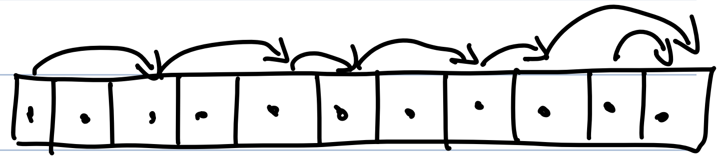
1. (a)

```
let L = [0] * (n+1)  // store the length of longest path      O(n)
Let P = [-1] * (n+1) // store the predecessor of each vertex   O(n)
for vertex u from 1 to n:              } m edges in total  O(m)
    for edges (u,v) with u<v:          }
        if L[u]+1 > L[v]:      O(1)
            L[v] = L[u]+1      O(1)
            P[v] = u           O(1)
V = 0     O(1)
for i in range (1, n+1):    O(n)
    if L[i] > L[v]:         O(1)
        v = i               O(1)
Path = [v]                  O(1)
While P[v] ≠ -1:            O(n)
    path = path.append (P[v])   O(1)
    v.P[v]                      O(1)
path.reverse()              O(n)
return path.                O(1)
In total : O(m+n)
```



(b).

We let the sequence be $A = [A_1, A_2, A_3, \cdots A_n]$ with vertex set $V = \{1,2,\cdots,n\}$

Then for each pair of vertices in V, (u,v) with $u<v$, add (u,v) to edge set E if $S_u < S_v$.

Now we run the algorithm in (a) for (V,E). Then we can find the longest increasing subsequence.

2. (a) let OPT[i] be the maximum revenue ending at position i. $O(n)$

for $i = -k-1, \cdots, -1$: $O(k) = O(n)$

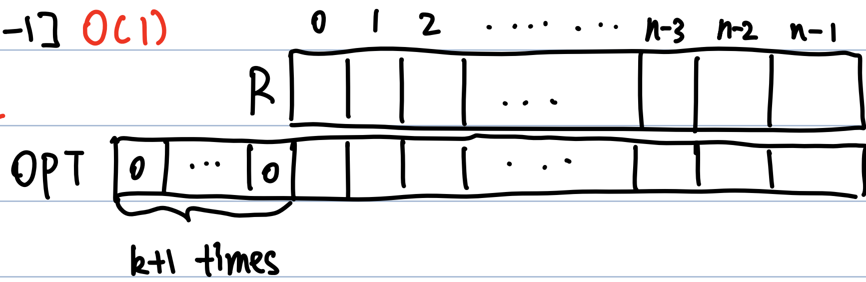
OPT[i] = 0. $O(1)$

for $i = 0, \cdots, n-1$: $O(n)$

OPT[i] = max(R[i] + OPT[i-k-1], OPT[i-1]) $O(1)$

return OPT[n-1] $O(1)$

In total $O(n)$.



For $i = -k-1, \ldots, -1$, we let OPT[i] = 0, which would not affect the revenues generated on real positions.

For $i = 0 \ldots k$, we have OPT[i] = max({R[0], R[1], ⋯, R[i]}), because we can only set one billboards from spot 0 to spot k, so maximum revenue including up to position i is max({R[i]+OPT[i-k-1], R[i-1]}).

(b). let OPT[i][j] be a 2d array $O(Mn)$

Here OPT[i][j] is the maximum revenue including up to position i and number j # of billboards.

for $i = 0, \ldots, n-1$: $O(n)$

OPT[i][0] = 0 # since no billboards. $O(1)$

for $i = -k-1, \ldots, -1$: $O(k) = O(n)$

for $j = 1, \cdots, M$: $O(M)$

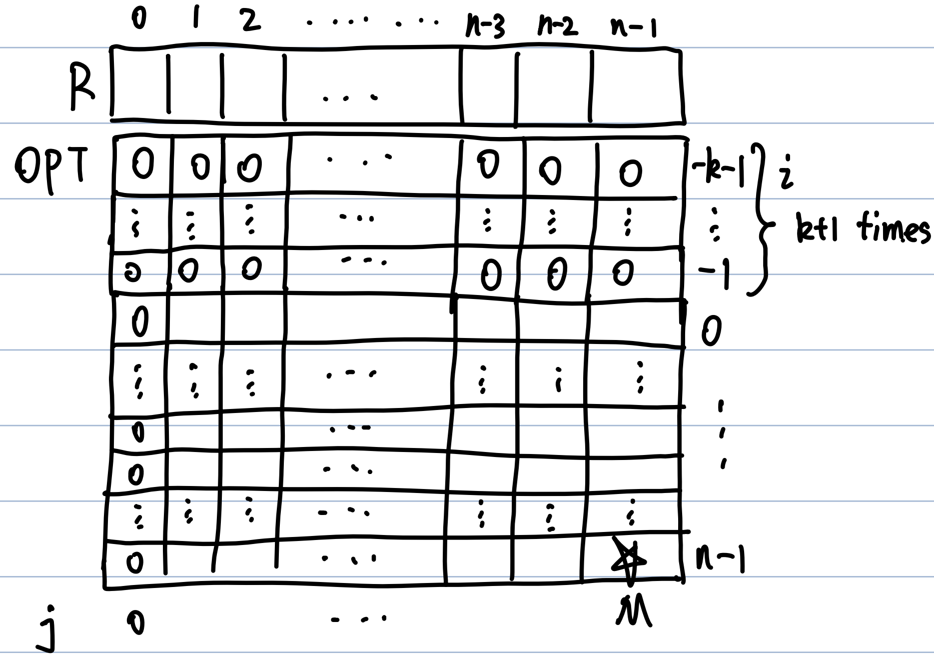
OPT[i][j] = 0 # since position is negative $O(1)$

for $i = 0, \cdots, n-1$: $O(n)$

for $j = 1, \cdots, M$: $O(M)$

OPT[i][j] = max(R[i] + OPT[i-k-1][j-1], OPT[i-1][j]) $O(1)$

return OPT[n-1][M] $O(1)$ In total: $O(Mn)$


0 1 2 n-3 n-2 n-1
R
OPT
-k-1
-1
0
n-1
i
k+1 times
j
M

3. let $OPT\_1[i]$ be the longest zigzagging subsequence with last 2 elements increasing $O(n)$

let $OPT\_2[i]$ be the longest zigzagging subsequence with last 2 elements decreasing $O(n)$

```
for i = 1, ..., n-1:                         O(n)
    for j = 0, ..., i-1:                     O(n)
        if A[i] ≥ A[j]:                      O(1)
            OPT_1[i] = max(OPT_1[i], OPT_2[j] + 1)   O(1)
        if A[i] ≤ A[j]:                      O(1)
            OPT_2[i] = max(OPT_2[i], OPT_1[j] + 1)   O(1)
return max(OPT_1[n-1], OPT_2[n-1]).          O(1).
```

In total: $O(n^2)$.

4.

(a) if there are n vertices, then there are $n!$ ways to traverse n vertices. Then in each way, we check whether the path is a Hamiltonian Path, so we have to check $n-1$ times for whether the edges exist.

In total: $n! \cdot O(n) = O(n! \cdot n)$

(b). let OPT[S][V] = {False}, with S as set of vertices and v be a vertex. $O(2^n \cdot n)$

```
for each vertex v in V(G):                    O(n)
    OPT[{v}][v] = True  // single vertex is a Hamiltonian circle.
for each set S of size k from 2 to n:         O(Σ_{k=2}^{n} (n k)) = O(2^n)
    for v in S:                               O(n)
        for u in S\{v}:                       O(n)
            if OPT[S\{v}][u] == True && edge (u,v) exists:   O(1)
                OPT[S][v] = True              O(1)
for v in V(G):                                O(n)
    if OPT[V][v] = True,                      O(1)
        return True                           O(1)
return False                                  O(1).
```

In total: $O(n^2 2^n)$.

Here, we use the fact that a path is Hamiltonian only if the prefix of it is Hamiltonian.

(c). $$\lim_{n\to\infty} \frac{2^n \cdot n^2}{n! \cdot n} = \lim_{n\to\infty} \frac{2^n}{(n-1)!} = \lim_{n\to\infty} \left(\frac{2}{n-1} \cdot \frac{2}{n-2} \cdot \frac{2}{n-3} \cdot \dots \cdot \frac{2}{2} \cdot \frac{2}{1} \cdot 2\right) = 0.$$

Thus, (b) is an asymptotic improvement from part (a).