EGMO Solutions

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Contents

1 Fundementals of Number Theory

1.1 Divisibility

No problems.

1.2 Divisibility Properties

Problem 1.2.1

Show that if $n > 1$ is an integer, $n \nmid 2n^2 + 3n + 1$.

Assume there exists such an n. Then, subtracting $n(2n+3)$ from the RHS of the condition, we find that $n \nmid 1$, so $n = 1$ or -1, which is a contradiction. $\square$

Problem 1.2.2

Let $a > b$ be natural numbers. Show that $a \nmid 2a + b$.

Assume for the sake of contradiction there exists $a > b$ where $a \mid 2a + b$. Then, $a \mid b$, implying that $a \leq b$, which is a contradiction. $\square$

Problem 1.2.3

For 2 fixed integers x, y, prove that

$$x - y \mid x^n - y^n$$

for any non-negative integer n.

Clearly, the statement is equivalent to $x^n - y^n \pmod{x - y} \equiv 0$. However, we can write that

$$x^n - y^n \equiv (x - (x - y))^n - y^n \equiv 0 \pmod{x - y}$$

as required. $\square$

1.3 Euclid's Division Lemma

No problems.

1.4 Primes

Problem 1.4.1

Find all positive integers n for which $3n - 4$, $4n - 5$, and $5n - 3$ are all prime numbers.

In order for $5n - 3$ to be prime, we must have n even or $n = 1$. Hence, make the transformation $n = 2n'$. Then, $3n - 4 \mapsto 6n' - 4$, which can never be prime other than when $n = 2$. Trying both $n = 1$ and $n = 2$, we find that only $n = \boxed{2}$ works. $\square$

Problem 1.4.2

If $p < q$ are two consecutive odd prime numbers, show that $p + q$ has at least 3 prime factors (not necessarily distinct).

Clearly, it cannot have zero or one prime factor. If it has two prime factors, then we can express

$$p + q = rs$$

for some primes r and s. However, we know that one of these has to be 2, hence WLOG assume it is r. Then,

$$\frac{p+q}{2} = s$$

which implies that there exists a prime between p and q, which contradicts the fact that they are consecutive, as required. $\square$

1.5 Looking at Numbers as Multisets

No problems.

1.6 GCD and LCM

Problem 1.6.1

Prove that $\gcd(a, b) = a$ if and only if $a \mid b$.

We start with the if direction. Clearly, if $a = 2^{a_1}3^{a_2}\dots$ and $b = 2^{b_1}3^{b_2}\dots$, then the divisibility condition implies $a_i \leq b_i$ for all $i \geq 1$. Hence,

$$\min(a_i, b - i) = a_i$$

which proves the claim.

For the only if direction, we know that $\min(a_i, b_i) = a_i$ for any $i \geq 1$, implying that $a_i \leq b_i$, which proves the desired result. $\square$