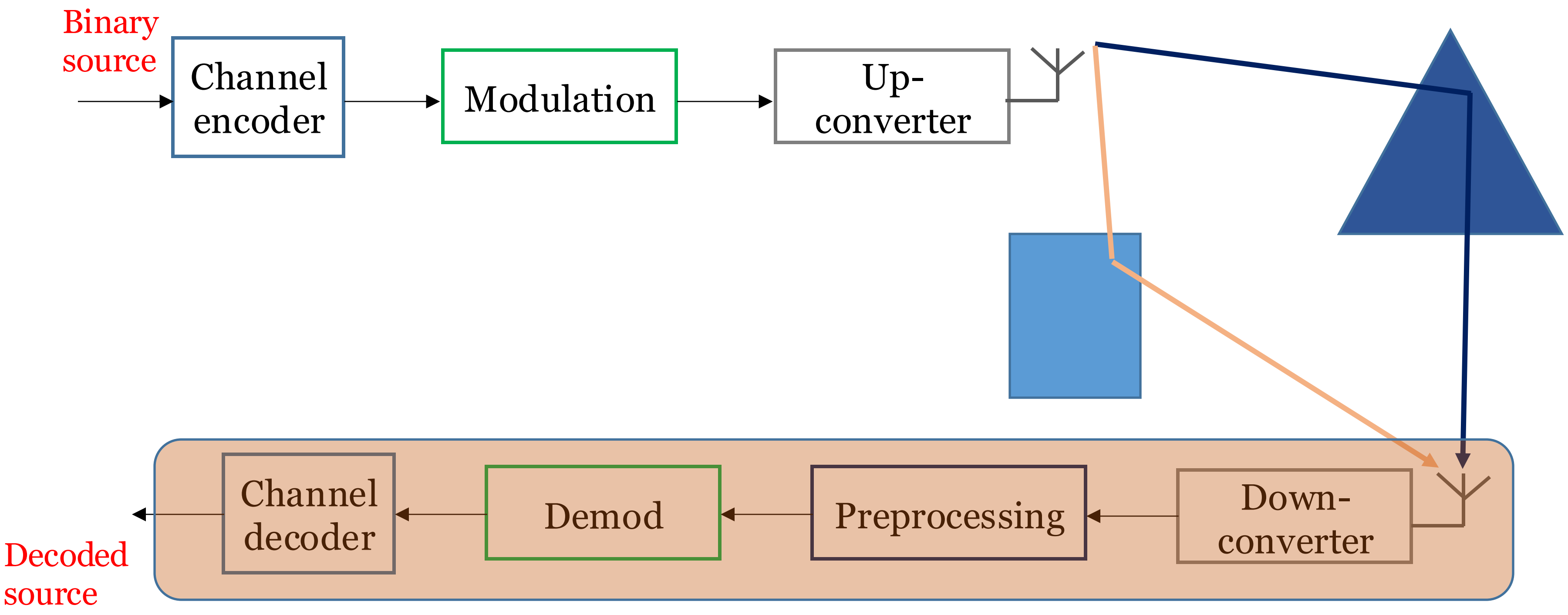
AI Lab for Wireless Communications

Week2 - Syndrome decoding & maximum likelihood decoding

A digital communication system over wireless channel



- Channel enc/dec allows error correction by adding redundancy
- Channel decoding is the part we focus on in the following courses

(n,k)-Linear Block Codes

- k-bit message $\boldsymbol{m}$, n-bit codeword $\boldsymbol{c}$
- Relationship: $\boldsymbol{c} = \boldsymbol{mG}$

- The code C contains all (2^k in total) such codewords
  - C is the row space of $\boldsymbol{G}$ $(k \times n)$
  - Call it a generator matrix
- There exists $\boldsymbol{H}$ $(n - k \times n)$ such that $\boldsymbol{cH}^T = \boldsymbol{Hc}^T = \boldsymbol{0}$
  - Rows of $\boldsymbol{H}$ span the nullspace of $\boldsymbol{G}$
  - Call it a parity check matrix

(7,4)-Hamming Coded

- *4*-bit message $\boldsymbol{m}$, *7*-bit codeword $\boldsymbol{c}$
- Relationship: $\boldsymbol{c} = \boldsymbol{mG}, \mathbf{c}\boldsymbol{H}^{\boldsymbol{T}} = \mathbf{0}$

- Generator matrix

$$\mathbf{G} := \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

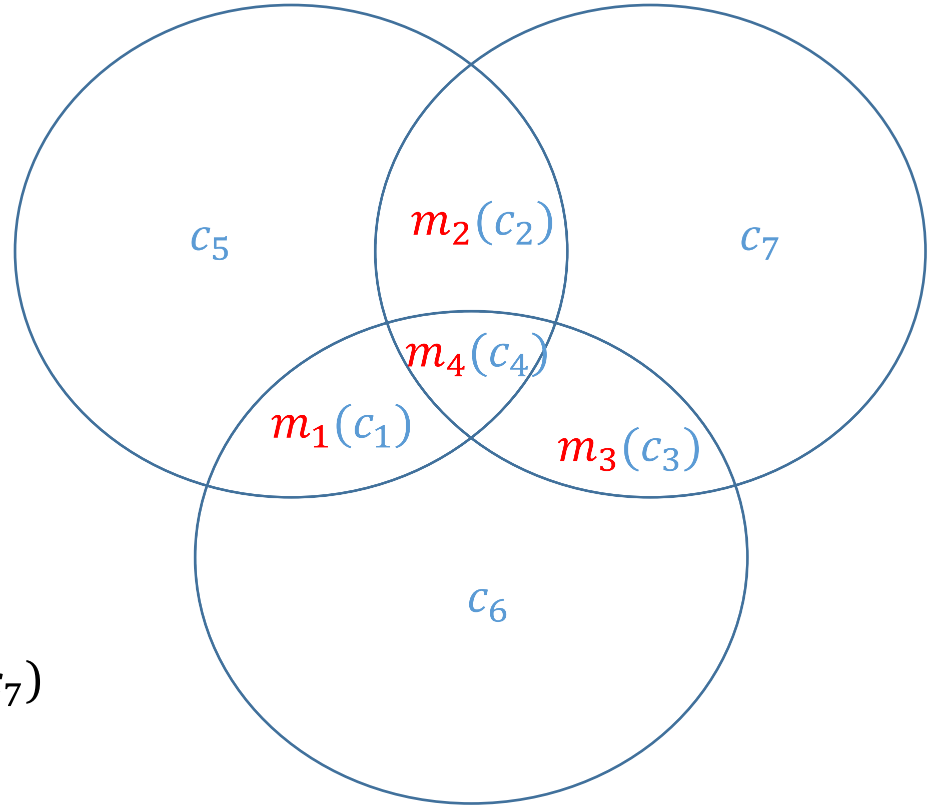
- Parity check matrix

$$\mathbf{H} := \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Illustration of Hamming Code

- The relationship of each bits in the codeword can be illustrated by the following figure
- $\boldsymbol{c} = \boldsymbol{mG}$

$$\mathbf{G} := \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$



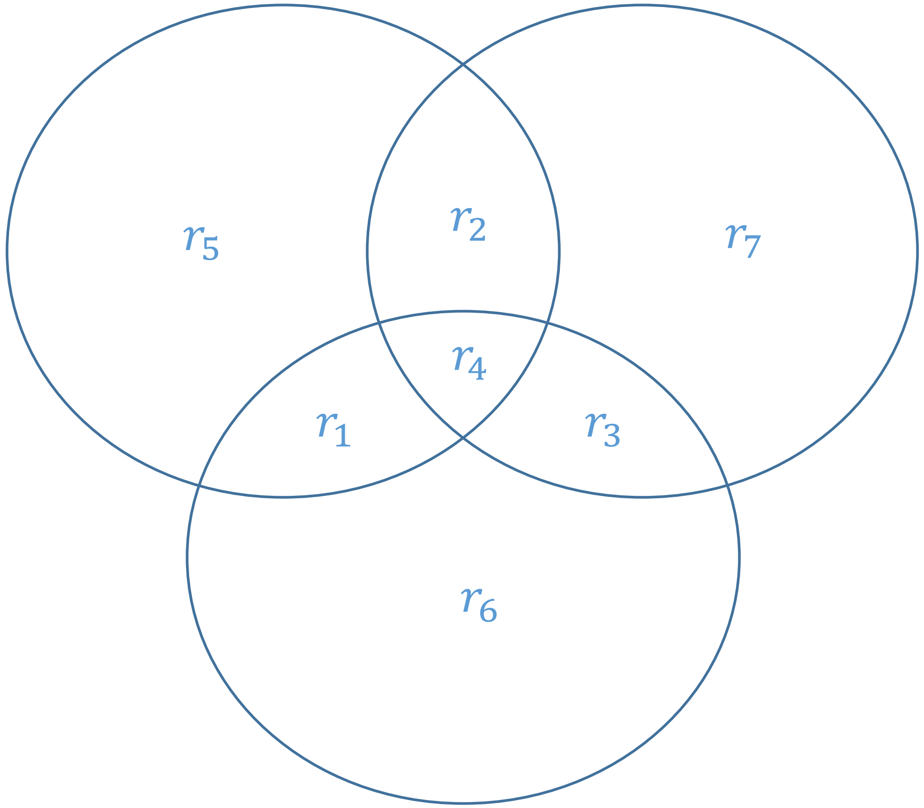
where $\boldsymbol{m} = (m_1, m_2, m_3, m_4)$ and $\boldsymbol{c} = (c_1, c_2, c_3, c_4, c_5, c_6, c_7)$

Illustration of Hamming Code

- The relationship of each bits in the codeword can be illustrated by the following figure
- $\boldsymbol{cH^T = 0}$

$$\mathbf{H} := \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

where $\boldsymbol{r} = (r_1, r_2, r_3, r_4, r_5, r_6, r_7)$



System Model

- When transmit, we map coded bits to baseband signal
- Binary phase shift keying (BPSK)

$$x_i = \sqrt{P}(2c_i - 1), \qquad \boldsymbol{x} = [x_1, \ldots, x_n]$$

- Additive white Gaussian noise (AWGN) channel

$$y_i = x_i + w_i,\ w_i \sim N(0, \frac{N_0}{2}),\ \boldsymbol{y} = [y_1, \ldots, y_n]$$

Noise in Communication System

- How to simulate the noisy channel?
  - ➢Set a target signal to noise ratio (SNR)
  - ➢Calculate the related signal power and noise power
  - ➢Generate the noise sequence and add it to the signal

$$SNR = 10\log_{10}\frac{\sigma_s^2}{\sigma_w^2} \Rightarrow \sigma_s^2 = \sigma_w^2 \times 10^{-\frac{SNR}{10}}$$

σ_s^2: signal power (variance) σ_w^2: noise power (variance)

- For complex signal, the noise is also complex. The calculated variance has to be divided by two for the generation of real or imaginary signal/noise

Syndrome decoding

Concept of Syndrome Decoding (1/2)

- First make a hard decision

$$\hat{z}_i = sign\,(y_i) \quad \text{and} \quad \hat{d}_i = (\hat{z}_i + 1)/2$$

- Assume $\boldsymbol{e}$ is the error, we have

$$\begin{aligned} \widehat{\boldsymbol{d}}\boldsymbol{H}^T &= \boldsymbol{s} \\ \Rightarrow \quad (\boldsymbol{x} + \boldsymbol{e})\boldsymbol{H}^T &= \boldsymbol{s} \\ \Rightarrow \quad \boldsymbol{x}\boldsymbol{H}^T + \boldsymbol{e}\boldsymbol{H}^T &= \boldsymbol{s} \\ \Rightarrow \quad \boldsymbol{0} + \boldsymbol{e}\boldsymbol{H}^T &= \boldsymbol{s} \end{aligned}$$

- **s** is the syndrome, if $\boldsymbol{s} \neq \boldsymbol{0}$, then there must exist error

Concept of Syndrome Decoding (2/2)

- How to recover the signal?

$\Rightarrow$ Define the error pattern $\hat{\boldsymbol{e}}$ that $(\hat{\boldsymbol{e}} + \boldsymbol{e})\boldsymbol{H}^T = \boldsymbol{0}$

- We can have

$$
\begin{aligned}
& \widehat{\boldsymbol{x}} = \widehat{\boldsymbol{d}} + \hat{\boldsymbol{e}} \\
\Rightarrow \quad & \widehat{\boldsymbol{x}} = (\boldsymbol{x} + \boldsymbol{e}) + \hat{\boldsymbol{e}} \\
\Rightarrow \quad & \widehat{\boldsymbol{x}} = \boldsymbol{x}
\end{aligned}
$$

Procedure of Syndrome Decoding

- Do hard decision
- Construct the standard array
- Compute the syndrome $\widehat{\boldsymbol{d}}\boldsymbol{H}^T = \boldsymbol{s}$
- Decide $\hat{\boldsymbol{e}} = \text{coset leader}(\boldsymbol{s})$
- Decode to $\widehat{\boldsymbol{x}} = \widehat{\boldsymbol{d}} + \hat{\boldsymbol{e}}$

Example (1/2)

- We have

$$\boldsymbol{y}' \quad = \quad \boldsymbol{x}' \quad + \ (0000100)$$

$$(1101000) = (1101010) + \ (0000100)$$

- Doing syndrome decoding

$$\boldsymbol{y}\boldsymbol{H}^T = \boldsymbol{s}$$

$$\Rightarrow \quad (\boldsymbol{x} + \boldsymbol{e})\boldsymbol{H}^T = \boldsymbol{s}$$

$$\Rightarrow \quad \boldsymbol{x}\boldsymbol{H}^T + \boldsymbol{e}\boldsymbol{H}^T = \boldsymbol{s}$$

$$\Rightarrow \quad \boldsymbol{0} + \boldsymbol{e}\boldsymbol{H}^T = \boldsymbol{s}$$

Example (2/2)

- Then $\boldsymbol{s} = (100)$
- By $\boldsymbol{eH}^T = \boldsymbol{s}$, there are several **e** that makes the equation holds

$$\boldsymbol{e_1} = (0000100), \boldsymbol{e_2} = (1101000)$$

- $\boldsymbol{e} = (0000100)$ is the most possible case since the probability of each bit flips is p

Standard array

Syndrome $s = rH^T$	**Error pattern** $\hat{e}$
(0,0,0)	(0,0,0,0,0,0,0)
(0,0,1)	(0,0,0,0,0,0,1)
(0,1,0)	(0,0,0,0,0,1,0)
(0,1,1)	(0,0,1,0,0,0,0)
(1,0,0)	(0,0,0,0,1,0,0)
(1,0,1)	(0,1,0,0,0,0,0)
(1,1,0)	(1,0,0,0,0,0,0)
(1,1,1)	(0,0,0,1,0,0,0)

Performance evaluation

- Block error rate (BLER) – If the received block has one or even more bits error, we say that this Block (codeword) is error. The block error rate is the number of block errors per unit block.

- Bit error rate (BER) – If any bit of a block is wrong, we say that there is an error. The bit error rate is the number of bit errors per unit bit.

Performance evaluation (2/2)

- Signal to Noise Ratio – The ratio of signal power to the noise power. $\left(SNR = 10\log_{10}\frac{P_{signal}}{P_{noise}} \right)$
- Go through all codewords and compare whether $\hat{\boldsymbol{m}} = \boldsymbol{m}$
- In communication system, we usually evaluate decoding scheme by observing block or bit error rate over different E_b/N_0

Maximum Likelihood Decoding

ML Decoding

MLD:

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x} \in C} ||\boldsymbol{y} - \boldsymbol{x}||^2$$

- This is optimal

- Complexity is very high, especially when n is large

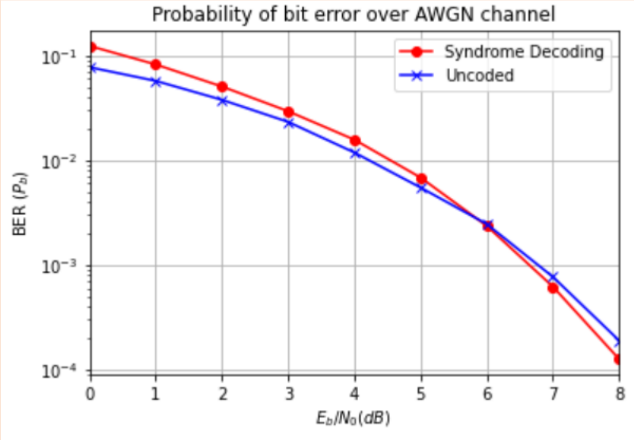
ML Decoding Steps

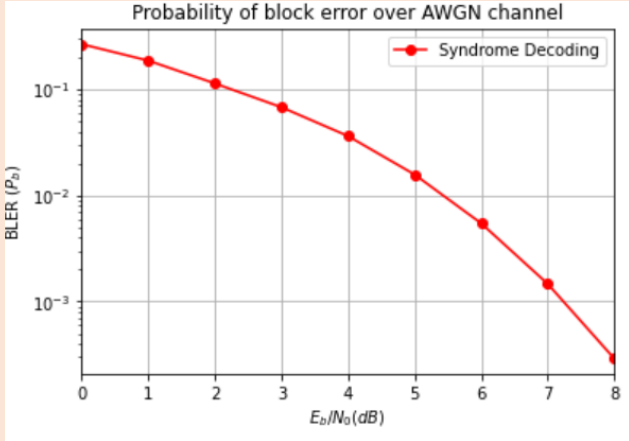
- Construct all possible codeword
- Calculate the norm of received codeword and possible codeword
- Find the codeword which has the minimum norm
- Calculate the BER and BLER

Lab for Today

- Implement the syndrome decoding and ML decoding
- Achieve the exact performance (BLER and BER)

Syndrome decoding





ML decoding