MAT195 TextBook Notes

Aman Bhargava

January 2019

Contents

0.1 Review: Memorizy Stuff

0.1.1 Trig Function Derivatives

$\frac{d}{dx}sin(x) = cos(x)$ $\frac{d}{dx}csc(x) = -csc(x)cot(x)$

$\frac{d}{dx}cos(x) = -sin(x)$ $\frac{d}{dx}sec(x) = sec(x)tan(x)$

$\frac{d}{dx}tan(x) = sec^2(x)$ $\frac{d}{dx}cot(x) = -csc^2(x)$

0.1.2 Inverse Trig Derivatives

$\frac{d}{dx}sin^{-1}(x) = \frac{1}{\sqrt{(1-x^2)}}$

$\frac{d}{dx}cos^{-1}(x) = \frac{-1}{\sqrt{(1-x^2)}}$

$\frac{d}{dx}tan^{-1}(x) = \frac{1}{1+x^2}$

0.1.3 How to complete the Square

1. Put $ax^2 + bx$ in brackets and forcefully factor out the a
2. Add $(\frac{b}{2})^2$ to the inside of the brackets and subtract it from the outside (you got it)
3. Factor and be happy that you've completed the square;

0.1.4 Trig Angle Sums

1. $sin(A + B) = sin(A)cos(B) + cos(A)sin(B)$
2. $cos(A + B) = cos(A)cos(B) - sin(A)sin(B)$
3. $sin(A - B) = sin(A)cos(B) - cos(A)sin(B)$
4. $cos(A - B) = cos(A)cos(B) + sin(A)sin(B)$

0.1.5 Hyperbolic Trig Functions

0.1.6 Inverse Hyperbolic Trig Function

0.2 Introduction and Course Description

Chapter 1

Techniques of Integration (Chapter 7 in Textbook)

1.1 Integration by Parts

Integration by parts is basically just the reverse product rule.

Product rule: $d/dx[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$

You could reverse this simply, but it wouldn't be that useful. The more useful form that *is* the integration by parts formula looks like this:

$$\int [f(x)g'(x)] = f(x)g(x) - \int [f'(x)g(x)]$$

You can think through that one pretty easily — you are just splitting up the initial integral, then moving half of it to the other side.

That's pretty useful, but there's an even more useful way to write the formula, and it looks like this:

$$\int udv = uv - \int vdu$$

This works because we let $u = f(x)$ and $v = g(x)$. $g'(x) = v' = dv/dx$, and same with u'. So to get to that formula we go like this:

$$\int uv'dx = uv - \int u'vdx \tag{1.1}$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx \tag{1.2}$$

$$\int udv = uv - \int vdu \tag{1.3}$$

1.1.1 Tips for Integration by Parts:

- When using the uv equation, it's useful to define things in this order:
  - u = ? dv = ?
  - du = ? v = ?
  - keeping in mind that $u'dx = du$ and $\int \frac{dv}{dx}dx = \int v' = v$
- Choose your u so that it becomes simpler when differentiated, and let your v be the thing that gets a little hairier.
- Practice a lot from the textbook, ya dingus

1.2 Trigonometric Integrals

There are a bunch of configurations of trig functions for which we need to learn the steps necessary to take the integral. It's important to practice this because recognizing the form of the integral is the most difficult thing to do here.

1.2.1 Strategy for $\int sin^m(x)cos^n(x)$

If the power of cosine is odd: "Save" one of the cosine terms, and then express it as $\int sin^m(x)cos^{2k+1}(x) * cos(x)$. Then turn the $cos^{2k+1}(x)$ into sin terms with pythagorean identity. Then substitute $u = sin(x)$ and solve.

If the power of sine is odd: Do the same thing but reverse sin and cos (save one $sin(x)$ and sub u for $cos(x)$

If both powers are even: Use the following identities to help you solve it:

$$sin^2(x) = 1/2(1 - cos(2x)) \tag{1.4}$$

$$cos^2(x) = 1/2(1 + cos(2x)) \tag{1.5}$$

$$sinxcosx = 1/2sin2x \tag{1.6}$$

1.2.2 Strategy for $\int tan^m(x)sec^n(x)$

If the power of $sec(x)$ is even, "save" a factor of $sec^2(x)$ and use identity $sec^2(x) = 1 + tan^2(x)$ to express the rest in terms of $tan(x)$. Then substitute $u = tan(x)$

If the power of tangent is odd, save a factor of $sec(x)tan(x)$ and convert the rest of the $tan(x)$'s using $tan^2(x) = sec^2(x) - 1$

Also note the following:

$$\int tan(x)dx = ln|sec(x)| + C$$

$$\int sec(x)dx = ln|sec(x) + tan(x)| + C$$

Remember this as well:

$$\frac{d}{dx}tan(x) = sec^2(x)$$

$$\frac{d}{dx}sec(x) = sec(x)tan(x)$$

1.2.3 Strategy for $\int sin(mx)cos(nx)dx$

Use the following identities:

$$sinAcosB = 1/2[sin(A-B) + sin(A+B)] \quad (1.7)$$

$$sinAsinB = 1/2[cos(A-B) - cos(A+B)] \quad (1.8)$$

$$cosAcosB = 1/2[cos(A-B) + cos(A+B)] \quad (1.9)$$

1.2.4 Strategy for $\int csc^m(x)cot^n(x)dx$

Know the following things:

$$\frac{d}{dx}csc(x) = -csc(x)cot(x) \quad (1.10)$$

$$cot^2(x) = csc^2(x) - 1 \quad (1.11)$$

$$\frac{d}{dx}cot(x) = -csc^2(x) \quad (1.12)$$

1.3 Trig Sub

What is trig sub? Trig sub is when you use the *inverse substitution* rule in conjunction with useful trigonometric identities and trigonometric integrals to solve integrals that you wouldn't otherwise be able to solve.

1.3.1 Inverse Substitution

Unlike u-substitution, you are substituting in a non-equivalent function $(g(x))$ for x instead of substituting a variable like u for the actual value of x. Hence, the following result arises:

$$\int f(x)dx = \int f(g(x))g'(x)dx$$

Qualificiations: g must have an inverse function, and g must be one-to-one.

1.3.2 List of Trig Subs

The main use of trig subs is to get rid of irritating radical signs that make integration hard. The following is a table of types (from the textbook):

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = asin\theta$, $-\pi/2 <= \theta <= \pi/2$	$1 - sin^2\theta = cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = atan\theta$, $-\pi/2 < \theta < \pi/2$	$1 + tan^2\theta = sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = asec\theta$, $-\pi/2 <= \theta <= \pi/2$	$sec^2\theta - 1 = tan^2\theta$

1.3.3 General Layout for Trig Sub

1. Make sure there is no other way (e.g. u-sub, etc.)
2. If there is a quadratic in the root, complete the square
3. Recognize the stuff in the root as one of the three.
4. Set $x = a * trig(\theta)$ and find dx in terms of θ
5. Solve the stuff.

1.4 Partial Fractions

This is just a way to break up rational functions in to little pieces that we can actually deal with. A proper rational function is one where the power on the top polynomial is lower than that of the bottom. Improper rationals are the other way around.

In order to use partial fractions, you need to make the rational function a proper one.

There are a few cases to consider for splitting things up into partial fractions:

1.4.1 Denominator is only distinct Linear Factors

1. Set an equality between the original rational and $\frac{A}{(root_1)} + \frac{B}{(root2)} + ...$
2. Multiply both sides by the denominator of the rational
3. Expand and solve for $A, B, ...$

1.4.2 Denominator is only Linear Factors but some are Repeated

It's roughly the same as last time, EXCEPT:

1. Suppose $(root_1)$ is repeated k times so $(root_1)^k$ is a factor
2. Then you need to use $\frac{A_1}{(root_1)} + \frac{A_2}{(root_1)^2} + ... + \frac{A_k}{(root_1)^k}$
3. Now solve as you did last time.

1.4.3 Denominator has non-repeated Quadratic Factors

Basically just have a linear term on top in the expansion, so $\frac{A_1x+B}{ax^2+bx+c}$ would be a term in the thing.

Also, $\int \frac{dx}{x^2+a^2} = \frac{1}{a}tan^{-1}(\frac{x}{a}) + C$

1.4.4 Denominator has repeated Quadratic Factors

Roughly the same as when you have linear repeated factors. if you have $(root_1)^k$, then you get

$$\frac{Ax+B}{(root_1)} + \frac{Cx+D}{(root_1)^2} + ... + \frac{Yx+Z}{(root_1)^k}$$

Then just solve as before.

1.4.5 Rationalizing Substitutions

Use substitutions to make annoying functions into rational functions and solve from there. For instance, if you have a radical in the numerator, make the subsitution u^2 = whatever was in the radical.

1.5 Strategy for Solving Integrals

1. Simplify the integrand with identities/algebra if possible.
2. Look for an obvious u-sub
3. Classify the integrand according to its form:
   (a) Trig integral ($sin^m(x)cos^n(x)$, etc.)
   (b) Rational Function
   (c) Integration by parts - especially with $polynomial * transcendental function$
   (d) Radicals → consider trig sub ($x = atan\theta$, etc.)
4. Try again, using several methods, and drawing from your MASSIVE PAST EXPERIENCE

1.6 Improper Integrals (Involving Infinity)

1.6.1 Definition of Improper Integral

$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$ Convergent improper integrals have an actual value. Divergent improper integrals don't. Integrals from negative infinity to positive infinity exist if and only if the integral from -infinity to a converges and the integral from a to +infinity converges (it is the sum of the two).

For $\int_1^\infty \frac{1}{x^p}dx$ is convergent if $p > 1$

For integrals where the y value goes to infinity , then you take limits around where it goes to infinity, ya dingus.

1.6.2 Comparison Theorem

You can prove that an integral is convergent if you can find another one for which the function is strictly of greater magnitude that is convergent (and likewise for non-convergency)

Chapter 2

Further Applications of Integration

2.1 Arc Length

2.1.1 Definition of Arc Length

Arc length is the actual length of a curve. It is defined as:

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} |P_{i-1}, P_i|$$

2.1.2 Arc Length Formula

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$