frac{\sin(\angle BAA_2)}{\sin(\angle CAA_2)} = \frac{\sin(\angle ABA_2)}{\sin(\angle ACA_2)} = \frac{\sin(\angle B + 45)}{\sin(\angle C + 45)}.$$

We get similar equations for the other points, and multiplying them all gives that

$$\frac{\sin(\angle BAA_2)}{\sin(\angle CAA_2)} \cdot \frac{\sin(\angle ACC_2)}{\sin(\angle BCC_2)} \cdot \frac{\sin(\angle CBB_2)}{\sin(\angle ABB_2)} = \frac{\sin(\angle B + 45)\sin(\angle C + 45)\sin(\angle A + 45)}{\sin(\angle C + 45)\sin(\angle A + 45)\sin(\angle B + 45)} = 1$$

as required. □

Problem 3.29 (TSTST 2011/4)
Acute triangle ABC is inscribed in circle ω. Let H and O denote its orthocenter and circumcenter, respectively. Let M and N be the midpoints of sides $\overline{AB}$ and $\overline{AC}$, respectively. Rays MH and NH meet ω at P and Q, respectively. Lines MN and PQ meet at R. Prove that $\overline{OA} \perp \overline{RA}$.



We begin with the following claim:

Claim
Quadrilateral $MNPQ$ is cyclic.

Proof. Let the reflection of H across M and N be denoted X and Y respectively. Then, we know that

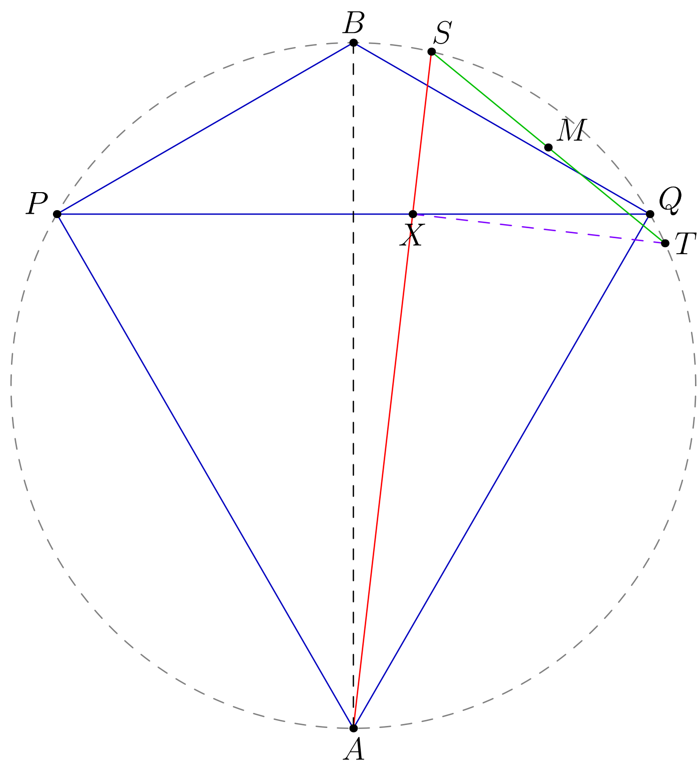
$$HQ \cdot HX = \tfrac{1}{2} HQ \cdot HN = \tfrac{1}{2} HP \cdot HM$$

so by the Converse of Power of a Point, we know that $MNPQ$ is cyclic, as required. $\square$

Now, consider the circles (ABC), (AMN), and $(MNPQ)$. Clearly, the radical center is R, and so because $\overline{AR}$ is tangent to (AMN), we know that $\overline{AR} \perp \overline{OA}$, as required. $\square$

Problem 3.30 (USAMO 2015/2, synthetic)

Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let X be a variable point on segment $\overline{PQ}$. Line AX meets ω again at S (other than A). Point T lies on arc AQB of ω such that XT is perpendicular to AX. Let M denote the midpoint of chord $\overline{ST}$. As X varies on segment $\overline{PQ}$, show that M moves along a circle.



Let N_9 be the nine-point circle of $\triangle AST$, and N be the midpoint of $\overline{AS}$. Then, we know that

$$\mathrm{Pow}_{N_9}(A) = AN \cdot AX = \frac{AS \cdot AX}{2} = \frac{AQ^2}{2}.$$

In addition since the radius of N_9 is constant (it is equal to $\frac{AO}{2}$), we know that the center of N_9 always lies on a circle centered at A. In addition, there exists a homothety taking N_9 to G centered at O (with scale factor $\frac{2}{3}$), so G also lies on some circle. Finally, there exists a homothety with scale factor $\frac{3}{2}$ sending G to M, centered at A. Hence, M always lies on a circle, as required. $\square$

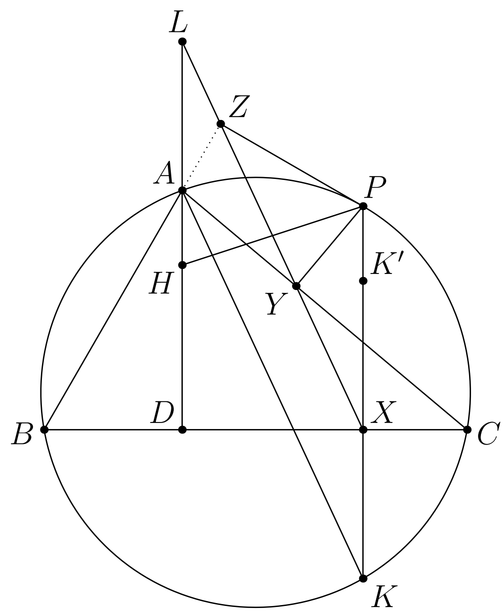
Remark

A complex bash can be found as Problem 6.35.

4 Assorted Configurations

Problem 4.1

Prove that the Simson line is parallel to $\overline{AK}$ in the notation of the figure below.



Clearly, $PYXC$ is cyclic. Hence,

$$\measuredangle AKP = \measuredangle ACP = \measuredangle YCP = \measuredangle YXP$$

so line $XY \parallel \overline{AK}$, and we are done. $\square$

Problem 4.2

Let K' be the reflection of K across $\overline{BC}$. Show that K' is the orthocenter of $\triangle PBC$.

Refer to the diagram above. The desired result follows from the fact that $PK' \perp BC$ and the reflection of K' lies on K. $\square$

Problem 4.3

Show that $LHXP$ is a parallelogram.

We have that

$$LH = LA + AH = XK + K'P = XK' + K'P = XP$$

and $LH \parallel PX$ so we are done. $\square$

Problem 4.5

Check $\angle IAI_B = 90°$ and $\angle IAI_C = 90°$ where I_B denotes the B-excenter and similarly for I_C.

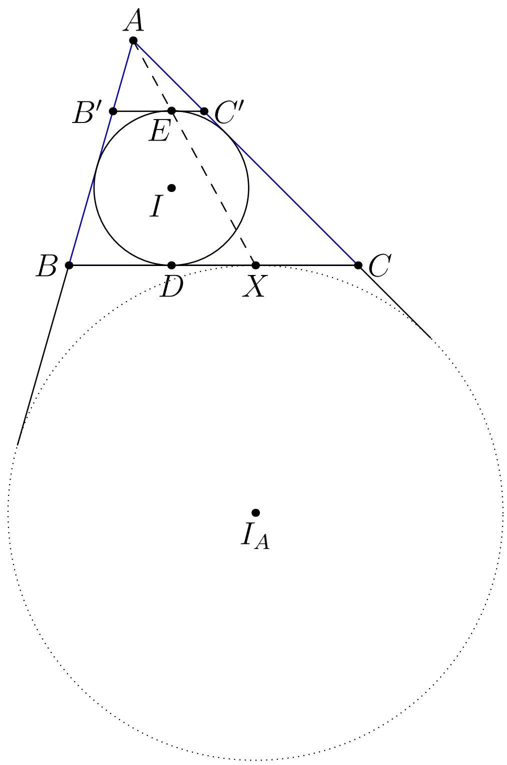
Since $I_C A$ bisects the external angle at A, we know that

$$\angle IAI_C = \angle I_C AB + \angle BAI = 90 - \frac{\angle A}{2} + \frac{A}{2} = 90°.$$

A similar computation gives the result for I_B, as required. □

Problem 4.8

Prove that A, E, and X are collinear and that $\overline{DE}$ is a diameter of the incircle.



Clearly, there exists a homothety at A sending $\triangle AB'C'$ to $\triangle ABC$. This homothety would also send E to X, so they are collinear, as required.

In addition, we know that E, I, D are collinear because:

$$\measuredangle EID = \measuredangle EC'C + \measuredangle C'CD = 0$$

since they $C'C$ is a transversal. Hence, we know that $\overline{DE}$ is a diameter of the incircle, so we are done. □

Problem 4.10

In the notation of the figure above, suppose $\overline{XY}$ is a diameter of the A-excircle. Show that D lies on $\overline{AY}$.

Clearly, there exists a homothety taking the incircle of $\triangle ABC$ to the excircle of $\triangle ABC$ centered at A. Hence, this homothety will take D to Y, so A, D, Y are collinear, implying the necessary claim. □

Problem 4.11

If M is the midpoint of $\overline{BC}$, prove that $\overline{AE} \parallel \overline{IM}$.

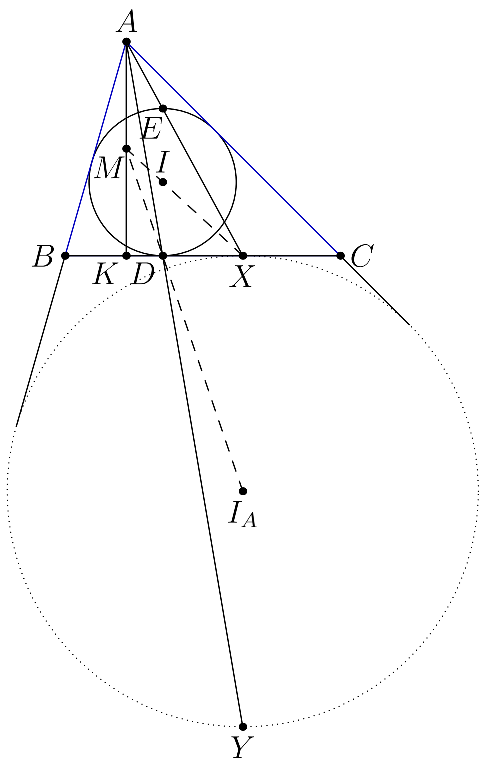
Obviously, there exists a homothety of scale factor 2 centered at D taking I to E and M to X. Hence,

$$\overline{IM} \parallel \overline{EX} \implies \overline{IM} \parallel \overline{AE}$$

as desired. $\square$

Problem 4.12

In the diagram below, prove that points X, I, M are collinear, if M is the midpoint of the altitude $\overline{AK}$.



Clearly, there exists a homothety sending $\triangle IDX$ to $\triangle MKX$ centered at X since they are similar. Hence, M, I, and X are collinear. $\square$

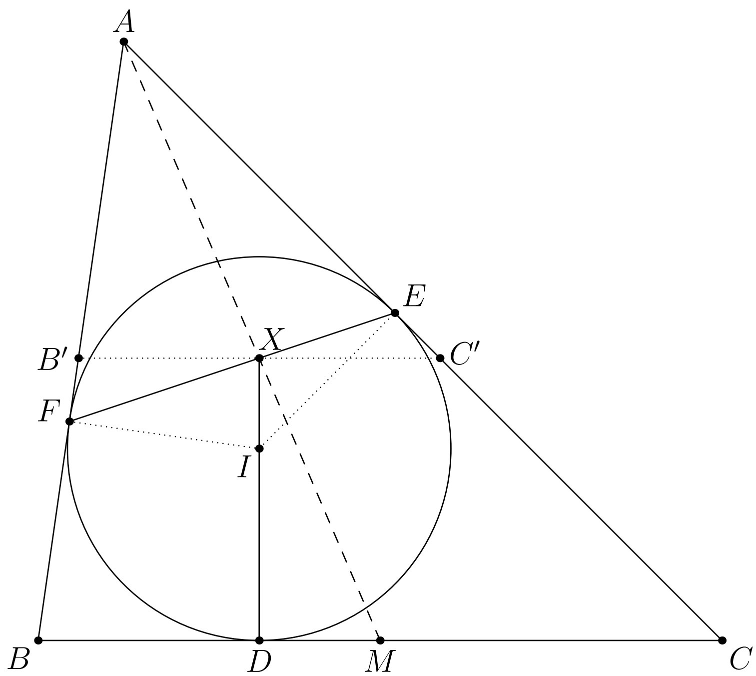
Problem 4.13

Show that D, I_A, M are collinear.

There exists a homothety centered at D sending $\triangle AKD$ to $\triangle YXD$. This homothety will also send M to I_A since they are both the midpoints of $\overline{AK}$ and $\overline{XY}$, respectively. Hence, M, D, and I_A are collinear. $\square$

Problem 4.15

Show that I must lie on $(AB'C')$.



Clearly, the feet of the altitudes from I are collinear, so by Simson Lines, we know that I lies on $(AB'C')$. $\square$

Problem 4.16

Prove that $XB' = XC'$.

Since $XB'FI$ and $XEC'I$ are cyclic, we can say that

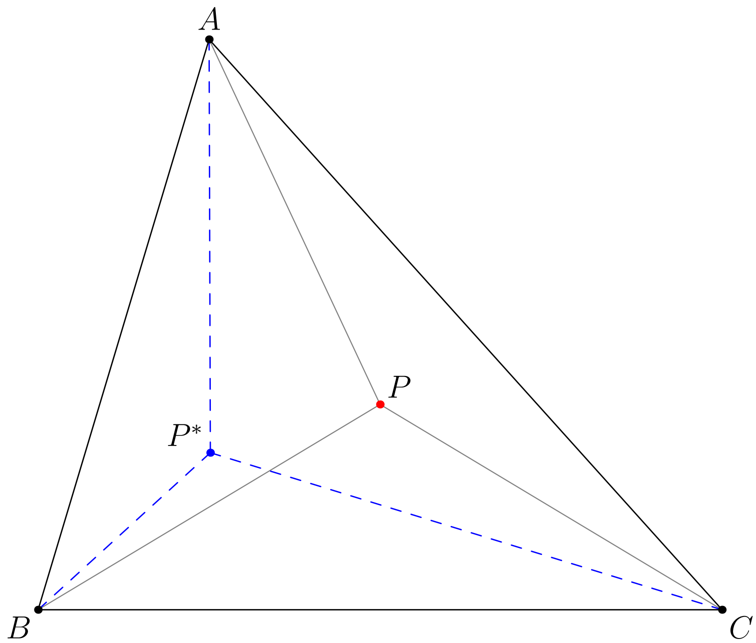
$$\measuredangle IB'X = \measuredangle IFX = \measuredangle IEX = \measuredangle IC'X.$$

Hence, $\triangle IB'X \cong \triangle IC'X$, from which the conclusion can easily be derived. $\square$

Problem 4.19

Show that if two of the angle relations below hold, then so does the third:

$$\measuredangle BAP = \measuredangle P^*AC, \quad \measuredangle CBP = \measuredangle P^*BA, \quad \measuredangle ACP = \measuredangle P^*CB.$$



WLOG say that we know the first two are true. By Trig Ceva, we know that

$$\frac{\sin(\measuredangle BAP^*)}{\sin(\measuredangle CAP^*)} \cdot \frac{\sin(\measuredangle ACP^*)}{\sin(\measuredangle BCP^*)} \cdot \frac{\sin(\measuredangle CBP^*)}{\sin(\measuredangle ABP^*)} = \frac{\sin(\measuredangle CAP)}{\sin(\measuredangle BAP)} \cdot \frac{\sin(\measuredangle ACP^*)}{\sin(\measuredangle BCP^*)} \cdot \frac{\sin(\measuredangle ABP)}{\sin(\measuredangle CBP)} = 1$$

and

$$\frac{\sin(\measuredangle BAP)}{\sin(\measuredangle CAP)} \cdot \frac{\sin(\measuredangle ACP)}{\sin(\measuredangle BCP)} \cdot \frac{\sin(\measuredangle CBP)}{\sin(\measuredangle ABP)} = 1.$$

Multiplying the two, we get that

$$\frac{\sin(\measuredangle ACP^*)}{\sin(\measuredangle BCP^*)} = \frac{\sin(\measuredangle BCP)}{\sin(\measuredangle ACP)}.$$

Now since they add to the same amount, and $\frac{\sin(a-x)}{\sin(x)}$ is monotonically decreasing, we must have $\measuredangle ACP^* = \measuredangle BCP$ and $\measuredangle BCP^* = \measuredangle ACP$, as required. $\square$

Problem 4.20

For a point P and triangle ABC, let X, Y, Z be the feet of the cevians through P. Let X' be the reflection of X about the midpoint of $\overline{BC}$ and define Y' and Z' similarly. Prove that the cevians $\overline{AX'}$, $\overline{BY'}$, and $\overline{CZ'}$ concur at a point P^t, the isotomic conjugate of P.

By Ceva's Theorem, we know that

$$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = 1.$$

This is equivalent to

$$1 = \frac{ZB}{AZ} \cdot \frac{XC}{BX} \cdot \frac{YA}{CY} = \frac{AZ'}{Z'B} \cdot \frac{BX'}{X'C} \cdot \frac{CY'}{Y'A}$$

so by Ceva's Theorem, they do concur. $\square$

Problem 4.21

Check that if Q is the isogonal conjugate of P, then P is the isogonal conjugate of Q.

The isogonal conjugate of P can be found by reflecting the cevians through P across the angle bisectors, so obviously repeating this again (finding Q^*) will take Q back to P. $\square$

Problem 4.22 (Isogonal Ratios)

Let D and E be points on $\overline{BC}$ so that $\overline{AD}$ and $\overline{AE}$ are isogonal. Then

$$\frac{BD}{DC} \cdot \frac{BE}{EC} = \left(\frac{AB}{AC}\right)^2.$$

By the Law of Sines, we know that

$$\frac{AB}{\sin(\angle BEA)} = \frac{BE}{\sin(\angle BAE)}$$

and

$$\frac{\sin(\angle AEC)}{AC} = \frac{\sin(\angle BEA)}{AC} = \frac{\sin(\angle EAC)}{EC}.$$

In addition, we have that

$$\frac{AB}{\sin(\angle BDA)} = \frac{BD}{\sin(\angle BAD)}$$

and

$$\frac{\sin(\angle ADC)}{AC} = \frac{\sin(\angle BDA)}{AC} = \frac{\sin(\angle DAC)}{DC}.$$

Multiplying the four equations together, we get that

$$\left(\frac{AB}{AC}\right)^2 = \frac{BD}{DC} \cdot \frac{BE}{EC} \cdot \frac{\sin(\angle EAC)\sin(\angle DAC)}{\sin(\angle BAE)\sin(\angle BAD)} = \frac{BD}{DC} \cdot \frac{BE}{EC} \cdot \frac{\sin(\angle BAD)\sin(\angle BAE)}{\sin(\angle BAE)\sin(\angle BAD)} = \frac{BD}{DC} \cdot \frac{BE}{EC}$$

as desired. $\square$

Problem 4.23

What is the isogonal conjugate of a triangle's circumcenter?

By Problem 1.7, it is the orthocenter. $\square$

Problem 4.25

Let X be the intersection of the tangents to (ABC) at B and C. In addition, let M be the intersection of the isogonal of $\overline{AX}$ on $\overline{BC}$. Show that

$$\frac{BM}{MC} = \frac{\sin(\angle B)\sin(\angle BAX)}{\sin(\angle C)\sin(\angle CAX)} = 1.$$

Due to the properties of tangents, we know that

$$\frac{\sin(\angle BAX)}{\sin(\angle CAX)} = \frac{\sin(\angle C)}{\sin(\angle B)}.$$

In addition, by the Law of Sines, we know that:

$$\frac{\sin(\angle BAM)}{BM} = \frac{\sin(\angle B)}{AM}$$
$$\frac{\sin(\angle CAM)}{CM} = \frac{\sin(\angle C)}{AM}.$$

Dividing the second equation by the first, we get that

$$\frac{BM\sin(\angle CAM)}{MC\sin(\angle BAM)} = \frac{BM\sin(\angle BAX)}{MC\sin(\angle CAX)} = \frac{\sin(\angle C)}{\sin(\angle B)}$$

as required. □

Problem 4.28

Let ABC be a triangle, and let the tangents to its circumcircle at B and C meet at point X. Let $\overline{AX}$ meet (ABC) again at K and $\overline{BC}$ at D. In addition, let M be the midpoint of $\overline{BC}$. Show that

$$\frac{AB}{BK} = \frac{AC}{CK}.$$

We know by angle chasing that

$$\triangle ABK \sim \triangle AMC$$

and

$$\triangle ACK \sim \triangle AMB.$$

Hence, we know that

$$\frac{AB}{BK} = \frac{AM}{MC} = \frac{AM}{MB} = \frac{AC}{CK}$$

so we are done. □

Problem 4.29

Use the configuration in Problem 4.28. Show that $\overline{BC}$ is the B-symmedian of $\triangle BAK$, and the C-symmedian of $\triangle CAK$.

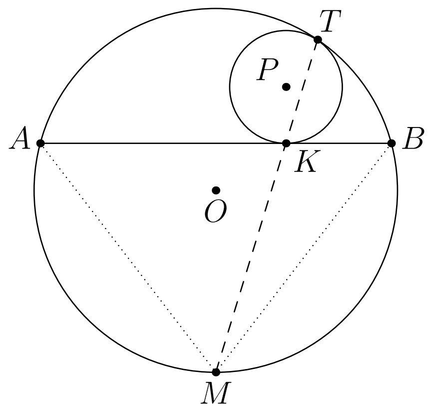
Clearly, we have that

$$\frac{BK}{KC} = \frac{BA}{AC}$$

and there can only be one point C that satisfies this on (ABC), so the first part is proved. The second part can easily be replicated with similar logic. □

Problem 4.31

Show that the homothety centered at T taking P to O, also takes K to M, and in particular that T , K, and M are collinear.



We know that

$$\triangle TPK \sim \triangle TOM$$

because they are both isosceles and share an angle. Hence, the same homothety taking P to O centered at T, takes K to M, as required. $\square$

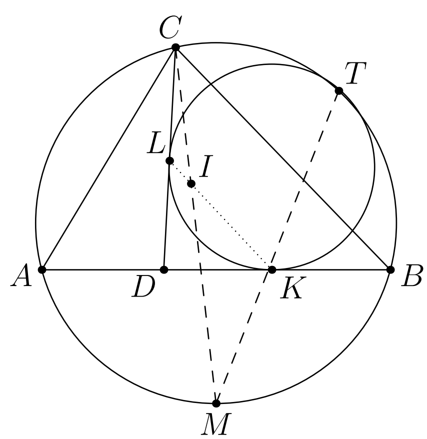
Problem 4.32

Show that $\triangle TMB \sim \triangle BMK$ in the diagram above.

Clearly, they share an angle, and we know that $\angle KBM = \angle BTM$ because of inscribed angles. Hence, they are similar, as required. $\square$

Problem 4.34

Prove that the points C, L, I, T are concyclic.



Consider the homothety centered at T sending the smaller circle to the bigger circle. We know that K will go to M, and so

$$\measuredangle TCM = \measuredangle TLK.$$

Hence,

$$\measuredangle TLI = \measuredangle TLK = \measuredangle TCM = \measuredangle TCI$$

implying that $CLIT$ is cyclic. $\square$

Problem 4.35

Show that $\triangle MKI \sim MIT$, in the diagram above.

They share an angle, and also

$$\measuredangle MIT = \measuredangle CIT = \measuredangle CLT = \measuredangle LKT = -\measuredangle MKI$$

so they are similar, and oppositely oriented.

Problem 4.37

Let ω_A refer to this A-mixtilinear circle. Let K and L the tangency points of ω_A on $\overline{AB}$ and $\overline{AC}$. Using the fact that I lies on $\overline{KL}$, check that I is in fact the midpoint of $\overline{KL}$.

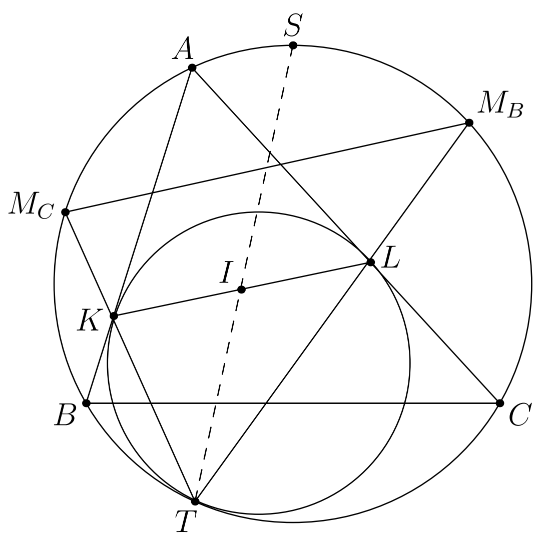
Because of the tangency, we know that

$$\triangle KIA \sim \triangle LIA.$$

Hence, we know that $KI = LI$, as desired. $\square$

Problem 4.38

Prove that $\angle ATK = \angle LTI$ in the diagram below.



We know that line TA is a symmedian of $\triangle KTL$ because of tangents at K and L intersect at A. Hence, lines TA and TI are isogonal, since $\overline{TI}$ is a median. This directly implies the necessary result due to the definition of isogonal. $\square$

Problem 4.39

Prove that S is the midpoint of the arc $\overset{\frown}{BC}$ containing A in the diagram above.

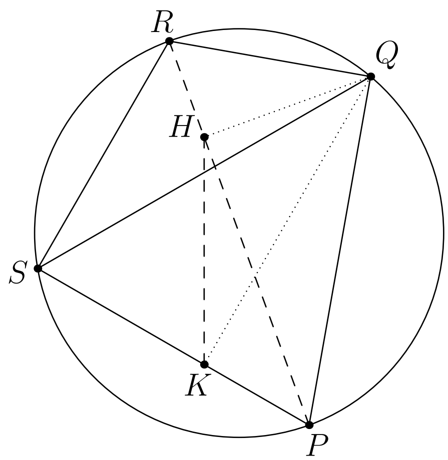
We know from Problem 4.34 that $BKIT$ is cyclic, and so is $CLIT$. Hence,

$$\measuredangle BTS = \measuredangle BTI = \measuredangle BKI = \measuredangle AKI = \measuredangle ALI = \measuredangle CLI = \measuredangle CTI = \measuredangle CTS.$$

Thus, we are done. $\square$

Problem 4.41 (Hong Kong 1998)

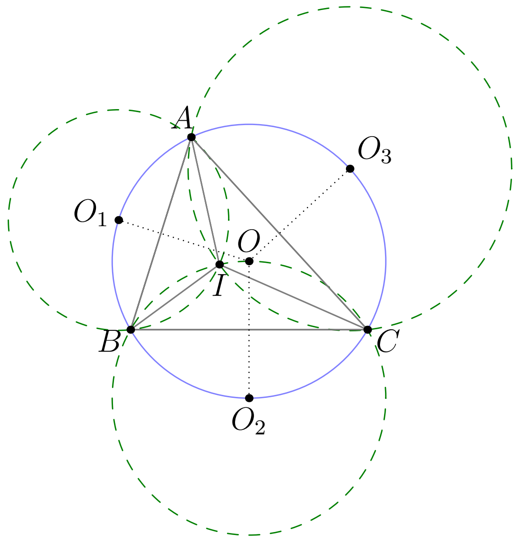
Let $PQRS$ be a cyclic quadrilateral with $\angle PSR = 90°$ and let H and K be the feet of the altitudes from Q to lines PR and PS. Prove that $\overline{HK}$ bisects $\overline{QS}$.



Since $\triangle PSR$ is right, we know that S is its orthocenter. Now since HK is a Simson Line, we know that $\overline{HK}$ bisects $\overline{QS}$ because of the Simson Lines properties. $\square$

Problem 4.42 (USAMO 1988/4)

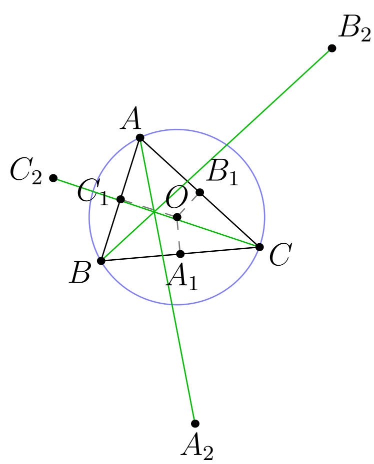
Suppose ABC is a triangle with incenter I. Show that the circumcenters of $\triangle IAB$, $\triangle IBC$, and $\triangle ICA$ lie on a circle whose center is the circumcenter of $\triangle ABC$.



Let the centers of the aforementioned circumcircles be denoted O_1, O_2, and O_3, as in the figure above. By the Incenter-Excenter Lemma, we know that these centers O_1, O_2, and O_3 lie on the midpoints of the arcs $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, and $\overset{\frown}{AC}$ respectively. Hence, they lie on (ABC), as required. $\square$

Problem 4.43 (USAMO 1995/3)

Given a nonisosceles, nonright triangle ABC, let O denote its circumcenter, and let A_1, B_1, and C_1 be the midpoints of sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Point A_2 is located on the ray $\overrightarrow{OA_1}$ so that $\triangle OAA_1$ is similar to $\triangle OA_2A$. Points B_2 and C_2 on rays $\overrightarrow{OB_1}$ and $\overrightarrow{OC_1}$, respectively, are defined similarly. Prove that lines AA_2, BB_2, and CC_2 are concurrent.



We begin with the following claim:

Claim
The lines AA_2, BB_2, and CC_2 are symmedians.

Proof. We will show that AA_2 is a symmedian. Similar methods can easily show the other two. By the similarity criterion, we know that

$$OA^2 = OA_1 \cdot OA_2.$$

Now, by the Law of Sines, we know that

$$\frac{OA_2}{\sin(\angle OCA_2)} = \frac{A_2C}{\sin(\angle A_2OC)}.$$

Substitution yields that

$$OA^2 = \frac{OA_1 \cdot A_2C}{\sin(\angle OCA_2)\sin(\angle A_2OC)} = \frac{OA_1 \cdot A_2C}{\sin(\angle OCA_2)\frac{A_1C}{OC}}.$$

Hence,

$$\frac{OA}{A_2C} = \frac{OC}{A_2C} = \frac{OA_1}{A_1C}.$$

Thus, we have that

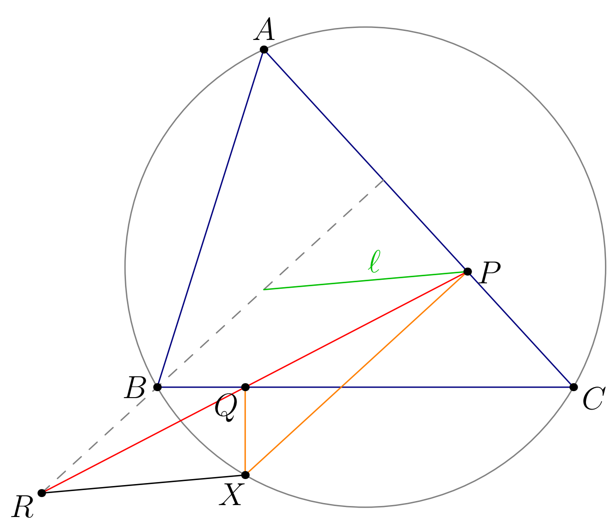
$$\triangle OA_1C \sim \triangle OCA_2.$$

This means that $\angle OCA_2 = 90°$ implying tangency. Hence, by the definition of symmedian, we can get that AA_2 is one. $\square$

We know that the symmedians concur at the symmedian point, so the existence of this point gives the required answer. $\square$

Problem 4.44 (USA TST 2014/1)

Let ABC be an acute triangle and let X be a variable interior point on the minor arc $\widehat{BC}$. Let P and Q be the feet of the perpendiculars from X to lines CA and CB, respectively. Let R be the intersection of line PQ and the perpendicular from B to $\overline{AC}$. Let ℓ be the line through P parallel to XR. Prove that as X varies along minor arc $\widehat{BC}$, the line ℓ always passes through a fixed point.



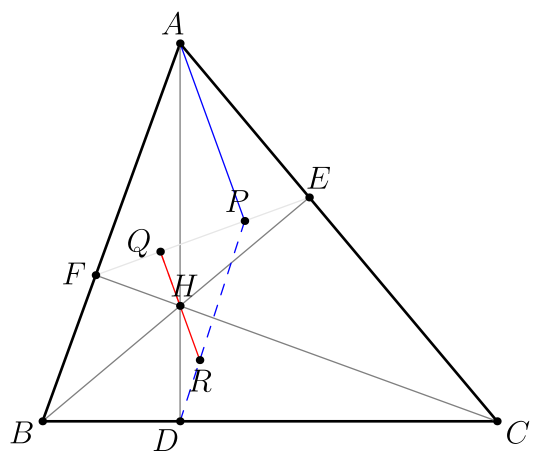
We claim the following:

> **Claim**
> The mentioned fixed point is H; the orthocenter of $\triangle ABC$.

Proof. Since PQ is a Simson Line, Simson Line properties tell us that $RXPH$ is a parallelogram, implying that the fixed point is indeed H. $\square$

Clearly, we have shown the existence of the fixed point by declaring what it is, so we are done. $\square$

> **Problem 4.45** (USA TST 2011/1)
> In an acute scalene triangle ABC, points D, E, F lie on sides BC, CA, AB, respectively, such that $AD \perp BC$, $BE \perp CA$, $CF \perp AB$. Altitudes AD, BE, CF meet at orthocenter H. Points P and Q lie on segment EF such that $AP \perp EF$ and $HQ \perp EF$. Lines DP and QH intersect at point R. Compute HQ/HR.

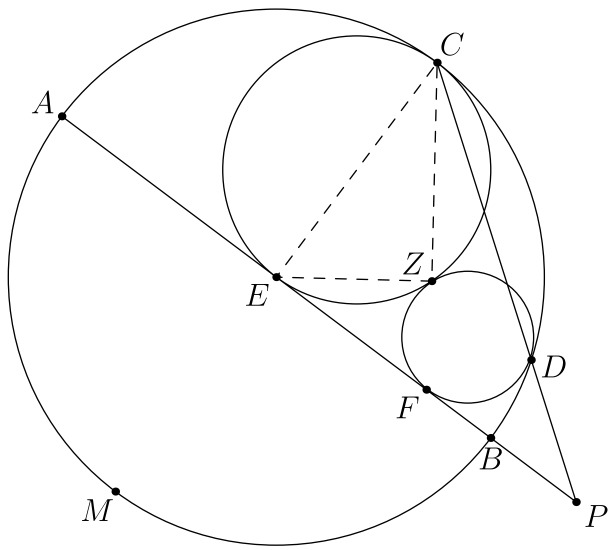


We claim that $\frac{HQ}{HR} = 1$. Note that H is the incenter of $\triangle DEF$, and Q is one of the contact points of the incircle. In addition, we know that A is the D-excenter of $\triangle DEF$, and P is the point of tangency of that circle with the triangle. Finally, we know that QR is a diameter of the incircle, because of the properties of the excircle. Hence, since H is the center of that incircle, we get the desired result. $\square$

Problem 4.46 (ELMO Shortlist 2012/G4)

Circles Ω and ω are internally tangent at point C. Chord AB of Ω is tangent to ω at E, where E is the midpoint of AB. Another circle, ω_1 is tangent to Ω, ω, and AB at D, Z, and F respectively. Rays CD and AB meet at P. If M is the midpoint of major arc AB, show that

$$\tan \angle ZEP = \frac{PE}{CM}.$$



Clearly, we can write that

$$\measuredangle ZEP = \measuredangle ZCE$$

due to the tangency. Hence, we just have to show that

$$\frac{EZ}{ZC} = \frac{PE}{CM}.$$

We will do this by showing that

$$\triangle CZM \sim \triangle EZP.$$

We already have one set of angles from the first line, thus we just need to show that

$$\angle CZM = \angle EZP \implies \angle CZE = 90^\circ = \angle PZM.$$

Let G be the point at which CP intersects ω_1. We know that

$$\triangle CPE \sim \triangle GPF$$

implying that there exists a homothety centered at P taking one triangle to the other. In addition, since CE and FG are both diameters, these homotheties also take the circles to each other. Hence, PZ passes through the centers of these circles. This means that

$$PZ \perp ZM$$

implying the necessary claim. $\square$

Problem 4.47 (USAMO 2011/5)

Let P be a given point inside quadrilateral $ABCD$. Points Q_1 and Q_2 are located within $ABCD$ such that

$$\angle Q_1BC = \angle ABP, \quad \angle Q_1CB = \angle DCP, \quad \angle Q_2AD = \angle BAP, \quad \angle Q_2DA = \angle CDP.$$

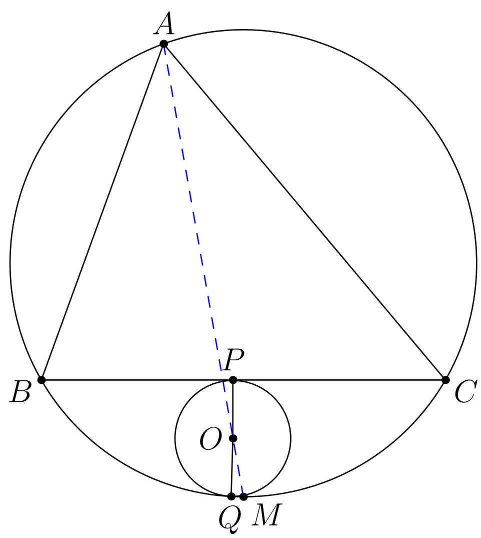
Prove that $\overline{Q_1Q_2} \parallel \overline{AB}$ if and only if $\overline{Q_1Q_2} \parallel \overline{CD}$.

If $AB \parallel CD$ then it is obvious.

Suppose that AB is not parallel to CD. Then, let R be the intersection of the two. Then, from the given conditions, we know that Q_1 and P are isogonal conjugates of $\triangle RBC$, and Q_2 and P are isogonal conjugates of $\triangle RAD$. Hence, Q_1, Q_2, and R are collinear, so neither of the statements can be true. $\square$

Problem 4.48 (Japan Olympiad 2009)

Triangle ABC is inscribed in circle ω. A circle with center O is drawn, tangent to side $\overline{BC}$ at a point P, and internally tangent to the arc BC not containing A at a point Q. Show that if $\angle BAO = \angle CAO$ then $\angle PAO = \angle QAO$.



We know that O must lie on the angle bisector of $\angle A$. In addition, extend AO to meet ω again at M. We know that M is the midpoint of arc BC.

Claim

Quadrilateral $APOQ$ is cyclic.

Proof. Let M_1 be the midpoint of arc BC containing A. Then, because of a homothety that sends both P to M_1 and O to the center of ω (call it O_1), we know that $PM_1 \parallel OO_1$.

Thus,

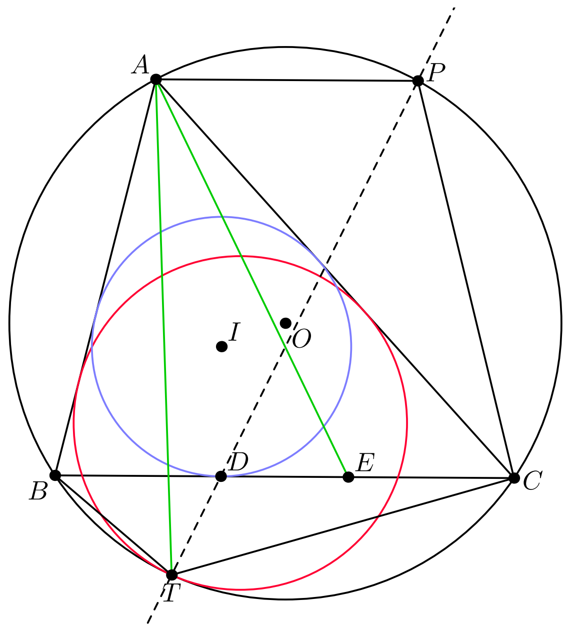
$$\measuredangle QAO = \measuredangle QAM = \measuredangle QM_1M = \measuredangle QM_1O = \measuredangle QPO.$$

$\square$

This implies that because $OP = OQ$, we have the desired result. $\square$

Problem 4.49

Let ABC be a triangle and let its incircle touch $\overline{BC}$ at D. Let T be the tangency point of the A-mixtilinear incircle with (ABC). Prove that $\angle BTA = \angle CTD$.



Let I be the incenter of $\triangle ABC$, E be a point on $\overline{BC}$ such that $BD = CE$, O be the circumcenter of $\triangle ABC$, and P be the reflection of A over the perpendicular bisector of $\overline{BC}$. In addition, define T' as the reflection of T over that same line. Then we can write that

$$\measuredangle EAC = \measuredangle BAT = \measuredangle T'AC$$

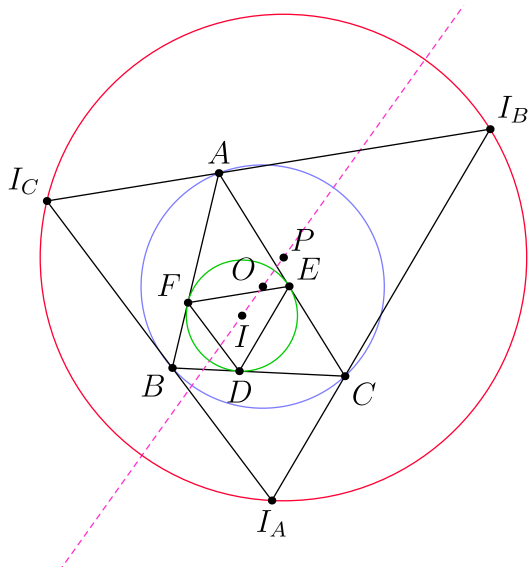
due to symmetry. Hence, A, E, and T' are collinear. In addition, by symmetry, we now know that arcs $AB = PC$, hence

$$\angle BTA = \angle BCA = \angle PBC = \angle PTC = \angle DTC$$

as required. $\square$

Problem 4.50 (Vietnam TST 2003/2)

Given a triangle ABC. Let O be the circumcenter of this triangle ABC. Let H, K, L be the feet of the altitudes of triangle ABC from the vertices A, B, C, respectively. Denote by A_0, B_0, C_0 the midpoints of these altitudes AH, BK, CL, respectively. The incircle of triangle ABC has center I and touches the sides BC, CA, AB at the points D, E, F, respectively. Prove that the four lines A_0D, B_0E, C_0F and OI are concurrent.



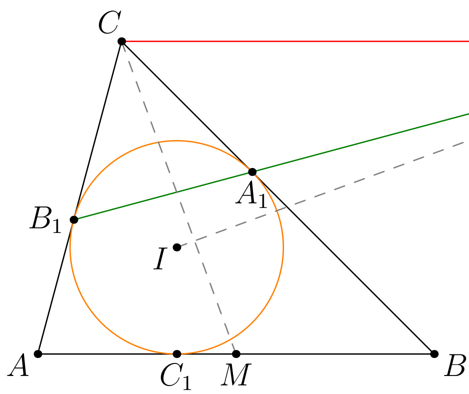
We know that the line A_0D passes through I_A (and likewise for the others), so we rename the lines to DI_A, EI_B, and FI_C. Now, note that $\triangle DEF$ and $\triangle I_AI_BI_C$ are homothetic because

$$\triangle DEF \sim \triangle I_AI_BI_C$$

and the sides are parallel to each other. Hence, the aforementioned lines concur. All that is left to show is that OI also concurs at this point. Now, note that (ABC) is the ninepoint circle of $\triangle I_AI_BI_C$. Hence, O is the ninepoint center, and I is the orthocenter of $\triangle I_AI_BI_C$. Finally, note that the center of homothety sends I to P, the circumcenter of $\triangle I_AI_BI_C$, and since IP is the Euler Line, we have the required result. $\square$

Problem 4.51 (Sharygin 2013)

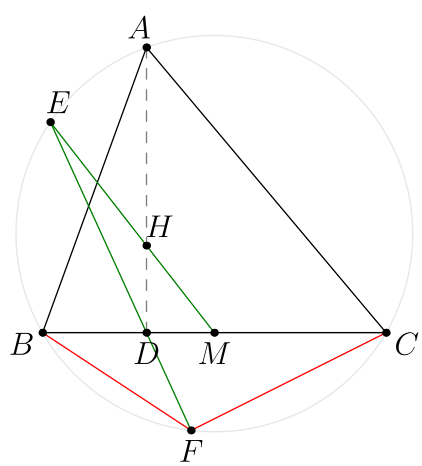
The incircle of triangle ABC touches $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at points A_1, B_1, C_1, respectively. The perpendicular from the incenter I to the median from vertex C meets the line A_1B_1 in point K. Prove that $CK \parallel AB$.



Let $H = CM \cap B_1K$. Then, we know that H is the orthocenter of $\triangle CIK$. However, since $HC_1 \perp AB$ and $IC \perp CK$, we have the required conclusion since it is well known that C_1, I, and H are collinear.

Problem 4.52 (APMO 2012/4)

Let ABC be an acute triangle. Denote by D the foot of the perpendicular line drawn from the point A to the side $\overline{BC}$, by M the midpoint of $\overline{BC}$, and by H the orthocenter of $\triangle ABC$. Let E be the point of intersection of the circumcircle Γ of the triangle ABC and the half line MH, and $F \neq E$ be the point of intersection of the line ED and the circle Γ. Prove that $\frac{BF}{CF} = \frac{AB}{AC}$.



We begin with the following claim:

Claim

Quadrilateral $AEDM$ is cyclic.

Proof. Consider the triangle formed by taking a homothety of scale factor two centered at H of

$\triangle HDM$. Clearly, this triangle is similar to $\triangle HEA$. Hence,

$$\triangle AEH \sim \triangle MDH$$

and we are done. $\square$

Because of this, we may now write that

$$\measuredangle AMC = \measuredangle AMD = \measuredangle AED = \measuredangle AEF = \measuredangle ABF$$

and

$$\measuredangle AFB = \measuredangle ACB = \measuredangle ACM.$$

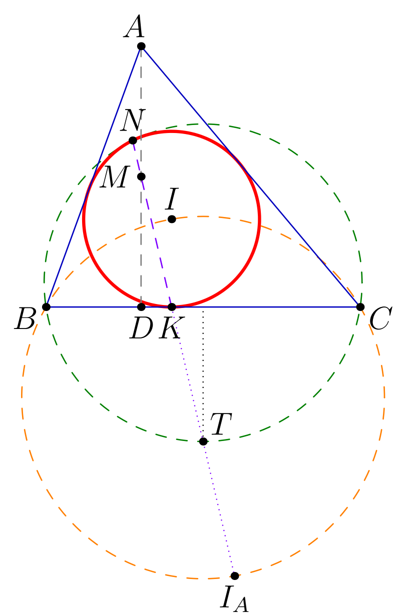
hence, this implies that

$$\triangle ABF \sim \triangle AMC$$

and since there can only be one point F on arc BC that satisfies this, we know that AF is a symmedian, implying the necessary result. $\square$

Problem 4.53 (ISL 2002/G7)

The incircle Ω of the acute-angled triangle ABC is tangent to its side $\overline{BC}$ at a point K. Let $\overline{AD}$ be an altitude of triangle ABC, and let M be the midpoint of the segment $\overline{AD}$. If $N \neq K$ is the common point of the circle Ω and the line KM, then prove that the incircle Ω and (BCN) are tangent to each other at the point N.



It is well known that M, K and I_A are collinear. Hence, define T to be the intersection of the perpendicular bisector of $\overline{BC}$ and $\overline{KI_A}$.

Claim

Quadrilateral $BNCT$ is cyclic.

Proof. Let r be the inradius and r_a the A-exradius. Then, we have that

$$KN \cdot KI_A = KI_A(KI_A + KN) - KI_A^2 = KI_A \cdot NI_A - KI_A^2 = \mathrm{Pow}_\Omega(I_A) - KI_A^2 = II_A^2 - r^2 - KI_A^2.$$

Now let X_A be the point of tangency of the A-excircle and $\overline{BC}$. Then, we can write that

$$KI_A^2 = KX_A^2 + X_AI_A^2 = (BC - 2BK)^2 + r_a^2$$

and

$$II_A^2 = KX_A^2 + (IK + XI_A)^2 = (BC - 2BK)^2 + (r + r_a)^2.$$

Hence,

$$KN \cdot KI_A = II_A^2 - r^2 - KI_A^2 = (r + r_a)^2 - r^2 - r_a^2 = 2rr_a.$$

This means that

$$KN \cdot KT = rr_a = (s - b)(s - c) = BK \cdot CK$$

implying the necessary result. $\square$

Finally, we can say that Ω and (BCT) are homothetic about N since T is the midpoint of arc BC and Ω is tangent to $\overline{BC}$ at K, implying the necessary conclusion. $\square$

5 Computational Geometry

Problem 5.5

Show that $[ABC] = sr$.

We have that

$$[ABC] = 2 \cdot \tfrac{r(s-a)}{2} + 2 \cdot \tfrac{r(s-b)}{2} + 2 \cdot \tfrac{r(s-c)}{2} = 3sr - r(a+b+c) = sr$$

as required. $\square$

Problem 5.6

In $\triangle ABC$ we have $AB = 13$, $BC = 14$, $CA = 15$. Find the length of the altitude from A onto $\overline{BC}$.

We may find that $[ABC] = 84$, and so the desired answer is $2 \cdot \frac{84}{14} = \boxed{12}$.

Problem 5.12

Complete the synthetic proof above of the stronger version of Ptolemy's theorem.

Let there be a cyclic quadrilateral $ABCD$ with $AB = a$, $BC = b$, $CD = c$, $DA = d$. Then by Ptolemy's, if $x = BD$, $y = AC$, and z is a third diagonal, we have that $xy = ac + bd$, $yz = ad + bc$, and $zx = ab + cd$. Hence,

$$x^2 = BD^2 = \frac{(ac+bd)(ab+cd)}{ad+bc}$$

and

$$y^2 = AC^2 = \frac{(ac+bd)(ad+bc)}{ab+cd}$$

as required. $\square$

Problem 5.16 (Star Theorem)

Let $A_1A_2A_3A_4A_5$ be a convex pentagon. Suppose rays A_2A_3 and A_4A_5 meet at X_1. Define X_2, X_3, X_4, X_5 similarly. Prove that

$$\prod_{i=1}^{5} X_iA_{i+2} = \prod_{i=1}^{5} X_iA_{i+3}$$

where indices are taken modulo 5.

By the Law of Sines, we know that

$$\frac{X_iA_{i+2}}{\sin(\angle X_iA_{i+3}A_{i+2})} = \frac{X_iA_{i+3}}{\sin(\angle X_iA_{i+2}A_{i+3})}.$$

In addition, note that

$$\angle X_iA_{i+3}A_{i+2} = \angle X_{i+1}A_{i+3}A_{i+4}.$$

Hence, taking the product of the first equation over all $i = 1, 2, \dots, 5$ gives the required result. $\square$

Problem 5.17

Let ABC be a triangle with inradius r. If the exradii of ABC are r_A, r_B, r_C, show that the triangle has area $\sqrt{r \cdot r_A \cdot r_B \cdot r_C}$.

We may write that:

$$\begin{aligned}
[ABC] &= \sqrt{s(s-a)(s-b)(s-c)} \\
&= \sqrt{\tfrac{[ABC]}{r} \cdot \tfrac{[ABC]}{r_a} \cdot \tfrac{[ABC]}{r_b} \cdot \tfrac{[ABC]}{r_c}} \\
&= [ABC]^2 \sqrt{\frac{1}{r \cdot r_a \cdot r_b \cdot r_c}} \\
\implies [ABC] &= \sqrt{r \cdot r_a \cdot r_b \cdot r_c}
\end{aligned}$$

so we are done. $\square$

Problem 5.18 (APMO 2013/1)

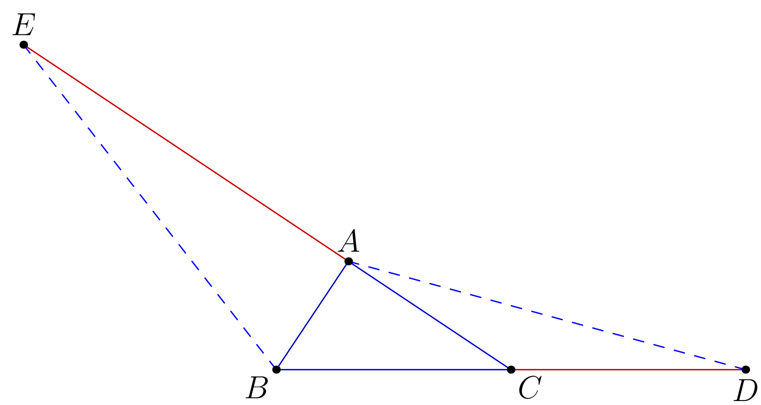
Let ABC be an acute triangle with altitudes AD, BE, and CF, and let O be the center of its circumcircle. Show that the segments OA, OF, OB, OD, OC, OE dissect the triangle ABC into three pairs of triangles that have equal areas.

We claim that $[AOE] = [BOD]$. As required,

$$\begin{aligned}
[AOE] &= \tfrac{1}{2} \cdot \sin(\angle OAE) \cdot OA \cdot AE \\
&= \tfrac{1}{2} \cdot \sin(\angle BAD) \cdot OB \cdot AE \\
&= \tfrac{1}{2} \cdot \tfrac{BD}{AB} \cdot OB \cdot AE \\
&= \tfrac{1}{2} \cdot \tfrac{AE}{AB} \cdot OB \cdot BD \\
&= \tfrac{1}{2} \cdot \sin(\angle EBA) \cdot OB \cdot BD \\
&= \tfrac{1}{2} \cdot \sin(\angle OBD) \cdot OB \cdot BD \\
&= [BOD]
\end{aligned}$$

where the angle equalities follow from the fact that AO and AH (the orthocenter of $\triangle ABC$) are isogonal. Similarly, we get that $[AFO] = [CDO]$, and $[BFO] = [COE]$. $\square$

Problem 5.19 (EGMO 2013/1, computational)
The side BC of the triangle ABC is extended beyond C to D so that $CD = BC$. The side CA is extended beyond A to E so that $AE = 2CA$. Prove that if $AD = BE$ then the triangle ABC is right-angled.



By the Law of Cosines, we have that

$$\frac{EC^2 + BC^2 - BE^2}{2 \cdot EC \cdot BC} = \cos(\angle ACB)$$

and also

$$\frac{AD^2 - AC^2 - CD^2}{2 \cdot AC \cdot CD} = \cos(\angle ACB).$$

Equating the two, we get that

$$\begin{aligned}
\frac{EC^2 + BC^2 - BE^2}{2 \cdot EC \cdot BC} &= \frac{AD^2 - AC^2 - CD^2}{2 \cdot AC \cdot CD} \\
\frac{EC^2 + BC^2 - BE^2}{3} &= AD^2 - AC^2 - CD^2 \\
9AC^2 + BC^2 - AD^2 &= 3(AD^2 - AC^2 - BC^2) \\
12AC^2 &= 4AD^2 - 4BC^2 \\
3AC^2 &= AD^2 - BC^2
\end{aligned}$$

However by Stewarts Theorem, we have that

$$AB^2 + AD^2 = 2BC^2 + 2AC^2.$$

Thus, adding this to the equation above, we have that

$$3AC^2 + AB^2 + AD^2 = AD^2 + BC^2 + 2AC^2 \implies AC^2 + AB^2 = BC^2$$

so we are done. □

Remark
The synthetic method can be found in Problem 3.26, and a barycentric bash as Problem 7.34.

Problem 5.20 (HMMT 2013)
Let triangle ABC satisfy $2BC = AB + AC$ and have incenter I and circumcircle ω. Let D be the intersection of AI and ω (with A, D distinct). Prove that I is the midpoint of $\overline{AD}$.

By Ptolemy, we have that

$$AC \cdot BD + AB \cdot DC = AD \cdot BC.$$

However, by the Incenter-Excenter Lemma, we know that

$$BD = CD = ID.$$

Hence,

$$\begin{aligned} AD \cdot BC &= AC \cdot BD + AB \cdot DC \\ &= AC \cdot ID + AB \cdot ID \\ &= 2 \cdot BC \cdot ID \\ \implies AD &= 2ID \end{aligned}$$

so we are done. $\square$

Problem 5.21 (USAMO 2010/4)
Let ABC be a triangle with $\angle A = 90^\circ$. Points D and E lie on sides AC and AB, respectively, such that $\angle ABD = \angle DBC$ and $\angle ACE = \angle ECB$. Segments BD and CE meet at I. Determine whether or not it is possible for segments AB, AC, BI, ID, CI, IE to all have integer lengths.

Assume for the sake of contradiction that it is possible. We know that

$$\angle BIC = 90 + \frac{\angle A}{2} = 135.$$

Now by the Law of Cosines, we know that

$$\begin{aligned} BI^2 + CI^2 - 2BI \cdot CI \cos(\angle BIC) &= BI^2 + CI^2 + BI \cdot CI\sqrt{2} \\ &= BC^2 \\ &= AC^2 + AB^2. \end{aligned}$$

However, in order for

$$BI^2 + CI^2 + BI \cdot CI\sqrt{2} = AC^2 + AB^2$$

we cannot have all of them be integer, contradiction. Thus, it is not possible for all of them to have integer lengths. $\square$

Problem 5.22 (Iran Olympiad 1999)
Let I be the incenter of triangle ABC and let ray AI meet the circumcircle of $\triangle ABC$ at D. Denote the feet of the perpendiculars from I to lines BD and CD by E and F, respectively. If $IE + IF = \frac{1}{2}AD$, calculate $\angle BAC$.

We have that

$$IE = ID\sin(\angle C)$$

and

$$IF = ID\sin(\angle B).$$

Hence, since $\sin(\angle B) = \frac{AC}{2R}$ and $\sin(\angle C) = \frac{AB}{2R}$ we may write that

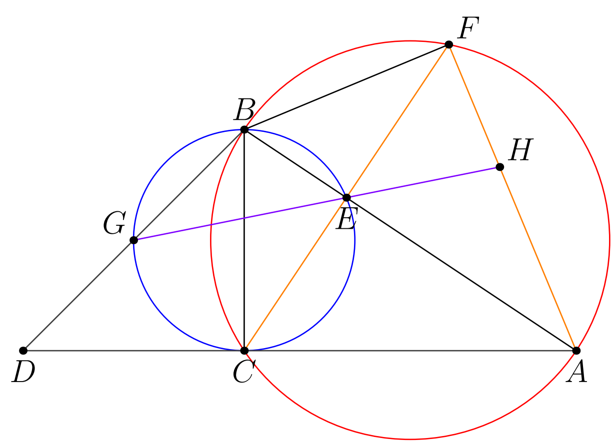
$$\begin{aligned} IE + IF &= ID\sin(\angle C) + ID\sin(\angle B) \\ &= \frac{AB \cdot CD}{2R} + \frac{AC \cdot BD}{2R} \\ &= \frac{AD \cdot BC}{2R} \\ &= \tfrac{1}{2}AD. \end{aligned}$$

Hence, $BC = R$, implying that $\angle BOC = 60^\circ$ if O is the circumcenter of $\triangle ABC$. Finally, Inscribed Angle Theorem tells us that $\angle BAC = \boxed{30^\circ}$ or $\boxed{150^\circ}$ as required. □

Problem 5.23 (CGMO 2002/4)

Circles O_1 and O_2 intersect at two points B and C, and $\overline{BC}$ is the diameter of circle O_1. Construct a tangent line of circle O_1 at C and intersecting circle O_2 at another point A. We join AB to intersect circle O_1 at point E, then join CE and extend it to intersect circle O_2 at point F. Assume H is an arbitrary point on line segment AF. We join HE and extend it to intersect circle O_1 at point G, and then join BG and extend it to intersect the extend line of AC at point D. Prove that

$$\frac{AH}{HF} = \frac{AC}{CD}.$$



We can write that

$$\frac{AH}{HF} = \frac{AE\sin(\angle AEH)}{EF\sin(\angle FEH)} = \frac{EC\sin(\angle BEG)}{EB\sin(\angle CEG)} = \frac{EC \cdot BG}{EB \cdot CG}.$$

In addition,

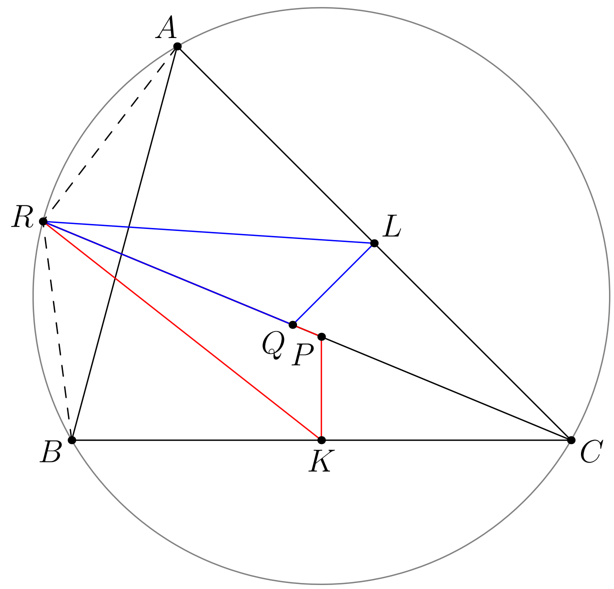
$$AC = \frac{BC \cdot CE}{BE}$$

and

$$CD = \frac{GC \cdot BC}{BG}$$

by similar triangles, and dividing gives the necessary result. □

Problem 5.24 (IMO 2007/4)
In triangle ABC the bisector of angle BCA intersects the circumcircle again at R, the perpendicular bisector of $\overline{BC}$ at P, and the perpendicular bisector of $\overline{AC}$ at Q. The midpoint of $\overline{BC}$ is K and the midpoint of $\overline{AC}$ is L. Prove that the triangles RPK and RQL have the same area.



We can write that

$$\frac{[RPK]}{[RQL]} = \frac{\frac{1}{2}\cdot RP \cdot PK \sin(\angle RPK)}{\frac{1}{2}\cdot RQ \cdot QL \sin(\angle RQL)} = \frac{RP \cdot PK}{RQ \cdot QL} = \frac{RP \cdot BC}{RQ \cdot AC}.$$

Now, since $\triangle ARB$ is isosceles, we know that

$$RA = \frac{AB}{2\cos\left(\frac{\angle C}{2}\right)}.$$

Hence, by Ptolemy, we have that

$$RA \cdot BC + RB \cdot AC = RA(BC + AC) = RC \cdot AB.$$

So,

$$RC = \frac{BC + AC}{2\cos\left(\frac{\angle C}{2}\right)}.$$

Finally,

$$\begin{aligned}
\frac{RP}{RQ} &= \frac{RC - PC}{RC - QC} \\
&= \frac{\frac{BC + AC}{2\cos\left(\frac{\angle C}{2}\right)} - \frac{KC}{\cos\left(\frac{\angle C}{2}\right)}}{\frac{BC + AC}{2\cos\left(\frac{\angle C}{2}\right)} - \frac{LC}{\cos\left(\frac{\angle C}{2}\right)}} \\
&= \frac{\frac{BC + AC}{2\cos\left(\frac{\angle C}{2}\right)} - \frac{BC}{2\cos\left(\frac{\angle C}{2}\right)}}{\frac{BC + AC}{2\cos\left(\frac{\angle C}{2}\right)} - \frac{AC}{2\cos\left(\frac{\angle C}{2}\right)}} \\
&= \frac{\frac{AC}{2\cos\left(\frac{\angle C}{2}\right)}}{\frac{BC}{2\cos\left(\frac{\angle C}{2}\right)}} \\
&= \frac{AC}{BC}.
\end{aligned}$$

Hence,

$$\frac{[RPK]}{[RQL]} = \frac{RP \cdot BC}{RQ \cdot AC} = \frac{AC \cdot BC}{BC \cdot AC} = 1$$

so we are done. □

Problem 5.25 (JMO 2013/5)

Quadrilateral $XABY$ is inscribed in the semicircle ω with diameter XY. Segments AY and BX meet at P. Point Z is the foot of the perpendicular from P to line XY. Point C lies on ω such that line XC is perpendicular to line AZ. Let Q be the intersection of segments AY and XC. Prove that

$$\frac{BY}{XP} + \frac{CY}{XQ} = \frac{AY}{AX}.$$

Let $\angle AYX = \alpha$ and $\angle BXY = \beta$. Then, we know that $\triangle BXY \sim \triangle ZXP$, so by the Law of Sines,

$$\frac{BY}{XP} = \frac{XY\sin(\beta)}{XP} = \frac{XY\sin(\beta)}{\frac{XY\sin(\alpha)}{\sin(\alpha+\beta)}} = \frac{\sin(\alpha+\beta)\sin(\beta)}{\sin(\alpha)}.$$

Now, we know that $AXPZ$ is cyclic, so

$$\begin{aligned}
\angle CXY &= \angle AXY - \angle AXC \\
&= 90 - \alpha - (90 - \angle XAZ) \\
&= 90 - \alpha - (90 - \angle XPZ) \\
&= 90 - \alpha - (90 - (90 - \beta)) \\
&= 90 - \alpha - \beta.
\end{aligned}$$

Hence,

$$CY = XY\sin(\angle CXY) = XY\sin(90 - \alpha - \beta) = XY\cos(\alpha + \beta).$$

In addition,

$$\frac{XQ}{\sin(\alpha)} = \frac{XY}{\sin(90+\beta)} = \frac{XY}{\cos(\beta)}$$

by the Law of Sines. Hence,

$$\frac{CY}{XQ} = \frac{XY\cos(\alpha+\beta)}{\frac{XY\sin(\alpha)}{\cos(\beta)}} = \frac{\cos(\alpha+\beta)}{\frac{\sin(\alpha)}{\cos(\beta)}} = \frac{\cos(\alpha+\beta)\cos(\beta)}{\sin(\alpha)}.$$

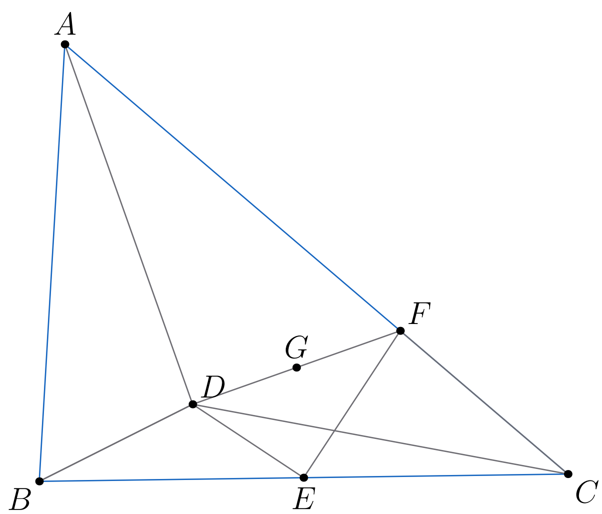
Finally, we can write that:

$$\begin{aligned}\frac{BY}{XP} + \frac{CY}{XQ} &= \frac{\sin(\alpha+\beta)\sin(\beta)}{\sin(\alpha)} + \frac{\cos(\alpha+\beta)\cos(\beta)}{\sin(\alpha)} \\ &= \frac{\sin(\alpha+\beta)\sin(\beta)+\cos(\alpha+\beta)\cos(\beta)}{\sin(\alpha)} \\ &= \frac{\cos(\alpha+\beta-\beta)}{\sin(\alpha)} \\ &= \tan(\alpha) \\ &= \frac{AY}{AX}\end{aligned}$$

as required. □

Problem 5.26 (CGMO 2007/5)

Point D lies inside triangle ABC such that $\angle DAC = \angle DCA = 30^\circ$ and $\angle DBA = 60^\circ$. Point E is the midpoint of segment $\overline{BC}$. Point F lies on segment $\overline{AC}$ with $AF = 2FC$. Prove that $\overline{DE} \perp \overline{EF}$.



Let G be the midpoint of $\overline{DF}$. Scale the figure down so that $AC = 3$. Then, we have that $AD = CD = \sqrt{3}$. In addition, since $AF = 2$ as well, we know that $\triangle ADF$ is right, and $DF = 1$.

Claim

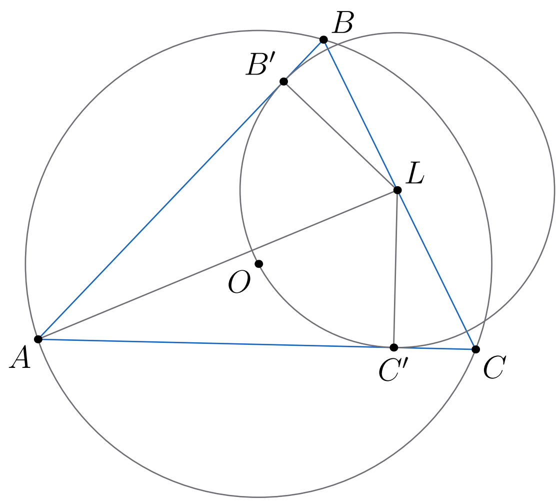
Quadrilateral $ODFC$ is a parallelogram.

Proof. Let O be the circumcenter of $\triangle ABD$. Then, $\angle ODC = 150^\circ$ and since $\angle ACD = 30^\circ$, we know that $OD \parallel FC$. In addition, we know that $OD = FC = 1$, so $ODFC$ is a parallelogram, as required. □

Hence, since G is the midpoint of DF and OC, we know that $GE = \frac{BO}{2} = 0.5$, and since $DG = DF = 0.5$, we are done. $\square$

Problem 5.27 (ISL 2011/G1)

Let ABC be an acute triangle. Let ω be a circle whose center L lies on the side $\overline{BC}$. Suppose that ω is tangent to $\overline{AB}$ at B' and $\overline{AC}$ at C'. Suppose also that the circumcenter O of triangle ABC lies on the shorter arc BC of ω. Prove that the circumcircle of $\triangle ABC$ and ω meet at two points.



We rephrase into the following claim:

Claim
We claim that
$$OL > \tfrac{1}{2}R.$$

Proof. Let the midpoint of $\overline{BC}$ be M. Then, we will show that

$$OL \geq OM > \tfrac{1}{2}R.$$

We know that $OM = \sqrt{R^2 - \frac{1}{4}BC^2}$ by the Pythagorean Theorem. Now, since

$$\angle B'OC' = 180 - \frac{\angle B'LC'}{2} = 90 + \frac{\angle A}{2} > \angle BOC = 2\angle A$$

Hence, $\angle A < 60°$.

We will now try to find the maximum value of $\frac{BC}{R}$. By the Law of Sines, we know that $\frac{BC}{R} = 2\sin(A)$, so

$$\max(\tfrac{BC}{R}) = \max(2\sin(A)) = 2\sin(60) = \sqrt{3}.$$

Hence,

$$\begin{aligned}
BC^2 &< 3R^2 \\
\tfrac{R^2}{4} + \tfrac{BC^2}{4} &< R^2 \\
\tfrac{R^2}{4} &< R^2 - \tfrac{BC^2}{4} \\
\tfrac{1}{2}R &< \sqrt{R^2 - \tfrac{1}{4}BC^2} \\
\tfrac{1}{2}R &< OM
\end{aligned}$$

so we have proved the claim. □

With this, it is clear to see that this directly implies the required result. □

Problem 5.28 (IMO 2001/1)
Consider an acute-angled triangle ABC. Let P be the foot of the altitude of triangle ABC issuing from the vertex A, and let O be the circumcenter of triangle ABC. Assume that $\angle C \geq \angle B + 30°$. Prove that $\angle A + \angle COP < 90°$.

We begin with a claim.

Claim
We claim that $CP < PO$.

Proof. We know that $CP = AC\cos(C)$. In addition, by the Law of Cosines,

$$OP^2 = R^2 + CP^2 - 2R \cdot CP\cos(90 - A) = R^2 + AC^2\cos^2(C) - 2R \cdot AC\cos(C)\cos(90 - A).$$

Thus,

$$\begin{aligned}
CP^2 &< OP^2 \\
AC^2\cos^2(C) &< R^2 + AC^2\cos^2(C) - 2R \cdot AC\cos(C)\cos(90 - A) \\
0 &< R^2 - 2R \cdot AC\cos(C)\cos(90 - A) \\
2 \cdot AC\cos(C)\cos(90 - A) &< R \\
&= \frac{AC}{2\sin(B)} \\
4\cos(C)\sin(B)\cos(90 - A) &< 1 \\
4\cos(C)\sin(B)\cos(90 - (180 - B - C)) &< 1 \\
4\cos(C)\sin(B)\sin(B + C) &< 1.
\end{aligned}$$

However, we can write that

$$4\cos(C)\sin(B)\sin(B+C) < 4\cos(C)\sin(B) = 2(\sin(B+C) - \sin(C-B)) \leq 2(\sin(B+C) - \tfrac{1}{2}) \leq 1$$

and since all steps are reversible, we have proved the result. □

From here, it is obvious that this implies that

$$\angle COP < \angle OCP = 90 - \angle A$$

so we are done. □

Problem 5.29 (IMO 2001/5)

Let ABC be a triangle. Let AP bisect $\angle BAC$ and let BQ bisect $\angle ABC$, with P on $\overline{BC}$ and Q on $\overline{AC}$. If $AB + BP = AQ + QB$ and $\angle BAC = 60°$, what are the angles of the triangle?

Let $\angle ABQ = \alpha$. Then, we know from the given condition

$$AB + \frac{AB}{2\sin(150-2\alpha)} = \frac{AB\sin(\alpha)}{\sin(120-\alpha)} + \frac{AB\sin(60)}{\sin(120-\alpha)}$$

so

$$1 + \frac{1}{2\sin(150-2\alpha)} = \frac{\sin(\alpha)+\sin(60)}{\sin(60+\alpha)} = \frac{2\sin(\frac{60+\alpha}{2})\cos(\frac{60-\alpha}{2})}{2\sin(\frac{60+\alpha}{2})\cos(\frac{60+\alpha}{2})} = \frac{\cos(\frac{60-\alpha}{2})}{\cos(\frac{60+\alpha}{2})}.$$

Now,

$$\frac{\cos(\frac{60-\alpha}{2})}{\cos(\frac{60+\alpha}{2})} - 1 = \frac{\cos(\frac{60-\alpha}{2}) - \cos(\frac{60+\alpha}{2})}{\cos(\frac{60+\alpha}{2})} = \frac{2\sin(30)\sin(\frac{\alpha}{2})}{\cos(\frac{60+\alpha}{2})} = \frac{\sin(\frac{\alpha}{2})}{\cos(\frac{60+\alpha}{2})}.$$

Hence,

$$\frac{1}{2\sin(150-2\alpha)} = \frac{\sin(\frac{\alpha}{2})}{\cos(\frac{60+\alpha}{2})}$$

which means that

$$\cos(30+\tfrac{\alpha}{2}) = 2\sin(\tfrac{\alpha}{2})\sin(150-2\alpha) = \cos(\tfrac{5}{2}\alpha-150) - \cos(150-\tfrac{3}{2}\alpha)$$

so

$$\cos(\tfrac{1}{2}\alpha+30) + \cos(\tfrac{5}{2}\alpha+30) = \cos(\tfrac{3}{2}\alpha+30).$$

Now, by sum to product, we know that

$$\frac{\cos(\frac{1}{2}\alpha+30)+\cos(\frac{5}{2}\alpha+30)}{2} = \cos(\tfrac{3}{2}\alpha+30)\cos(\alpha) = \frac{\cos(\frac{3}{2}\alpha+30)}{2}$$

implying that

$$\cos(\alpha) = \frac{1}{2} \text{ or } \cos(\tfrac{3}{2}\alpha+30) = 0.$$

It clearly must be the latter since we cannot have $\angle B = 120$, so we know that $\alpha = 40$, giving angles: $(\angle A, \angle B, \angle C) = \boxed{(60°, 80°, 40°)}$. □

Problem 5.30 (IMO 2001/6)

Let $a > b > c > d$ be positive integers and suppose that

$$ac + bd = (b+d+a-c)(b+d-a+c).$$

Prove that $ab + cd$ is not prime.

Expanding, we get that

$$\begin{aligned} ac + bd &= (b+d+a-c)(b+d-a+c) \\ &= b^2 + bd - ab + bc + bd + d^2 - ad + cd + ab + ad - a^2 + ac - bc - cd + ac - c^2 \\ &= b^2 + bd + bd + d^2 - a^2 + ac + ac - c^2 \end{aligned}$$

so

$$a^2 + c^2 - ac = b^2 + d^2 + bd. \tag{1}$$

Now construct a quadrilateral $WXYZ$ with side lengths a, c, b, d in that order (define $WX = a$, $XY = c$, $YZ = b$, $ZW = d$).

Clearly, this quadrilateral is cyclic because the Law of Cosines on (1) tells us that the opposite angles are indeed supplementary. Now by Ptolemy's,

$$WY^2 = \frac{(ab+cd)(ad+bc)}{ac+bd}.$$

Now assume for the sake of contradiction that $ab + cd$ is prime. Then, by the rearrangement inequality,

$$ab + cd > ac + bd > ad + bc.$$

However, this implies that the quantity above is not an integer, which is a contradiction. □

6 Complex Numbers

Problem 6.3

Show that the foot of the altitude from Z to $\overline{AB}$ is given by

$$\frac{(\overline{a}-\overline{b})z+(a-b)\overline{z}+\overline{a}b-a\overline{b}}{2(\overline{a}-\overline{b})}.$$

Make the transformation $z \mapsto \frac{z-a}{b-a}$. Then, the required answer is

$$\frac{\frac{z-a}{b-a}+\frac{\overline{z}-\overline{a}}{\overline{b}-\overline{a}}}{2}.$$

We then revert the transformation. Hence, the answer is:

$$\begin{aligned}
\frac{(b-a)(\frac{z-a}{b-a}+\frac{\overline{z}-\overline{a}}{\overline{b}-\overline{a}})}{2}+a &= \frac{z+a+(b-a)(\frac{\overline{z}-\overline{a}}{\overline{b}-\overline{a}})}{2} \\
&= \frac{(z+a)(\overline{b}-\overline{a})+(b-a)(\overline{z}-\overline{a})}{2(\overline{b}-\overline{a})} \\
&= \frac{z\overline{b}-z\overline{a}+a\overline{b}-a\overline{a}+\overline{z}b-\overline{z}a-\overline{a}b+a\overline{a}}{2(\overline{b}-\overline{a})} \\
&= \frac{(\overline{b}-\overline{a})z+(b-a)\overline{z}+a\overline{b}-\overline{a}b}{2(\overline{b}-\overline{a})} \\
&= \boxed{\frac{(\overline{a}-\overline{b})z+(a-b)\overline{z}+\overline{a}b-a\overline{b}}{2(\overline{a}-\overline{b})}}
\end{aligned}$$

so we are done. □

Problem 6.8

Prove that complex numbers z, a, b are collinear if and only if

$$\frac{z-a}{z-b}=\overline{\left(\frac{z-a}{z-b}\right)}.$$

If they are collinear, then we know that the quantity

$$\frac{z-a}{z-b}$$

is real, since it describes sending the point a to 0, and sending z, b onto the real axis (if and only if they are collinear). The finish is clear. □

Problem 6.14

Let H be the orthocenter of $\triangle ABC$. Let X be the reflection of H over $\overline{BC}$ and Y the reflection over the midpoint of $\overline{BC}$. Prove that X and Y lie on (ABC), and $\overline{AY}$ is a diameter.

We can express X as:

$$\begin{aligned} X &= 2 \cdot \tfrac{1}{2}(a+b+(a+b+c)-ab(\overline{a+b+c}))-(a+b+c) \\ &= a+b-ab(\tfrac{1}{a}+\tfrac{1}{b}+\tfrac{1}{c}) \\ &= a+b-a-b-\tfrac{ab}{c} \\ &= -\tfrac{ab}{c}. \end{aligned}$$

which indeed has magnitute 1. □

Now, we can also express Y as

$$Y = 2 \cdot (\tfrac{b+c}{2}) - (a+b+c) = -a$$

so we are done. □

Remark

The synthetic solution can be found as Problem 1.17.

Problem 6.20

Let a, b, c, d be distinct complex numbers, not all collinear. Then A, B, C, D are concyclic if and only if

$$\frac{b-a}{c-a} \div \frac{b-d}{c-d}$$

is a real number.

This is equivalent to $\angle BAC = \angle BDC$, which is the criterion for a cyclic quadrilateral. □

Remark

It is stated in the EGMO errata that this is supposed to be the correct statement, not Theorem 6.16.

Problem 6.21

Show that if a, b, c are complex numbers, then the signed area of triangle ABC is given by

$$\frac{i}{4}\begin{vmatrix} a & \overline{a} & 1 \\ b & \overline{b} & 1 \\ c & \overline{c} & 1 \end{vmatrix}.$$

We may write that:

$$\begin{aligned} \frac{i}{4}\begin{vmatrix} a & \overline{a} & 1 \\ b & \overline{b} & 1 \\ c & \overline{c} & 1 \end{vmatrix} &= \frac{i}{4}\begin{vmatrix} 2\Re(a) & \overline{a} & 1 \\ 2\Re(b) & \overline{b} & 1 \\ 2\Re(c) & \overline{c} & 1 \end{vmatrix} \\ &= \frac{i}{4}\begin{vmatrix} 2\Re(a) & -i\Im(a) & 1 \\ 2\Re(b) & -i\Im(b) & 1 \\ 2\Re(c) & -i\Im(c) & 1 \end{vmatrix} \\ &= \frac{1}{2}\begin{vmatrix} \Re(a) & \Im(a) & 1 \\ \Re(b) & \Im(b) & 1 \\ \Re(c) & \Im(c) & 1 \end{vmatrix} \end{aligned}$$

which is the so called “Shoelace Formula”. □

Problem 6.22

Let ABC be a triangle with orthocenter H and let P be a point on (ABC).

(a) Show that the Simson line exists.

(b) Show that the Simson line at P bisects $\overline{PH}$.

We begin with part (a). Let the foot of P to $\overline{AB}$ be X, to $\overline{AC}$ be Y, and analogously for Z. Then, we may write that

$$x = \tfrac{1}{2}(a + b + p - ab\overline{p}) = \tfrac{1}{2}(a + b + p - \tfrac{ab}{p})$$

and similarly,

$$y = \tfrac{1}{2}(a + c + p - \tfrac{ac}{p})$$
$$z = \tfrac{1}{2}(b + c + p - \tfrac{bc}{p}).$$

Now, x, y, and z are collinear if and only if

$$\frac{x - y}{x - z}$$

is real. Hence,

$$\begin{aligned}
\frac{x-y}{x-z} &= \frac{\frac{1}{2}(a+b+p-\frac{ab}{p}) - \frac{1}{2}(a+c+p-\frac{ac}{p})}{\frac{1}{2}(a+b+p-\frac{ab}{p}) - \frac{1}{2}(b+c+p-\frac{bc}{p})} \\
&= \frac{a+b+p-\frac{ab}{p} - (a+c+p-\frac{ac}{p})}{a+b+p-\frac{ab}{p} - (b+c+p-\frac{bc}{p})} \\
&= \frac{b - \frac{ab}{p} - c + \frac{ac}{p}}{a - \frac{ab}{p} - c + \frac{bc}{p}} \\
&= \frac{bp - ab - cp + ac}{ap - ab - cp + bc} \\
&= \frac{(b-c)(p-a)}{(a-b)(p-c)}
\end{aligned}$$

however we know that this quantity is real since this is exactly the criteria for the points being concyclic, so we are done. $\square$

We now continue with part (b). We will show that the midpoint of $\overline{PH}$, X and Y are collinear:

$$\begin{aligned}
\begin{vmatrix} x & \overline{x} & 1 \\ y & \overline{y} & 1 \\ \frac{h+p}{2} & \overline{\frac{h+p}{2}} & 1 \end{vmatrix} &= \begin{vmatrix} \frac{1}{2}(a+b+p-\frac{ab}{p}) & \frac{1}{2}(\frac{1}{a}+\frac{1}{b}+\frac{1}{p}-\frac{p}{ab}) & 1 \\ \frac{1}{2}(a+c+p-\frac{ac}{p}) & \frac{1}{2}(\frac{1}{a}+\frac{1}{c}+\frac{1}{p}-\frac{p}{ac}) & 1 \\ \frac{1}{2}(a+b+c+p) & \frac{1}{2}(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{p}) & 1 \end{vmatrix} \\
&= \frac{1}{4}\begin{vmatrix} a+b+p-\frac{ab}{p} & \frac{1}{a}+\frac{1}{b}+\frac{1}{p}-\frac{p}{ab} & 1 \\ a+c+p-\frac{ac}{p} & \frac{1}{a}+\frac{1}{c}+\frac{1}{p}-\frac{p}{ac} & 1 \\ a+b+c+p & \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{p} & 1 \end{vmatrix} \\
&= \frac{1}{4}\begin{vmatrix} -c-\frac{ab}{p} & -\frac{1}{c}-\frac{p}{ab} & 1 \\ -b-\frac{ac}{p} & -\frac{1}{b}-\frac{p}{ac} & 1 \\ 0 & 0 & 1 \end{vmatrix} \\
&= \frac{1}{4}\begin{vmatrix} c+\frac{ab}{p} & \frac{1}{c}+\frac{p}{ab} & 1 \\ b+\frac{ac}{p} & \frac{1}{b}+\frac{p}{ac} & 1 \\ 0 & 0 & 1 \end{vmatrix} \\
&= \frac{1}{4}\left(\left(c+\frac{ab}{p}\right)\left(\frac{1}{b}+\frac{p}{ac}\right)-\left(\frac{1}{c}+\frac{p}{ab}\right)\left(b+\frac{ac}{p}\right)\right) \\
&= 0
\end{aligned}$$

so we are done. □

Remark

A synthetic proof of part (a) can be found as Problem 1.48. In addition, a synthetic proof of part (b) can be found as a consequence of Problem 4.3.

Problem 6.29

Give a proof of the inscribed angle theorem using complex numbers.

Let A, B, and C be points on the unit circle with center O. Then, we wish to show that

$$\arg\left(\frac{a}{b}\right) = 2\arg\left(\frac{a-c}{b-c}\right) = \arg\left(\left(\frac{a-c}{b-c}\right)^2\right).$$

This is equivalent to showng that 0, $\frac{a}{b}$, and $(\frac{a-c}{b-c})^2$ are collinear. Indeed,

$$\begin{aligned}
\overline{\left(\frac{(\frac{a-c}{b-c})^2}{\frac{a}{b}}\right)} &= \overline{\left(\frac{b(a-c)^2}{a(b-c)^2}\right)} \\
&= \frac{\frac{1}{b}(\frac{1}{a}-\frac{1}{c})^2}{\frac{1}{a}(\frac{1}{b}-\frac{1}{c})^2} \\
&= \frac{a(\frac{1}{a}-\frac{1}{c})^2}{b(\frac{1}{b}-\frac{1}{c})^2} \\
&= \frac{a\left(\frac{1}{a^2}-\frac{2}{ac}+\frac{1}{c^2}\right)}{b\left(\frac{1}{b^2}-\frac{2}{bc}+\frac{1}{c^2}\right)} \\
&= \frac{ab^2c^2\left(c^2-2ac+a^2\right)}{a^2bc^2\left(c^2-2bc+b^2\right)} = \frac{b(a-c)^2}{a(b-c)^2}
\end{aligned}$$

so we are done. □

Problem 6.30 (Complex Chord)

Show that a point P lies on a chord $\overline{AB}$ of the unit circle if and only if $p + ab\overline{p} = a + b$.

This is equivalent to showing that if the foot from P to $\overline{AB}$ is P, then $p+ab\overline{p} = a+b$. However, it is well known that the foot can be expressed as $\frac{1}{2}(a+b+p-ab\overline{p})$. Hence, all we need to solve is:

$$\tfrac{1}{2}(a+b+p-ab\overline{p}) = p \implies p + ab\overline{p} = a + b$$

as required. $\square$

Problem 6.31

Let $ABCD$ be a cyclic quadrilateral. Let H_A, H_B, H_C, H_D denote the orthocenters of triangles BCD, CDA, DAB, and ABC, respectively. Prove that $\overline{AH_A}$, $\overline{BH_B}$, $\overline{CH_C}$, and $\overline{DH_D}$ concur.

Toss onto the complex plane. Note that the point $\frac{1}{2}(a+b+c+d)$ is the midpoint of all the aforementioned segments, so we are done. $\square$

Problem 6.32

Let $ABCD$ be a quadrilateral circumscribed around a circle with center I. Prove that I lies on the line joining the midpoints of $\overline{AC}$ and $\overline{BD}$.

Toss onto the complex plane, let $I = 0$ and scale so that the circle has radius 1. In addition, let the tangency points of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ to the circle be W, X, Y, and Z, respectively. Then, we know that

$$\begin{aligned} A &= \frac{2wz}{w+z} \\ B &= \frac{2wx}{w+x} \\ C &= \frac{2xy}{x+y} \\ D &= \frac{2yz}{y+z}. \end{aligned}$$

Thus, all we need to show is that B, C, and $I = 0$ are collinear. Hence,

$$\begin{aligned} \frac{\frac{2wz}{w+z} + \frac{2xy}{x+y}}{\frac{2wx}{w+x} + \frac{2yz}{y+z}} &= \frac{\frac{wz}{w+z} + \frac{xy}{x+y}}{\frac{wx}{w+x} + \frac{yz}{y+z}} \\ &= \frac{\frac{1}{\frac{1}{w}+\frac{1}{z}} + \frac{1}{\frac{1}{x}+\frac{1}{y}}}{\frac{1}{\frac{1}{w}+\frac{1}{x}} + \frac{1}{\frac{1}{y}+\frac{1}{z}}} \\ &= \frac{\frac{1}{\left(\frac{1}{w}+\frac{1}{z}\right)\left(\frac{1}{x}+\frac{1}{y}\right)}}{\frac{1}{\left(\frac{1}{w}+\frac{1}{x}\right)\left(\frac{1}{y}+\frac{1}{z}\right)}} \\ &= \frac{\left(\frac{1}{w}+\frac{1}{x}\right)\left(\frac{1}{y}+\frac{1}{z}\right)}{\left(\frac{1}{w}+\frac{1}{z}\right)\left(\frac{1}{x}+\frac{1}{y}\right)} = \frac{(w+x)(y+z)}{(w+z)(x+y)} \end{aligned}$$

and the last line implies that the expression is real, as required. □

Problem 6.33 (China TST 2011)

Let ABC be a triangle, and let A', B', C' be points on its circumcircle, diametrically opposite A, B, C, respectively. Let P be any point inside $\triangle ABC$ and let D, E, F be the feet of the altitudes from P onto $\overline{BC}$, $\overline{CA}$, $\overline{AB}$, respectively. Let X, Y, Z denote the reflections of A', B', C' over D, E, F, respectively. Show that triangles XYZ and ABC are similar to each other.

Toss onto the complex plane so that the circumcircle is the unitcircle. Then, we know that $a' = -a$, $b' = -b$, and $c' = -c$. In addition,

$$\begin{aligned} d &= \tfrac{1}{2}(b+c+p-bc\overline{p}) \\ e &= \tfrac{1}{2}(a+c+p-ac\overline{p}) \\ f &= \tfrac{1}{2}(a+b+p-ab\overline{p}). \end{aligned}$$

Thus,

$$\begin{aligned} x &= a+b+c+p-bc\overline{p} \\ y &= a+b+c+p-ac\overline{p} \\ z &= a+b+c+p-ab\overline{p}. \end{aligned}$$

This means it suffices to show that the triangle formed by a, b, c and ab, ac, bc are similar. Now, the condition for oppositely similar triangles is the same as directly similar, but showing it is equal to the conjugate of the other side. Hence:

$$\overline{\frac{b-a}{c-a}} = \frac{\frac{1}{b}-\frac{1}{a}}{\frac{1}{c}-\frac{1}{a}} = \frac{ac(a-b)}{ab(a-c)} = \frac{c(a-b)}{b(a-c)} = \frac{ac-bc}{ab-bc} = \frac{bc-ac}{bc-ab}$$

so we are done. □

Problem 6.34 (Napoleon's Theorem)

Let ABC be a triangle and erect equilateral triangles on sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ outside of ABC with centers O_A, O_B, O_C. Prove that $\triangle O_AO_BO_C$ is equilateral and that its center coincides with the centroid of triangle ABC.

We may find that $\frac{o_c-b}{a-b} = \frac{1}{2} + \frac{\sqrt{3}}{6}i$ and similar equations for the others. Hence, we know that:

$$\begin{aligned} o_a &= (\tfrac{1}{2}+\tfrac{\sqrt{3}}{6}i)(b-c)+c \\ o_b &= (\tfrac{1}{2}+\tfrac{\sqrt{3}}{6}i)(c-a)+a \\ o_c &= (\tfrac{1}{2}+\tfrac{\sqrt{3}}{6}i)(a-b)+b. \end{aligned}$$

Summing all of these equations and dividing by three gives that

$$\frac{o_a+o_b+o_c}{3} = \frac{a+b+c}{3}$$

so the centroids coincide. □

Next, we will show that $\triangle O_AO_BO_C$ is equilateral. We will do this by showing that

$$(o_a+o_b+o_c)^2 = (a+b+c)^2 = 3(o_ao_b+o_ao_c+o_bo_c).$$

We know that

$$\begin{aligned} o_a o_b &= \left(\left(\tfrac{1}{2} + \tfrac{\sqrt{3}}{6}i\right)(b-c)+c\right)\left(\left(\tfrac{1}{2} + \tfrac{\sqrt{3}}{6}i\right)(c-a)+a\right) \\ &= \left(\tfrac{1}{2} + \tfrac{\sqrt{3}}{6}i\right)^2 (bc - ab - c^2 + ac) + \left(\tfrac{1}{2} + \tfrac{\sqrt{3}}{6}i\right)(c^2 + ab - 2ac) + ac. \end{aligned}$$

Cyclicly summing this gives that

$$\begin{aligned} &\left(\tfrac{1}{2} + \tfrac{\sqrt{3}}{6}i\right)^2 (ab + ac + bc - a^2 - b^2 - c^2) \\ &+ \left(\tfrac{1}{2} + \tfrac{\sqrt{3}}{6}i\right)(a^2 + b^2 + c^2 - ab - ac - bc) + ab + ac + bc = \frac{(a+b+c)^2}{3} \end{aligned}$$

after expanding and collecting terms. Hence, we are done. □

Problem 6.35 (USAMO 2015/2, complex)

Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let X be a variable point on segment $\overline{PQ}$. Line AX meets ω again at S (other than A). Point T lies on arc AQB of ω such that XT is perpendicular to AX. Let M denote the midpoint of chord $\overline{ST}$. As X varies on segment $\overline{PQ}$, show that M moves along a circle.

Let the circumcircle of $APBQ$ be the unitcircle, $A = -1$, and $B = 1$. Then, we know that

$$x = \tfrac{1}{2}(a + s + t - \tfrac{as}{t}) = \tfrac{1}{2}(s + t - 1 + \tfrac{s}{t}).$$

Then, note that

$$4\Re(x) + 2 = s + \tfrac{1}{s} + t + \tfrac{1}{t} + \tfrac{s}{t} + \tfrac{t}{s}.$$

Now, note that

$$4\left|\tfrac{1}{2} - \tfrac{s+t}{2}\right|^2 = |s+t-1|^2 = (s+t-1)\left(\tfrac{1}{s} + \tfrac{1}{t} - 1\right) = 4\Re(x) + 5$$

which does not depend on X, so we are done. □

Remark

A synthetic proof can be found as Problem 3.30.

Problem 6.36 (China TST 2006)

Point H is the orthocenter of triangle ABC. Points D, E, and F lie on the circumcircle of triangle ABC such that $\overline{AD} \parallel \overline{BE} \parallel \overline{CF}$. Points S, T, and U are the respective reflections of D, E, and F across the lines BC, CA, and AB. Prove that S, T, U, and H are concyclic.

Let the circumcircle be the unitcircle, and rotate so that the segments $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are perpendicular to the real axis. Then, we know that:

$$\begin{aligned} d &= \overline{a} = \tfrac{1}{a} \\ e &= \overline{b} = \tfrac{1}{b} \\ f &= \overline{c} = \tfrac{1}{c}. \end{aligned}$$

Thus, we can say that:

$$\begin{aligned} s &= b + c - \tfrac{bc}{d} = b + c - abc \\ t &= a + c - \tfrac{ac}{e} = a + c - abc \\ u &= a + b - \tfrac{ab}{f} = a + b - abc. \end{aligned}$$

Finally,

$$\begin{aligned}\frac{\frac{a+c-abc-(b+c-abc)}{a+b-abc-(b+c-abc)}}{\frac{a+c-abc-(a+b+c)}{a+b-abc-(a+b+c)}} &= \frac{\frac{a-b}{a-c}}{\frac{abc+b}{abc+c}} \\ &= \frac{(a-b)(abc+c)}{(a-c)(abc+b)}.\end{aligned}$$

The conjugate of this is

$$\frac{\left(\frac{1}{a}-\frac{1}{b}\right)\left(\frac{1}{abc}+\frac{1}{c}\right)}{\left(\frac{1}{a}-\frac{1}{c}\right)\left(\frac{1}{abc}+\frac{1}{b}\right)} = \frac{\left(\frac{1}{a}-\frac{1}{b}\right)(ab+1)}{\left(\frac{1}{a}-\frac{1}{c}\right)(ac+1)} = \frac{\left(\frac{b-a}{ab}\right)(ab+1)}{\left(\frac{c-a}{ac}\right)(ac+1)} = \frac{(a-b)(abc+c)}{(a-c)(abc+b)}$$

so we are done. $\square$

Remark
In the book, this is cited as MOP 2006, when it is more accurately from China TST 2006.

Problem 6.37 (USA January TST for IMO 2014)

Let $ABCD$ be a cyclic quadrilateral, and let E, F, G, and H be the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$, respectively. Let W, X, Y, and Z be the orthocenters of triangles AHE, BEF, CFG, and DGH, respectively. Prove that quadrilaterals $ABCD$ and $WXYZ$ have the same area.

Toss onto the complex plane so that the circumcircle of $ABCD$ is the unitcircle. Then, note that:

$$\begin{aligned} e &= \tfrac{a+b}{2} \\ f &= \tfrac{b+c}{2} \\ g &= \tfrac{c+d}{2} \\ h &= \tfrac{a+d}{2}. \end{aligned}$$

Hence, we can find w by taking a homothety of scale factor 2 centered at A. This homothety will send E to B and H to D. Hence, the orthocenter of $\triangle AHE$ is

$$w = a + \tfrac{1}{2}(b+d).$$

Similarly, we get that:

$$\begin{aligned} x &= b + \tfrac{1}{2}(a+c) \\ y &= c + \tfrac{1}{2}(b+d) \\ z &= d + \tfrac{1}{2}(a+c). \end{aligned}$$

We now begin the grueling task of expansion:

$$\begin{aligned}[ABCD] &= [ABC] + [CDA] \\ &= \begin{vmatrix} a & \frac{1}{a} & 1 \\ b & \frac{1}{b} & 1 \\ c & \frac{1}{c} & 1 \end{vmatrix} + \begin{vmatrix} c & \frac{1}{c} & 1 \\ d & \frac{1}{d} & 1 \\ a & \frac{1}{a} & 1 \end{vmatrix} \\ &= \left(\tfrac{a}{b} - \tfrac{a}{c} - \tfrac{b}{a} + \tfrac{c}{a} + \tfrac{b}{c} - \tfrac{c}{b}\right) + \left(\tfrac{c}{d} - \tfrac{c}{a} - \tfrac{d}{c} + \tfrac{a}{c} + \tfrac{d}{a} - \tfrac{a}{d}\right) \\ &= \left(\tfrac{a}{b} - \tfrac{b}{a} + \tfrac{b}{c} - \tfrac{c}{b}\right) + \left(\tfrac{c}{d} - \tfrac{d}{c} + \tfrac{d}{a} - \tfrac{a}{d}\right)\end{aligned}$$

and

$$
\begin{aligned}
[WXYZ] &= [WXY] + [YZW] \\
&= \begin{vmatrix} a+\frac{1}{2}(b+d) & \frac{1}{a}+\frac{1}{2}\left(\frac{1}{b}+\frac{1}{d}\right) & 1 \\ b+\frac{1}{2}(a+c) & \frac{1}{b}+\frac{1}{2}\left(\frac{1}{a}+\frac{1}{c}\right) & 1 \\ c+\frac{1}{2}(b+d) & \frac{1}{c}+\frac{1}{2}\left(\frac{1}{b}+\frac{1}{d}\right) & 1 \end{vmatrix} + \begin{vmatrix} c+\frac{1}{2}(b+d) & \frac{1}{c}+\frac{1}{2}\left(\frac{1}{b}+\frac{1}{d}\right) & 1 \\ d+\frac{1}{2}(a+c) & \frac{1}{d}+\frac{1}{2}\left(\frac{1}{a}+\frac{1}{c}\right) & 1 \\ a+\frac{1}{2}(b+d) & \frac{1}{a}+\frac{1}{2}\left(\frac{1}{b}+\frac{1}{d}\right) & 1 \end{vmatrix} \\
&= \begin{vmatrix} \frac{1}{2}a-\frac{1}{2}c & \frac{1}{2}\cdot\frac{1}{a}-\frac{1}{2}\cdot\frac{1}{c} & 1 \\ \frac{1}{2}b-\frac{1}{2}d & \frac{1}{2}\cdot\frac{1}{b}-\frac{1}{2}\cdot\frac{1}{d} & 1 \\ \frac{1}{2}c-\frac{1}{2}a & \frac{1}{2}\cdot\frac{1}{c}-\frac{1}{2}\cdot\frac{1}{a} & 1 \end{vmatrix} + \begin{vmatrix} \frac{1}{2}c-\frac{1}{2}a & \frac{1}{2}\cdot\frac{1}{c}-\frac{1}{2}\cdot\frac{1}{a} & 1 \\ \frac{1}{2}d-\frac{1}{2}b & \frac{1}{2}\cdot\frac{1}{d}-\frac{1}{2}\cdot\frac{1}{b} & 1 \\ \frac{1}{2}a-\frac{1}{2}c & \frac{1}{2}\cdot\frac{1}{a}-\frac{1}{2}\cdot\frac{1}{c} & 1 \end{vmatrix} \\
&= \frac{1}{4}\left(\begin{vmatrix} a-c & \frac{1}{a}-\frac{1}{c} & 1 \\ b-d & \frac{1}{b}-\frac{1}{d} & 1 \\ c-a & \frac{1}{c}-\frac{1}{a} & 1 \end{vmatrix} + \begin{vmatrix} c-a & \frac{1}{c}-\frac{1}{a} & 1 \\ d-b & \frac{1}{d}-\frac{1}{b} & 1 \\ a-c & \frac{1}{a}-\frac{1}{c} & 1 \end{vmatrix}\right) \\
&= \frac{1}{2}\left(\frac{2a}{b}-\frac{2b}{a}+\frac{2b}{c}-\frac{2c}{b}-\frac{2a}{d}+\frac{2c}{d}+\frac{2d}{a}-\frac{2d}{c}\right) \\
&= \left(\frac{a}{b}-\frac{b}{a}+\frac{b}{c}-\frac{c}{b}\right)+\left(\frac{c}{d}-\frac{d}{c}+\frac{d}{a}-\frac{a}{d}\right)
\end{aligned}
$$

so we are done. □

Problem 6.38 (OMO Fall 2013)

Let ABC be a triangle with $AB = 13$, $AC = 25$, and $\tan A = \frac{3}{4}$. Denote the reflections of B, C across AC, AB by D, E, respectively, and let O be the circumcenter of triangle ABC. Let P be a point such that $\triangle DPO \sim \triangle PEO$, and let X and Y be the midpoints of the major and minor arcs BC of the circumcircle of triangle ABC. Find $PX \cdot PY$.

Toss onto the complex plane. Let $O = 0$ and (ABC) be the unitcircle. Then, we get that:

$$
\begin{aligned}
d &= a + c - \tfrac{ac}{b} \\
e &= a + b - \tfrac{ab}{c}.
\end{aligned}
$$

Now, we know that

$$p^2 = de$$

and

$$x + y = 0 \qquad xy = -bc.$$

Hence,

$$
\begin{aligned}
PX \cdot PY &= |p-x|\,|p-y| \\
&= \sqrt{|p-x|^2\,|p-y|^2} \\
&= \sqrt{(p-x)(p-y)(\overline{p}-\overline{x})(\overline{p}-\overline{y})} \\
&= \sqrt{(p^2-(x+y)p+xy)(\overline{p^2}-(\overline{x+y})\overline{p}+\overline{xy})} \\
&= \sqrt{(de-bc)(\overline{de}-\overline{bc})} \\
&= \sqrt{de\cdot\overline{de}-bc\cdot\overline{de}-\overline{bc}\cdot de+|bc|^2} \\
&= \sqrt{\overline{de}(de-bc)-\tfrac{de}{bc}+1} \\
&= \sqrt{\overline{\left(a+c-\tfrac{ac}{b}\right)\left(a+b-\tfrac{ab}{c}\right)}\left(\left(a+c-\tfrac{ac}{b}\right)\left(a+b-\tfrac{ab}{c}\right)-bc\right)-\tfrac{de}{bc}+1}
\end{aligned}
$$

$$
\begin{aligned}
&= \sqrt{\left(\tfrac{1}{a}+\tfrac{1}{c}-\tfrac{b}{ac}\right)\left(\tfrac{1}{a}+\tfrac{1}{b}-\tfrac{c}{ab}\right)\left(\left(a+c-\tfrac{ac}{b}\right)\left(a+b-\tfrac{ab}{c}\right)-bc\right)-\tfrac{de}{bc}+1} \\
&= \sqrt{\left(\tfrac{2}{a^2}-\tfrac{c}{a^2b}+\tfrac{1}{bc}-\tfrac{b}{a^2c}\right)\left(a^2+ab-\tfrac{a^2b}{c}+ac+bc-ab-\tfrac{a^2c}{b}-ac+a^2-bc\right)-\tfrac{de}{bc}+1} \\
&= \sqrt{\left(\tfrac{2}{a^2}-\tfrac{c}{a^2b}+\tfrac{1}{bc}-\tfrac{b}{a^2c}\right)\left(2a^2-\tfrac{a^2b}{c}-\tfrac{a^2c}{b}\right)-\tfrac{de}{bc}+1} \\
&= \sqrt{4-\tfrac{2b}{c}-\tfrac{2c}{b}-\tfrac{2c}{b}+1+\tfrac{c^2}{b^2}+\tfrac{2a^2}{bc}-\tfrac{a^2}{c^2}-\tfrac{a^2}{b^2}-\tfrac{2b}{c}+\tfrac{b^2}{c^2}+1-\tfrac{de}{bc}+1} \\
&= \sqrt{7-\tfrac{4b}{c}-\tfrac{4c}{b}+\tfrac{c^2}{b^2}+\tfrac{2a^2}{bc}-\tfrac{a^2}{c^2}-\tfrac{a^2}{b^2}+\tfrac{b^2}{c^2}-\frac{\left(a+c-\frac{ac}{b}\right)\left(a+b-\frac{ab}{c}\right)}{bc}} \\
&= \sqrt{7-\tfrac{4b}{c}-\tfrac{4c}{b}+\tfrac{c^2}{b^2}+\tfrac{2a^2}{bc}-\tfrac{a^2}{c^2}-\tfrac{a^2}{b^2}+\tfrac{b^2}{c^2}-\left(\frac{a^2-\frac{a^2b}{c}+bc-\frac{a^2c}{b}+a^2}{bc}\right)} \\
&= \sqrt{7-\tfrac{4b}{c}-\tfrac{4c}{b}+\tfrac{c^2}{b^2}+\tfrac{2a^2}{bc}-\tfrac{a^2}{c^2}-\tfrac{a^2}{b^2}+\tfrac{b^2}{c^2}-\tfrac{a^2}{bc}+\tfrac{a^2}{c^2}-1+\tfrac{a^2}{b^2}-\tfrac{a^2}{bc}} \\
&= \sqrt{6-\tfrac{4b}{c}-\tfrac{4c}{b}+\tfrac{c^2}{b^2}+\tfrac{b^2}{c^2}} \\
&= \tfrac{c}{b}+\tfrac{b}{c}-2 \\
&= -(b-c)(\overline{b}-\overline{c}) \\
&= BC^2.
\end{aligned}
$$

Now, we are in the final stretch:

$$\tan^2 A+1=\sec^2 A \implies \tfrac{25}{9}=\sec^2 A \implies \cos A=\pm\tfrac{3}{5}.$$

Finally, the Law of Cosines gives that

$$BC^2=AB^2+AC^2-2AB\cdot AC\cos(A)=13^2+25^2\mp 2\cdot 13\cdot 25\cdot \tfrac{3}{5}=\boxed{404}\text{ or }\boxed{1184}$$

as required. □

Problem 6.39 (Tangent Addition)

Consider angles A, B, C in the open interval $(-90°, 90°)$.

(a) Let $x=\tan A$, $y=\tan B$, $z=\tan C$. Prove that

$$\tan(A+B+C)=\frac{(x+y+z)-xyz}{1-(xy+yz+zx)}$$

if $xy+yz+zx\neq 1$, and is undefined otherwise.

(b) Generalize to multiple variables.

We solve both problems at once by immediately generalizing to k variables. Let

$$a_n=1+\tan(r_n)i$$

for $1\le n\le k$. Now let e_n denote the nth elementary symmetric sum (of k variables/tangents in this case). Then, note that

$$
\begin{aligned}
\arg(a_1a_2\dots a_k)&=\arg((1+\tan(r_1)i)(1+\tan(r_2)i)\dots(1+\tan(r_k)i)) \\
&=\arg((1-e_2+e_4-\dots)+(e_1-e_3+e_5-\dots)i).
\end{aligned}
$$

Hence,

$$\begin{aligned}\tan\left(\sum_{n=1}^{k} r_n\right) &= \tan\left(\arg(a_1 a_2 \dots a_k)\right) \\ &= \tan(\arg((1 - e_2 + e_4 - \dots) + (e_1 - e_3 + e_5 - \dots)i)) \\ &= \frac{e_1 - e_3 + e_5 - \dots}{1 - e_2 + e_4 - \dots}\end{aligned}$$

so we are done. □

Problem 6.40 (Schiffler Point)

Let ABC be a triangle with incenter I. Prove that the Euler lines of triangles AIB, BIC, CIA, and ABC are concurrent (called the Schiffler point of ABC).

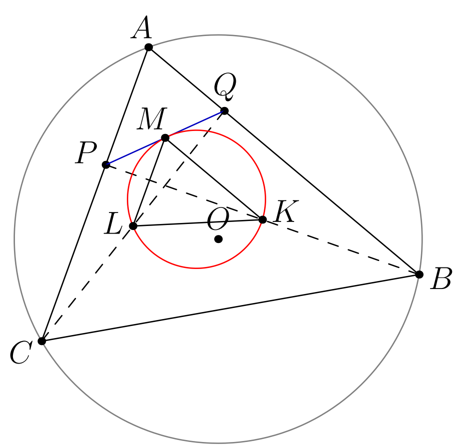
Toss onto the complex plane, with (ABC) as the unit circle. Denote $A = a^2$, and analogously for B and C. Then, we know by the Incenter-Excenter Lemma, that the circumcenter of $\triangle AIB$ is the midpoint of arc AB not containing C, or $-ab$. In addition, the centroid of $\triangle AIB$ is

$$\tfrac{a^2+b^2-ab-bc-ac}{3}$$

Now, we are trying to find the intersection of the line through 0 and $a+b+c$, and the former two points. We omit the routine (but very annoying) calculation using the intersection of lines formula, and state the result as a symmetric polynomial in a, b, c. Hence, we have the required result. □

Problem 6.41 (IMO 2009/2, complex)

Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB, respectively. Let K, L, and M be the midpoints of the segments BP, CQ, and PQ, respectively, and let Γ be the circle passing through K, L, and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$.



Toss onto the complex plane. Let (ABC) be the unit circle. Since p and q lie on chords, we know that

$$p + ac\overline{p} = a + c \quad \text{and} \quad q + ab\overline{q} = a + b.$$

We now make the following claim:

Claim
Triangles APQ and MKL are similar.

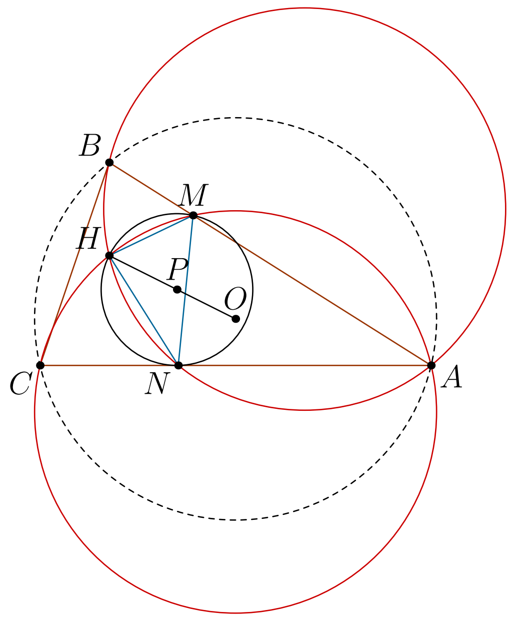
Proof. Note that
$$\triangle CQP \sim \triangle LQM$$
and
$$\triangle BPQ \sim \triangle KPM$$
for obvious reasons (note that they are both in the ratio $2:1$). In addition, we can say that
$$\angle APQ = 180 - \angle CPQ = 180 - \angle LMQ = \angle K$$
and similarly,
$$\angle AQP = 180 - \angle BQP = 180 - \angle KMP = \angle L.$$
Thus, we know that $\triangle APQ \sim \triangle MKL$, as required. □

Using this claim, we know that:
$$\begin{aligned}
\overline{\frac{p-a}{q-a}} &= \frac{k-m}{l-m} \\
&= \frac{\frac{p+b}{2} - \frac{p+q}{2}}{\frac{c+q}{2} - \frac{p+q}{2}} \\
&= \frac{\frac{b-q}{2}}{\frac{c-p}{2}} \\
&= \frac{b-q}{c-p} \\
\frac{\overline{p} - \frac{1}{a}}{\overline{q} - \frac{1}{a}} &= \frac{b-q}{c-p} \\
(c-p)\left(\overline{p} - \frac{1}{a}\right) &= (b-q)\left(\overline{q} - \frac{1}{a}\right) \\
c\overline{p} - \frac{c}{a} - |p|^2 + \frac{p}{a} &= b\overline{q} - \frac{b}{a} - |q|^2 + \frac{q}{a} \\
\frac{a+c-p}{a} - \frac{c}{a} - |p|^2 + \frac{p}{a} &= \frac{a+b-q}{a} - \frac{b}{a} - |q|^2 + \frac{q}{a} \\
|p|^2 &= |q|^2
\end{aligned}$$
so we are done. □

Remark
A synthetic solution can be found as Problem 2.35.

Problem 6.42 (APMO 2010/4)

Let ABC be an acute triangle with $AB > BC$ and $AC > BC$. Denote by O and H the circumcenter and orthocenter of ABC. Suppose that the circumcircle of triangle AHC intersects the line AB at $M \neq A$, and the circumcircle of triangle AHB intersects the line AC at $N \neq A$. Prove that the circumcenter of triangle MNH lies on line OH.



We begin with the following claim:

Claim

We claim that M is the reflection of B across line CH, and N is the reflection of C across line BH.

Proof. We will prove the former; the latter can be proven in a similar fashion. Let the reflection of B across CH be B'. Clearly, B' lies on AB. In addition, we know that

$$\angle CMA = 180 - \angle CMB = 180 - \angle B = \angle CHA$$

so we have shown that $CHMA$ is cyclic. Since there is only one point other than A that can satisfy both of these conditions, we know that $B' = M$, as required. □ □

We proceed by complex bash. Let (ABC) be the unit circle. Then, we know that

$$m = a + c - \tfrac{ab}{c} \quad \text{and} \quad n = a + b - \tfrac{ac}{b}$$

due to the claim. We now make the transformations $m \mapsto m - a - b - c$ and similarly for n and h. Then, the circumcenter of these points can be expressed as:

$$\begin{aligned}\frac{\left(-b - \frac{ab}{c}\right)\left(-c - \frac{ac}{b}\right)\left(\overline{-b - \frac{ab}{c}} - \overline{-c - \frac{ac}{b}}\right)}{\left(\overline{-b - \frac{ab}{c}}\right)\left(-c - \frac{ac}{b}\right) - \left(-b - \frac{ab}{c}\right)\left(\overline{-c - \frac{ac}{b}}\right)} &= \frac{\left(b + \frac{ab}{c}\right)\left(c + \frac{ac}{b}\right)\left(\overline{c + \frac{ac}{b}} - \left(\overline{b + \frac{ab}{c}}\right)\right)}{\left(\overline{b + \frac{ab}{c}}\right)\left(c + \frac{ac}{b}\right) - \left(b + \frac{ab}{c}\right)\left(\overline{c + \frac{ac}{b}}\right)} \\ &= \frac{\left(b + \frac{ab}{c}\right)\left(c + \frac{ac}{b}\right)\left(\frac{1}{c} + \frac{b}{ac} - \frac{1}{b} - \frac{c}{ab}\right)}{\left(\frac{1}{b} + \frac{c}{ab}\right)\left(c + \frac{ac}{b}\right) - \left(b + \frac{ab}{c}\right)\left(\frac{1}{c} + \frac{b}{ac}\right)}\end{aligned}$$

$$= \frac{(a+c)(a+b)\left(\frac{1}{c}+\frac{b}{ac}-\frac{1}{b}-\frac{c}{ab}\right)}{\left(\frac{1}{b}+\frac{c}{ab}\right)\left(c+\frac{ac}{b}\right)-\left(b+\frac{ab}{c}\right)\left(\frac{1}{c}+\frac{b}{ac}\right)}$$
$$= \frac{(a+c)(a+b)\left(ab+b^2-ac-c^2\right)}{\frac{c}{b}(a+c)(bc+ac)-\frac{b}{c}(bc+ab)(a+b)}$$
$$= \frac{(a+b)\left(ab+b^2-ac-c^2\right)}{\frac{c^2}{b}(a+b)-\frac{b^2}{c}(a+b)}$$
$$= \frac{\left(ab+b^2-ac-c^2\right)}{\frac{c^2}{b}-\frac{b^2}{c}}$$
$$= \frac{(a+b+c)(b-c)}{\frac{c^3-b^3}{bc}}$$
$$= -\frac{bc}{b^2+bc+c^2}\cdot(a+b+c).$$

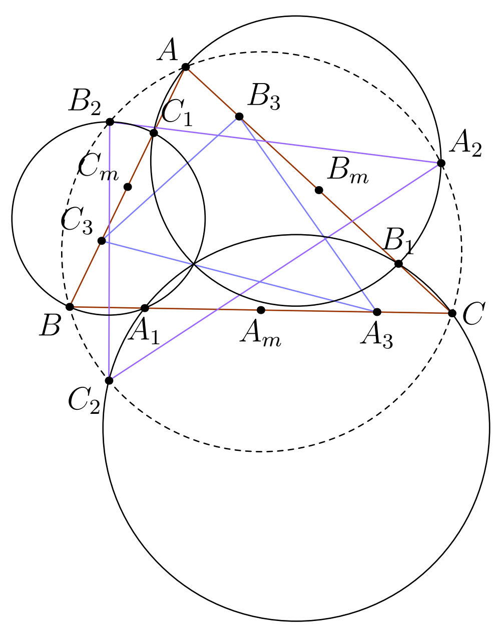
Now, we can write that:

$$\overline{\left(\frac{bc}{b^2+bc+c^2}\right)} = \frac{\frac{1}{bc}}{\frac{1}{b^2}+\frac{1}{bc}+\frac{1}{c^2}} = \frac{bc}{b^2+bc+c^2}$$

so this quantity is real. Hence, the circumcenter of the transformed version of $\triangle MNH$ is a scalar multiple of $a+b+c$, which implies the result. □

Problem 6.43 (ISL 2006/G9)

Points A_1, B_1, C_1 are chosen on the sides BC, CA, AB of a triangle ABC respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, C_2 respectively ($A_2 \neq A, B_2 \neq B, C_2 \neq C$). Points A_3, B_3, C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of the sides BC, CA, AB respectively. Prove that the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.



Toss onto the complex plane. Then, we have that:

$$a_2 = \frac{bb_1-cc_1}{b+b_1-c-c_1}$$

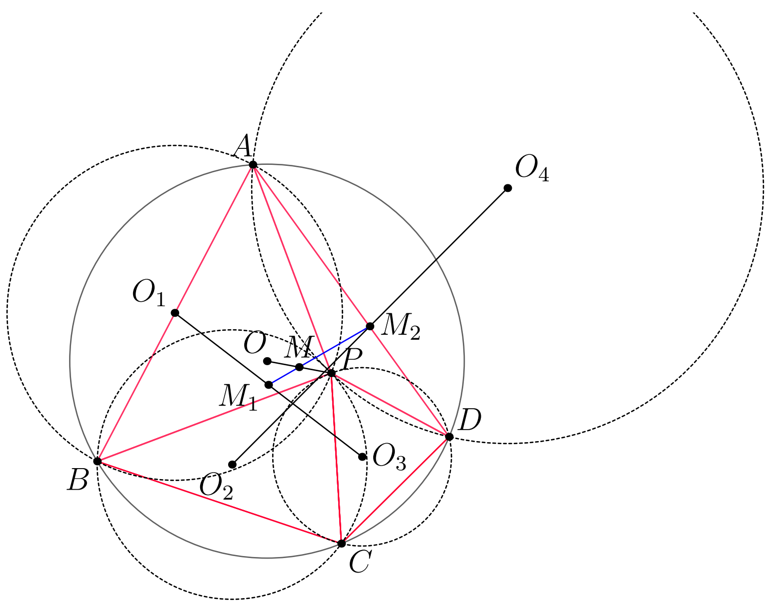
and analogously for b_2 and c_2. Hence,

$$\begin{aligned}
\frac{a_2-b_2}{a_2-c_2} &= \frac{\frac{bb_1-cc_1}{b+b_1-c-c_1} - \frac{aa_1-cc_1}{a+a_1-c-c_1}}{\frac{bb_1-cc_1}{b+b_1-c-c_1} - \frac{aa_1-bb_1}{a+a_1-b-b_1}} \\
&= \frac{\frac{bb_1-cc_1}{b+b_1-c-c_1} - \frac{cc_1-aa_1}{c+c_1-a-a_1}}{\frac{bb_1-cc_1}{b+b_1-c-c_1} - \frac{aa_1-bb_1}{a+a_1-b-b_1}} \\
&= \frac{\frac{aa_1(b+b_1-c-c_1)+bb_1(c+c_1-a-a_1)+cc_1(a+a_1-b-b_1)}{(c+c_1-a-a_1)}}{\frac{aa_1(c+c_1-b-b_1)+bb_1(a+a_1-c-c_1)+cc_1(b+b_1-a-a_1)}{(a+a_1-b-b_1)}} \\
&= \frac{b+b_1-a-a_1}{c+c_1-a-a_1} \\
&= \frac{a_3-b_3}{a_3-b_3}
\end{aligned}$$

as required. □

Problem 6.44 (MOP 2006)

Given a cyclic quadrilateral $ABCD$ with circumcenter O and a point P on the plane, let O_1, O_2, O_3, O_4 denote the circumcenters of triangles PAB, PBC, PCD, PDA respectively. Prove that the midpoints of segments $\overline{O_1O_3}$, $\overline{O_2O_4}$, and $\overline{OP}$ are collinear.



Toss onto the complex plane with $(ABCD)$ as the unitcircle. Denote M_1 as the midpoint of $\overline{O_1O_3}$, M_2 as the midpoint of $\overline{O_2O_4}$, and M as the midpoint of $\overline{OP}$. We will compute o_1. Using the circumcenter formula, we get that:

$$\begin{aligned}
o_1 &= \begin{vmatrix} a & a\overline{a} & 1 \\ b & b\overline{b} & 1 \\ p & p\overline{p} & 1 \end{vmatrix} \div \begin{vmatrix} a & \overline{a} & 1 \\ b & \overline{b} & 1 \\ p & \overline{p} & 1 \end{vmatrix} \\
&= \begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ p & p\overline{p} & 1 \end{vmatrix} \div \begin{vmatrix} a & \frac{1}{a} & 1 \\ b & \frac{1}{b} & 1 \\ p & \overline{p} & 1 \end{vmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{a - ap\overline{p} - b + bp\overline{p}}{\frac{a}{b} - a\overline{p} - \frac{b}{a} + \frac{p}{a} + b\overline{p} - \frac{p}{b}} \\
&= \frac{ab(b-a)(p\overline{p}-1)}{(a-b)(a+b) + ab\overline{p}(b-a) + p(b-a)} \\
&= \frac{ab(p\overline{p}-1)}{-a-b+ab\overline{p}+p}.
\end{aligned}
$$

Hence, we know that:

$$
\begin{aligned}
m_1 &= \frac{\frac{ab(p\overline{p}-1)}{ab\overline{p}-a-b+p} + \frac{cd(p\overline{p}-1)}{cd\overline{p}-c-d+p}}{2} \\
&= \frac{ab(p\overline{p}-1)}{2(ab\overline{p}-a-b+p)} + \frac{cd(p\overline{p}-1)}{2(cd\overline{p}-c-d+p)}.
\end{aligned}
$$

Similarly,

$$
m_2 = \frac{ad(p\overline{p}-1)}{2(ad\overline{p}-a-d+p)} + \frac{bc(p\overline{p}-1)}{2(bc\overline{p}-b-c+p)}.
$$

Finally, we perform the required calculation:

$$
\begin{aligned}
\overline{\frac{m_2-m}{m_1-m}} &= \overline{\frac{\frac{ad(p\overline{p}-1)}{2(ad\overline{p}-a-d+p)} + \frac{bc(p\overline{p}-1)}{2(bc\overline{p}-b-c+p)} - \frac{p}{2}}{\frac{ab(p\overline{p}-1)}{2(ab\overline{p}-a-b+p)} + \frac{cd(p\overline{p}-1)}{2(cd\overline{p}-c-d+p)} - \frac{p}{2}}} \\
&= \overline{\frac{\frac{ad(p\overline{p}-1)}{ad\overline{p}-a-d+p} + \frac{bc(p\overline{p}-1)}{bc\overline{p}-b-c+p} - p}{\frac{ab(p\overline{p}-1)}{ab\overline{p}-a-b+p} + \frac{cd(p\overline{p}-1)}{cd\overline{p}-c-d+p} - p}} \\
&= \overline{\frac{\frac{p\overline{p}-1}{\overline{p}-\frac{1}{a}-\frac{1}{d}+\frac{p}{ad}} + \frac{p\overline{p}-1}{\overline{p}-\frac{1}{b}-\frac{1}{c}+\frac{p}{bc}} - p}{\frac{p\overline{p}-1}{\overline{p}-\frac{1}{a}-\frac{1}{b}+\frac{p}{ab}} + \frac{p\overline{p}-1}{\overline{p}-\frac{1}{c}-\frac{1}{d}+\frac{p}{cd}} - p}} \\
&= \frac{\frac{p\overline{p}-1}{p-a-d+ad\overline{p}} + \frac{p\overline{p}-1}{p-b-c+bc\overline{p}} - \overline{p}}{\frac{p\overline{p}-1}{p-a-b+ab\overline{p}} + \frac{p\overline{p}-1}{p-c-d+cd\overline{p}} - \overline{p}}.
\end{aligned}
$$

We omit the calculation, but state the result that the conjugate of the penultimate expression and last expression are equal, implying that the quantity is real. □

Problem 6.45 (Shortlist 1998/G6)

Let $ABCDEF$ be a convex hexagon such that $\angle B + \angle D + \angle F = 360^\circ$ and

$$
\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1.
$$

Prove that

$$
\frac{BC}{CA} \cdot \frac{AE}{EF} \cdot \frac{FD}{DB} = 1.
$$

Toss onto the complex plane so that $(ABCDEF)$ is the unit circle. Then, we can encompass the two conditions above in the single condition:

$$
\frac{b-a}{b-c} \cdot \frac{d-c}{d-e} \cdot \frac{f-e}{f-a} = 1.
$$

Now, this implies that

$$
-ace + acf + ade - adf + bce - bcf - bde + bdf = -abd + abe + acd - ace + bdf - bef - cdf + cef
$$

by expansion. Moving around terms, we have that

$$-abe + abf + ade - adf + bce - bcf - cde + cdf = -abd + abf + acd - acf + bde - bef - cde + cef$$

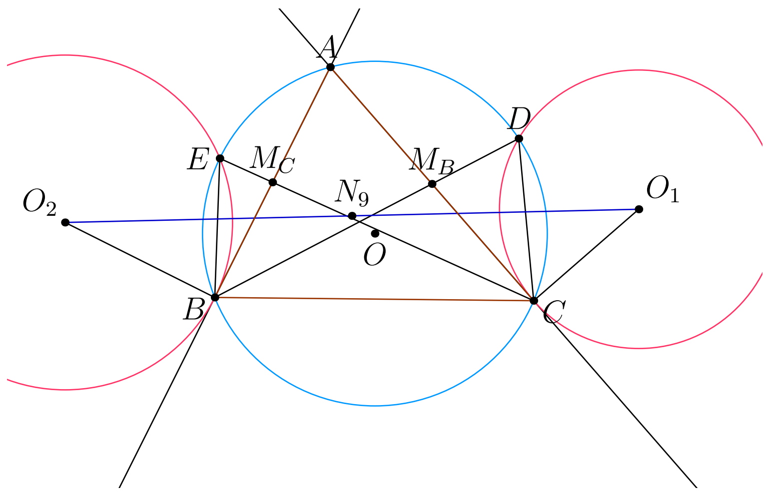
which directly implies

$$\frac{b-c}{a-c} \cdot \frac{a-e}{f-e} \cdot \frac{f-d}{b-d} = 1$$

as required. □

Problem 6.46 (ELMO SL 2013)

Let ABC be a triangle inscribed in circle ω, and let the medians from B and C intersect ω at D and E respectively. Let O_1 be the center of the circle through D tangent to AC at C, and let O_2 be the center of the circle through E tangent to AB at B. Prove that O_1, O_2, and the nine-point center of $\triangle ABC$ are collinear.



Toss onto the complex plane with (ABC) as the unit circle and $O = 0$. Let the midpoint of $\overline{AC}$ be M_B, and similarly for $\overline{AB}$. Then, we have that:

$$\begin{aligned} m_b = \frac{a+c}{2} &= \frac{ac(b+d) - bd(a+c)}{ac - bd} \\ (a+c)(ac-bd) &= 2ac(b+d) - 2bd(a+c) \\ a^2c + ac^2 - 2abc &= 2acd - abd - bcd \\ \frac{a^2c + ac^2 - 2abc}{2ac - ab - bc} &= d \end{aligned}$$

and similarly,

$$\frac{a^2b + ab^2 - 2abc}{2ab - ac - bc} = e.$$

Claim

We claim that $\triangle AOD \sim \triangle CO_1D$.

Proof. Clearly, they are both isosceles. In addition,

$$\angle AOD = 2\angle ABD = 2\angle ACD = 2(90 - \angle DCO_1) = \angle CO_1D$$

so we are done. □ □

Hence, by the claim, we know that

$$\frac{o-d}{a-d}=\frac{o_1-d}{c-d} \implies o_1=\frac{-d(c-d)}{a-d}+d \quad \text{and} \quad o_2=\frac{-e(b-e)}{a-e}+e.$$

Now, we wish to show that

$$\frac{o_2-n_9}{o_1-n_9}=\frac{\frac{-e(b-e)}{a-e}+e-\frac{a+b+c}{2}}{\frac{-d(c-d)}{a-d}+d-\frac{a+b+c}{2}}$$

is real. Finally,

$$\begin{aligned}
\frac{\frac{-e(b-e)}{a-e}+e-\frac{a+b+c}{2}}{\frac{-d(c-d)}{a-d}+d-\frac{a+b+c}{2}} &= \frac{(a-d)\left(-e(b-e)+ae-e^2-\frac{(a+b+c)(a-e)}{2}\right)}{(a-e)\left(-d(c-d)+ad-d^2-\frac{(a+b+c)(a-d)}{2}\right)} \\
&= \frac{(a-d)\left(ae-be-\frac{(a+b+c)(a-e)}{2}\right)}{(a-e)\left(ad-cd-\frac{(a+b+c)(a-d)}{2}\right)} \\
&= \frac{(a-\frac{a^2c+ac^2-2abc}{2ac-ab-bc})\left(ae-be-\frac{(a+b+c)(a-e)}{2}\right)}{(a-e)\left(\frac{a(a^2c+ac^2-2abc)}{2ac-ab-bc}-\frac{c(a^2c+ac^2-2abc)}{2ac-ab-bc}-\frac{(a+b+c)\left(a-\frac{a^2c+ac^2-2abc}{2ac-ab-bc}\right)}{2}\right)} \\
&= \frac{(a(2ac-ab-bc)-a^2c-ac^2+2abc)\left(ae-be-\frac{(a+b+c)(a-e)}{2}\right)}{(a-e)\left((a-c)(a^2c+ac^2-2abc)-\frac{(a+b+c)\left(a(2ac-ab-bc)-a^2c-ac^2+2abc\right)}{2}\right)} \\
&= \frac{(a(a-c)(c-b))\left(ae-be-\frac{(a+b+c)(a-e)}{2}\right)}{(a-e)\left(ac(a-c)(a+c-2b)-\frac{(a+b+c)(a(a-c)(c-b))}{2}\right)} \\
&= \frac{(c-b)\left(ae-be-\frac{(a+b+c)(a-e)}{2}\right)}{(a-e)\left(c(a+c-2b)-\frac{(a+b+c)(c-b)}{2}\right)} \\
&= \frac{(c-b)\left((a-b)\left(\frac{a^2b+ab^2-2abc}{2ab-ac-bc}\right)-\frac{(a+b+c)\left(a-\frac{a^2b+ab^2-2abc}{2ab-ac-bc}\right)}{2}\right)}{\left(a-\frac{a^2b+ab^2-2abc}{2ab-ac-bc}\right)\left(c(a+c-2b)-\frac{(a+b+c)(c-b)}{2}\right)} \\
&= \frac{(c-b)\left(ab(a-b)\,(a+b-2c)-\frac{a(a+b+c)(a-b)(b-c)}{2}\right)}{a(a-b)(b-c)\left(c(a+c-2b)-\frac{(a+b+c)(c-b)}{2}\right)} \\
&= -\frac{2b\,(a+b-2c)-(a+b+c)(b-c)}{2c(a+c-2b)-(a+b+c)(c-b)} \\
&= -1
\end{aligned}$$

which is a real quantity, as required. □

Remark

It turns out that N_9 is the midpoint of $\overline{O_1O_2}$.

7 Barycentric Coordinates

Problem 7.5

Find the barycentric coordinates for the midpoint of $\overline{AB}$.

It is clearly $\boxed{\left(\frac{1}{2}, \frac{1}{2}, 0\right)}$.

Problem 7.6 (Barycentric Coordinates)

Let $P = (x : y : z)$ be a point with x, y, $z \neq 0$. Show that the isogonal conjugate of P is given by

$$P^* = \left(\frac{a^2}{x} : \frac{b^2}{y} : \frac{c^2}{z}\right)$$

and the isotomic conjugate is given by

$$P^t = \left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z}\right).$$

We begin with the first problem. Denote A' as the intersection of AP with BC and similarly for B' and C'. Then, we have that

$$A' = (0 : y : z), \quad B' = (x : 0 : z), \quad C' = (x : y : 0).$$

Now, let A^* be on BC so that AA' is isogonal to AA^*. Then, by Problem 4.22 we know that

$$\frac{BA'}{A'C} \cdot \frac{BA^*}{A^*C} = \left(\frac{AB}{AC}\right)^2.$$

However, we have that $\frac{BA'}{A'C} = \frac{z}{y}$. Hence, we must have that

$$\frac{BA^*}{A^*C} = \left(\frac{c}{b}\right)^2 \cdot \frac{y}{z} \implies A^* = \left(0 : b^2z : c^2y\right).$$

We find similar equations for B^* and C^*. Hence, we find that the point

$$\left(\frac{a^2}{x} : \frac{b^2}{y} : \frac{c^2}{z}\right)$$

can be scaled further by yz, xz, and xy to show that it lies on AA^*, BB^*, and CC^*, respectively, so we have the required point.

Now, we will find the isotomic conjugate. Denote A', B', and C' as in the last problem. Then, let A^t be a point on BC such that AA' and AA^t are isotomic to each other, and define B^t and C^t respectively. Then, note that

$$A^t = (0 : z : y), \quad B^t = (z : 0 : x), \quad C^t = (y : x : 0).$$

Hence, we find the point that lies on all of these is the given point, $\left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z}\right)$ so we are done. □

Problem 7.8

Using the areal definition, show that $I = (a : b : c)$. Deduce the angle bisector theorem.

We have that
$$[PBC] : [PAC] : [PAB] = a : b : c$$
due to the base and height formula for the area of a triangle. Hence, we have the first part. The second part follows because the points on the sides are of the form $(ua : vb : wc)$ where (u, v, w) is some permutation of $(0, 1, 1)$. $\square$

Problem 7.9

Find the barycentric coordinates for the intersection of the symmedian from A and the median from B.

Let the point be $P = (x : y : z)$. Then, we have that
$$y : z = b^2 : c^2$$
and
$$x : z = 1 : 1.$$
Hence, the point is
$$P = \boxed{(c^2 : b^2 : c^2)}$$
as required. $\square$

Problem 7.17 (Barycentric Circumcircle)

The circumcircle (ABC) of the reference triangle has equation
$$a^2yz + b^2zx + c^2xy = 0.$$

The equation for any circle is
$$-a^2yz - b^2zx - c^2xy + (ux + vy + wz)(x + y + z) = 0$$
for some reals u, v, and w. Now, we have that the circle must pass through $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. Hence, plugging these in, we find that $u = v = w = 0$, implying the necessary conclusion. $\square$

Problem 7.18

Consider a displacement vector $\overrightarrow{PQ} = (x_1, y_1, z_1)$. Show that $PQ \perp BC$ if and only if
$$0 = a^2(z_1 - y_1) + x_1(c^2 - b^2).$$

The displacement vector $\overrightarrow{BC}$ is $(0, -1, 1)$. Hence, plugging this into the criteria for perpendicular displacement vectors, we have that:
$$0 = a^2(y_1 - z_1) + b^2x_1 - c^2x_1$$
which is precisely the necessary statement. $\square$

Problem 7.19 (Barycentric Perpendicular Bisector)

The perpendicular bisector of $\overline{BC}$ has equation

$$0 = a^2(z-y) + x(c^2-b^2).$$

Let $P = (x, y, z)$, and $M = \left(0, \frac{1}{2}, \frac{1}{2}\right)$ be the midpoint of $\overline{BC}$. Then, the displacement vector $\overrightarrow{PM} = \left(-x, \frac{1}{2}-y, \frac{1}{2}-z\right)$. So, by Problem 7.18, we have the required conclusion. $\square$

Problem 7.31

Let ABC be a triangle with altitude $\overline{AL}$ and let M be the midpoint of $\overline{AL}$. If K is the symmedian point of triangle ABC, prove that $\overline{KM}$ bisects $\overline{BC}$.

We know that if $L = (L_x, L_y, L_z)$, then

$$0 = a^2(L_y - L_z) + (L_x - 1)(b^2 - c^2).$$

However, we know that $L_x = 0$. Hence,

$$0 = a^2(L_y - L_z) - (b^2 - c^2) \implies L_y - L_z = \frac{b^2 - c^2}{a^2}.$$

Now, since $L_y + L_z = 1$, we can solve for both of them, getting that

$$\begin{aligned} L_z &= \frac{a^2 + b^2 - c^2}{2a^2} \\ L_y &= \frac{a^2 + c^2 - b^2}{2a^2}. \end{aligned}$$

Now, the midpoint of $\overline{AL}$ is then

$$M = \left(\tfrac{1}{2}, \tfrac{a^2+b^2-c^2}{4a^2}, \tfrac{a^2+c^2-b^2}{4a^2}\right).$$

In addition, we know that $K = (a^2, b^2, c^2)$. Now, the line through K and M can be represented as

$$ux + vy + z = 0$$

for u, v satisfying

$$\frac{u}{2} + \frac{a^2v + b^2v - c^2v}{4a^2} + \frac{a^2 + c^2 - b^2}{4a^2} = 0$$

and

$$a^2u + b^2v + c^2 = 0 \implies u = \frac{-b^2v - c^2}{a^2}.$$

Hence,

$$\begin{aligned} \frac{u}{2} + \frac{a^2v + b^2v - c^2v}{4a^2} + \frac{a^2 + c^2 - b^2}{4a^2} &= \frac{-b^2v - c^2}{2a^2} + \frac{a^2v + b^2v - c^2v}{4a^2} + \frac{a^2 + c^2 - b^2}{4a^2} \\ &= \frac{-2b^2v - 2c^2}{4a^2} + \frac{a^2v + b^2v - c^2v}{4a^2} + \frac{a^2 + c^2 - b^2}{4a^2} \\ &= \frac{a^2v - b^2v - c^2v}{4a^2} + \frac{a^2 - b^2 - c^2}{4a^2} \\ &= 0. \end{aligned}$$

Thus, $v = -1$, which implies the required conclusion. $\square$

Problem 7.32

Let I and G denote the incenter and centroid of a triangle ABC and let N denote the Nagel point; this is the intersection of the cevians that join A to the contact point of the A-excircle on BC, and similarly for B and C. Prove that I, G, N are collinear and that $NG = 2GI$.

Call the contact point of the A-excircle with BC as $A' = (0, A'_y, 1 - A'_y)$. Then, since $I_A = \left(\frac{-a}{b+c-a}, \frac{b}{b+c-a}, \frac{c}{b+c-a}\right)$, we know that $\overrightarrow{I_A A'} = \left(\frac{a}{b+c-a}, A'_y - \frac{b}{b+c-a}, 1 - A'_y - \frac{c}{b+c-a}\right)$. Hence,

$$0 = a^2\left(2A'_y - 1 + \frac{c-b}{b+c-a}\right) + \frac{a}{b+c-a}\left(b^2 - c^2\right).$$

Hence,

$$A'_y = \frac{a-b+c}{2a}$$

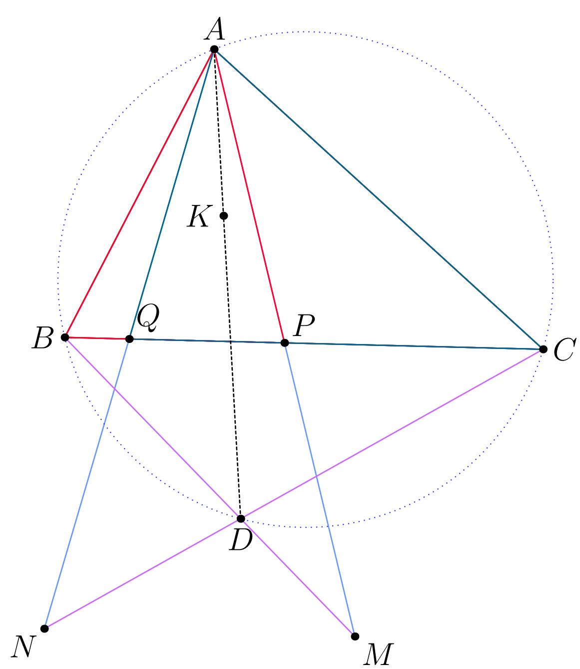
and we get similar equations for B' and C'. After solving a quick system of ratios, we find that

$$N = (b+c-a : a+c-b : a+b-c) = \left(1 - \frac{2a}{a+b+c}, 1 - \frac{2b}{a+b+c}, 1 - \frac{2c}{a+b+c}\right).$$

After more computations, we then can find the required conclusion since $I = \left(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c}\right)$ and $G = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. $\square$

Problem 7.33 (IMO 2014/4)

Let P and Q be on segment $\overline{BC}$ of an acute triangle ABC such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Let M and N be the points on AP and AQ, respectively, such that P is the midpoint of $\overline{AM}$ and Q is the midpoint of $\overline{AN}$. Prove that the intersection of BM and CN is on the circumference of triangle ABC.



By similar triangles, we have that

$$\frac{BP}{AB} = \frac{AB}{BC} \implies BP = \frac{AB^2}{BC}.$$

Similarly,

$$CQ = \frac{AC^2}{BC}.$$

Hence, using barycentric coordinates, we have that

$$P = \left(0, 1 - \tfrac{c^2}{a^2}, \tfrac{c^2}{a^2}\right) \qquad Q = \left(0, \tfrac{b^2}{a^2}, 1 - \tfrac{b^2}{a^2}\right).$$

Thus, we find that

$$M = \left(-1, 2 - \tfrac{2c^2}{a^2}, \tfrac{2c^2}{a^2}\right) \qquad N = \left(-1, \tfrac{2b^2}{a^2}, 2 - \tfrac{2b^2}{a^2}\right).$$

Now, let $D = BM \cap CN = (D_x, D_y, D_z)$. Then, we find that

$$D_x : D_y = -1 : \tfrac{2b^2}{a^2} \qquad D_x : D_z = -1 : \tfrac{2c^2}{a^2}$$

implying that

$$D = \left(-1 : \tfrac{2b^2}{a^2} : \tfrac{2c^2}{a^2}\right) = \left(\tfrac{-a^2}{2b^2+2c^2-a^2}, \tfrac{2b^2}{2b^2+2c^2-a^2}, \tfrac{2c^2}{2b^2+2c^2-a^2}\right).$$

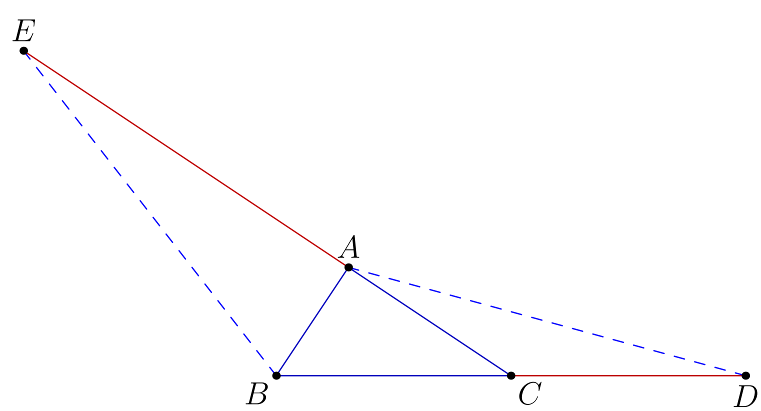
Finally,

$$\begin{aligned} & a^2\left(\tfrac{2b^2}{2b^2+2c^2-a^2}\right)\left(\tfrac{2c^2}{2b^2+2c^2-a^2}\right) + b^2\left(\tfrac{-a^2}{2b^2+2c^2-a^2}\right)\left(\tfrac{2c^2}{2b^2+2c^2-a^2}\right) + c^2\left(\tfrac{-a^2}{2b^2+2c^2-a^2}\right)\left(\tfrac{2b^2}{2b^2+2c^2-a^2}\right) \\ =& \left(\tfrac{1}{2b^2+2c^2-a^2}\right)^2 (4a^2b^2c^2 - 2a^2b^2c^2 - 2a^2b^2c^2) \\ =& 0 \end{aligned}$$

as desired. $\square$

Remark
The symmedian point K lies on AD.

Problem 7.34 (EGMO 2013/1, barycentric)
The side BC of the triangle ABC is extended beyond C to D so that $CD = BC$. The side CA is extended beyond A to E so that $AE = 2CA$. Prove that if $AD = BE$ then the triangle ABC is right-angled.



Clearly, $D = (0, -1, 2)$ and $E = (3, 0, -2)$. Now, $\overrightarrow{AD} = (-1, -1, 2)$ and $\overrightarrow{BE} = (3, -1, -2)$. Then,

$$AD = \left|a^2(-1)(2) + b^2(-1)(2) + c^2(-1)(-1)\right| = \left|c^2 - 2a^2 - 2b^2\right| \overset{\triangle \text{ Ineq}}{=} 2a^2 + 2b^2 - c^2.$$

Similarly,

$$BE = \left|a^2(-1)(-2) + b^2(3)(-2) + c^2(3)(-1)\right| = \left|2a^2 - 6b^2 - 3c^2\right| \overset{\triangle \text{ Ineq}}{=} 6b^2 + 3c^2 - 2a^2.$$

Equating, we have that

$$a^2 + b^2 = c^2$$

as required. $\square$

Remark

For a complex bash, look to Problem 5.19, and for a synthetic method, look to Problem 3.26.

Problem 7.35 (ELMO SL 2013/G3)

In $\triangle ABC$, a point D lies on line BC. The (ABD) meets AC at $F \neq A$, and (ADC) meets AB at $E \neq A$. Prove that as D varies, (AEF) always passes through a fixed point other than A, and that this point lies on the A-median.

Denote $D = (0, d, 1-d)$. Then, we have that

$$(ABD) : -a^2yz - b^2zx - c^2xy + wz(x + y + z) = 0.$$

Now, it also passes through D, so

$$-a^2d(1-d) + w(1-d) = 0.$$

Since $D \neq B$, we can divide out $1 - d$ to get that $w = a^2d$. Hence,

$$(ABD) : -a^2yz - b^2zx - c^2xy + a^2dz(x + y + z) = 0.$$

Then, let $F = (f, 0, 1-f)$ and since it lies on (ABD), we have that

$$-b^2f(1-f) + a^2d(1-f) = 0 \implies f = \tfrac{a^2d}{b^2}.$$

Now, let

$$(ADC) : -a^2yz - b^2zx - c^2xy + vy(x + y + z) = 0.$$

Since it passes through D, we then have that

$$-a^2d(1-d) + vd = 0 \implies v = a^2(1-d).$$

Hence,

$$(ADC) : -a^2yz - b^2zx - c^2xy + a^2y(1-d)(x + y + z) = 0.$$

Since $E = (e, 1-e, 0)$ lies on (ADC), we find that

$$-c^2e(1-e) + a^2(1-e)(1-d) = 0 \implies e = \tfrac{a^2(1-d)}{c^2}.$$

Now, note that

$$(AEF) : -a^2yz - b^2zx - c^2xy + (my + nz)(x + y + z) = 0$$

and since it passes through E and F, we have that

$$-c^2e(1-e)+m(1-e)=0 \implies m=c^2e$$
$$-b^2f(1-f)+n(1-f)=0 \implies n=b^2f.$$

Hence,

$$(AEF): -a^2yz-b^2zx-c^2xy+(c^2ey+b^2fz)(x+y+z)=0.$$

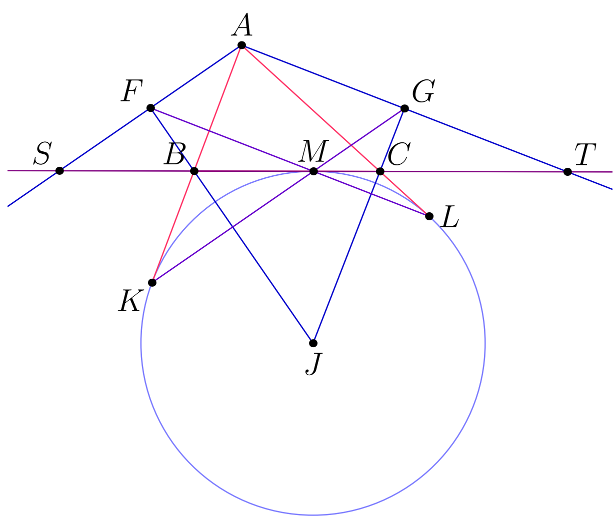
Now, any point on the A-median can be characterized by $\left(t:\frac{1}{2}:\frac{1}{2}\right)$. Plugging this in, we have that

$$\begin{aligned}
-\tfrac{a^2}{4}-\tfrac{b^2t}{2}-\tfrac{c^2t}{2}+\left(\frac{c^2e}{2}+\tfrac{b^2f}{2}\right)(t+1)=0& \\
a^2+2b^2t+2c^2t&=(2c^2e+2b^2f)(t+1)\\
a^2+2b^2t+2c^2t&=2c^2et+2b^2ft+2c^2e+2b^2f\\
2b^2t+2c^2t-2c^2et-2b^2ft&=2c^2e+2b^2f-a^2\\
t&=\frac{2c^2e+2b^2f-a^2}{2b^2+2c^2-2c^2e-2b^2f}\\
t&=\frac{2a^2(1-d)+2a^2d-a^2}{2b^2+2c^2-2a^2(1-d)-2a^2d}\\
t&=\frac{a^2}{2b^2+2c^2-2a^2}
\end{aligned}$$

which does not depend on D, as required. □

Problem 7.36 (IMO 2012/1)

Given triangle ABC, the point J is the center of the excircle opposite the vertex A. This excircle is tangent to side $\overline{BC}$ at M, and to lines AB and AC at K and L, respectively. Lines LM and BJ meet at F, and lines KM and CJ meet at G. Let S be the point of intersection of lines AF and BC, and let T be the point of intersection of lines AG and BC. Prove that M is the midpoint of $\overline{ST}$.



We have that $M=(0,m,1-m)$. Then, since $J=\left(\frac{-a}{b+c-a},\frac{b}{b+c-a},\frac{c}{b+c-a}\right)$, we have that

$\overrightarrow{JM} = \left(\frac{a}{b+c-a}, m - \frac{b}{b+c-a}, 1 - m - \frac{c}{b+c-a}\right)$. Hence,

$$0 = a^2\left(2m - 1 + \tfrac{c-b}{b+c-a}\right) + \tfrac{a}{b+c-a}(b^2 - c^2).$$

So,

$$m = \frac{a-b+c}{2a}.$$

Now, we know that $AK = \frac{a+b+c}{2}$, and $K = (t : 1 : 0) = \left(\frac{t}{t+1}, \frac{1}{t+1}, 0\right)$ for some t. Hence, $\overrightarrow{KA} = \left(\frac{-1}{t+1}, \frac{1}{t+1}, 0\right)$ which means that

$$c^2\left(\tfrac{1}{t+1}\right)^2 = \left(\tfrac{a+b+c}{2}\right)^2$$

so

$$t = \frac{-a-b+c}{a+b+c}.$$

Hence, we have that

$$K = \left(\frac{-a-b+c}{2c}, \frac{a+b+c}{2c}, 0\right).$$

Now, the line through C and J can be expressed as

$$bx + ay = 0$$

and the line through K and M can be expressed as

$$ux + vy + wz = 0.$$

However, we have that

$$u(-a-b+c) + v(a+b+c) = 0 \implies u = \frac{v(a+b+c)}{a+b-c}$$

since it passes through K and

$$v(a-b+c) + w(a+b-c) = 0 \implies w = \frac{v(a-b+c)}{-a-b+c}$$

since it passes through M. Hence, the line is

$$\frac{x(a+b+c)}{a+b-c} + y + \frac{z(a-b+c)}{-a-b+c} = 0.$$

Now, if $G = (G_a, G_b, G_c)$, then,

$$bG_a + aG_b = 0 \qquad \frac{G_a(a+b+c)}{a+b-c} + G_b + \frac{G_c(a-b+c)}{-a-b+c} = 0 \qquad G_a + G_b + G_c = 1.$$

Solving this, we find that

$$G = \left(\tfrac{1}{2}, -\tfrac{b}{2a}, \tfrac{a+b}{2a}\right).$$

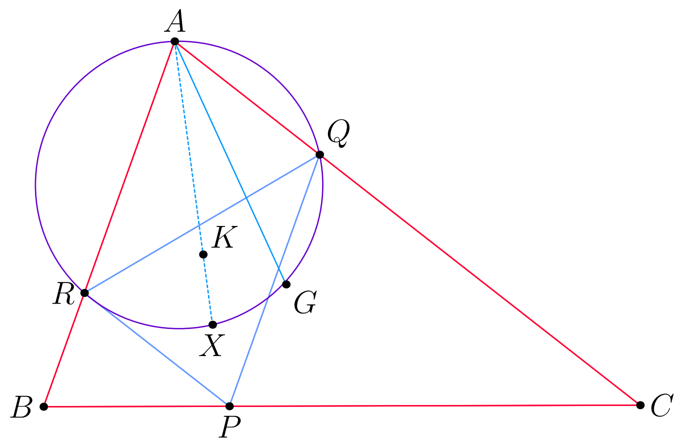
Now, clearly, the intersection of AG with BC is

$$T = \left(0, -\tfrac{c}{a}, \tfrac{a+c}{a}\right).$$

Hence, we have that $CT = b$, which implies $MT = MS$ after some elementary addition of lengths, and the fact that due to symmetry (or the result can be replicated), we have that $BS = c$. $\square$

Problem 7.38 (USA TST 2008/7)

Let ABC be a triangle with G as its centroid. Let P be a variable point on $\overline{BC}$. Points Q and R lie on sides AC and AB respectively, such that $PQ \parallel AB$ and $PR \parallel AC$. Prove that, as P varies along $\overline{BC}$, (AQR) passes through a fixed point X such that $\angle BAG = \angle CAX$.



Let K be the symmedian point of $\triangle ABC$. Then, we make the following claim:

Claim
Point X lies on the A-symmedian (line AK).

Proof. Due to the given angle condition, we know that lines AX and AG are isogonal. However, AK and AG are also isogonal, implying that A, K, and X are collinear. $\square$

Now, define $P = (0, p, 1-p)$. Then, we clearly have that

$$BR = \frac{AB \cdot BP}{BC} = \frac{c\sqrt{a^2(1-p)(1-p)}}{a} = c(1-p).$$

Hence, $R = (1-p, p, 0)$. Similarly, we find that $Q = (p, 0, 1-p)$. Now,

$$(AQR) : -a^2yz - b^2xz - c^2xy + (vy + wz)(x + y + z) = 0.$$

However, since it passes through Q, we know that

$$-b^2p(1-p) + w(1-p) = 0 \implies w = b^2p$$

and since it passes through R,

$$-c^2p(1-p) + vp = 0 \implies v = c^2(1-p).$$

Hence,

$$(AQR) : -a^2yz - b^2xz - c^2xy + (c^2(1-p)y + b^2pz)(x + y + z) = 0.$$

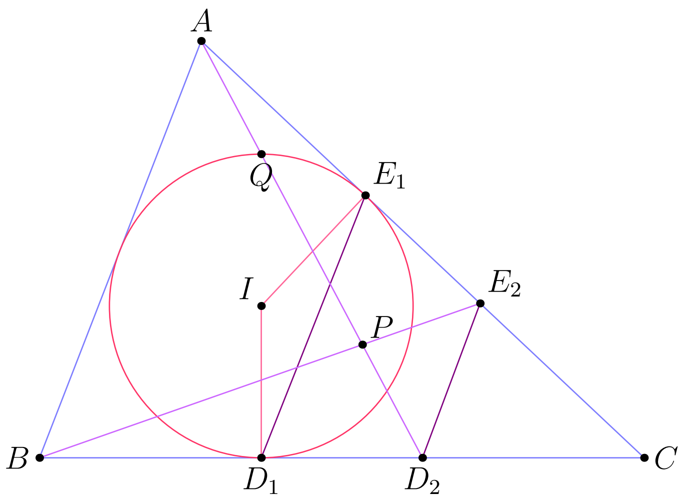
Now, a point on AK can be characterized as $\left(t : \frac{b^2}{a^2+b^2+c^2} : \frac{c^2}{a^2+b^2+c^2}\right) = (t(a^2+b^2+c^2) : b^2 : c^2)$. Substituting, we have that:

$$-a^2b^2c^2 - 2b^2c^2t(a^2+b^2+c^2) + b^2c^2(t(a^2+b^2+c^2) + b^2 + c^2) = 0$$

so t doesn't depend on p, as required. $\square$

Problem 7.39 (USAMO 2001/2)

Let ABC be a triangle and let ω be its incircle. Denote D_1 and E_1 as the points where ω is tangent to $\overline{BC}$ and $\overline{AC}$, respectively. Denote D_2 and E_2 as the points on $\overline{BC}$ and $\overline{AC}$, respectively, such that $CD_2 = BD_1$ and $CE_2 = AE_1$, and denote P as the point of intersection of $\overline{AD_2}$ and $\overline{BE_2}$. Circle ω intersects segment AD_2 at two points, the closer of which to the vertex A is denoted by Q. Prove that $AQ = D_2P$.



We let $D_1 = (0, d, 1-d)$. Then, since $I = \left(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c}\right)$, we know that

$$\overrightarrow{D_1I} = \left(\frac{a}{a+b+c}, \frac{b}{a+b+c} - d, \frac{c}{a+b+c} + d - 1\right).$$

Hence, because $D_1I \perp BC$, we know that

$$0 = a^2\left(2d - 1 + \frac{c-b}{a+b+c}\right) + \frac{a}{a+b+c}(c^2 - b^2) \implies d = \frac{a+b-c}{2a}.$$

Thus,

$$D_1 = \left(0, \frac{a+b-c}{2a}, \frac{a-b+c}{2a}\right).$$

Similarly, we find that

$$E_1 = \left(\frac{a+b-c}{2b}, 0, \frac{-a+b+c}{2b}\right).$$

However, clearly then, we have that

$$D_2 = \left(0, \frac{a-b+c}{2a}, \frac{a+b-c}{2a}\right)$$

and

$$E_2 = \left(\frac{-a+b+c}{2b}, 0, \frac{a+b-c}{2b}\right).$$

Then,

$$P = (-a+b+c : a-b+c : a+b-c) = \left(\frac{-a+b+c}{a+b+c}, \frac{a-b+c}{a+b+c}, \frac{a+b-c}{a+b+c}\right).$$

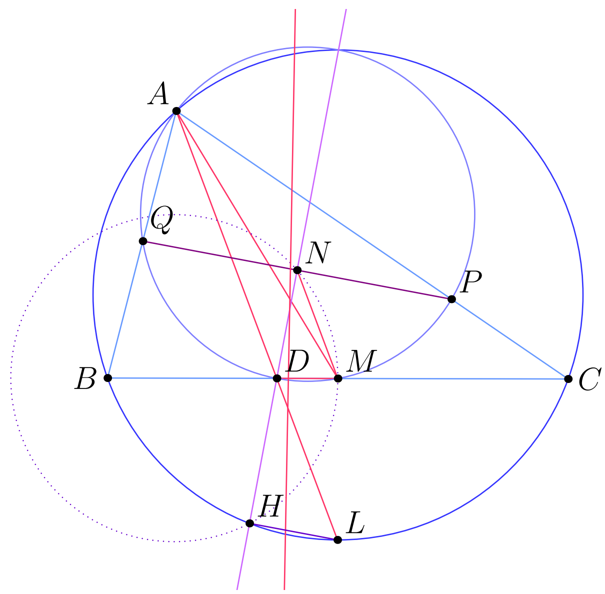
Now, it is well known that if D_2 is contact point of the A-excenter to $\overline{BC}$, (which it is), then the point where AD_2 intersects the incircle (closer to A) is diametrically opposite to D_1. Hence,

$$Q = \left(\frac{2a}{a+b+c}, \frac{2b}{a+b+c} - \frac{a+b-c}{2a}, \frac{2c}{a+b+c} - \frac{a-b+c}{2a}\right).$$

Finally, if one computes the displacement vectors, you find that $\overrightarrow{AQ} = \overrightarrow{PD_2}$, so we are done. $\square$

Problem 7.40 (USA TSTST 2012/7)

Let there be a triangle ABC such that the angle bisector of angle A intersects $\overline{BC}$ and (ABC) at D and $L \neq A$, respectively. Let M be the midpoint of $\overline{BC}$, and (ADM) intersect $\overline{AB}$ and $\overline{AC}$ again at Q and $P \neq A$, respectively. In addition, let N be the midpoint of $\overline{PQ}$, and let H be the foot of the perpendicular from L to line ND. Prove that line ML is tangent to (HMN).



We make the following claim:

Claim
The lines AD and NM are parallel.

Proof. We clearly have that $M = (0, 0.5, 0.5)$, and $D = (0 : b : c) = \left(0, \frac{b}{b+c}, \frac{c}{b+c}\right)$. In addition,

$$(ADM) : -a^2yz - b^2xz - c^2xy + (vy + wz)(x + y + z) = 0$$

since it passes through A. In addition, since it passes through M, we have that

$$-\tfrac{a^2}{4} + 0.5v + 0.5w = 0 \implies v + w = \tfrac{a^2}{2}$$

and since it passes through D,

$$-a^2bc + (vb + wc)(b + c) = 0.$$

Solving, we find that $v = \frac{a^2c}{2b+2c}$, and $w = \frac{a^2b}{2b+2c}$. Hence,

$$(ADM) : -a^2yz - b^2xz - c^2xy + \left(\tfrac{a^2cy}{2b+2c} + \tfrac{a^2bz}{2b+2c}\right)(x + y + z) = 0.$$

Now, $Q \neq A$ is the intersection of this circle with AB. Hence, let $Q = (1 - q, q, 0)$ and substitute this into the circle equation to get that

$$-c^2q(1 - q) + \tfrac{a^2cq}{2b+2c} = 0 \implies 1 - q = \tfrac{a^2}{2bc+2c^2}.$$

So, $Q = \left(\frac{a^2}{2bc+2c^2}, 1 - \frac{a^2}{2bc+2c^2}, 0\right)$. Similarly, letting $P = (1-p, 0, p)$, we have that

$$-b^2p(1-p) + \frac{a^2bp}{2b+2c} = 0 \implies 1 - p = \frac{a^2}{2bc+2b^2}.$$

Hence, $P = \left(\frac{a^2}{2bc+2b^2}, 0, 1 - \frac{a^2}{2bc+2b^2}\right)$. Thus, the midpoint of $\overline{PQ}$,

$$N = \left(\frac{a^2}{4bc}, \frac{1}{2} - \frac{a^2}{4bc+4c^2}, \frac{1}{2} - \frac{a^2}{4bc+4b^2}\right).$$

Now, consider the displacement vectors $\overrightarrow{AD} = \left(-1, \frac{b}{b+c}, \frac{c}{b+c}\right)$ and $\overrightarrow{NM} = \left(\frac{a^2}{4bc}, \frac{-a^2}{4bc+4c^2}, \frac{-a^2}{4bc+4b^2}\right)$. They are clearly scalar multiples of each other, proving the desired statement. $\square$

Now, since $\angle DML + \angle DHL = 90 + 90 = 180$, we know that $DMLH$ is cyclic. Thus,

$$\angle HNM \overset{AD \parallel NM}{=} \angle HDL \overset{(DMLH)}{=} \angle HML$$

implying the tangency, as required. $\square$

Problem 7.41

Let ABC be a triangle with incenter I. Let P and Q denote the reflections of B and C across CI and BI, respectively. Show that $PQ \perp OI$, where O is the circumcenter of ABC.

Due to the definition of the angle bisector, P is the point on $\overrightarrow{CA}$ such that $BC = BP$. Similarly, Q is the point on $\overrightarrow{BA}$ such that $BQ = BC$. Hence, $P = (a : 0 : b - a) = \left(\frac{a}{b}, 0, 1 - \frac{a}{b}\right)$ and $Q = \left(\frac{a}{c}, 1 - \frac{a}{c}, 0\right)$. We have the displacement vector

$$\overrightarrow{PQ} = \left(\frac{a}{c} - \frac{a}{b}, 1 - \frac{a}{c}, \frac{a}{b} - 1\right) = \left(\frac{a}{c} - \frac{a}{b}\right)\overrightarrow{A} + \left(1 - \frac{a}{c}\right)\overrightarrow{B} + \left(\frac{a}{b} - 1\right)\overrightarrow{C}$$

and the psuedo-displacment vector

$$\overrightarrow{OI} = \overrightarrow{I} - \overrightarrow{O} = a\overrightarrow{A} + b\overrightarrow{B} + c\overrightarrow{C}$$

after scaling. Now, the quantity that we need to check is:

$$a^2\left(c\left(1 - \frac{a}{c}\right) + b\left(\frac{a}{b} - 1\right)\right) + b^2\left(c\left(\frac{a}{c} - \frac{a}{b}\right) + a\left(\frac{a}{b} - 1\right)\right) + c^2\left(b\left(\frac{a}{c} - \frac{a}{b}\right) + a\left(1 - \frac{a}{c}\right)\right)$$

which is indeed equal to zero after expanding. $\square$

Problem 7.42

Let ABC be a triangle with circumcircle Ω and let T_A denote the tangency points of the A-mixtilinear incircle to Ω. Define T_B and T_C similarly. Prove that lines AT_A, BT_B, CT_C, IO are concurrent, where I and O denote the incenter and circumcenter of $\triangle ABC$.

We know that all of the lines AT_A, BT_B, and CT_C pass through the isogonal conjugate of the Nagel Point since AT_A is isogonal to the cevian through A and the contact point of the A-excircle and likewise for the other lines. Hence, all we have to show is that IO passes through this point as well. The Nagel Point is just $(s-a : s-b : s-c)$. The isogonal conjugate of this point is

$$\left(\frac{a^2}{s-a} : \frac{b^2}{s-b} : \frac{c^2}{s-c}\right).$$

Thus, all we have to show is that the determinant

$$\begin{vmatrix} a & b & c \\ a^2 S_a & b^2 S_b & c^2 S_c \\ \frac{a^2}{s-a} & \frac{b^2}{s-b} & \frac{c^2}{s-c} \end{vmatrix}$$

is zero. If it is zero, then we divide each of the columns by their gcd to get the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ aS_a & bS_b & cS_c \\ \frac{a}{s-a} & \frac{b}{s-b} & \frac{c}{s-c} \end{vmatrix}.$$

Expanding, we have that:

$$\begin{aligned}
\begin{vmatrix} 1 & 1 & 1 \\ aS_a & bS_b & cS_c \\ \frac{a}{s-a} & \frac{b}{s-b} & \frac{c}{s-c} \end{vmatrix} &= \frac{bcS_b}{s-c} - \frac{bcS_c}{s-b} - \frac{acS_a}{s-c} + \frac{acS_c}{s-a} + \frac{abS_a}{s-b} - \frac{abS_b}{s-a} \\
&= \frac{a}{s-a}(cS_c - bS_b) + \frac{b}{s-b}(aS_a - cS_c) + \frac{c}{s-c}(bS_b - aS_a) \\
&= \frac{a}{s-a}\left(\frac{a^2c+b^2c-c^3}{2} - \frac{a^2b+bc^2-b^3}{2}\right) \\
&\quad + \frac{b}{s-b}\left(\frac{ab^2+ac^2-a^3}{2} - \frac{a^2c+b^2c-c^3}{2}\right) \\
&\quad + \frac{c}{s-c}\left(\frac{a^2b+bc^2-b^3}{2} - \frac{ab^2+ac^2-a^3}{2}\right).
\end{aligned}$$

Now, we multiply by a factor of four to get rid of the denominators and the s term which means that the last expression is equal to:

$$\begin{aligned}
&\frac{a}{-a+b+c}(a^2c+b^2c-c^3-a^2b-bc^2+b^3) + \frac{b}{a-b+c}(ab^2+ac^2-a^3-a^2c-b^2c+c^3) \\
&\quad + \frac{c}{a+b-c}(a^2b+bc^2-b^3-ab^2-ac^2+a^3) \\
&= -a(-ab+ac-b^2+c^2) - b(ab-bc+a^2-c^2) - c(bc-ac+b^2-a^2) \\
&= 0
\end{aligned}$$

so they are indeed collinear. □

Problem 7.43 (USA December TST for IMO 2012)

In acute triangle ABC, $\angle A < \angle B$ and $\angle A < \angle C$. Let P be a variable point on $\overline{BC}$. Points D and E lie on $\overline{AB}$ and $\overline{AC}$, respectively, such that $BP = PD$ and $CP = PE$. Prove that as P moves along $\overline{BC}$, the (ADE) passes through a fixed point other than A.

We claim this fixed point is the orthocenter H.

Since $PB = PD$,

$$BD = 2BP\sin\left(\tfrac{\angle BPD}{2}\right) = 2BP\sin(90 - B) = 2BP\cos(B)$$

and similarly

$$CE = 2CP\cos(C) = 2a\cos(C) - 2BP\cos(C).$$

Hence,

$$D = (2BP\cos(B) : c - 2BP\cos(B) : 0)$$

and

$$E = (2a\cos(C) - 2BP\cos(C) : 0 : b - 2a\cos(C) + 2BP\cos(C)).$$

Now, since (ADE) passes through A, we know that

$$(ADE) : -a^2yz - b^2xz - c^2xy + (vy + wz)(x + y + z) = 0.$$

Since it passes through D,

$$-c^2(2BP\cos(B))(c - 2BP\cos(B)) + cv(c - 2BP\cos(B)) = 0$$

implying that

$$v = 2cBP\cos(B).$$

In addition, since it passes through E,

$$-b^2(2a\cos(C) - 2BP\cos(C))(b - 2a\cos(C) + 2BP\cos(C)) + bw(b - 2a\cos(C) + 2BP\cos(C)) = 0$$

implying

$$w = b(2a\cos(C) - 2BP\cos(C)) = 2ab\cos(C) - 2bBP\cos(C).$$

Hence,

$$(ADE) : -a^2yz - b^2xz - c^2xy + ((2cBP\cos(B))y + (2ab\cos(C) - 2bBP\cos(C))z)(x + y + z) = 0.$$

Now, since $H = (\tan(A), \tan(B), \tan(C))$,

$$\begin{aligned}
&- a^2\tan(B)\tan(C) - b^2\tan(A)\tan(C) - c^2\tan(A)\tan(B)\\
&+ ((2cBP\cos(B))\tan(B) + (2ab\cos(C) - 2bBP\cos(C))\tan(C))(\tan(A) + \tan(B) + \tan(C))\\
=&- a^2\tan(B)\tan(C) - b^2\tan(A)\tan(C) - c^2\tan(A)\tan(B)\\
&+ \tan(A)\tan(B)\tan(C)(2cBP\sin(B) + 2ab\sin(C) - 2bBP\sin(C))\\
=&- a^2\tan(B)\tan(C) - b^2\tan(A)\tan(C) - c^2\tan(A)\tan(B)\\
&+ \tan(A)\tan(B)\tan(C)(2cBP\sin(B) + 2ab\sin(C) - 2bBP\sin(C))\\
=&- a^2\tan(B)\tan(C) - b^2\tan(A)\tan(C) - c^2\tan(A)\tan(B) + 2ab\sin(C)\tan(A)\tan(B)\tan(C)\\
=&- a^2\tan(B)\tan(C) - b^2\tan(A)\tan(C) - c^2\tan(A)\tan(B) + 4K\tan(A)\tan(B)\tan(C)\\
=&- a^2\tan(B)\tan(C) - b^2\tan(A)\tan(C) - c^2\tan(A)\tan(B) + \frac{abc\tan(A)\tan(B)\tan(C)}{R}.
\end{aligned}$$

Dividing by $\tan(A)\tan(B)\tan(C)$, we have that

$$\begin{aligned}
-\frac{a^2}{\tan(A)} - \frac{b^2}{\tan(B)} - \frac{c^2}{\tan(C)} + \frac{abc}{R} &= -\frac{a^2\cos(A)}{\sin(A)} - \frac{b^2\cos(B)}{\sin(B)} - \frac{c^2\cos(C)}{\sin(C)} + \frac{abc}{R}\\
&= -2(aR\cos(A) + bR\cos(B) + cR\cos(C)) + \frac{abc}{R}\\
&= -2(2R^2\sin(A)\cos(A) + 2R^2\sin(B)\cos(B)\\
&\quad + 2R^2\sin(C)\cos(C)) + \frac{abc}{R}\\
&= -2(R^2\sin(2A) + R^2\sin(2B) + R^2\sin(2C)) + \frac{abc}{R}\\
&= -8R^2(\sin(A)\sin(B)\sin(C)) + \frac{abc}{R}\\
&= 0
\end{aligned}$$

as required, where the fourth line follows from sum to product. $\square$

Problem 7.44 (Sharygin 2013)

Let C_1 be an arbitrary point on side $\overline{AB}$ of $\triangle ABC$. Points A_1 and B_1 are on rays BC and AC such that $\angle AC_1B_1 = \angle BC_1A_1 = \angle ACB$. The lines AA_1 and BB_1 meet in point C_2. Prove that all the lines C_1C_2 have a common point.

Let $C_1 = (c_1, 1-c_1, 0)$. Then, notice that A_1 is just the intersection of (AC_1C) with BC, and similarly for B_1. Computing, we find that

$$(AC_1C) : -a^2yz - b^2xz - c^2xy + c^2c_1y(x+y+z) = 0.$$

Hence, $A_1 = \left(0, 1 - \frac{c^2c_1}{a^2}, \frac{c^2c_1}{a^2}\right)$. Similarly, we find that $B_1 = \left(1 - \frac{c^2(1-c_1)}{b^2}, 0, \frac{c^2(1-c_1)}{b^2}\right)$. Now, we have that C_2 is just

$$C_2 = \left(\frac{b^2}{c^2(1-c_1)} - 1 : \frac{a^2}{c^2c_1} - 1 : 1\right).$$

Now, we wish to show that there exists a point $P = (x, y, z)$ such that:

$$\begin{aligned}\begin{vmatrix} x & y & z \\ c_1 & 1-c_1 & 0 \\ \dfrac{b^2}{c^2(1-c_1)} - 1 & \dfrac{a^2}{c^2c_1} - 1 & 1 \end{vmatrix} &= x(1-c_1) - yc_1 + z\left(\frac{a^2}{c^2} - \frac{b^2}{c^2}\right) \\ &= 0.\end{aligned}$$

Now, notice that the point $P = (a^2 - b^2 - c^2 : b^2 + c^2 - a^2 : c^2)$ annihilates all the terms, so we have the required point. $\square$

Problem 7.45 (APMO 2013/5)

Let $ABCD$ be a quadrilateral inscribed in a circle ω, and let P be a point on AC such that $\overline{PB}$ and $\overline{PD}$ are tangent to ω. The tangent at C intersects PD at Q and the line AD at R. Let $E \neq A$ be the intersection between AQ and ω. Prove that B, E, R are collinear.

We use barycentrics on $\triangle BCD$, where $B = (1,0,0)$, $C = (0,1,0)$, and $D = (0,0,1)$. Notice that CA is a symmedian, so $A = (b^2 : \lambda : d^2)$, and since A lies on $(BCD) : b^2yz + c^2xz + d^2xy = 0$, we know that $A = (2b^2 : -c^2 : 2d^2)$. In addition, we may compute that $P = (b^2 : -c^2 : d^2)$. Since Q is the intersection of PD and the tangent at C, we know that $Q = (b^2 : -c^2 : -d^2)$. Similarly, since R is the intersection of AD and the tangent at C, we know that $R = (-2b^2 : -c^2 : 2d^2)$. Now, the line AQ $(ux + vy + wz = 0)$ satisfies

$$2b^2u - c^2v + 2d^2w = 0$$

and

$$b^2u - c^2v - d^2w = 0.$$

Solving, we get that:

$$AQ : \frac{3x}{b^2} + \frac{4y}{c^2} - \frac{z}{d^2} = 0.$$

Now,

$$(BCD) : \frac{yz}{c^2d^2} + \frac{xz}{b^2d^2} + \frac{xy}{b^2c^2} = 0.$$

We can let $\alpha = \dfrac{x}{b^2}$ and similar, to get that:

$$\begin{aligned} 3\alpha + 4\beta - \gamma &= 0 \\ \alpha + \beta + \gamma &= 0. \end{aligned}$$

Solving this, we find that $E = (2b^2 : -3c^2 : 6d^2)$. We can then write out the relevant collinearity equation:

$$\begin{aligned} \frac{1}{b^2c^2d^2} \begin{vmatrix} 2b^2 & -c^2 & 2d^2 \\ -2b^2 & -c^2 & 2d^2 \\ 2b^2 & -3c^2 & 6d^2 \end{vmatrix} &= -12 + 12 - 12 + 4 + 6 + 2 \\ &= 0 \end{aligned}$$

as required. □

8 Inversion

Problem 8.9 (Inverting an Orthocenter)

Let ABC be a triangle with orthocenter H and altitudes $\overline{AD}$, $\overline{BE}$, $\overline{CF}$. Perform an inversion around C with radius $\sqrt{CH \cdot CF}$. Where do the six points each go?

The points H and F swap places, as do A and E, as do C and F by Power of a Point. □

Problem 8.10 (Inverting a Circumcenter)

Let ABC be a triangle with circumcenter O. Invert around C with radius 1. What is the relation between O^*, C, A^*, and B^*?

We claim that A^*B^* has the property that C and O^* are reflections about this line. Clearly, $\angle CA^*B^* = \angle CBA$ and

$$\angle O^*A^*B^* = \angle CA^*O^* - \angle CA^*B^* = \angle COA - \angle CBA = \angle CBA$$

as required. We can get similar results for the other set of angles ($\angle CB^*A^*$ and $\angle O^*B^*A^*$), so we are done. □

Problem 8.11 (Inverting an Incircle)

Let ABC be a triangle with circumcircle Γ and contact triangle DEF. Consider an inversion with respect to the incircle of triangle ABC. Show that Γ is sent to the nine-point circle of triangle DEF.

Since AE and AF are tangent to the incircle, we know that A^* is just the midpoint of $\overline{EF}$ and similarly for B^* and C^*. Hence, the circumcircle of these points is just the nine-point circle, as required. □

Problem 8.17 (Force-Overlayed Inversion)

Let ABC be a triangle. Consider the transformation consisting of an inversion about A with radius $\sqrt{AB \cdot AC}$, followed by a reflection around the angle bisector of $\angle BAC$. Show that this transformation fixes B and C.

Clearly, the distance $AB^* = AC$ and since B^* lies on AB, we know that B^* and C are reflections about the angle bisector of $\angle BAC$. A similar argument suffices for C^*. □

Problem 8.19 (Inversion Distance Formula)

Let A and B be points other than O and consider an inversion about O with radius r. Then show that:

$$A^*B^* = \frac{r^2}{OA \cdot OB} \cdot AB.$$

We know that

$$OA \cdot OA^* = r^2$$

and

$$\frac{AB}{A^*B^*} = \frac{OB}{OA^*}$$

by similar triangles. Hence,

$$\frac{r^2}{OA \cdot OB} \cdot AB = \frac{OA^*}{OB} \cdot AB = A^*B^*$$

as required. □

Problem 8.20 (Ptolemy's Inequality)

For any four distinct points A, B, C, and D in a plane, no three collinear, prove that

$$AB \cdot CD + BC \cdot DA \geq AC \cdot BD.$$

Moreover, show that equality holds if and only if A, B, C, D lie on a circle in that order.

Invert about a circle centered at D with radius DA. Then, by the triangle inequality,

$$A^*B^* + B^*C^* \geq A^*C^*$$

with equality if and only if the points are collinear. Now applying the inversion distance formula,

$$\frac{DA^2}{DA \cdot DB} \cdot AB + \frac{DA^2}{DB \cdot DC} \cdot BC \geq \frac{DA^2}{DA \cdot DC} \cdot AC.$$

Simplifying,

$$\frac{1}{DB} \cdot AB + \frac{DA}{DB \cdot DC} \cdot BC \geq \frac{1}{DC} \cdot AC$$

and multiplying both sides by $DB \cdot DC$, we find that

$$DC \cdot AB + DA \cdot BC \geq DB \cdot AC$$

as required. □

Problem 8.23

Let ABC be a right triangle with $\angle C = 90°$ and let X and Y be points in the interiors of $\overline{CA}$ and $\overline{CB}$, respectively. Construct four circles passing through C, centered at A, B, X, Y. Prove that the four points lying on at exactly two of these four circles are concyclic.

Consider inverting about a circle with radius 1, centered at C. Then, all of the circles will be sent to lines perpendicular to the side that they are on. This means that the required intersections just make a rectangle, which is indeed cyclic. □

Problem 8.24

Let ω_1, ω_2, ω_3, ω_4 be circles with consecutive pairs tangent at A, B, C, D. Prove that quadrilateral $ABCD$ is cyclic.

Consider an inversion about A. Then, the conclusion is clear by noticing the similar triangles. □

Problem 8.25

Let A, B, C be three collinear points and P be a point not on this line. Prove that the circumcenters of $\triangle PAB$, $\triangle PBC$, and $\triangle PCA$ lie on a circle passing through P.

Inverting about P, it suffices to show by Problem 8.10, that the reflections of P across A^*B^*, B^*C^*, and A^*C^* are collinear, where A^*, B^*, C^*, and P are concyclic. Then, by the existence of the Simson Line, we can just consider a homothety of scale factor 2 centered at P, which takes the Simson Line to the necessary points, so we are done. $\square$

Problem 8.26 (BAMO 2008/6)

A point D lies inside triangle ABC. Let A_1, B_1, C_1 be the second intersection points of the lines AD, BD, and CD with (BDC), (CDA), and (ADB), respectively. Prove that

$$\frac{AD}{AA_1} + \frac{BD}{BB_1} + \frac{CD}{CC_1} = 1.$$

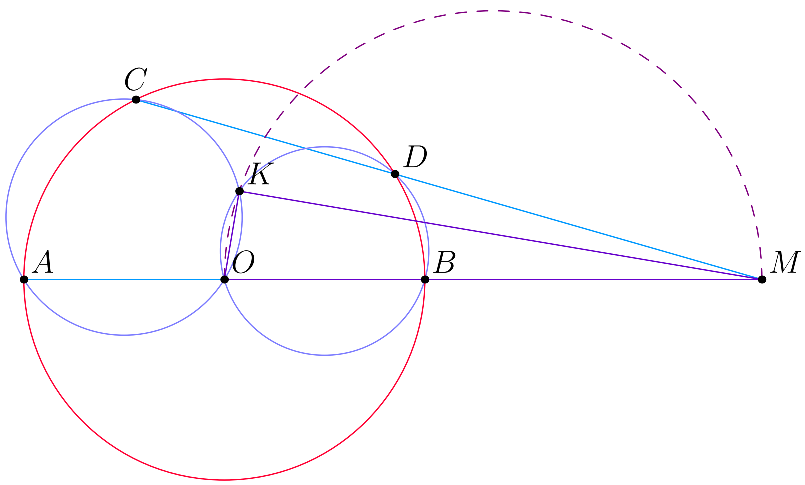
Inverting about D with a radius of 1, we find that

$$\frac{AD}{AA_1} = \frac{\frac{1}{A^*D}}{\frac{A^*B^*}{DA^* \cdot DB^*}} = \frac{DB^*}{A^*B^*}$$

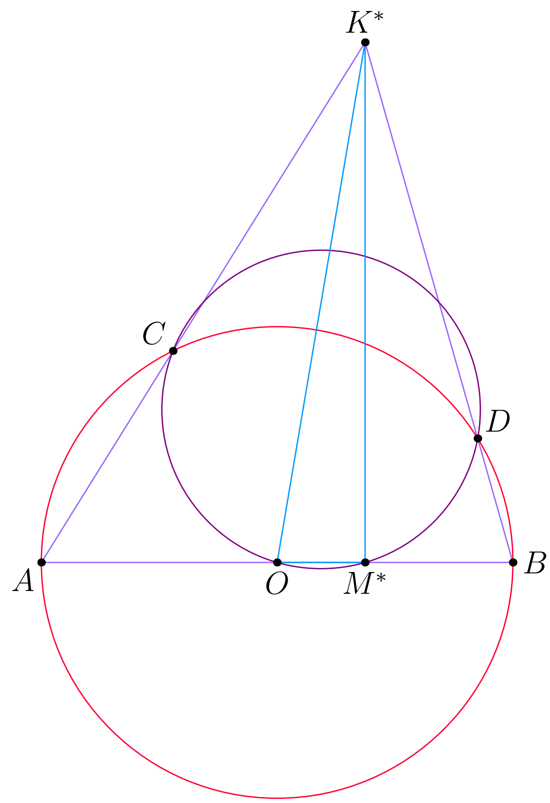
by the distance formula. So, the problem is in fact equivalent to Problem 3.18, and we are done. $\square$

Problem 8.27 (Iran Olympiad 1996)

Consider a semicircle with center O and diameter $\overline{AB}$. A line intersects line AB at M and the semicircle at C and D such that $MC > MD$ and $MB < MA$. Suppose (AOC) and (BOD) meet at a point $K \neq O$. Prove that $\angle MKO = 90^\circ$.



We invert about the circle centered at O with diameter $\overline{AB}$. Then, we wish to show that $\angle K^*M^*O$ is right, in the diagram below:



However, notice that since C is the foot of the B-altitude of $\triangle ABK^*$, D is the foot of the C-altitude, and O is the midpoint of $\overline{AB}$, the circle (COD) is the nine-point circle of $\triangle ABK^*$. Hence, M^* is the foot of the K^*-altitude, and we are done. $\square$

Problem 8.28 (ISL 2003/G4)

Let Γ_1, Γ_2, Γ_3, Γ_4 be distinct circles such that Γ_1, Γ_3 are externally tangent at P, and Γ_2, Γ_4 are externally tangent at the same point P. Suppose that Γ_1 and Γ_2; Γ_2 and Γ_3; Γ_3 and Γ_4; Γ_4 and Γ_1 meet at A, B, C, D, respectively, and that all these points are different from P. Prove that

$$\frac{AB \cdot BC}{AD \cdot DC} = \frac{PB^2}{PD^2}.$$

Invert about P. Then, $A^*B^*C^*D^*$ is a parallelogram. Hence, by Problem 8.19, we have that

$$\frac{AB}{PA \cdot PB} = \frac{CD}{PC \cdot PD}$$

and

$$\frac{AD}{PA \cdot PD} = \frac{BC}{PB \cdot PD}.$$

Taking the quotient suffices. $\square$

Problem 8.29

Let ABC be a triangle with incenter I and circumcenter O. Prove that line IO passes through the centroid G_1 of the contact triangle.

Consider an inversion about its incircle. We know that O, I, and O^* are collinear. However, by Problem 8.11, O^* is the nine-point center of the contact triangle. Hence, by the properties of the Euler line, we know that I, G_1 and O^* are also collinear, which implies the necessary conclusion. $\square$

Problem 8.30

Let ABC be a triangle and let Q be a point such that $AB \perp QB$ and $AC \perp QC$. The incircle of $\triangle ABC$ is tangent to BC, CA, and AB at points D, E, and F, respectively. If $\overrightarrow{QI}$ intersects EF at P, prove that $DP \perp EF$.

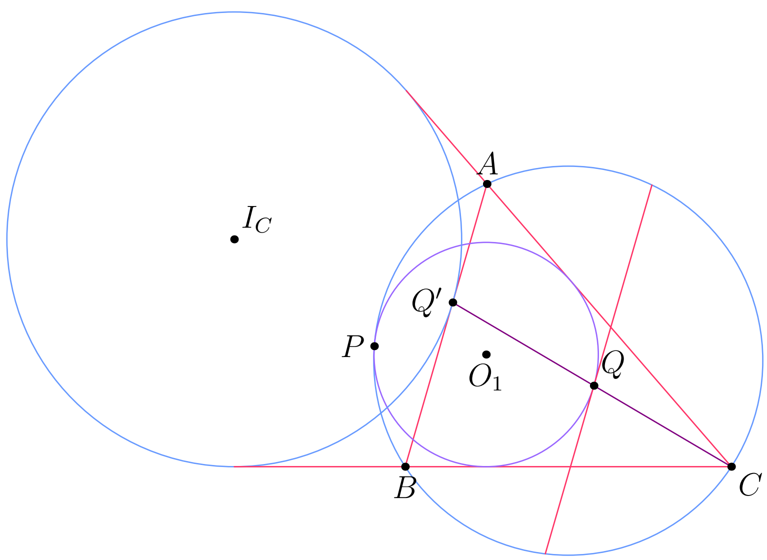
Let X be the intersection of $\overrightarrow{QI}$ with (ABC). Then, an inversion about the incircle sends (ABC) to the nine-point circle of $\triangle DEF$, and $\triangle ABC$ to the medial triangle of $\triangle DEF$. However, since

$$90^\circ = \angle QXA = \angle IXA = \angle IA^*X^*$$

we know that $X \neq A$ gets sent to the intersection of the nine-point circle with EF, which is the foot of the altitude from D to EF, as required. $\square$

Problem 8.31 (EGMO 2013/5)

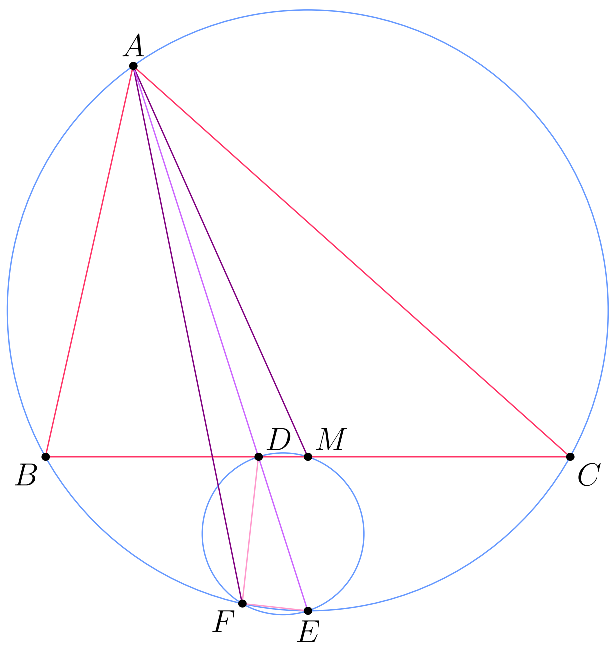
There exists a triangle ABC such that the circle ω is tangent to the sides AC and BC, and it is internally tangent to (ABC) at the point P. A line parallel to AB intersecting the interior of $\triangle ABC$ is tangent to ω at Q. Prove that $\angle ACP = \angle QCB$.



Consider a $\sqrt{ab}$ inversion about C follow by a reflection about the angle bisector of $\angle ACB$. This will swap A and B, and also send AB to (ABC) and vice-versa. As a result, it will take the C-excircle to the C-mixtilinear circle, and vice-versa. Thus, it sends P to Q', which is the intersection of CQ and AB. However, since C, Q, and Q' are collinear, we are done. $\square$

Problem 8.32 (Russian Olympiad 2009)

In triangle ABC, the internal angle bisector of $\angle A$ intersects $\overline{BC}$ at D and (ABC) again at E. The circle with diameter $\overline{DE}$ meets (ABC) again at F. Prove that $\overline{AF}$ is a symmedian of $\triangle ABC$.



Consider a force-overlay inversion Ψ about A (a $\sqrt{bc}$ inversion followed by a reflection about the angle bisector of $\angle A$). Then, Ψ will swap B and C, (ABC) and BC, as well as D and E. The last fact implies that the circle with diameter $\overline{DE}$ stays fixed. Hence, Ψ swaps F and M as well, implying that $\overline{AF}$ and $\overline{AM}$ are isogonal, as required. $\square$

Problem 8.33 (ISL 1997/9)

Let $A_1A_2A_3$ be a scalene triangle with incenter I. Let C_i, $i = 1, 2, 3$, be the smaller circle through I tangent to A_iA_{i+1} and A_iA_{i+2} (taken modulo 3). Let B_i, $i = 1, 2, 3$, be the second point of intersection of C_{i+1} and C_{i+2}. Prove that the circumcenters of the triangles A_1B_1I, A_2B_2I, A_3B_3I are collinear.

By an inversion about the incircle of $\triangle A_1B_1C_1$, it suffices to show that $\triangle A_1'A_2'A_3'$ and $\triangle B_1'B_2'B_3'$ are homothetic where X' is the image of point X after the inversion. We know that $\triangle A_1'A_2'A_3'$ is the medial triangle of the contact triangle of $\triangle A_1A_2A_3$. In addition, because of perpendicularity, we know that $\triangle B_1'B_2'B_3'$ is homothetic to the contact triangle. Thus, $\triangle A_1'A_2'A_3'$ and $\triangle B_1'B_2'B_3'$ are homothetic, which implies the required conclusion. $\square$

Problem 8.34 (IMO 1993/2)

Let A, B, C, D be four points in the plane, with C and D on the same side of the line AB, such that $AC \cdot BD = AD \cdot BC$ and $\angle ADB = 90^\circ + \angle ACB$. Find the ratio $\frac{AB \cdot CD}{AC \cdot BD}$, and prove that (ACD) and (BCD) are orthogonal.

Invert about A with radius r. Then, the conditions change to $B^*D^* \perp B^*C^*$ and $B^*D^* = B^*C^*$.

We start with the first part. We note that

$$\frac{AB \cdot CD}{AC \cdot BD} = \frac{AC^* \cdot CD}{AB^* \cdot BD} = \frac{AC^* \cdot AB^* \cdot C^*D^*}{AB^* \cdot AC^* \cdot B^*D^*} = \frac{C^*D^*}{B^*D^*} = \sqrt{2}$$

as required. □

We finish with the second part. We wish to show that the line C^*D^* passes through the center of circle $(B^*C^*D^*)$. However, this is clear, since $B^*D^* \perp B^*C^*$, implying that $\overline{C^*D^*}$ is the diameter of this circle, as required. □

Problem 8.35 (IMO 1996/2)

Let P be a point inside $\triangle ABC$ such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Let D, E be the incenters of triangles APB, APC, respectively. Show that the lines AP, BD, CE concur.

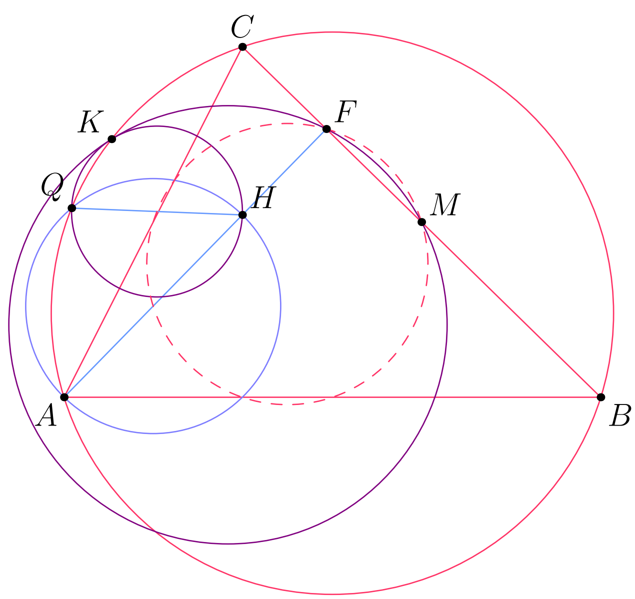
Invert about A with a radius of r. Then, the angle condition translates to $\angle C^*B^*P^* = \angle B^*C^*P^*$, so $B^*P^* = C^*P^*$. Hence,

$$\frac{r^2}{AB \cdot AP} \cdot BP = \frac{r^2}{AC \cdot AP} \cdot CP \implies \frac{BP}{AB} = \frac{CP}{AC}.$$

Hence, the angle bisectors of $\angle ABP$ and $\angle ACP$ concur at the same point on AP, by the Angle Bisector Theorem, so we are done. □

Problem 8.36 (IMO 2015/3)

Let ABC be an acute triangle with $AB > AC$. Let H be its orthocenter, and F the foot of the altitude from A. Let M be the midpoint of $\overline{BC}$. Let Q be the point on (ABC) such that $\angle HQA = 90°$ and let K be the point on (ABC) such that $\angle HKQ = 90°$. Assume that the points A, B, C, K, and Q are all different and lie on (ABC) in this order. Prove that (KQH) and (FKM) are tangent to each other.



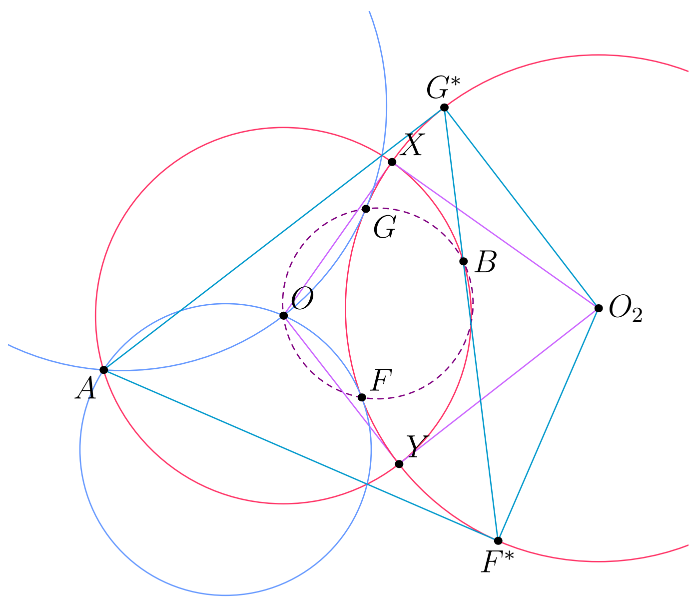
Consider a negative inversion Φ taking the nine-point circle to (ABC). Then, notice that $\Phi(F) = A$ and $\Phi(M) = Q$. As a result,

$$\Phi((KQH)) = K^*M \qquad \Phi((FKM)) = (AK^*Q).$$

Thus, we wish to show that K^*M is tangent to (AK^*Q). Now, we know that $\angle QKH = \angle K^*MH = 90^\circ$, and we know that $AQ \perp QM$, so as a result, $AQ \parallel K^*M$. Thus, we wish to show that $K^*A = K^*Q$. Now, notice that the midpoints of $\overline{AH}$ and $\overline{QH}$ both lie on the nine-point circle, and the midpoint of $\overline{AH}$ is the antipode of M. Thus, the angle with K^*, the midpoint of $\overline{AH}$, and the midpoint of $\overline{QH}$ form a right angle, and so K^* lies on the perpendicular bisector of $\overline{AQ}$ as required. $\square$

Problem 8.37 (ELMO SL 2013)

Let ω_1 and ω_2 be two orthogonal circles, and let the center of ω_1 be O. Diameter $\overline{AB}$ of ω_1 is selected so that B lies strictly inside ω_2. The two circles tangent to ω_2 through both O and A touch ω_2 at F and G. Prove that quadrilateral $FOGB$ is cyclic.



We wish to show that $\angle OGB = \angle OFB = 90^\circ$. Consider an inversion Φ about ω_1. If X and Y are the intersection points of ω_1 and ω_2, then they stay fixed under Φ. Now, we wish to show that F^*G^* passes through B. However, notice that F^* and G^* are just the tangency points of the tangents from A to ω_2. Now, let the midpoint of AF^* be M, and the midpoint of AG^* be N. Then, MN is the radical axis of ω_2 and the circle with radius 0 centered at A. Now, if O lies on this radical axis, then,

$$OO_2^2 - O_2Y^{*2} = OY^2$$

where O_2 is the center of ω_2, which is obviously true, as desired. $\square$