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$$\begin{pmatrix} \xi_{11} & \frac{\xi_{12}+i\eta_{12}}{\sqrt{2}} \\ \frac{\xi_{12}-i\eta_{12}}{\sqrt{2}} & \xi_{22} \end{pmatrix}$$

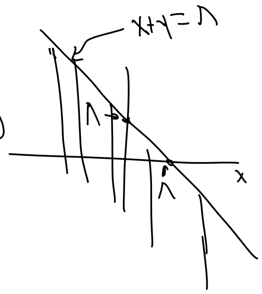
a & b : each iid.

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

$P(a+b<\Lambda)$

$$\{(a,b) : a+b<\Lambda\}$$

D



$$\iint_D \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} dx\,dy = \mathbb{P}\left((a+b)<\Lambda\right).$$

$$P(\Lambda) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\Lambda-x} dy\; e^{-\left(\frac{x^2+y^2}{2}\right)}$$

$$p'(\Lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \; e^{-\frac{1}{2}\left(x^2 + \underbrace{(\Lambda - x)^2}_{x^2 - 2\Lambda x + \Lambda^2}\right)}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \; e^{-\frac{1}{2}\left(2x^2 + \Lambda^2 - 2\Lambda x\right)}$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}\Lambda^2} \int dx \, e^{-(x^2 - \Lambda x)}$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}\Lambda^2} \int dx \; e^{-\left(\left(x - \frac{\Lambda}{2}\right)^2 - \frac{\Lambda^2}{4}\right)}$$

$$= \frac{1}{2\pi} e^{-\frac{\Lambda^2}{4}} \int dx \, e^{-x^2}$$

$$= \frac{1}{2\sqrt{\pi}} e^{-\Lambda^2/4}$$

$$\mathbb{P}(a+b < \Lambda) = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\Lambda} e^{-t^2/4} \, dt$$

$$\mathbb{P}(a+b < 2\Lambda) = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{2\Lambda} e^{-t^2/4} \, dt$$

$t = 2s$

$dt = 2ds$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\Lambda} e^{-s^2} \, ds$$

$$\mathbb{P}\left(\frac{\xi}{\sqrt{2}} < \lambda\right) = \mathbb{P}\left(\xi < \sqrt{2}\lambda\right)$$

$x = \sqrt{2}s$
$dx = \sqrt{2}\,ds$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\sqrt{2}\lambda} e^{-x^2/2}\,dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\lambda} e^{-s^2}\,ds$$

Statistically $\frac{a+b}{2}$ and $\frac{\xi}{\sqrt{2}}$ should be considered the same (in distribution.)

Returning to our Laboratory:

1. Do we believe from experimentation that:

$$\mathbb{E}\left(\#\,(\text{evals in } (a,b))\right) \text{ is diff'ble?}$$

Did we compute an approximation to

$$\frac{1}{\Delta}\,\mathbb{E}\left(\#\text{ evals in } (b, b+\Delta)\right)$$

for $\Delta = \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}$?

What is it about our plot that suggest <u>diff'bility</u>?

OK Let's be careful.

Divide the last quantity plotted by Δ, and then

$\Delta = .01$ repeat.

See the semicircle!

Now we are pretty confident!

Now Let's fix edges $-6:0.05:6$

and crank up N.

Stop the code to check it a few times

$N = 4, 8, 16, 32, 64, 128, 256$

You'll have to rescale it, because the evals accumulate on growing sets

Try to determine empirically how the rescaling works. $\sim (2\sqrt{N}, 2\sqrt{N})$.

Last time we provided a mathematical definition of the average density of eigenvalues, a fcn. of 1 variable:

$$G(b) = \frac{d}{db} \mathbb{E}\left(\#(\text{evals in } (a,b))\right)$$

We have now seen, roughly, how this fcn. behaves as N grows.

Being careful: What is

$$\int_a^b G(s)\,ds \;?$$

should be $\mathbb{E}(\#\text{ evals in }(a,b))$

So how about $\int_{-\infty}^{\infty} G(s)\,ds$? $(= N)$

That means our natural rescaling should be (if there is justice in the world)

$$\frac{1}{N^{1/2}}\, G(\sqrt{N}\, x) \underset{N\to\infty}{\cong} c\sqrt{A^2 - x^2} \quad \text{explain!}$$

$\int_{-\infty}^{\infty} G\,dt = \int N^{1/2} c\sqrt{A^2 - t^2/N}\,dt$

A fundamental quantity: given a subset of $\mathbb{R}$

$B = (a,b)$ or any subset,

$$n(B) = \#\{\text{evals in } B\}$$

When you encounter such a r.v., you ask
- avg.?
- var?
- probabilities?

$\text{var} = \mathbb{E}(n(B)^2) - (\mathbb{E}(n(B)))^2$

$\mathbb{P}(n(B) = k) = ?$

Interest: behavior as $N \to \infty$.

In fact the set B may be chose to depend on N as well, to see interesting limits.

for example, as we have seen,

$$\mathbb{E}(\#\text{ evals in } (0, s))$$

$$\int_{\mathbb{R}} \frac{1}{N} G \, dt = 1.$$

t effectively on $(-c\sqrt{N}, c\sqrt{N})$

$$\int_{\mathbb{R}} \frac{1}{N} G(u\sqrt{N}) \sqrt{N} du \qquad t = u\sqrt{N}$$

$$= \int \frac{1}{N^{1/2}} G(u\sqrt{N}) du \qquad u \in (-c, c)$$

this has nice limit?

$$\frac{1}{N^{1/2}} G(u\sqrt{N}) \cong \psi(u)$$

$$G(t) = N^{1/2} \psi(t/\sqrt{N})$$

$$= N^{1/2} \sqrt{A^2 - t^2/N}$$

$$\mathbb{E}(\#\text{ in } (0,s))$$

$$= \int_0^s G \, d\lambda = \int_0^s c N^{1/2} \sqrt{A^2 - \lambda^2/N} \, d\lambda$$

$$\lambda = \sqrt{N} u$$

$$= \int_0^{s/\sqrt{N}} c N \sqrt{A^2 - u^2} \, du$$

$$= cN \int_0^{s/\sqrt{N}} \sqrt{A^2 - u^2} \, du$$

to get $\mathbb{E}(\#\text{ in } (0, s)) \cong 1$

how large should s be?

$$\frac{s}{\sqrt{N}} \cong \frac{1}{N}$$

$$s \cong \frac{1}{\sqrt{N}}$$

So if we study sets of size $\frac{1}{\sqrt{N}}$,
we should expect typically to see a finite # evals. This is essentially the smallest interesting scale for eigenvalues, referred to as the <u>microscopic scale.</u>

On the other side of the universe, how large should an interval be in order to see a positive fraction of the eigenvalues?

$$\mathbb{E}(\#\text{ in }(a,b)) \simeq \int_a^b c N^{1/2} \psi(y/\sqrt{N})\, dy$$

$\frac{y}{\sqrt{N}} = u$

$$= \int_{a/\sqrt{N}}^{b/\sqrt{N}} c N \psi(u)\, du = N \cdot c \int_{a/\sqrt{N}}^{b/\sqrt{N}} \psi(u) du$$

so we should expect $\frac{a}{\sqrt{N}} \simeq \alpha$ fixed

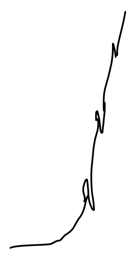
to see a fraction of the evals in an interval. $\frac{b}{\sqrt{N}} \simeq \beta$ fixed

$\alpha < \beta$

Note we are assuming the sets are not too large! Otherwise we are checking for eigenvalues in regions where, in fact, they are very unlikely.

This scaling is referred to as the macroscopic scaling, or bulk scaling. It is thought of as the largest interesting scaling for eigenvalue behavior.

What about the edges? What is going on there?



Challenge: use the codes to investigate "precisely" how the edge scales.

Discuss... What about studying λ_{max}?