0.1 Construirea jocurilor Bayesiene

When, using the informational extended strategies, the matrices $A(\alpha,\beta)$ and $B\,(\alpha,\beta)$ were already built, we use the following notations: $\left\|a_{i_j^\alpha j_i^\beta}\right\|_{i\in I}^{j\in J} \equiv \left\|a_{ij}^{\alpha\beta}\right\|_{i\in I}^{j\in J}$ and $\left\|b_{i_j^\alpha j_i^\beta}\right\|_{i\in I}^{j\in J} \equiv \left\|b_{ij}^{\alpha\beta}\right\|_{i\in I}^{j\in J}$ for all $\alpha=\overline{1,\kappa_1}$ and $\beta=\overline{1,\kappa_2}$. Every player knows that the utilities are determined by the set of matrices $\{AB(\alpha,\beta)\}_{\alpha=\overline{1,\kappa_1}}^{\beta=\overline{1,\kappa_2}}$, were

$$AB(\alpha,\beta)=\begin{pmatrix} \left(a_{11}^{\alpha\beta},b_{11}^{\alpha\beta}\right) & \cdots & \left(a_{1j}^{\alpha\beta},b_{1j}^{\alpha\beta}\right) & \cdots & \left(a_{1m}^{\alpha\beta},b_{1m}^{\alpha\beta}\right) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \left(a_{i1}^{\alpha\beta},b_{i1}^{\alpha\beta}\right) & \cdots & \left(a_{ij}^{\alpha\beta},b_{ij}^{\alpha\beta}\right) & \cdots & \left(a_{im}^{\alpha\beta},b_{im}^{\alpha\beta}\right) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \left(a_{nm}^{\alpha\beta},b_{nm}^{\alpha\beta}\right) & \cdots & \left(a_{nj}^{\alpha\beta},b_{nj}^{\alpha\beta}\right) & \cdots & \left(a_{nm}^{\alpha\beta},b_{nm}^{\alpha\beta}\right) \end{pmatrix},$$

but they do not know which matrix from this set will be used.

So, the game of imperfect information on the set of informational extended strategies generates the following normal form incomplete information game

$$\widetilde{\mathbf{\Gamma}}=\left\langle \{1,2\},I,J,\left\{AB(\alpha,\beta)=\left\|\left(a_{ij}^{\alpha\beta},b_{ij}^{\alpha\beta}\right)\right\|_{i\in I}^{j\in J}\right\}_{\alpha=\overline{1,\kappa_1}}^{\beta=\overline{1,\kappa_2}}\right\rangle. \tag{1}$$

We will denote the pure strategy of player 1 by $\widetilde{\mathbf{i}}=i_1i_2...i_\beta...i_{\kappa_2}$ and it has the following meaning: the player will chose the line $i_1\in I$ from the utility matrix $A(\alpha,1)$, the line $i_2\in I$ from the utility matrix $A(\alpha,2)$ and so on, line $i_{\kappa_2}\in I$ from the utility matrix $A(\alpha,\kappa_2)$. Then the set of all pure strategy of player 1 will be determined by the set of all corteges of type $i_1i_2...i_\beta...i_{\kappa_2}$ for all $i_\beta\in I$ and will be denoted by $\widetilde{\mathbf{I}}(\alpha)$. By the same way we will denote the pure strategy of player 2 by $\widetilde{\mathbf{j}}=j_1j_2...j_\alpha...j_{\kappa_2}$ and it has the following meaning: the player will chose column $j_1\in J$ from utility matrix $B(1,\beta)$, column $j_2\in J$ from utility matrix $B(2,\beta)$ and so on he will chose column $j_{\kappa_1}\in J$ from utility matrix $B(\kappa_1,\beta)$. Then the set of all pure strategy of player 2 will be determined by the set of all corteges of type $j_1j_2...j_\alpha...j_{\kappa_2}$ for all $j_\alpha\in J$ and will be denoted by $\widetilde{\mathbf{J}}(\beta)$.

A payoff function specifies each player's expected payoff matrices for every possible combination of all player's actions and types. Hence, if the player 1 of type α chooses the pure strategy $\widetilde{\mathbf{i}}\in\widetilde{\mathbf{I}}(\alpha)$, and the player 2 plays some strategy $\widetilde{\mathbf{j}}\in\widetilde{\mathbf{J}}(\beta)$ for all $\alpha\beta=\overline{1,\kappa_1}$, then expected payoffs of player 1 is the following matrix

$$\mathbf{A}(\alpha)=\left\|\mathbf{a}_{\widetilde{\mathbf{i}}\widetilde{\mathbf{j}}}\right\|_{\widetilde{\mathbf{i}}\in\widetilde{\mathbf{I}}(\alpha)}^{\widetilde{\mathbf{j}}\in\widetilde{\mathbf{J}}(\beta)} \tag{2}$$

where $a_{\widetilde{\mathbf{i}}\widetilde{\mathbf{j}}} = \sum\limits_{\beta=\overline{1,\kappa_2}} p(\beta/\alpha) a^{\alpha\beta}_{i_\beta j_\alpha}$. Similarly, if player 2 of type β chooses the pure strategy $\widetilde{\mathbf{j}} \in \widetilde{\mathbf{J}}(\beta)$ and the player 1 plays some strategy $\widetilde{\mathbf{i}} \in \widetilde{\mathbf{I}}(\alpha)$ for all $\alpha = \overline{1,\kappa_1}$, then expected payoffs of player 2 of type β is

$$\mathbf{B}(\beta) = \left\| \mathbf{b}_{\widetilde{\mathbf{i}}\widetilde{\mathbf{j}}} \right\|_{\widetilde{\mathbf{i}} \in \widetilde{\mathbf{I}}(\alpha)}^{\widetilde{\mathbf{j}} \in \widetilde{\mathbf{J}}(\beta)} \tag{3}$$

where $b_{\widetilde{\mathbf{i}}\widetilde{\mathbf{j}}} = \sum\limits_{\alpha=\overline{1,\kappa_1}} q(\alpha|\beta) b^{\alpha\beta}_{i_\beta j_\alpha}$. Here $p(\beta|\alpha) = \dfrac{p(\beta \cap \alpha)}{p(\alpha)}$ (Bayes'Rule) (respectively $q(\alpha|\beta) = \dfrac{q(\alpha \cap \beta)}{q(\beta)}$) is the conditional probability assigned to the type $\beta = \overline{1,\kappa_2}$ (respectively $\alpha = \overline{1,\kappa_1}$) when the type of the player 1 is α (respectively of the player 2 is β).

So, we can introduce the following definition.

Definition 1 *For the incomplete information game $\widetilde{\mathbf{\Gamma}}$ from (1) the normal form game*

$$\Gamma_{Bayes} = \left\langle \{1,2\}, \widetilde{\mathbf{I}}, \widetilde{\mathbf{J}}, \mathcal{A}, \mathcal{B} \right\rangle, \tag{4}$$

where $\widetilde{\mathbf{I}} = \bigcup\limits_{\alpha=\overline{1,\kappa_1}} \widetilde{\mathbf{I}}(\alpha)$, $\widetilde{\mathbf{J}} = \bigcup\limits_{\beta=\overline{1,\kappa_2}} J(\beta)$ and the utility matrices are $\mathcal{A} = \|\mathbf{A}(\alpha)\|_{\alpha=\overline{1,\kappa_1}}$ and $\mathcal{B} = \|\mathbf{B}(\beta)\|_{\beta=\overline{1,\kappa_2}}$, is called the associated Bayesian game in the non informational extended strategies.

Example 2 *The construction of the Bayesian game for the 2×3 bimatrix games in incomplete information, generated by the informational extended strategies.*

Consider a bimatrix game in incomplete information for which the utilities are:

$$\mathbf{AB}(\alpha,\beta) = \begin{pmatrix} \begin{pmatrix} a^{\alpha\beta}_{11}, b^{\alpha\beta}_{11} \\ a^{\alpha\beta}_{21}, b^{\alpha\beta}_{21} \end{pmatrix} & \begin{pmatrix} a^{\alpha\beta}_{12}, b^{\alpha\beta}_{12} \\ a^{\alpha\beta}_{22}, b^{\alpha\beta}_{22} \end{pmatrix} & \begin{pmatrix} a^{\alpha\beta}_{13}, b^{\alpha\beta}_{13} \\ a^{\alpha\beta}_{23}, b^{\alpha\beta}_{23} \end{pmatrix} \end{pmatrix}.$$

The Bayesian game will contain the elements:a) the set of players $\{1,2\}$;b) the set of actions of the players $I = \{1,2\}$, $J = \{1,2,3\}$; c) the set of types of the player 1 is $\Delta_1 = \{\alpha = 1,2\}$ and of the player 2 is $\Delta_2 = \{\beta = 1,2\}$; d) denote the type probability for player 1 by $p(\beta|\alpha)$, respectively $q(\alpha|\beta)$ for player 2; e) for any fixed α the set of pure strategies of the player 1 is $I(\alpha) = \{i_\beta i_\gamma : i_\beta \in I, i_\gamma \in I, \forall \beta,\gamma \in \Delta_2, \beta \neq \gamma\} = \{1_1 1_2, 1_1 2_2, 2_1 1_2, 2_1 2_2\}$ and for any fixed β the set of pure strategies of the player 2 is $J(\beta) = \{j_\alpha j_\delta : j_\alpha \in J, j_\delta \in J, \forall \alpha,\delta \in \Delta_1, \alpha \neq \delta\}$ $= \{1_1 1_2, 1_1 2_2, 2_1 1_2, 2_1 2_2, 1_1 3_2, 3_1 1_2, 2_1 3_2, 3_1 2_2, 3_1 3_2\}$. The players do not know the exact type of the partners and supply this lack of information by the belief probabilities. Thus the player 1, being of type α, will assume with the probability $p(\beta = 1|\alpha)$ that he has the payoff matrice $\begin{pmatrix} a^{\alpha 1}_{11} & a^{\alpha 1}_{12} & a^{\alpha 1}_{13} \\ a^{\alpha 1}_{21} & a^{\alpha 1}_{22} & a^{\alpha 1}_{23} \end{pmatrix}$ and, with probabilities $p(\beta = 2|\alpha)$, the payoff matrice$\begin{pmatrix} a^{\alpha 2}_{11} & a^{\alpha 2}_{12} & a^{\alpha 2}_{13} \\ a^{\alpha 2}_{21} & a^{\alpha 2}_{22} & a^{\alpha 2}_{23} \end{pmatrix}$. Respectively, the player 2, being of type β, will assume with the probability $q(\alpha = 1|\beta)$ that he has the payoff matrice$\begin{pmatrix} b^{1\beta}_{11} & b^{1\beta}_{12} & b^{1\beta}_{13} \\ b^{1\beta}_{21} & b^{1\beta}_{22} & b^{1\beta}_{23} \end{pmatrix}$ and, with the probability $q(\alpha = 2|\beta)$, the payoff matrice$\begin{pmatrix} b^{2\beta}_{11} & b^{2\beta}_{11} & b^{2\beta}_{13} \\ b^{2\beta}_{21} & b^{2\beta}_{22} & b^{2\beta}_{23} \end{pmatrix}$. We denote by $E_1(a^{\alpha 1}_{i_1 j_\alpha}, a^{\alpha 2}_{i_2 j_\alpha}) = p(\beta = 1|\alpha)a^{\alpha 1}_{i_1 j_\alpha} + p(\beta = 2|\alpha)a^{\alpha 2}_{i_2 j_\alpha}$ $E_2(b^{1\beta}_{i_\beta j_1}, b^{2\beta}_{i_\beta j_2}) = q(\alpha = 1|\beta)b^{1\beta}_{i_\beta j_1} + q(\alpha = 2|\beta)b^{2\beta}_{i_\beta j_2}$ for any $i \in I$, $j \in J$, $\alpha = \overline{1, \kappa_1}$, $\alpha\beta = \overline{1, \kappa_1}$, the average value if the player 1, respectively the player 2, knows the belief probabilities (or the probabilities setted by the Nature). We will construct the utility matrices when the player 1 is of type α and, at the same time, the player 2 is of type β. Based on the facts mentioned above we will obtain the next bimatrix game in which the utility of the players is described by the following matrices with 4 lines and 9 columns:

$\mathbf{A}(\alpha) =$

$\tilde{\mathbf{i}} \diagdown \tilde{\mathbf{j}}$	$1_1 1_2$	$1_1 2_2$	$2_1 1_2$	$2_1 2_2$	$1_1 3_2$
$1_1 1_2$	$E_1\left(a^{\alpha 1}_{1j_\alpha}, a^{\alpha 2}_{1j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{1j_\alpha}, a^{\alpha 2}_{1j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{1j_\alpha}, a^{\alpha 2}_{1j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{1j_\alpha}, a^{\alpha 2}_{1j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{1j_\alpha}, a^{\alpha 2}_{1j_\alpha}\right)$
$1_1 2_2$	$E_1\left(a^{\alpha 1}_{1j_\alpha}, a^{\alpha 2}_{2j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{1j_\alpha}, a^{\alpha 2}_{2j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{1j_\alpha}, a^{\alpha 2}_{2j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{1j_\alpha}, a^{\alpha 2}_{2j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{1j_\alpha}, a^{\alpha 2}_{2j_\alpha}\right)$
$2_1 1_2$	$E_1\left(a^{\alpha 1}_{2j_\alpha}, a^{\alpha 2}_{1j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{2j_\alpha}, a^{\alpha 2}_{1j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{2j_\alpha}, a^{\alpha 2}_{1j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{2j_\alpha}, a^{\alpha 2}_{1j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{2j_\alpha}, a^{\alpha 2}_{1j_\alpha}\right)$
$2_1 2_2$	$E_1\left(a^{\alpha 1}_{2j_\alpha}, a^{\alpha 2}_{2j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{2j_\alpha}, a^{\alpha 2}_{2j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{2j_\alpha}, a^{\alpha 2}_{2j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{2j_\alpha}, a^{\alpha 2}_{2j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{2j_\alpha}, a^{\alpha 2}_{2j_\alpha}\right)$

$\tilde{\mathbf{i}} \diagdown \tilde{\mathbf{j}}$	$3_1 1_2$	$2_1 3_2$	$3_1 2_2$	$3_1 3_2$
$1_1 1_2$	$E_1\left(a^{\alpha 1}_{1j_\alpha}, a^{\alpha 2}_{1j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{1j_\alpha}, a^{\alpha 2}_{1j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{1j_\alpha}, a^{\alpha 2}_{1j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{1j_\alpha}, a^{\alpha 2}_{1j_\alpha}\right)$
$1_1 2_2$	$E_1\left(a^{\alpha 1}_{1j_\alpha}, a^{\alpha 2}_{2j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{1j_\alpha}, a^{\alpha 2}_{2j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{1j_\alpha}, a^{\alpha 2}_{2j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{1j_\alpha}, a^{\alpha 2}_{2j_\alpha}\right)$
$2_1 1_2$	$E_1\left(a^{\alpha 1}_{2j_\alpha}, a^{\alpha 2}_{1j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{2j_\alpha}, a^{\alpha 2}_{1j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{2j_\alpha}, a^{\alpha 2}_{1j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{2j_\alpha}, a^{\alpha 2}_{1j_\alpha}\right)$
$2_1 2_2$	$E_1\left(a^{\alpha 1}_{2j_\alpha}, a^{\alpha 2}_{2j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{2j_\alpha}, a^{\alpha 2}_{2j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{2j_\alpha}, a^{\alpha 2}_{2j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{2j_\alpha}, a^{\alpha 2}_{2j_\alpha}\right)$

(5)

$\mathbf{B}(\beta) \quad =$

$\tilde{\mathbf{i}} \diagdown \tilde{\mathbf{j}}$	$1_1 1_2$	$1_1 2_2$	$2_1 1_2$	$2_1 2_2$	$1_1 3_2$
$1_1 1_2$	$E_2\left(b^{1\beta}_{i_\beta 1}, b^{2\beta}_{i_\beta 1}\right)$	$E_2\left(b^{1\beta}_{i_\beta 1}, b^{2\beta}_{i_\beta 2}\right)$	$E_2\left(b^{1\beta}_{i_\beta 2}, b^{2\beta}_{i_\beta 1}\right)$	$E_2\left(b^{1\beta}_{i_\beta 2}, b^{2\beta}_{i_\beta 2}\right)$	$E\left({}_2 b^{1\beta}_{i_\beta 1}, b^{2\beta}_{i_\beta 3}\right)$
$1_1 2_2$	$E_2\left(b^{1\beta}_{i_\beta 1}, b^{2\beta}_{i_\beta 1}\right)$	$E_2\left(b^{1\beta}_{i_\beta 1}, b^{2\beta}_{i_\beta 2}\right)$	$E_2\left(b^{1\beta}_{i_\beta 2}, b^{2\beta}_{i_\beta 1}\right)$	$E_2\left(b^{1\beta}_{i_\beta 2}, b^{2\beta}_{i_\beta 2}\right)$	$E_2\left(b^{1\beta}_{i_\beta 1}, b^{2\beta}_{i_\beta 3}\right)$
$2_1 1_2$	$E_2\left(b^{1\beta}_{i_\beta 1}, b^{2\beta}_{i_\beta 1}\right)$	$E_2\left(b^{1\beta}_{i_\beta 1}, b^{2\beta}_{i_\beta 2}\right)$	$E_2\left(b^{1\beta}_{i_\beta 2}, b^{2\beta}_{i_\beta 1}\right)$	$E_2\left(b^{1\beta}_{i_\beta 2}, b^{2\beta}_{i_\beta 2}\right)$	$E_2\left(b^{1\beta}_{i_\beta 1}, b^{2\beta}_{i_\beta 3}\right)$
$2_1 2_2$	$E_2\left(b^{1\beta}_{i_\beta 1}, b^{2\beta}_{i_\beta 1}\right)$	$E_2\left(b^{1\beta}_{i_\beta 1}, b^{2\beta}_{i_\beta 2}\right)$	$E_2\left(b^{1\beta}_{i_\beta 2}, b^{2\beta}_{i_\beta 1}\right)$	$E_2\left(b^{1\beta}_{i_\beta 2}, b^{2\beta}_{i_\beta 2}\right)$	$E_2\left(b^{1\beta}_{i_\beta 1}, b^{2\beta}_{i_\beta 3}\right)$

$\tilde{\mathbf{i}} \diagdown \tilde{\mathbf{j}}$	$3_1 1_2$	$2_1 3_2$	$3_1 2_2$	$3_1 3_2$
$1_1 1_1$	$E_2\left(b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 1}\right)$	$E_2\left(b^{1\beta}_{i_\beta 2}, b^{2\beta}_{i_\beta 3}\right)$	$E\left({}_2 b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 2}\right)$	$E_2\left(b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 3}\right)$
$1_1 2_2$	$E_2\left(b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 1}\right)$	$E_2\left(b^{1\beta}_{i_\beta 2}, b^{2\beta}_{i_\beta 3}\right)$	$E_2\left(b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 2}\right)$	$E_2\left(b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 3}\right)$
$2_1 1_2$	$E_2\left(b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 1}\right)$	$E_2\left(b^{1\beta}_{i_\beta 2}, b^{2\beta}_{i_\beta 3}\right)$	$E_2\left(b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 2}\right)$	$E_2\left(b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 3}\right)$
$2_1 2_2$	$E_2\left(b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 1}\right)$	$E_2\left(b^{1\beta}_{i_\beta 2}, b^{2\beta}_{i_\beta 3}\right)$	$E_2\left(b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 2}\right)$	$E_2\left(b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 3}\right)$

(6)

What is the meaning, for example, of elements at the intersection of line $1_1 2_2$ and column $1_1 3_2$? Using the belief probabilities for types of the players, we get that player 1, being of type α, will chose the line $i = 1$ from the matrix $\begin{pmatrix} a_{11}^{\alpha 1} & a_{12}^{\alpha 1} & a_{13}^{\alpha 1} \\ a_{21}^{\alpha 1} & a_{22}^{\alpha 1} & a_{23}^{\alpha 1} \end{pmatrix}$ (when the player 2 is of type $\beta = 1$) and line $i = 2$ from the matrix $\begin{pmatrix} a_{11}^{\alpha 2} & a_{12}^{\alpha 2} & a_{13}^{\alpha 2} \\ a_{21}^{\alpha 2} & a_{22}^{\alpha 2} & a_{23}^{\alpha 2} \end{pmatrix}$ (when player 2 is of type $\beta = 2$), and correspondingly, the player 2 being of type β, will chose the column $j = 1$ from the matrix $\begin{pmatrix} b_{11}^{1\beta} & b_{12}^{1\beta} & b_{13}^{1\beta} \\ b_{21}^{1\beta} & b_{22}^{1\beta} & b_{23}^{1\beta} \end{pmatrix}$ (when the type of the player 1 is $\alpha = 1$) and the column $j = 3$ in the matrix $\begin{pmatrix} b_{11}^{2\beta} & b_{12}^{2\beta} & b_{13}^{2\beta} \\ b_{21}^{2\beta} & b_{22}^{2\beta} & b_{23}^{2\beta} \end{pmatrix}$ (when the type of the player 1 is $\alpha = 2$), then the average value of the payoff for the player 1 is $E_1\left(a_{1j_\alpha}^{\alpha 1}, a_{2j_\alpha}^{\alpha 2}\right) = p(\beta = 1|\alpha) a_{1j_\alpha}^{\alpha 1} + p(\beta = 2|\alpha) a_{2j_\alpha}^{\alpha 2}$ and respectively, for the player 2 is $E_2\left(b_{i_\beta 1}^{1\beta}, b_{i_\beta 3}^{2\beta}\right) = q(\alpha = 1|\beta) b_{i_\beta 1}^{1\beta} + q(\alpha = 2|\beta) b_{i_\beta 3}^{2\beta}$.

Finally, for $\alpha = 1, \alpha = 2, \beta = 1$ and $\beta = 2$ we obtain:

$A(1) =$

$\widetilde{\mathbf{i}} \diagdown \widetilde{\mathbf{j}}$	$1_1 1_2$	$1_1 2_2$	$2_1 1_2$	$2_1 2_2$	$1_1 3_2$
$1_1 1_2$	$E_1\left(a_{11}^{11}, a_{11}^{12}\right)$	$E_1\left(a_{11}^{11}, a_{11}^{12}\right)$	$E_1\left(a_{12}^{11}, a_{12}^{12}\right)$	$E_1\left(a_{12}^{11}, a_{12}^{12}\right)$	$E_1\left(a_{11}^{11}, a_{11}^{12}\right)$
$1_1 2_2$	$E_1\left(a_{11}^{11}, a_{21}^{12}\right)$	$E_1\left(a_{11}^{11}, a_{21}^{12}\right)$	$E_1\left(a_{12}^{11}, a_{22}^{12}\right)$	$E_1\left(a_{12}^{11}, a_{22}^{12}\right)$	$E_1\left(a_{11}^{11}, a_{21}^{12}\right)$
$2_1 1_2$	$E_1\left(a_{21}^{11}, a_{11}^{12}\right)$	$E_1\left(a_{21}^{11}, a_{11}^{12}\right)$	$E_1\left(a_{22}^{11}, a_{12}^{12}\right)$	$E_1\left(a_{22}^{11}, a_{12}^{12}\right)$	$E_1\left(a_{21}^{11}, a_{11}^{12}\right)$
$2_1 2_2$	$E_1\left(a_{21}^{11}, a_{21}^{12}\right)$	$E_1\left(a_{21}^{11}, a_{21}^{12}\right)$	$E_1\left(a_{22}^{11}, a_{22}^{12}\right)$	$E_1\left(a_{22}^{11}, a_{22}^{12}\right)$	$E_1\left(a_{21}^{11}, a_{21}^{12}\right)$

$\widetilde{\mathbf{i}} \diagdown \widetilde{\mathbf{j}}$	$3_1 1_2$	$2_1 3_2$	$3_1 2_2$	$3_1 3_2$
$1_1 1_2$	$E_1\left(a_{13}^{11}, a_{13}^{12}\right)$	$E_1\left(a_{12}^{11}, a_{12}^{12}\right)$	$E_1\left(a_{13}^{11}, a_{13}^{12}\right)$	$E_1\left(a_{13}^{11}, a_{13}^{12}\right)$
$1_1 2_2$	$E_1\left(a_{13}^{11}, a_{23}^{12}\right)$	$E_1\left(a_{12}^{11}, a_{22}^{12}\right)$	$E_1\left(a_{13}^{11}, a_{23}^{12}\right)$	$E_1\left(a_{13}^{11}, a_{23}^{12}\right)$
$2_1 1_2$	$E_1\left(a_{23}^{11}, a_{13}^{12}\right)$	$E_1\left(a_{22}^{11}, a_{12}^{12}\right)$	$E_1\left(a_{23}^{11}, a_{13}^{12}\right)$	$E_1\left(a_{23}^{11}, a_{13}^{12}\right)$
$2_1 2_2$	$E_1\left(a_{23}^{11}, a_{23}^{12}\right)$	$E_1\left(a_{22}^{11}, a_{22}^{12}\right)$	$E_1\left(a_{23}^{11}, a_{23}^{12}\right)$	$E_1\left(a_{23}^{11}, a_{23}^{12}\right)$

$A(2) =$

$\widetilde{\mathbf{i}} \diagdown \widetilde{\mathbf{j}}$	$1_1 1_2$	$1_1 2_2$	$2_1 1_2$	$2_1 2_2$	$1_1 3_2$
$1_1 1_2$	$E_1\left(a_{11}^{21}, a_{11}^{22}\right)$	$E_1\left(a_{12}^{21}, a_{12}^{22}\right)$	$E_1\left(a_{11}^{21}, a_{11}^{22}\right)$	$E_1\left(a_{12}^{21}, a_{12}^{22}\right)$	$E_1\left(a_{13}^{21}, a_{13}^{22}\right)$
$1_1 2_2$	$E_1\left(a_{11}^{21}, a_{21}^{22}\right)$	$E_1\left(a_{12}^{21}, a_{22}^{22}\right)$	$E_1\left(a_{11}^{21}, a_{21}^{22}\right)$	$E_1\left(a_{12}^{21}, a_{22}^{22}\right)$	$E_1\left(a_{13}^{21}, a_{23}^{22}\right)$
$2_1 1_2$	$E_1\left(a_{21}^{21}, a_{11}^{22}\right)$	$E_1\left(a_{22}^{21}, a_{12}^{22}\right)$	$E_1\left(a_{21}^{21}, a_{11}^{22}\right)$	$E_1\left(a_{22}^{21}, a_{12}^{22}\right)$	$E_1\left(a_{23}^{21}, a_{13}^{22}\right)$
$2_1 2_2$	$E_1\left(a_{21}^{21}, a_{21}^{22}\right)$	$E_1\left(a_{22}^{21}, a_{22}^{22}\right)$	$E_1\left(a_{21}^{21}, a_{21}^{22}\right)$	$E_1\left(a_{22}^{21}, a_{22}^{22}\right)$	$E_1\left(a_{23}^{21}, a_{23}^{22}\right)$

$\widetilde{\mathbf{i}} \diagdown \widetilde{\mathbf{j}}$	$3_1 1_2$	$2_1 3_2$	$3_1 2_2$	$3_1 3_2$
$1_1 1_2$	$E_1\left(a_{11}^{21}, a_{11}^{22}\right)$	$E_1\left(a_{13}^{21}, a_{13}^{22}\right)$	$E_1\left(a_{12}^{21}, a_{12}^{22}\right)$	$E_1\left(a_{13}^{21}, a_{13}^{22}\right)$
$1_1 2_2$	$E_1\left(a_{11}^{21}, a_{21}^{22}\right)$	$E_1\left(a_{13}^{21}, a_{23}^{22}\right)$	$E_1\left(a_{12}^{21}, a_{22}^{22}\right)$	$E_1\left(a_{13}^{21}, a_{23}^{22}\right)$
$2_1 1_2$	$E_1\left(a_{21}^{21}, a_{11}^{22}\right)$	$E_1\left(a_{23}^{21}, a_{13}^{22}\right)$	$E_1\left(a_{22}^{21}, a_{12}^{22}\right)$	$E_1\left(a_{23}^{21}, a_{13}^{22}\right)$
$2_1 2_2$	$E_1\left(a_{21}^{21}, a_{21}^{22}\right)$	$E_1\left(a_{23}^{21}, a_{23}^{22}\right)$	$E_1\left(a_{22}^{21}, a_{22}^{22}\right)$	$E_1\left(a_{23}^{21}, a_{23}^{22}\right)$

$B(1) =$

$\widetilde{\mathbf{i}} \diagdown \widetilde{\mathbf{j}}$	$1_1 1_2$	$1_1 2_2$	$2_1 1_2$	$2_1 2_2$	$1_1 3_2$
$1_1 1_2$	$E_2\left(b_{11}^{11}, b_{11}^{21}\right)$	$E_2\left(b_{11}^{11}, b_{12}^{21}\right)$	$E_2\left(b_{12}^{11}, b_{11}^{21}\right)$	$E_2\left(b_{12}^{11}, b_{12}^{21}\right)$	$E\left({}_2 b_{11}^{11}, b_{13}^{21}\right)$
$1_1 2_2$	$E_2\left(b_{11}^{11}, b_{11}^{21}\right)$	$E_2\left(b_{11}^{11}, b_{12}^{21}\right)$	$E_2\left(b_{12}^{11}, b_{11}^{21}\right)$	$E_2\left(b_{12}^{11}, b_{12}^{21}\right)$	$E_2\left(b_{11}^{11}, b_{13}^{21}\right)$
$2_1 1_2$	$E_2\left(b_{21}^{11}, b_{21}^{21}\right)$	$E_2\left(b_{21}^{11}, b_{22}^{21}\right)$	$E_2\left(b_{22}^{11}, b_{21}^{21}\right)$	$E_2\left(b_{22}^{11}, b_{22}^{21}\right)$	$E_2\left(b_{21}^{11}, b_{23}^{21}\right)$
$2_1 2_2$	$E_2\left(b_{21}^{11}, b_{21}^{21}\right)$	$E_2\left(b_{21}^{11}, b_{22}^{21}\right)$	$E_2\left(b_{22}^{11}, b_{21}^{21}\right)$	$E_2\left(b_{22}^{11}, b_{22}^{21}\right)$	$E_2\left(b_{21}^{11}, b_{23}^{21}\right)$

$\widetilde{\mathbf{i}} \diagdown \widetilde{\mathbf{j}}$	$3_1 1_2$	$2_1 3_2$	$3_1 2_2$	$3_1 3_2$
$1_1 1_1$	$E_2\left(b_{13}^{11}, b_{11}^{21}\right)$	$E_2\left(b_{12}^{11}, b_{13}^{21}\right)$	$E\left({}_2 b_{13}^{11}, b_{12}^{21}\right)$	$E_2\left(b_{13}^{11}, b_{13}^{21}\right)$
$1_1 2_2$	$E_2\left(b_{13}^{11}, b_{11}^{21}\right)$	$E_2\left(b_{12}^{11}, b_{13}^{21}\right)$	$E_2\left(b_{13}^{11}, b_{12}^{21}\right)$	$E_2\left(b_{13}^{11}, b_{13}^{21}\right)$
$2_1 1_2$	$E_2\left(b_{23}^{11}, b_{21}^{21}\right)$	$E_2\left(b_{22}^{11}, b_{23}^{21}\right)$	$E_2\left(b_{23}^{11}, b_{22}^{21}\right)$	$E_2\left(b_{23}^{11}, b_{23}^{21}\right)$
$2_1 2_2$	$E_2\left(b_{23}^{11}, b_{21}^{21}\right)$	$E_2\left(b_{22}^{11}, b_{23}^{21}\right)$	$E_2\left(b_{23}^{11}, b_{22}^{21}\right)$	$E_2\left(b_{23}^{11}, b_{23}^{21}\right)$

$B(2) =$

$\widetilde{\mathbf{i}} \diagdown \widetilde{\mathbf{j}}$	$1_1 1_2$	$1_1 2_2$	$2_1 1_2$	$2_1 2_2$	$1_1 3_2$
$1_1 1_2$	$E_2\left(b_{11}^{12}, b_{11}^{22}\right)$	$E_2\left(b_{11}^{12}, b_{12}^{22}\right)$	$E_2\left(b_{12}^{12}, b_{11}^{22}\right)$	$E_2\left(b_{12}^{12}, b_{12}^{22}\right)$	$E\left({}_2 b_{11}^{12}, b_{13}^{22}\right)$
$1_1 2_2$	$E_2\left(b_{21}^{12}, b_{21}^{22}\right)$	$E_2\left(b_{21}^{12}, b_{22}^{22}\right)$	$E_2\left(b_{22}^{12}, b_{i_\beta 1}^{22}\right)$	$E_2\left(b_{22}^{12}, b_{i_\beta 2}^{22}\right)$	$E_2\left(b_{21}^{12}, b_{23}^{22}\right)$
$2_1 1_2$	$E_2\left(b_{11}^{12}, b_{11}^{22}\right)$	$E_2\left(b_{11}^{12}, b_{12}^{22}\right)$	$E_2\left(b_{12}^{12}, b_{11}^{22}\right)$	$E_2\left(b_{12}^{12}, b_{12}^{22}\right)$	$E_2\left(b_{11}^{12}, b_{13}^{22}\right)$
$2_1 2_2$	$E_2\left(b_{21}^{12}, b_{21}^{22}\right)$	$E_2\left(b_{21}^{12}, b_{22}^{22}\right)$	$E_2\left(b_{22}^{12}, b_{21}^{22}\right)$	$E_2\left(b_{22}^{12}, b_{22}^{22}\right)$	$E_2\left(b_{21}^{12}, b_{23}^{22}\right)$

$\widetilde{\mathbf{i}} \diagdown \widetilde{\mathbf{j}}$	$3_1 1_2$	$2_1 3_2$	$3_1 2_2$	$3_1 3_2$
$1_1 1_2$	$E_2\left(b_{13}^{12}, b_{11}^{22}\right)$	$E_2\left(b_{12}^{12}, b_{13}^{22}\right)$	$E\left({}_2 b_{13}^{12}, b_{12}^{22}\right)$	$E_2\left(b_{13}^{12}, b_{13}^{22}\right)$
$1_1 2_2$	$E_2\left(b_{23}^{12}, b_{21}^{22}\right)$	$E_2\left(b_{22}^{12}, b_{23}^{22}\right)$	$E_2\left(b_{23}^{12}, b_{22}^{22}\right)$	$E_2\left(b_{23}^{12}, b_{23}^{22}\right)$
$2_1 1_2$	$E_2\left(b_{13}^{12}, b_{11}^{22}\right)$	$E_2\left(b_{12}^{12}, b_{13}^{22}\right)$	$E_2\left(b_{13}^{12}, b_{12}^{22}\right)$	$E_2\left(b_{13}^{12}, b_{13}^{22}\right)$
$2_1 2_2$	$E_2\left(b_{23}^{12}, b_{21}^{22}\right)$	$E_2\left(b_{22}^{12}, b_{23}^{22}\right)$	$E_2\left(b_{23}^{12}, b_{22}^{22}\right)$	$E_2\left(b_{23}^{12}, b_{23}^{22}\right)$

So,the normal form of the Bayesian game from is

$$\begin{aligned}\Gamma_{Bayes} &= \Big\langle \{1,2\}, \widetilde{\mathbf{I}} = \widetilde{\mathbf{I}}(\alpha = 1) \cup \widetilde{\mathbf{I}}(\alpha = 2), \widetilde{\mathbf{J}} = \widetilde{\mathbf{J}}(\beta = 1) \cup \widetilde{\mathbf{J}}(\beta = 1)\ , \\ &\quad \mathcal{A} = \|\mathbf{A}(\alpha = 1), \mathbf{A}(\alpha = 1)\|\ , \mathcal{B} = \|\mathbf{B}(\beta = 1), \mathbf{B}(\beta = 2)\| \Big\rangle\ .\end{aligned}$$

Bimatrix games $\langle \mathbf{A}(1), \mathbf{B}(1) \rangle$, $\langle \mathbf{A}(1), \mathbf{B}(2) \rangle$, $\langle \mathbf{A}(2), \mathbf{B}(1) \rangle$ and $\langle \mathbf{A}(2), \mathbf{B}(2) \rangle$ are subgames of the constructed above Bayesian game.

As a particular case we will examine the next example. We consider the following bimatrix game $H_1 = \begin{pmatrix} 3 & 5 & 4 \\ 6 & 7 & 2 \end{pmatrix}$, $H_2 = \begin{pmatrix} 0 & 5 & 1 \\ 4 & 3 & 2 \end{pmatrix}$ for which we construct the normal form of the Bayesian game associated to the informational extended game.

For example, suppose that the informational extended strategies of the player 1 are $\theta_1^1(j) = \begin{cases} 1 \text{ if } j = 1,2 \\ 2 \text{ if } j = 3 \end{cases}$, $\theta_1^2(j) = \begin{cases} 1 \text{ if } j = 1,3 \\ 2 \text{ if } j = 2 \end{cases}$ and respectively, for the player 2 are $\theta_2^1(i) = \begin{cases} 1 \text{ if } i = 1 \\ 2 \text{ if } i = 2 \end{cases}$, $\theta_2^2(i) = \begin{cases} 1 \text{ if } i = 2 \\ 2 \text{ if } i = 1 \end{cases}$.

As mentioned above, the informational extended strategies $\left\{\theta_1^1, \theta_1^2, \theta_2^1, \theta_2^2\right\}$ generate an incomplete information game in which the payoff matrix may be one of the following matrices (one in which the utility of the players is determined by one of the matrix bellow):

$$AB\left(\theta_1^1,\theta_2^1\right) = \begin{pmatrix} (3,0) & (3,0) & (6,4) \\ (5,5) & (5,5) & (7,3) \end{pmatrix}, \; AB\left(\theta_1^2,\theta_2^1\right) = \begin{pmatrix} (3,0) & (6,4) & (3,0) \\ (5,5) & (7,3) & (5,5) \end{pmatrix} \quad (7)$$

$$AB\left(\theta_1^1,\theta_2^2\right) = \begin{pmatrix} (5,5) & (5,5) & (7,3) \\ (3,0) & (3,0) & (6,0) \end{pmatrix}, \; AB\left(\theta_1^2,\theta_2^2\right) = \begin{pmatrix} (5,5) & (7,3) & (5,5) \\ (3,0) & (6,4) & (3,0) \end{pmatrix}.$$

We will construct the Bayesian game for the game in incomplete and imperfect information over the set of informational non extended strategies I, J from (7). The set of types of the player 1 is $\alpha \in \Delta_1 = \{1,2\}$ and of the player 2 is $\beta \in \Delta_2 = \{1,2\}$. Let's consider that the belief probabilities of the types are: for the player 1 : $p(\beta|\alpha) = \begin{cases} p \text{ for } \beta = 1 \\ 1-p \text{ for } \beta = 2 \end{cases}$ and for the player 2 : $q(\alpha|\beta) = \begin{cases} q \text{ for } \alpha = 1 \\ 1-q \text{ for } \alpha = 2 \end{cases}$, $0 \le p \le 1$, $0 \le q \le 1$. Thus we get a Bayesian game in which the utility functions of the players, depending of their types, will be:

$$\mathbf{A}(\alpha=1) = \begin{pmatrix} 5-2p & 5-2p & 5-2p & 5-2p & 5-2p & 6 & 3 & 6 & 6 \\ 3 & 3 & 3 & 3 & 3 & 6p & 6-3p & 3+3p & 6 \\ 5 & 5 & 5 & 5 & 5 & 7 & 5 & 7 & 7 \\ 3+2p & 3+2p & 3+2p & 3+2p & 3+2p & 6+p & 3+2p & 6+p & 6+p \end{pmatrix},$$

$$\mathbf{B}(\beta=1) = \begin{pmatrix} 0 & 4 & 0 & 4 & 0 & 4q & 0 & 4 & 4q \\ 0 & 4 & 0 & 4 & 0 & 4q & 0 & 4 & 4q \\ 5 & 3+2q & 5 & 3+2q & 5 & 5-q & 5 & 3 & 5-2q \\ 5 & 3+2q & 5 & 3+2q & 5 & 5-2q & 5 & 3 & 5-2q \end{pmatrix},$$

$$\mathbf{A}(\alpha=2) = \begin{pmatrix} 5-2p & 7-p & 5-2p & 7-p & 5-2p & 5-2p & 5-2p & 7-p & 5-2p \\ 3 & 6 & 3 & 6 & 3 & 3 & 3 & 6 & 3 \\ 3+2p & 7 & 5 & 7-4p & 5 & 5 & 5 & 7 & 5 \\ 3+2p & 6+p & 3+2p & 3+2p & 6+p & 3+2p & 3+2p & 6+p & 3+2p \end{pmatrix},$$

$$B(\beta=2) = \begin{pmatrix} 5 & 3+2q & 5 & 3+2q & 5 & 5-2q & 5 & 3 & 5-2q \\ 0 & 4-4q & 0 & 4-4q & 0 & 0 & 0 & 4-4q & 0 \\ 5 & 3+2q & 5 & 3+2q & 5 & 5-2q & 5 & 3 & 5-2q \\ 0 & 4-4q & 0 & 4-4q & 0 & 0 & 0 & 4-4q & 0 \end{pmatrix}.$$

Algorithm 1

1. Using the MPI programming model generate the virtual medium of MPI-process comunication with linear topologie and dimension $\varkappa_1 \cdot \varkappa_2$. Root process broadcast to all $\varkappa_1 \cdot \varkappa_2$ MPI process the initial matrices $A = ||a_{ij}||_{i\in I}^{j\in J}$, and $B = ||b_{ij}||_{i\in I}^{j\in J}$ of the bimatrix game $\Gamma = \langle A, B\rangle$.

2. MPI proces with rank $k_1 = \alpha$,for all $\alpha = \overline{1,\varkappa_1}$ generate the "beliver-probabilities" $p(\beta/\alpha)$ for all $\beta = \overline{1,\varkappa_2}$ and MPI proces with rank $k_2 = \beta$, for all $\beta = \overline{1,\varkappa_2}$ generate the "beliver-probabilities" $q(\alpha/\beta)$ for all $\alpha = \overline{1,\varkappa_1}$.

3. Using the MPI programming model and open source library ScaLAPACK-BLACS, initialize the processes grid $\{(\alpha,\beta)\}_{\alpha=\overline{1,\varkappa_1}}^{\beta=\overline{1,\varkappa_2}}$. All fixed MPI process (α,β) using the OpenMP directives and combinatorial algoritm construct the sets $\widetilde{\mathbf{I}}(\alpha), \widetilde{\mathbf{J}}(\beta)$.

4. MPI proces with rank α, for all $\alpha = \overline{1,\kappa_1}$ construct payoff matrix $A(\alpha)$ from (2) and MPI proces with rank β, for all $\beta = \overline{1,\varkappa_2}$ construct payoff matrix $B(\beta)$ from (3).

5. Using open source library ScaLAPACK-BLACS MPI proces with rank α, for all $\alpha = \overline{1,\kappa_1}$ send the matrix $A(\alpha)$ to proces with rank β, for all $\beta = \overline{1,\varkappa_2}$, and proces with rank β, for all $\beta = \overline{1,\varkappa_2}$ send the matrix $B(\beta)$ to proces with rank α, for all $\alpha = \overline{1,\kappa_1}$. So all process (α,β) have a pair of matrices $(\mathbf{A}(\alpha),\ \mathbf{B}(\beta))$.