Conceptual Exercises

February 25, 2019

```
In [1]: # import the packages
        import numpy as np
        import matplotlib.pyplot as plt
        import pandas as pd
        import statsmodels.api as sm

        from sklearn.metrics import mean_squared_error
        from sklearn.preprocessing import PolynomialFeatures
        from sklearn.model_selection import KFold, cross_val_score
        from sklearn.linear_model import LinearRegression
```

1 Conceptual exercises

1.1 Backfitting approach for GAMs

1.1.1 Generate a response Y and two predictors X_1, X_2 with $n = 100$.

```
In [2]: np.random.seed(30100)

        X1 = np.random.normal(size=100)
        X2 = np.random.normal(size=100)
        eps = np.random.normal(size=100)
        beta = np.random.uniform(0,4,size=3)

        Y = beta[0] + beta[1]*X1 + beta[2]*X2 + eps

In [3]: print("The simulated betas are: {}".format(beta))
```

```
The simulated betas are: [3.11845525 0.04921224 2.39379572]
```

1.1.2 Initialize $\hat{\beta}_1$ to take on a value of your choice. It does not matter what value you choose

```
In [4]: beta1 = 1
```

1.1.3 Keeping $\hat{\beta}_1$ fixed, fit the model

```
In [5]: y_reduced = (Y - beta1*X1)
```

```
In [6]: ls = sm.OLS(y_reduced, sm.add_constant(X2)).fit()
        print("The estimated coefficient of X2 is: {:.4f}".format(ls.params[1]))
```

```
The estimated coefficient of X2 is: 2.4621
```

1.1.4 Keeping $\hat{\beta}_2$ fixed, fit the model

```
In [7]: beta2 = ls.params[1]
        y_reduced = (Y - beta2*X2)
        ls = sm.OLS(y_reduced, sm.add_constant(X1)).fit()
        print("The estimated coefficient of X1 is: {:.4f}".format(ls.params[1]))
```

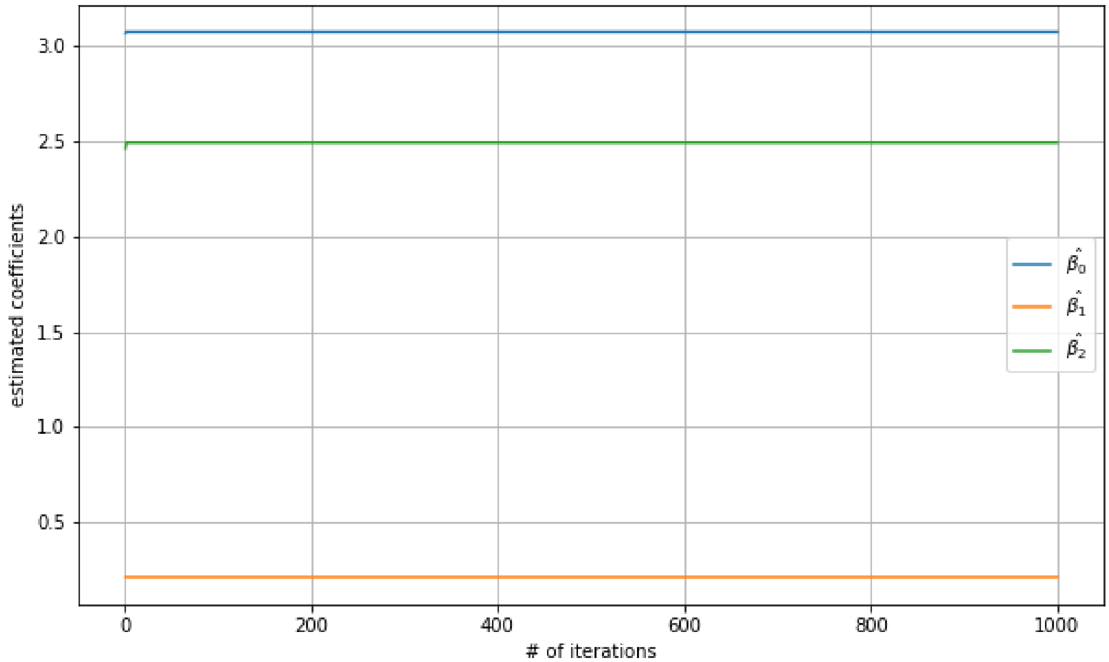
```
The estimated coefficient of X1 is: 0.2134
```

1.1.5 Write a for loop to repeat (c) and (d) 1000 times. Report the estimates for $\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2$ at each iteration of the for loop. Create a plot in which each of these values is displayed, with $\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2$ each shown in a different color.

```
In [8]: X = sm.add_constant(np.vstack((X1, X2)).T)
```

```
In [9]: beta0_lst = []; beta1_lst = []; beta2_lst = []
        beta0 = beat1 = beta2 = 1
        for i in range(1000):
            # estimate beta2
            beta2 = sm.OLS(Y-beta1*X[:,1], X[:,[0,2]]).fit().params[1]
            # estimate beta0, beta1
            beta0, beta1 = sm.OLS(Y-beta2*X[:,2], X[:,:2]).fit().params
            beta0_lst.append(beta0); beta1_lst.append(beta1); beta2_lst.append(beta2)
```

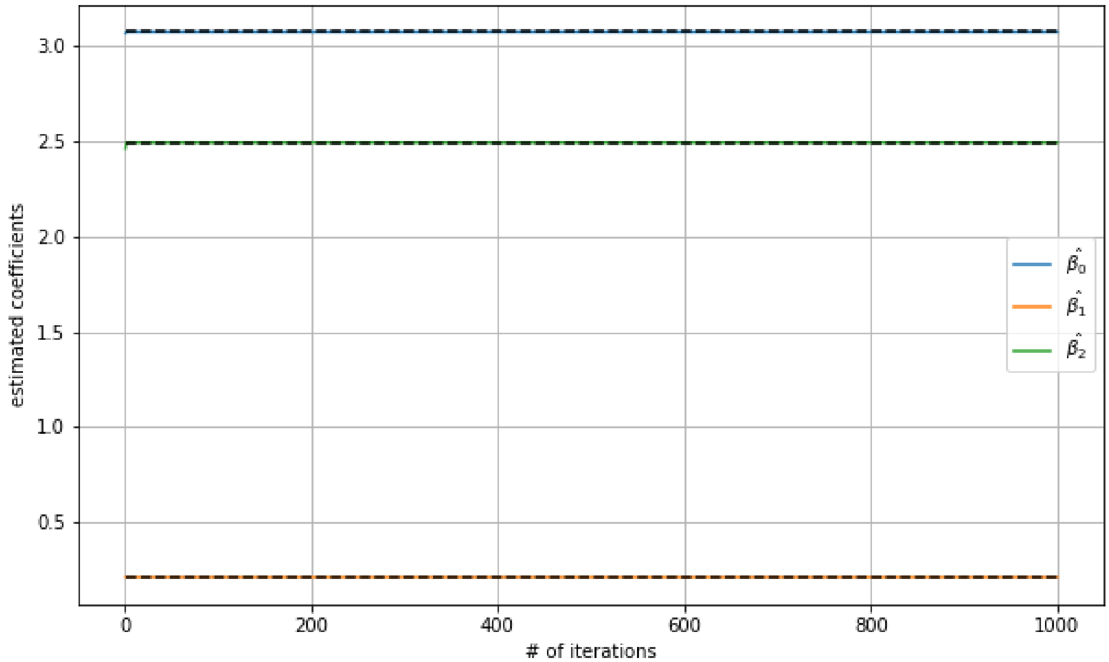
```
In [10]: plt.figure(figsize=(10,6))
         plt.plot(range(1,1001), beta0_lst, label = r'$\hat{\beta_{0}}$')
         plt.plot(range(1,1001), beta1_lst, label = r'$\hat{\beta_{1}}$')
         plt.plot(range(1,1001), beta2_lst, label = r'$\hat{\beta_{2}}$')
         plt.xlabel("# of iterations"); plt.ylabel("estimated coefficients")
         plt.grid()
         plt.legend()
         plt.show()
```



1.1.6 Compare your answer in (e) to the results of simply performing multiple linear regression to predict Y using X_1 and X_2. Overlay those multiple linear regression coefficient estimates on the plot obtained in (e).

```
In [11]: params = sm.OLS(Y, X).fit().params
```

```
In [12]: plt.figure(figsize=(10,6))
         plt.plot(range(1,1001), beta0_lst, label = r'$\hat{\beta_{0}}$')
         plt.plot(range(1,1001), beta1_lst, label = r'$\hat{\beta_{1}}$')
         plt.plot(range(1,1001), beta2_lst, label = r'$\hat{\beta_{2}}$')
         plt.plot(range(1,1001), [params[0]]*1000, color = 'black', linestyle = 'dashed')
         plt.plot(range(1,1001), [params[1]]*1000, color = 'black', linestyle = 'dashed')
         plt.plot(range(1,1001), [params[2]]*1000, color = 'black', linestyle = 'dashed')
         plt.xlabel("# of iterations"); plt.ylabel("estimated coefficients")
         plt.grid()
         plt.legend()
         plt.show()
```



As we can observe from the plot, the estimates derived from back-fitting methods will approximately converge to the OLS estimation.

1.1.7 On this data set, how many backfitting iterations were required in order to obtain a “good” approximation to the multiple regression coefficient estimates?

```
In [13]: print("Number of iterations for beta0 to converge is: {}".format(beta0_lst.index(beta0
         print("Number of iterations for beta1 to converge is: {}".format(beta1_lst.index(beta1
         print("Number of iterations for beta2 to converge is: {}".format(beta2_lst.index(beta2
```

```
Number of iterations for beta0 to converge is: 5
Number of iterations for beta1 to converge is: 5
Number of iterations for beta2 to converge is: 5
```

It takes only five iterations for the backfitting to obtain a "good" approximation to the OLS estimates.

1.2 Backfitting with large P

```
In [14]: X = sm.add_constant(np.random.normal(size = (1000, 100)))
         eps = np.random.normal(size = 1000)
         beta = np.random.uniform(0, 10, 101)

         Y = np.dot(X, beta) + eps
```

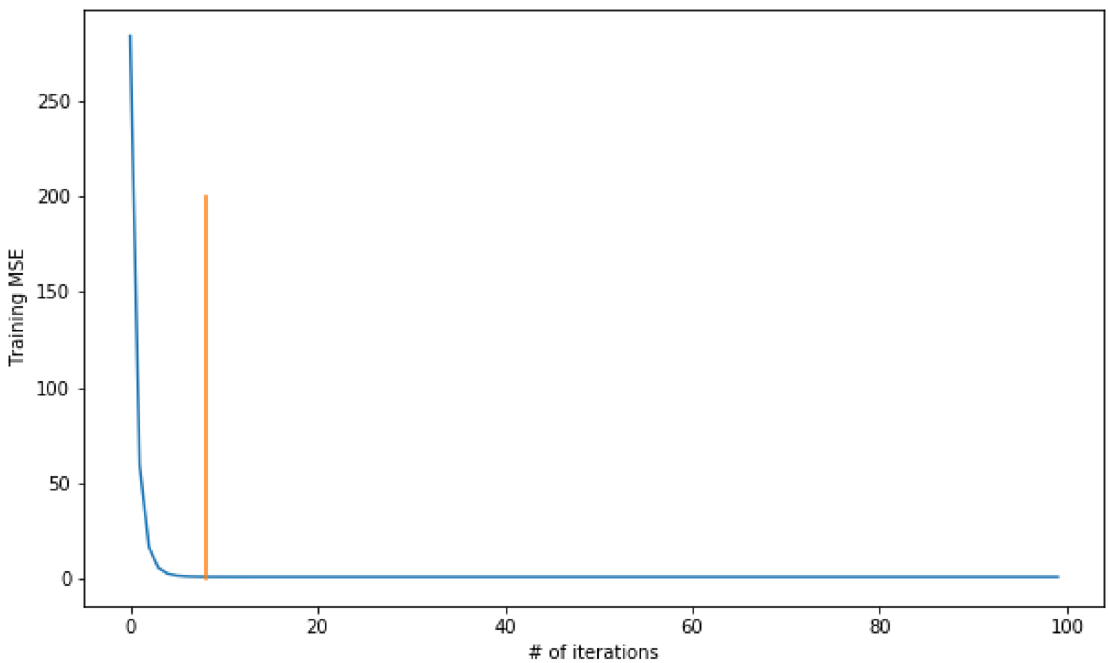
```
In [15]: # do the back-fitting process
         beta_lst = np.ones(shape = (101,101))
         mse_lst = []
         for i in range(1,101):
             for j in range(101):
                 beta_lst[j,i] = sm.OLS(Y - np.dot(X[:,np.arange(101)!=j], beta_lst[np.arange(
             mse_lst.append(mean_squared_error(Y, np.dot(X, beta_lst[:,i])))

In [16]: # do the regular OLS estimation
         res = sm.OLS(Y,X).fit()
         mse_ols = mean_squared_error(Y, res.fittedvalues)

In [17]: print("The MSE using OLS regression is {}".format(mse_ols))

The MSE using OLS regression is 0.8388775268277755


In [18]: plt.figure(figsize=(10,6))
         plt.plot(range(100), mse_lst)
         idx = list(np.array(mse_lst) <= 0.91).index(True)
         plt.plot([idx, idx], [0,200])
         plt.xlabel("# of iterations"); plt.ylabel("Training MSE")
         plt.show()
```



It took approximately 12 iterations to reach a good approximation to the OLS estimation, in terms of training set MSE.