Integrability and Beyond!!! Week 2.

$$H(t,A) = 1 - t\int_A K_N(x,x)\,dx + \frac{t^2}{2}\int_A\int_A \det\begin{pmatrix} K_N(x,x) & K_N(x,y) \\ K_N(y,x) & K_N(y,y)\end{pmatrix}$$

$$+ \dots + \frac{(-1)^N}{N!}\int\cdots\int \det\left[K_N(x_i,x_j)\right] d^N x$$

$$H(1,A) = \mathbb{P}(\text{no evals in } A)$$

$$H'(1,A) = \mathbb{P}(\text{exactly } 1 \text{ eval in } A)$$

$$H''(1,A) = 2\,\mathbb{P}(\text{exactly } 2 \text{ evals in } A)$$

$$H^{(j)}(1,A) = j!\,\mathbb{P}(\text{exactly } j \text{ evals in } A)$$

$$-H'(0) = \mathbb{E}(\#\text{ evals in } A)$$

$$\left.\frac{d^2 \ln H}{dt^2}\right|_{t=0} - H'(0) = \operatorname{Var}(\#\text{ evals in } A)$$

$$\mathbb{E}(\#\text{ evals in } A) = \int_A K_N(x,x)\,dx$$

$$\operatorname{Var}(\#\text{ in } A) = H''(0) - (H'(0))^2 - H'(0)$$

$$= \int_A\int_A \det\begin{pmatrix} K_N(x,x) & K_N(x,y) \\ K_N(y,x) & K_N(y,y)\end{pmatrix} dx\,dy - \int_A\int_A K_N(x,x)\,K_N(y,y)\,dx\,dy + \int_A K_N(x,x)\,dx$$

$$= \int_A\int_A -K_N(x,y)^2\,dx\,dy + \int_A K_N(x,x)\,dx$$

$$\mathbb{E}(\#\text{ in }A) = \int_A K_N(x,x)\,dx = \int_A \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}(1+x^2)\,dx$$

$$\operatorname{Var}(\#\text{ in }A) = \frac{d^2}{dt^2}\ln H(t,A)\Big|_{t=0} - H'(0)$$

$$= H''(0) - (H'(0))^2 - H'(0)$$

$$= -\frac{1}{2\pi}\int_A\int_A e^{-\frac{1}{2}(x^2+y^2)}(1+xy)^2\,dx\,dy$$

$$+ \frac{1}{\sqrt{2\pi}}\int_A e^{-\frac{1}{2}x^2}(1+x^2)\,dx$$

How does $\mathbb{E}(\#\text{ evals in }(a,b))$ grow with N?

$\mathrm{Var}(\#\text{ evals in }(a,b))$ behave?

How do these two quantities depend on a & b?

Now take $N = 3, 4, 5, \ldots$ fix $(a,b)\ (= A)$

consider $n(A) = \#$ of evals in A.

Describe $\mathbb{P}(n(A) = j)$, $j = 1, \ldots, N$.

$N=2$: $\mathbb{P}(n(A)=0) = 1 - \int_A K_N(x,x)\,dx + \frac{1}{2}\int_{A\times A} \det\begin{pmatrix} K(x,x) & K(x,y) \\ K(y,x) & K(y,y)\end{pmatrix}$

$\mathbb{P}(n(A)=1) = \int_A K_N(x,x) - \int_{A\times A}\det(\quad)$

$\mathbb{P}(n(A)=2) = \frac{1}{2}\int_{A\times A}\det(\quad)$

$N=3$ $\mathbb{P}(n(A)=0) = 1 - \int_A K_N + \frac{1}{2}\int \det(\quad)_{2\times2} - \frac{1}{6}\int\det(\quad)_{3\times3}$

$1 = \int K_N - \int\det(\quad)_{2\times2} + \frac{1}{2}\int\det(\quad)_{3\times3}$

$2 = \frac{1}{2}\left(\int\det(\quad)_{2\times2} - \int\det(\quad)_{3\times3}\right)$

$3 = \frac{1}{3!}\int\det(\quad)_{3\times3}$

General N

$$\mathbb{P}(n(A) = j) = \sum_{k=j}^{N} \frac{(-1)^k (t^k)^{(j)}}{j!\,k!} \int_A \cdots \int_A \det(\ )_{k\times k}$$

$$= \sum_{k=j}^{N} \frac{(-1)^k}{j!\,k!} \frac{k!}{(k-j)!} \int_A \cdots \int_A \det(\ )_{k\times k}$$

$$= \sum_{k=j}^{N} \frac{(-1)^k}{j!(k-j)!} \int_A \cdots \int_A \det(\ )_{k\times k}$$

Quest: plot this for $N \sim 100$, $j = 0, 1, \ldots, 100$ several different choices of A.

should concentrate around $j \cong \int_A K_N(x,x)\,dx$.

There is a central limit theorem!

$$\mathbb{P}(\text{no evals in } (a,b)) = H(1, (a,b))$$

$$1 - \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2}(1+x^2)\,dx + \frac{1}{4\pi} \int_a^b \int_a^b e^{-\left(\frac{x^2+y^2}{2}\right)} (x-y)^2\,dx\,dy$$

$b \to \infty$:

$$\mathbb{P}(\lambda_{max} < a) = 1 - \frac{1}{\sqrt{2\pi}} \int_a^\infty e^{-x^2/2}(1+x^2)\,dx + \frac{1}{4\pi}\int_a^\infty \int_a^\infty e^{-\frac{x^2+y^2}{2}} (x-y)^2\,dx\,dy$$ ⑤

density of λ_{max} :

$$\frac{d}{da} : \frac{1}{\sqrt{2\pi}} e^{-\frac{a^2}{2}}(1+a^2) - \frac{1}{2\pi}\int_a^\infty e^{-\frac{a^2+x^2}{2}} (x-a)^2\,dx$$

$\mathbb{P}$(1 eval in A, 1 in B), $A \cap B \neq \emptyset$.

$$\{2 \text{ in } A\cup B\} = \{2 \text{ in } A\} \cup \{2 \text{ in } B\} \cup \{1 \text{ in } A \;\&\; 1 \text{ in } B\}$$

$$\mathbb{P}(1 \text{ in } A \;\&\; 1 \text{ in } B) = \mathbb{P}(2 \text{ in } A\cup B) - \mathbb{P}(2 \text{ in } A) - \mathbb{P}(2 \text{ in } B)$$

$\mathbb{P}$(1 eval in (a,b) | 1 eval in (c,d))

with $d < a$ and

(c,d) very small.

Conditional probability:

$\mathbb{P}$(1 eval in (a,b) | 1 eval in (c,d))

$$= \frac{\mathbb{P}(1 \text{ in } (a,b) \;\&\; 1 \text{ in } (c,d))}{\mathbb{P}(1 \text{ in } (c,d))}$$

We can compute $\mathbb{P}(1 \text{ in } (c,d))$.

What about 1 in (a,b) & 1 in (c,d)?

$$\{2 \text{ in } A\cup B\} = \{2 \text{ in } A\} \cup \{2 \text{ in } B\} \cup \{1 \text{ in } A \;\&\; 1 \text{ in } B\}$$

These we know

take limit $d-c \to 0$:

$$\mathbb{P}(1 \text{ eval in } (a,b) \mid 1 \text{ eval at } c^*)$$

$$= \frac{\mathbb{P}(2 \text{ in } (a,b)\cup(c,d)) - \mathbb{P}(2 \text{ in } (a,b)) - \mathbb{P}(2 \text{ in } (c,d))}{\mathbb{P}(1 \text{ eval in } (c,d))}$$

$$= \frac{\frac{1}{2}\frac{d^2}{dt^2}H(t,(a,b)\cup(c,d))\Big|_{t=1} - \frac{1}{2}\frac{d^2}{dt^2}H(t,(a,b))\Big|_{t=1} - \frac{1}{2}\frac{d^2}{dt^2}H(t,(c,d))\Big|_{t=1}}{-\frac{d}{dt}H(t,(c,d))\Big|_{t=1}}$$

$$= \frac{1}{2}\,\frac{\int_{A\cup B}\int_{A\cup B}\frac{1}{2\pi}(x-y)^2 e^{-\frac{1}{2}(x^2+y^2)}\,dx\,dy - \int_A\int_A \frac{1}{2\pi}(x-y)^2 e^{-\frac{1}{2}(x^2+y^2)} - \frac{1}{2}\int_B\int_B \frac{(x-y)^2 e^{-\frac{1}{2}(x^2+y^2)}}{2\pi}}{\frac{1}{\sqrt{2\pi}}\int_B e^{-\frac{1}{2}x^2}(1+x^2)\,dx - \frac{1}{2\pi}\int_B\int_B (x-y)^2 e^{-\frac{1}{2}(x^2+y^2)}\,dx\,dy}$$

$$= \frac{1}{2} \frac{2\int_B \int_A \frac{1}{2\pi}(x-y)^2 e^{-\frac{1}{2}(x^2+y^2)}\, dx\, dy}{\frac{1}{\sqrt{2\pi}}\int_B e^{-\frac{1}{2}x^2}(1+x^2)\, dx - O(|B|^2)}$$

$$\cong \frac{1}{\sqrt{2\pi}} \left(\frac{\int_A (x-c)^2 e^{-\frac{1}{2}x^2} dx\, e^{-\frac{1}{2}c^2} \cancel{(d-c)}}{\cancel{(d-c)}\, e^{-\frac{1}{2}c^2}(1+c^2)} + O((d-c)) \right)$$

$$\cong \frac{1}{\sqrt{2\pi}\,(1+c^2)} \int_A (x-c)^2 e^{-\frac{1}{2}x^2} dx.$$

$\mathbb{P}($ 1 eval $> a \mid$ 1 eval at $c)$

$$= \frac{1}{\sqrt{2\pi}\,(1+c^2)} \int_a^\infty (x-c)^2 e^{-\frac{1}{2}x^2} dx$$

<u>Challenge</u>: 3×3 case, 4×4 case ...

\# evals in $A = (a,b)$

$$\mathbb{E}(\#\text{ in } A) = \int_A K_N(x,x)\, dx = \int_A \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}(1+x^2)\, dx$$

$$\mathrm{Var}(\#\text{ in } A) = \frac{d^2}{dt^2} \ln H(t,A)\Big|_{t=0} - H'(0)$$