Problem 1.

Proof:

$$\text{LHS} = \mathbb{P}r(\alpha_1, \cdots, \alpha_n \mid \beta)$$

$$= \frac{\mathbb{P}r(\alpha_1 \wedge \cdots \wedge \alpha_n \wedge \beta)}{\mathbb{P}r(\beta)}$$

$$= \frac{\mathbb{P}r(\alpha_1 \wedge \cdots \wedge \alpha_n \wedge \beta)}{\mathbb{P}r(\alpha_2 \wedge \cdots \wedge \alpha_n \wedge \beta)} \cdot \frac{\mathbb{P}r(\alpha_2 \wedge \cdots \wedge \alpha_n \wedge \beta)}{\mathbb{P}r(\alpha_3 \wedge \cdots \wedge \alpha_n \wedge \beta)} \cdot \cdots \cdot \frac{\mathbb{P}r(\alpha_{n-1} \wedge \alpha_n \wedge \beta)}{\mathbb{P}r(\alpha_n \wedge \beta)} \cdot \frac{\mathbb{P}r(\alpha_n \wedge \beta)}{\mathbb{P}r(\beta)}$$

$$= \mathbb{P}r(\alpha_1 \mid \alpha_2, \cdots, \alpha_n, \beta) \cdot \mathbb{P}r(\alpha_2 \mid \alpha_3, \cdots, \alpha_n, \beta) \cdot \cdots \cdot \mathbb{P}r(\alpha_{n-1} \mid \alpha_n, \beta) \cdot \mathbb{P}r(\alpha_n \mid \beta)$$

$$= \text{RHS}$$

Proved. $\square$

Problem 2.

From the problem context, we have:

$$\begin{cases} \Pr(\text{oil}) = 0.5 \\ \Pr(\text{gas}) = 0.2 \\ \Pr(\neg\text{oil} \wedge \neg\text{gas}) = 0.3 \\ \Pr(+ \mid \text{oil}) = 0.9 \\ \Pr(+ \mid \text{gas}) = 0.3 \\ \Pr(+ \mid \neg\text{oil} \wedge \neg\text{gas}) = 0.1 \end{cases}$$

, and we want to compute $\Pr(\text{oil} \mid +)$.

$$\Pr(\text{oil} \mid +) = \frac{\Pr(\text{oil} \wedge +)}{\Pr(+)} = \frac{\Pr(+ \wedge \text{oil})}{\Pr(\text{oil})} \cdot \frac{\Pr(\text{oil})}{\Pr(+)} = \Pr(+ \mid \text{oil}) \cdot \Pr(\text{oil}) / \Pr(+)$$

Here we namely want to compute $\Pr(+)$.

We have $\Pr(+) = \Pr(+ \mid \text{oil}) \cdot \Pr(\text{oil}) + \Pr(+ \mid \text{gas}) \cdot \Pr(\text{gas}) + \Pr(+ \mid \neg\text{oil} \wedge \neg\text{gas}) \cdot \Pr(\neg\text{oil} \wedge \neg\text{gas})$

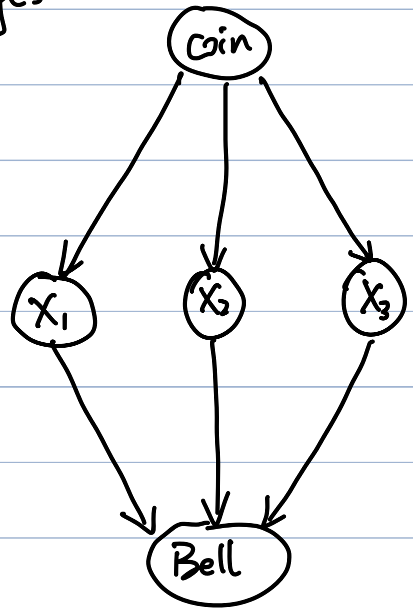
$$= 0.9 \cdot 0.5 + 0.3 \cdot 0.2 + 0.1 \cdot 0.3$$

$$= 0.54$$

So $\Pr(\text{oil} \mid +) = 0.9 \cdot 0.5 / 0.54 = \frac{5}{6} \approx 0.8333$

Problem 3.

Bayes Network:



CPT for node "Bell":

X_1	X_2	X_3	Bell	$Pr(Bell \mid X_1, X_2, X_3)$
H	H	H	✓	1
H	H	H	X	0
H	H	$\bar{H}$	✓	0
H	H	$\bar{H}$	X	1
H	$\bar{H}$	H	✓	0
H	$\bar{H}$	H	X	1
H	$\bar{H}$	$\bar{H}$	✓	0
H	$\bar{H}$	$\bar{H}$	X	1
$\bar{H}$	H	H	✓	0
$\bar{H}$	H	H	X	1
$\bar{H}$	H	$\bar{H}$	✓	0
$\bar{H}$	H	$\bar{H}$	X	1
$\bar{H}$	$\bar{H}$	H	✓	0
$\bar{H}$	$\bar{H}$	H	X	1
$\bar{H}$	$\bar{H}$	$\bar{H}$	✓	1
$\bar{H}$	$\bar{H}$	$\bar{H}$	X	0

CPT for node "Coin":

Coin	Pr(coin)
a	1/3
b	1/3
c	1/3

CPT for node "X_1":

coin	X_1	$Pr(X_1 \mid Coin)$
a	H	0.2
a	$\bar{H}$	0.8
b	H	0.4
b	$\bar{H}$	0.6
c	H	0.8
c	$\bar{H}$	0.2

CPT for node "X_2":

coin	X_2	$Pr(X_2 \mid Coin)$
a	H	0.2
a	$\bar{H}$	0.8
b	H	0.4
b	$\bar{H}$	0.6
c	H	0.8
c	$\bar{H}$	0.2

CPT for node "X_3":

coin	X_3	$Pr(X_3 \mid Coin)$
a	H	0.2
a	$\bar{H}$	0.8
b	H	0.4
b	$\bar{H}$	0.6
c	H	0.8
c	$\bar{H}$	0.2

Problem 4:

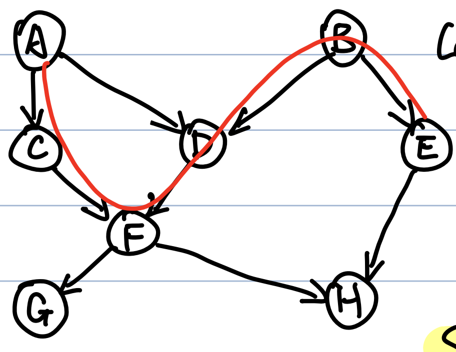
(a) $I(A, \emptyset, BE)$ $I(E, B, ACDFG)$

$I(B, \emptyset, AC)$ $I(F, CD, ABE)$

$I(C, A, BDE)$ $I(G, F, ABCDEH)$

$I(D, AB, CE)$ $I(H, EF, ABCDG)$

(b) ①. d_separated(A, F, E):



Consider the path —: $A \to C \to F$ is open;

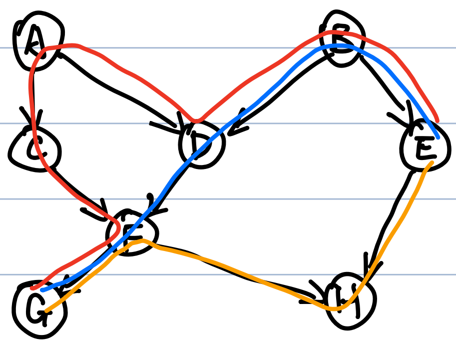
$C \to F \leftarrow D$ is open, as $F \in \mathbf{Z}$; $F \leftarrow D \leftarrow B$ is open,

$D \leftarrow B \to F$ is open.

So this path is not blocked.

So d-seperated(A, F, E) is false.

②. d_separated(G, B, E):



Consider path —:

$D \leftarrow B \to E$ is closed, as $B \in \mathbf{Z}$. So this path is blocked.

Consider path —:

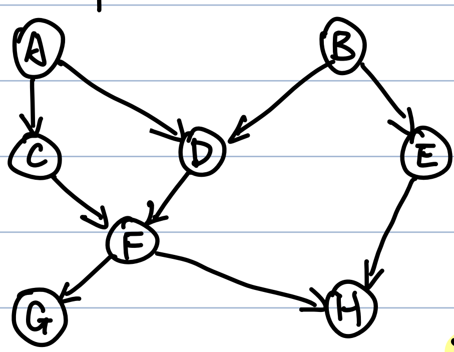
$D \leftarrow B \to E$ is closed, as $B \in \mathbf{Z}$. So this path is blocked.

Consider path —:

$F \to H \leftarrow E$ is closed, as $H \notin \mathbf{Z}$. So this path is blocked.

Since all these 3 paths are blocked, d-separated(G, B, E) is true.

③. d_separated(AB, CDE, GH)



Any possible path must go along at least one of $A \to C \to F$, $A \to D \to F$, $B \to D \to F$, or $B \to E \to H$. Since $\{C, D, E\} \subseteq \mathbf{Z}$, and all the values stated above are squential, all of these values are closed, so all possible paths are blocked.

Thus, d-separated(AB, CDE, GH) is true.

(c).

$$\mathbb{P}r(a,b,c,d,e,f,g,h) = \mathbb{P}r(a)\mathbb{P}r(b|a)\mathbb{P}r(c|a,b)\mathbb{P}r(d|a,b,c)\mathbb{P}r(e|a,b,c,d)$$

$$\cdot \mathbb{P}r(f|a,b,c,d,e)\mathbb{P}r(g|a,b,c,d,e,f)\mathbb{P}r(h|a,b,c,d,e,f,g)$$

according to the assumptions we made in (a)

$$= \mathbb{P}r(a)\cdot\mathbb{P}r(b)\cdot\mathbb{P}r(c|a)\cdot\mathbb{P}r(d|a,b)\cdot\mathbb{P}r(e|b)$$

$$\cdot\mathbb{P}r(f|c,d)\cdot\mathbb{P}r(g|f)\cdot\mathbb{P}r(h|e,f)$$

(d). ① $\mathbb{P}r(A=1, B=1)$

Since we have $I(A,\emptyset,BE)$, we have A and B are independent

Thus, $\mathbb{P}r(A=1,B=1)=\mathbb{P}r(A=1)\cdot\mathbb{P}r(B=1)$

$$= 0.2\times 0.7$$

$$= 0.14$$

② $\mathbb{P}r(E=0 \mid A=0)$

Since we have $I(E,B,ACDFG)$, we have E and A are independent given B.

Thus, $\mathbb{P}r(E=0|A=0) = \mathbb{P}r(E=0|B=0,A=0)\cdot\mathbb{P}r(B=0)+\mathbb{P}r(E=0|B=1,A=0)\cdot\mathbb{P}r(B=1)$

$$= \mathbb{P}r(E=0|B=0)\cdot\mathbb{P}r(B=0)+\mathbb{P}r(E=0|B=1)\cdot\mathbb{P}r(B=1)$$

$$= 0.1\times 0.3+0.9\times 0.7$$

$$= 0.66$$

Problem 5.

(a) We have

	A	B	$\mathbb{P}r(A,B)$
w_0	T	T	0.3
w_1	T	F	0.2
w_2	F	T	0.1
w_3	F	F	0.4

as $\alpha: A \Rightarrow B \equiv \neg A \vee B$, we see α is valid as in worlds w_0, w_2, w_3

so $Models(\alpha) = \{w_0, w_2, w_3\}$

(b) $\mathbb{P}r(\alpha) = \sum_i \{\mathbb{P}r(w_i) \mid w_i \in Models(\alpha)\}$

$= \mathbb{P}r(w_0) + \mathbb{P}r(w_2) + \mathbb{P}r(w_3)$

$= 0.3 + 0.1 + 0.4$

$= 0.8$

(c). $\mathbb{P}r(A,B|\alpha) = \mathbb{P}r(A \wedge B \wedge \alpha) / \mathbb{P}r(\alpha) = \mathbb{P}r(w_0) / \mathbb{P}r(\alpha) = 0.3/0.8 = \frac{3}{8} = 0.375$

(d). $A \Rightarrow \neg B \equiv \neg A \vee \neg B = \neg(A \wedge B)$

$\mathbb{P}r(A \Rightarrow \neg B | \alpha) = \mathbb{P}r(\neg(A \wedge B) \wedge \alpha) / \mathbb{P}r(\alpha) = (\mathbb{P}r(w_2) + \mathbb{P}r(w_3)) / \mathbb{P}r(\alpha)$

$= (0.1 + 0.4) / 0.8$

$= \frac{5}{8} = 0.625$