Integrability and Beyond!!!...

statistical quantities in RMT

Fredholm Det. representations for (a) spectral gaps

(b) largest eigenvalue, (c) number statistics.

$\{\xi_{ij}, \eta_{ij}\}$ iid standard Gaussian

$$\begin{pmatrix} \sqrt{2}\,\xi_{11} & \xi_{12} & \cdots & \xi_{1n} \\ \vdots & \ddots & & \\ \xi_{1n} & & & \sqrt{2}\,\xi_{nn} \end{pmatrix} \in \mathcal{S}(n, \mathbb{R})$$

Gaussian Orthog. Ensemble

$$\begin{pmatrix} \xi_{11} & \frac{\xi_{12}+i\eta_{12}}{\sqrt{2}} & \\ \frac{\xi_{12}-i\eta_{12}}{\sqrt{2}} & \xi_{22} & \\ & & \ddots \end{pmatrix} \in \mathcal{S}(n, \mathbb{C})$$

GUE

2x2 example: $\begin{pmatrix} \xi_{11} & \frac{\xi_{12}+i\eta_{12}}{\sqrt{2}} \\ \frac{\xi_{12}-i\eta_{12}}{\sqrt{2}} & \xi_{22} \end{pmatrix}$

There are many different examples. We will focus on a collection of examples that have an integrable structure, the Prototypical example is "GUE" defined above.

We will need the Hermite polynomials

$$\int P_j(x)\, P_k(x)\, e^{-x^2/2}\, dx = \delta_{jk}$$

$$P_j = \kappa_j x^j + \text{l.o.t.}, \quad \kappa_j > 0.$$

given a function $f(x)$, $x \in \mathbb{R}$, smooth & decaying,

You can represent it using Hermite fcns.

$$f(x) = e^{-\frac{x^2}{4}} \sum_{j=0}^{\infty} a_j P_j(x)$$

$$a_k = \int_{\mathbb{R}} f(x)\, P_k(x)\, e^{-x^2/4}\, dx$$

We express this as an integral operator

$$\mathcal{K}_N(f) = \int_{-\infty}^{\infty} K_N(x,y)\, f(y)\, dy$$

$$K_N(x,y) = e^{-\left(\frac{x^2+y^2}{4}\right)} \sum_{l=0}^{N-1} P_l(x) P_l(y)$$

$\mathcal{K}_N(f)$ is a projection operator onto the first N orthogonal functions.

$$\mathcal{K}_N(f) = e^{-\frac{x^2}{4}} \sum_{l=0}^{N-1} P_l(x) \int_{\mathbb{R}} e^{-y^2/4} P_l(y) f(y)\, dy.$$

It is a fact that $\mathcal{K}_N(f) \xrightarrow[N\to\infty]{} f$.

Now let $(a,b) = B$, an interval in $\mathbb{R}$.

$\mathcal{K}_N^B$ is the integral operator acting on functions in $L^2((a,b))$ with kernel $K_N(x,y)$.

$$\mathcal{K}_N^B(f) = \int_a^b K_N(x,y) f(y)\, dy,$$

This is a finite rank operator, whose range is

$$\text{Span}\left\{ e^{-x^2/4} P_0, \dots, e^{-x^2/4} P_{N-1}(x) \right\}.$$

You could pick a basis $\left\{ e^{-x^2/4} \tilde{p}_\ell \right\}_{\ell=0}^{\infty}$ where $\tilde{p}_\ell$ are "truncated Hermite" $\tilde{\varphi}_\ell$

$$f \longleftrightarrow (\tilde{a}_0, \tilde{a}_1, \dots, \quad)^T.$$

$$f = \sum_{\ell=0}^{\infty} \tilde{a}_\ell \tilde{\varphi}_\ell$$

$$K_N^B f = \sum_{\ell=0}^{N-1} \tilde{a}_\ell \, K_N^B(\tilde{\varphi}_\ell)$$

$$K_N^B(\tilde{\varphi}_\ell) = \sum_{m=0}^{N-1} \varphi_m \int \tilde{\varphi}_\ell \varphi_m \, dy$$

$$= \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \tilde{\varphi}_n \langle \varphi_m, \tilde{\varphi}_n \rangle \langle \tilde{\varphi}_\ell \varphi_m \rangle$$

$$K_N^B f = \sum_\ell \left(\sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \tilde{\varphi}_n \langle \tilde{\varphi}_\ell \varphi_m \rangle \langle \varphi_m \tilde{\varphi}_n \rangle \right) a_\ell$$

$$= \sum_{n=0}^{N-1} \tilde{\varphi}_n \sum_{m=0}^{N-1} \langle \varphi_m \tilde{\varphi}_n \rangle \sum_{\ell=0}^{N-1} \langle \varphi_m \tilde{\varphi}_\ell \rangle a_\ell$$

$$(\ )_n = \sum_{m,\ell} \langle \varphi_m \tilde{\varphi}_n \rangle \langle \varphi_m \tilde{\varphi}_\ell \rangle a_\ell$$

$$= \sum_{m,\ell} \int \tilde{\varphi}_n(x) \varphi_m(x) \, dx \int \varphi_m(y) \tilde{\varphi}_\ell(y) \, dy \; a_\ell$$

$$A_{n,\ell} = \sum_m \int \tilde{\varphi}_n(x)\, \varphi_m(x)\, \varphi_m(y)\, \tilde{\varphi}_\ell(y)\, dx\, dy = \iint \tilde{\varphi}_n\, K_N(x,y)\, \varphi_\ell$$

There is a basis so that K_N is expressible as

$$\langle K_N \rangle = \left[\begin{array}{c|c} \underset{(N\times N)}{K} & 0 \\ \hline 0 & 0 \end{array}\right].$$

Now consider the operator

$$1 - tK_N \quad , \qquad t \text{ is a parametre near } 1.$$

using the above basis,

$$\langle 1 - tK_N \rangle = \left(\begin{array}{c|c} 1 - tK & 0 \\ \hline 0 & I \end{array}\right)$$

in other words, on most of the space, $1-tK_N$ acts as the identity operator.

$$H(t) = \det(1 - t K_N) = \det(I - t K)$$

$H(1) =$ Probability that there are 0 eigenvalues of M in (a,b).

$-H'(1) = \mathbb{P}$ (exactly 1 eval in (a,b)).

$(-1)^j H^{(j)}(1) = \mathbb{P}$ (exactly j evals in (a,b)).

$-H'(0) = \mathbb{E}$ (# evals in (a,b))

$\left(\text{or } -\frac{d}{dt} \ln H(t) \Big|_{t=0} \right)$

$$\frac{d^2}{dt^2} \ln H(t) \Big|_{t=0} - H'(0) = \mathrm{Var}(\text{\# evals in } (a,b))$$

The case of 2×2 random matrices

Hermite polynomials $P_0(x)$, $P_1(x)$

$$\int P_0^2(x) e^{-x^2/2} dx = 1$$

$$P_0^2 = \frac{1}{\sqrt{2\pi}}, \quad P_0 = \frac{1}{(2\pi)^{1/4}}$$

$P_1 = \kappa_1 x$ (odd so that $\int P_0 P_1 e^{-x^2/2} dx = 0$

$$K_1^2 \int_{-\infty}^{\infty} x^2 e^{-x^2/2}\, dx = 1.$$

$$K_1^2 = \frac{1}{\sqrt{2\pi}} \qquad K_1 = \frac{1}{(2\pi)^{1/4}}$$

$$K_2(x,y) = e^{\frac{-(x^2+y^2)}{4}} \left(P_0(x) P_0(y) + P_1(x) P_1(y) \right)$$

$$= \frac{e^{-\left(\frac{x^2+y^2}{4}\right)}}{\sqrt{2\pi}} \left(1 + xy \right)$$

$$H_2(t, a, b) = 1 - t \int_a^b K_2(x,x)\, dx$$

$$+ \frac{t^2}{2} \int_a^b \int_a^b \det \begin{pmatrix} K(x,x) & K(x,y) \\ K(y,x) & K(y,y) \end{pmatrix} dx\,dy$$

$$1 - \frac{t}{\sqrt{2\pi}} \int_a^b e^{\frac{-x^2}{2}} (1 + x^2)\, dx + \frac{t^2}{2} \int_a^b \int_a^b \frac{e^{\frac{-(x^2+y^2)}{2}}}{2\pi} \det \begin{pmatrix} 1+x^2 & 1+xy \\ 1+xy & 1+y^2 \end{pmatrix}$$

$$1 - \frac{t}{\sqrt{2\pi}} \int_a^b e^{-\frac{x^2}{2}} (1 + x^2)\, dx + \frac{t^2}{2} \int_a^b \int_a^b \frac{e^{\frac{-x^2+y^2}{2}}}{2\pi} \det \begin{pmatrix} 1+x^2 & 1+xy \\ x(y-x) & y(y-x) \end{pmatrix}$$

$$\begin{pmatrix} 1+x^2 & x(y-x) \\ x(y-x) & (y-x)^2 \end{pmatrix}$$

$$1-\frac{t}{\sqrt{2\pi}}\int_{a}^{b} e^{-\frac{x^2}{2}}(1+x^2)\,dx+\frac{t^2}{2}\int_a^b\int_a^b \frac{e^{-\left(\frac{x^2+y^2}{2}\right)}}{2\pi}(y-x)^2\,dx\,dy$$

$b-a\to 0$.

$\mathbb{P}$(no evals in (a,b)): $\to 1$

$$\mathbb{P}(\text{exactly 1 eval in } (a,b)) := -\left(-\int_a^b \frac{e^{-x^2}(1+x^2)\,dx}{\sqrt{2\pi}} + \frac{1}{4\pi}\int_a^b\int_a^b e^{-\left(\frac{x^2+y^2}{2}\right)}(y-x)^2\,dx\,dy\right)$$

$$=\frac{1}{\sqrt{2\pi}}\int_a^b e^{-x^2}(1+x^2)\,dx-\frac{1}{4\pi}\int_a^b\int_a^b e^{-\left(\frac{x^2+y^2}{2}\right)}(x^2+y^2-2xy)\,dx\,dy$$

$$=\frac{1}{\sqrt{2\pi}}(\mu_0+\mu_2)-\frac{1}{4\pi}\left(2\mu_2\mu_0-2\mu_1^2\right)$$

$$=\frac{1}{\sqrt{2\pi}}\mu_0\left(1-\frac{1}{2\sqrt{2\pi}}\mu_2\right)+\frac{1}{\sqrt{2\pi}}\mu_2\left(1-\frac{1}{2\sqrt{2\pi}}\mu_0\right)$$

$\mathbb{P}$(no evals in (a,∞)) =

$$1-\frac{1}{\sqrt{2\pi}}\int_a^\infty e^{-\frac{x^2}{2}}(1+x^2)\,dx+\frac{1}{4\pi}\int_a^\infty\int_a^\infty e^{-\left(\frac{x^2+y^2}{2}\right)}(x-y)^2\,dx\,dy$$

$$\mathbb{P}(\text{exactly 1 eval in } (a,\infty))=\frac{1}{\sqrt{2\pi}}\int_a^\infty e^{-x^2/2}(1+x^2)\,dx-\frac{1}{4\pi}\int_a^\infty\int_a^\infty e^{-\left(\frac{x^2+y^2}{2}\right)}(x-y)^2.$$

(smallest eval $< a$) $\longleftrightarrow$ largest eval $> -a$

$\mathbb{P}$(exactly 2 evals in (a,∞)): no evals $< a$ $\quad(-\infty,a)$