



Term Structure of Interest Rates

Financial Engineering and Computations

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此章內容

- Financial Engineering & Computation教課書的第五章 Term Structure of Interest Rates
- C++財務程式設計的第四章 (4-1,4-2)

Outline

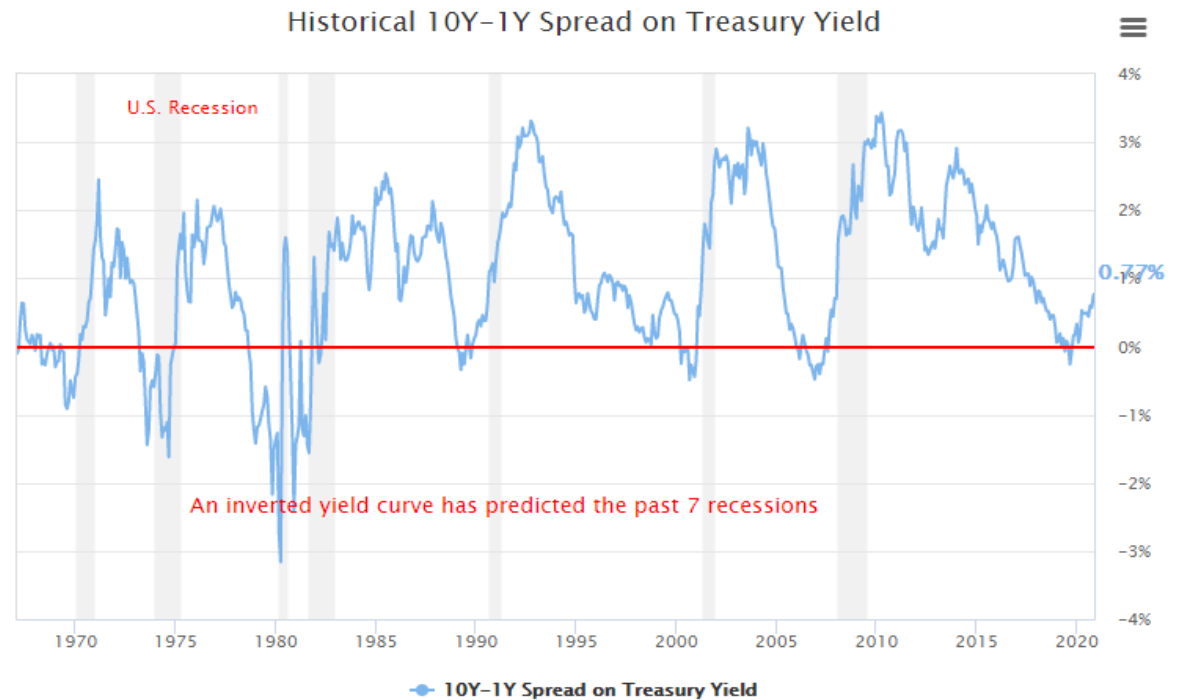
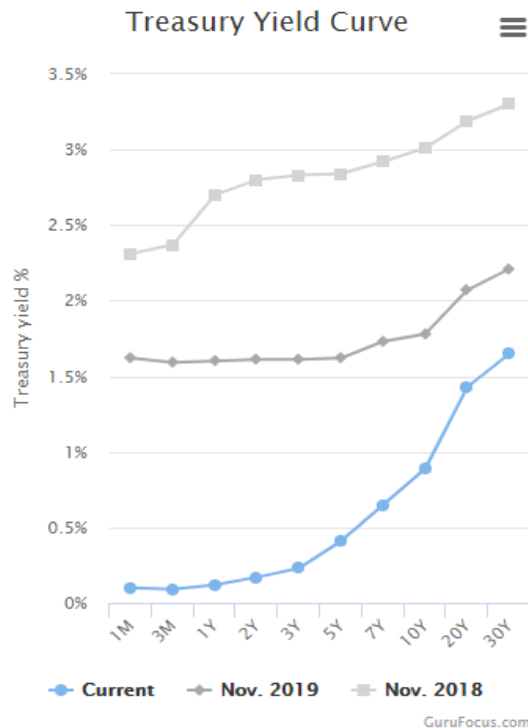


- Introduction
- Spot Rates
- Extracting Spot Rates from Yield Curves
- Spot Rate Curve , Forward Rate Curve, Yield Curve
- Forward Rates
- Locking in the Forward Rates
- Term Structure Theory

Term Structure of Interest Rates

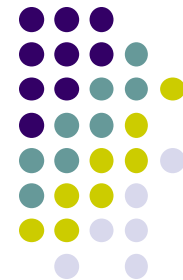


- The interest rates vary with maturity.



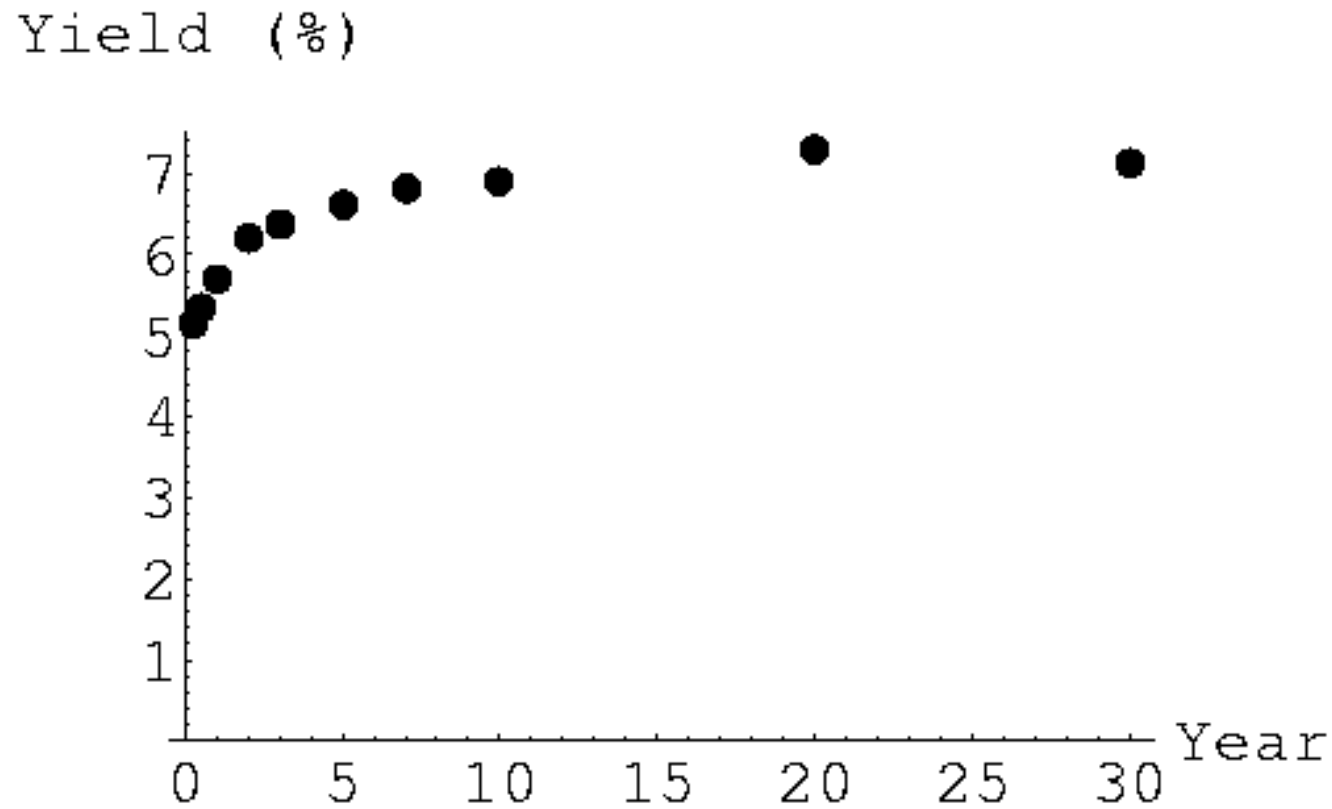
https://www.gurufocus.com/yield_curve.php

Term Structure of Interest Rates



- Concerned with how interest rates change with maturity.
- The set of yields to maturity for bonds forms the term structure.
 - The bonds must be of equal quality.
 - Credit spread.
 - They differ solely in their terms to maturity.

Yield Curve

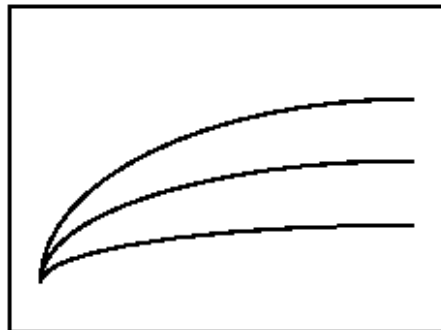


Four Shapes



- A normal yield curve is upward sloping.
- An inverted yield curve is downward sloping.
- A flat yield curve is flat.
- A humped yield curve is upward sloping at first but then turns downward sloping.

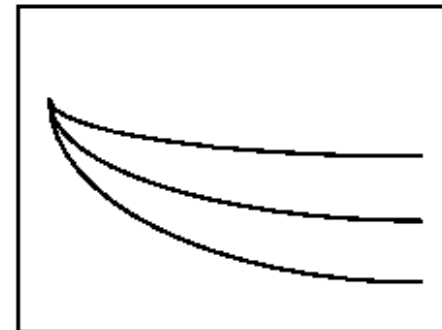
Normal curve



forward rate curve
spot rate curve
yield curve

(a)

Inverted curve



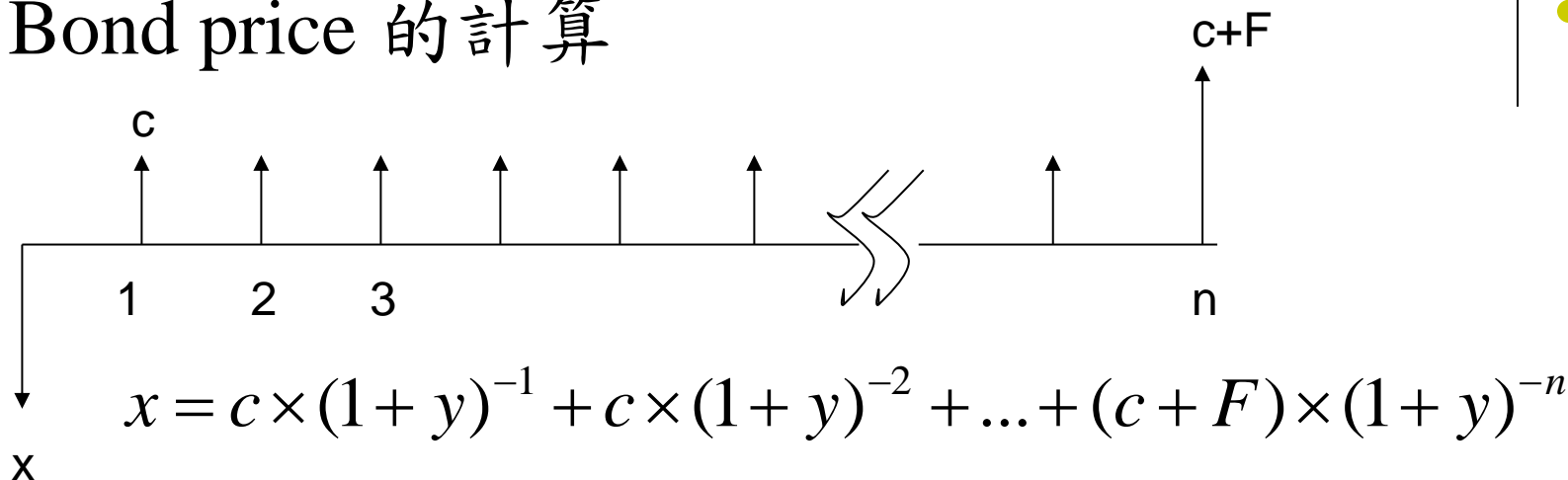
yield curve
spot rate curve
forward rate curve

(b)

Bond yield 和 Zero Rate



- Bond price 的計算



- 不同的時間現金流用相同的 y 折現

- y 為市場交易的價格

- 不同的現金流結構無法用來折現

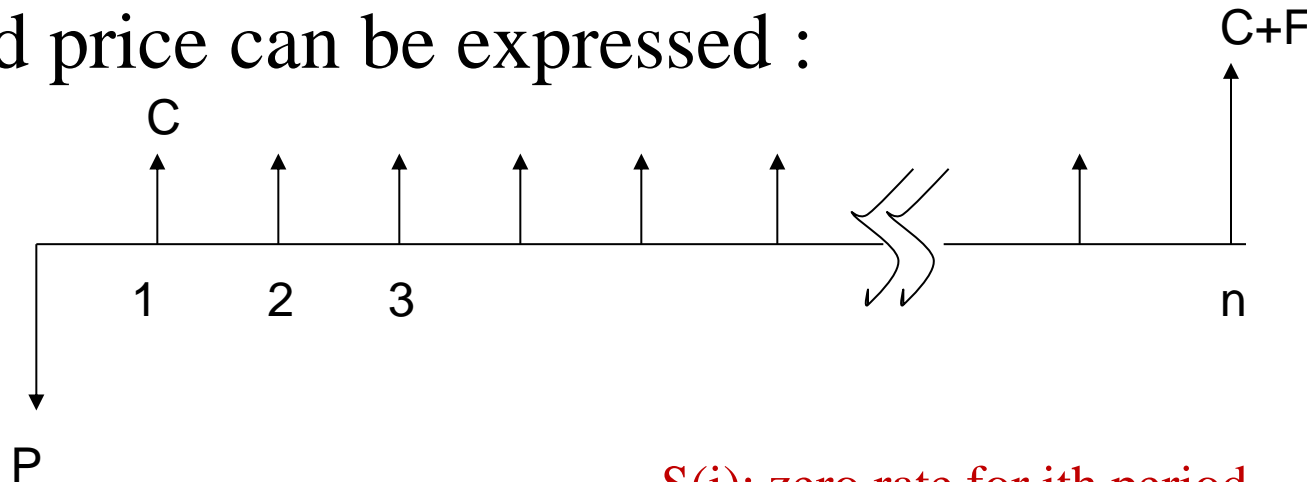
- 使用zero rate 替代

- n -year zero rate (zero coupon rate): 可視為 n 年後到期的零息債券的yield rate

Spot Rate (Zero rate)



- It is defined as yield of zero coupon bond
- A spot rate curve (zero-coupon yield curve) is a plot of spot rates against maturity.
- Bond price can be expressed :



$S(i)$: zero rate for i th period

$$P = C \times (1 + S(1))^{-1} + C \times (1 + S(2))^{-2} + \dots + (C + F) \times (1 + S(n))^{-n}$$

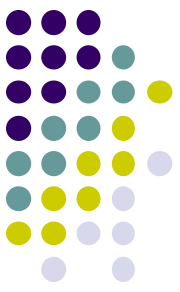
y : bond yield

$$P = C \times (1 + y)^{-1} + C \times (1 + y)^{-2} + \dots + C \times (1 + y)^{-n} + F \times (1 + y)^{-n}$$

市場上無法直接觀察長天期Spot rate



- 市場上交易的債券中
 - 短天期的債券多為zero coupon bond
 - 長天期的皆為coupon bearing bond
- 無法觀察長天期的Spot rate
- 可是市場折現利率應和maturity相關,
 - 求長天期的Spot rate實屬必須
- 藉由coupon bearing bond的拆解可得



方法簡介

- 考慮一個二期的模型,假定有兩個債券 B1,B2到期日分別為第一和第二期. Bond yield分別為 y_1, y_2 , 債券價格如下
$$P_1 = \frac{c + F}{(1 + y_1)} \quad P_2 = \frac{c}{(1 + y_2)} + \frac{c + F}{(1 + y_2)^2}$$
- 考慮在第一期到期的零息債券,其zero rate $S(1)=y_1$
- 考慮在第二期到期的零息債券,其zero rate $S(2)$ 計算如下:
 - B2可拆解成兩個零息債券,
 - 一個在第一期到期,面值為 $c \Rightarrow$ 用 $S(1)$ 折現
 - 另一個在第二期到期,面值為 $c+F \Rightarrow$ 用 $S(2)$ 折現
 - 可得以下算式:

$$P_2 = \frac{c}{(1 + S(1))} + \frac{c + F}{(1 + S(2))^2}$$

因 $S(1)$ 已知,可求出 $S(2)$

n期的Zero Rate計算



- 假定 $S(1), S(2), S(3), \dots, S(n-1)$ 皆已知
- $S(n)$ 滿足以下式子:

$$P_n = \frac{c}{(1+S(1))} + \frac{c}{(1+S(2))^2} + \dots + \frac{c+F}{(1+S(n))^n}$$

- 移項可得 $(1+S(n))^n = \frac{c+F}{P_n - \frac{c}{(1+S(1))} - \frac{c}{(1+S(2))^2} - \dots - \frac{c}{(1+S(n-1))^{n-1}}}$

$$S_n = \sqrt[n]{\frac{c+F}{P_n - \frac{c}{(1+S(1))} - \frac{c}{(1+S(2))^2} - \dots - \frac{c}{(1+S(n-1))^{n-1}}} - 1}$$

The procedure is called **bootstrapping**.

```

float ZeroRate[5];
float Yield[5];
float C;
scanf("%f",&C);
for(int i=0;i<5;i=i+1)
{
    printf("輸入Yield rate %d:",i+1);
    scanf("%f",&Yield[i]);
}
ZeroRate[0]=Yield[0];

```

程式宣告

輸入殖利率

第一期Zero rate=Yield

```

for(i=1;i<=4;i++)
{
    float BondValue=0;

```

計算第i+1期zero rate

```

for(int j=0;j<=i;j=j+1)
{
    float Discount=1;
    for(int k=0;k<=j;k++)
    {
        Discount=Discount/(1+Yield[i]);
    }
    BondValue=BondValue+Discount*C;
    if(j==i)
    {
        BondValue=BondValue+Discount*100;
    }
}

```

計算債券價格Pi+1

```

for(j=0;j<i;j=j+1)
{
    float PV=C;
    for(int k=0;k<=j;k++)
    {
        PV=PV/(1+ZeroRate[j]);
    }
    BondValue=BondValue-PV;
}

```

$$Bn - \frac{c}{(1+Z1)} - \frac{c}{(1+Z2)^2} - \dots - \frac{c}{(1+Zn-1)^{n-1}}$$

```

ZeroRate[i]=pow((C+100)/BondValue,1.0/(i+1))-1;
}

```

```

for(i=0;i<=4;i++)
{
    printf("第%d期zero rate=%f\n",i,ZeroRate[i]);
}

```

列印zero rate

見第四章 ZeroCurve project

Compare Yield Curve and Spot rate curve



- Spot rate curve is zero coupon yield curve or zero curve.
- Spot rate curve is consisted of zero rate.
 - zero coupon bond
- Yield curve is consisted of bond yield.
 - coupon bearing bond 、 zero coupon bond

Example



- Suppose the 1-year T-bill has yield of 8%. Because this security is a zero-coupon bond, the 1-year spot rate is 8%. When the 2-year 10% T-note is trading at 90, the 2-year spot rate satisfies

$$\therefore 90 = \frac{10}{1.08} + \frac{110}{(1 + S(2))^2} \Rightarrow S(2) = 0.1672 \text{ or } 16.72\%$$

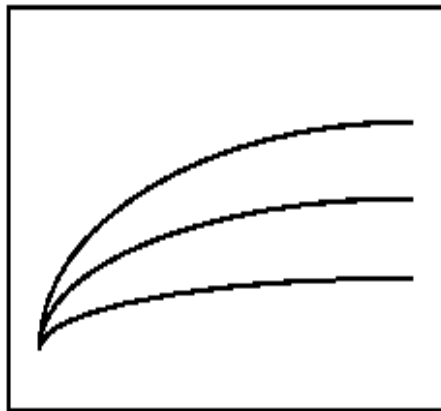
Spot Rate Curve and Yield Curve



- If the yield curve is flat, the spot rate curve coincides with the yield curve.
- y_k : yield to maturity for the k -period coupon bond.
- $S(k) \geq y_k$, if $y_1 < y_2 < \dots$ (yield curve is normal).
- $S(k) \leq y_k$, if $y_1 > y_2 > \dots$ (yield curve is inverted).
- $S(k) \geq y_k$, if $S(1) < S(2) < \dots$ (spot rate curve is normal).
- $S(k) \leq y_k$, if $S(1) > S(2) > \dots$ (spot rate curve is inverted).

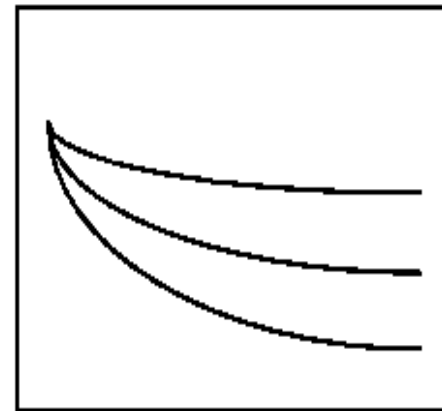


Figure 5.6: Shapes of Curves



forward rate curve
spot rate curve
yield curve

(a)



yield curve
spot rate curve
forward rate curve

(b)

Forward rate curves will be discussed later.

Shapes



- The spot rate curve often has the same shape as the yield curve.
 - If the spot rate curve is inverted, then the yield curve is inverted, and vice versa.
 - However, a normal yield curve does not guarantee a normal spot rate curve.
- When the final principal payment is relatively insignificant, the spot rate curve and the yield curve do not share the same shape.
(Bonds of high coupon rates and long maturities.)

Shapes



- Consider a 3-period coupon bond that pays \$1 per period and repays the principal of \$100 at maturity.
- Assume spot rates $S(1) = 0.1$, $S(2) = 0.9$, and $S(3) = 0.901$.
- Yields to maturity are $y_1 = 0.1$, $y_2 = 0.8873$, and $y_3 = 0.8851$, not strictly increasing!

Yield Spread



- Yield spread is the difference between the IRR of the risky bond and that of a risk-free bond with comparable same maturity.

$$P_{risk-free} = \sum_{i=1}^n C(1+y)^{-i} + F \times (1+y)^{-n}$$

$$P_{risky} = \sum_{i=1}^n c \times (1+y+y')^{-i} + F \times (1+y+y')^{-n}$$

Where y' is the yield spread.

Static Spread



- The static spread is the amount s by which the spot rate curve has to shift in parallel in order to price the risky bond correctly,
$$P_{risky} = \sum_{t=1}^n \frac{C_t}{(1 + s + S(t))^t}$$
- Unlike the yield spread, the static spread incorporates information from the zero rate structure.
- The amount of static spread can be considered as the **constant credit spread** to the Treasury spot rate curve that **reflects the risk premium of a corporate bond**.

#Homework 5



假定A公司發行一個 n 期的債券，每一期所要償還的票息為 C ，到期日時還需償還票面價值100元。市場上 n 期債券的報酬率為 R ，該債券的信用風險可用yield spread (S)來表示，所以該債券價格可示如下

$$\text{債券價格} = \sum_{i=1}^n \frac{C}{(1+R+S)^i} + \frac{100}{(1+R+S)^n}$$

#Homework 5



假定市場上的第*i*期的零息利率可用 Z_i 表示，
該公司的static spread用*s*表示，則債券價格為

$$\text{債券價格} = \sum_{i=1}^n \frac{C}{(1 + Z_i + s)^i} + \frac{100}{(1 + Z_n + s)^n}$$

請使用上述關係式，撰寫程式只輸入每一期的yield rate、期數、債息、跟yield spread差等訊息來計static spread。

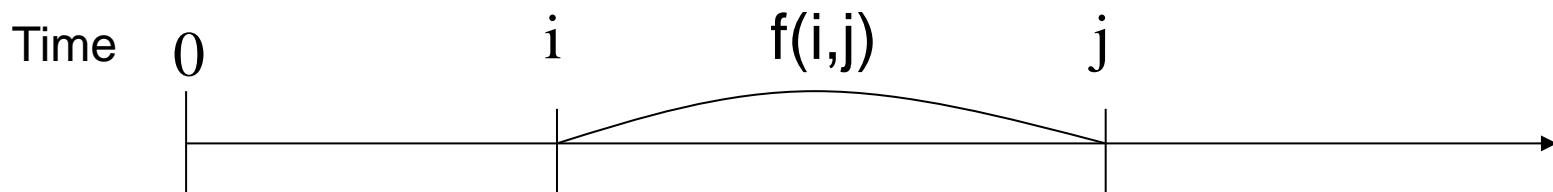
可參考C++財務設式設計3-5.2、4-1.3、4-1.4、4-1.5等章節

Forward Rate



- The forward rate reflect information regarding future interest rates implied by the market.
- If we invest \$1 from now to jth period.

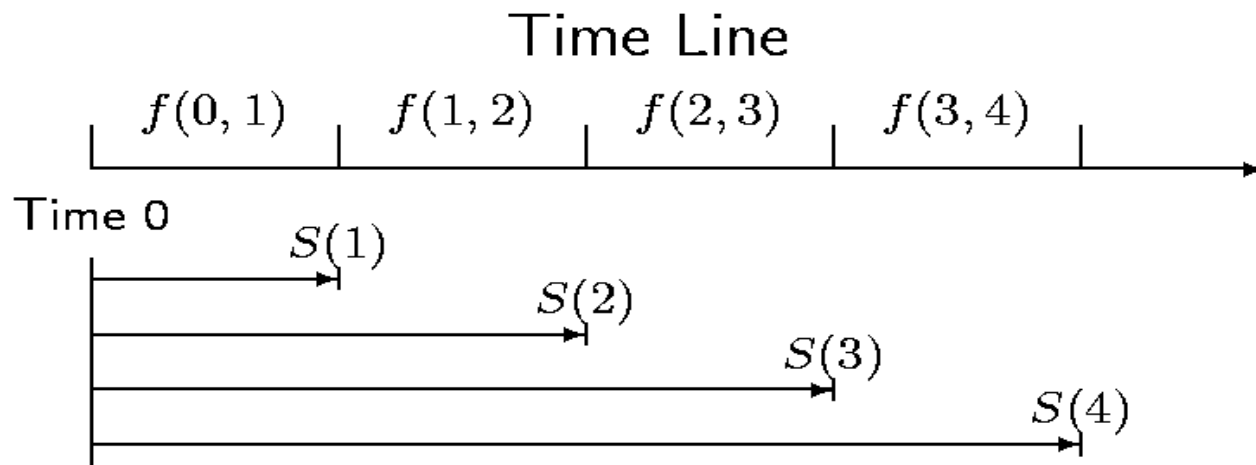
$$(1 + S(j))^j = (1 + S(i))^i (1 + f(i, j))^{j-i} \Rightarrow f(i, j) = \left(\frac{(1 + S(j))^j}{(1 + S(i))^i} \right)^{i-j} - 1$$



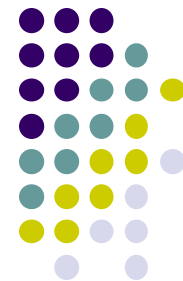
Forward Rates



- By definition, $f(0, j) = S(j)$.
- $f(i, j)$ is called the (implied) forward rates.
 - More precisely, the $(j-i)$ -period forward rate i periods from now.



Example: Spot and Forward Rate



- In this example, if \$1 is invested in 5-period zero-coupon bond (**maturity strategy**), it will grow to be $\$1 \times (1 + S(5))^5$
- An alternative strategy is to invest \$1 in one-period zero-coupon bonds and then reinvest at the one-period forward rates (**rollover strategy**). The result is exactly the same as expected.

$$(\$1 \times (1 + f(0,1))) \times (1 + f(1,2)) \times (1 + f(2,3)) \times (1 + f(3,4)) \times (1 + f(4,5))$$

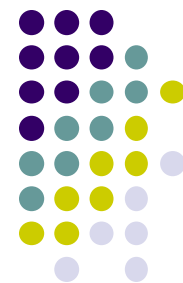
Forward rate

Forward Rate and Future Zero Rate



- We did not assume any a priori relation between $f(i, j)$ and future spot rate $S(i, j)$.
 - This is the subject of the term structure theories.
- Term structure theories have different explanation.
 - Unbiased expectation theory.
 - $f(i, j) = E(S(i, j))$
 - Liquidity preference theory.
 - $f(i, j) > E(S(i, j))$

Unbiased Expectations Theory



- Forward rate equals the average future spot rate,

$$f(a, b) = E[S(a, b)]$$

- Implies that a normal spot rate curve is due to the fact that the market expects the future spot rate to rise.
 - $f(j, j + 1) > S(j + 1)$ **if and only if** $S(j + 1) > S(j)$.
- Therefore, $E[S(j, j + 1)] > S(j + 1) > \dots > S(1)$ if and only if $S(j + 1) > \dots > S(1)$.
- Conversely, the spot rate is expected to fall if and only if the spot rate curve is inverted.

Liquidity Preference Theory



- The liquidity preference holds that investors demand a risk premium for holding long-term bonds.
- This implies that $f(a,b) > E(S(a,b))$.
- Even if people expect the interest rate to decline and rise equally, the theory asserts that the curve is upward sloping more often.

Spot and Forward Rate under Continuous Compounding



- The formula for the forward rate:

$$\because e^{-j \times S(j)} = e^{-i \times S(i)} e^{-(j-i) \times f(i,j)}$$

$$\Rightarrow -jS(j) = -iS(i) - (j-i)f(i,j) \Rightarrow f(i,j) = \frac{jS(j) - iS(i)}{j-i}$$

- The spot rate is an arithmetic average of forward rates. $\because e^{-j \times S(j)} = e^{-S(1)} e^{-f(1,2)} e^{-f(2,3)} \dots e^{-f(j-1,j)}$

$$\Rightarrow -jS(j) = -S(1) - f(1,2) - f(2,3) \dots - f(j-1,j)$$

$$\Rightarrow S(j) = \frac{f(0,1) + f(1,2) + f(2,3) \dots + f(j-1,j)}{j}$$

Spot and Forward Rate under Continuous Compounding



- The one-period forward rate:

$$f(j, j+1) = (j+1)S(j+1) - jS(j) \quad (5.10)$$

- Under continuous time instead of discrete time, the instantaneous forward rate at T time equals

$$\begin{aligned} \because f(T, T + \Delta T) &= S(T + \Delta T) + (S(T + \Delta T) - S(T)) \frac{T}{\Delta T} \\ \Rightarrow f(T) &\equiv \lim_{\Delta T \rightarrow 0} f(T, T + \Delta T) = S(T) + T \frac{\partial S}{\partial T} \end{aligned} \quad (5.11)$$

Note that $f(T) > S(T)$ if and only if $(\partial S / \partial T) > 0$

Example: Spot and Forward Rate



- Compute the one-period forward rates from this spot rate curve:

$S(1):2.0\%$, $S(2):2.5\%$, $S(3):3.0\%$, $S(4):3.5\%$, $S(5):4.0\%$.

$$\therefore \frac{2 + f(1,2)}{2} = 2.5 \Rightarrow f(1,2) = 3\%$$

- Answer:
$$\therefore \frac{2 + 3 + f(2,3)}{3} = 3 \Rightarrow f(2,3) = 4\%$$

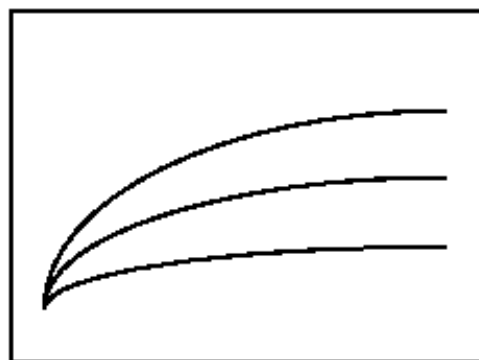
$$\therefore \frac{2 + 3 + 4 + f(3,4)}{4} = 3.5 \Rightarrow f(3,4) = 5\%$$

$$\therefore \frac{2 + 3 + 4 + 5 + f(4,5)}{5} = 4 \Rightarrow f(4,5) = 6\%$$

殖利率曲線，零息利率曲線，遠期利率曲線關係

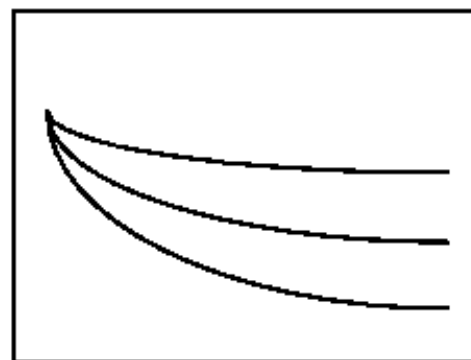


Normal curve



(a)

Inverted curve

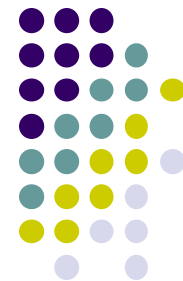


(b)

$$(1 + Z_b)^b = (1 + Z_a)^a (1 + f(a, b))^{b-a}$$

當 $Z_b > Z_a \Rightarrow f(a, b) > Z_b > Z_a$

Spot Rate and Forward Rate



- When the spot rate curve is normal, the forward rate dominates the spot rates,
$$f(i, j) > S(j) > \dots > S(i).$$
- When the spot rate curve is inverted, the forward rate is dominated by the spot rates,

$$f(i, j) < S(j) < \dots < S(i).$$

Locking in the Forward Rates



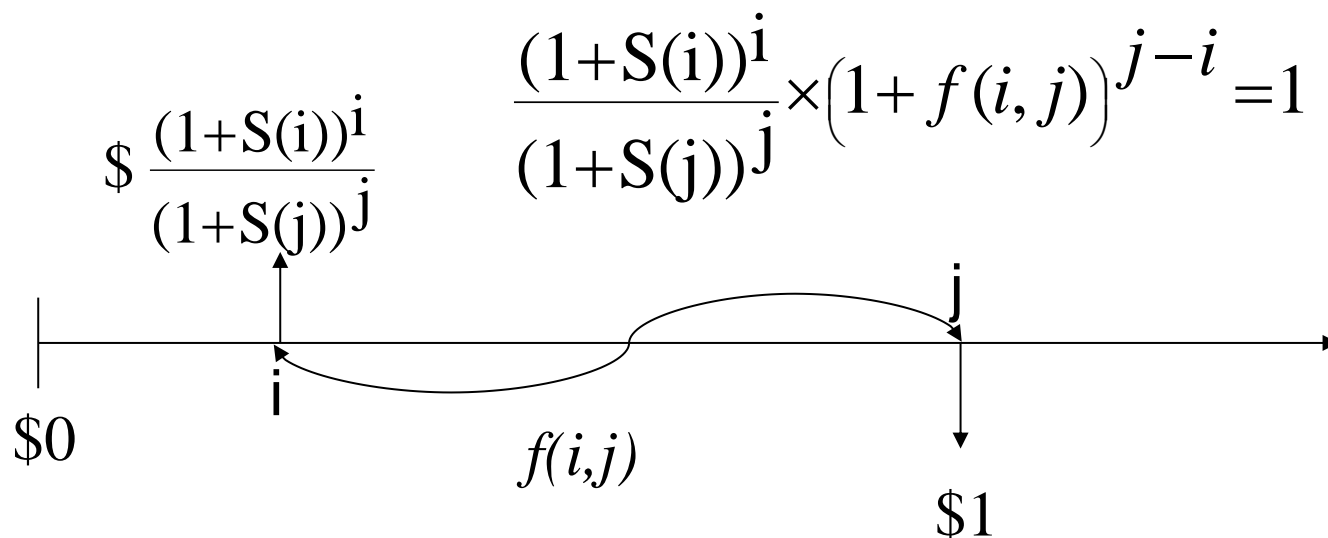
- Forward rates may not be realized in the future ($f(i,j) \neq S(i,j)$), but we can lock in any forward rate $f(i,j)$.
- Now we can make following strategies.
 - Buy **1** unit *j-year* zero-coupon bond.
 - Sell $\frac{(1+S(i))^i}{(1+S(j))^j}$ units *i-year* zero-coupon bonds.
- No net initial investment, because

$$\frac{(1+S(i))^i}{(1+S(j))^j} \times \frac{1}{(1+S(i))^i} - 1 \times \frac{1}{(1+S(j))^j} = 0$$

Locking in the Forward Rates



- At time j there will be a cash inflow of \$1.
- At time i there will be a cash outflow of \$ $\frac{(1+S(i))^i}{(1+S(j))^j}$
- The cash flow stream implies the rate $f(i,j)$ between times i and j .



Homework 6



- The fact that forward rate can be locked in today means that future spot rates must equal today's forward rates, or $S(a,b)=f(a,b)$, in a certain economy. Why? How about an uncertain economy?
(Hint:可舉一個簡單實例，用套利的觀念來說明)