Topics Course on Random Matrices and eigenvalues

1. Introduction: Random matrices with your bare hands.

   Developing intuition about eigenvalue statistics using the computer, and understanding definitions.

2. A generating function that contains fundamental information about random eigenvalues. Fredholm Determinants & integrability.

3. How to explicitly describe random eigenvalues. The analytical version of the introduction.

4. Open research problems: eigenvalue number statistics, when matrix size gets large.

5. Why is it true, part a.

   From the random eigenvalues to Fredholm determinants

6. Why is it true, part b.

   From random matrices to random eigenvalues.

$\{\xi_{ij}, \eta_{ij}\}$ iid standard Gaussians.

$$\begin{pmatrix} \xi_{11} & \frac{\xi_{12}+i\eta_{12}}{\sqrt{2}} & \frac{\xi_{13}+i\eta_{13}}{\sqrt{2}} \\ \frac{\xi_{12}-i\eta_{12}}{\sqrt{2}} & \xi_{22} & \frac{\xi_{23}+i\eta_{23}}{\sqrt{2}} \\ \frac{\xi_{13}-i\eta_{13}}{\sqrt{2}} & \frac{\xi_{23}-i\eta_{23}}{\sqrt{2}} & \ddots \end{pmatrix}_{N\times N}$$

Primary interest is in eigenvalues and their behavior as $N\to\infty$.

They are random so we want to think about them appropriately using probability.

Eigenvalues are all real.

First question: how do we generate these matrices? matlab command

randn : each call generates one (or more) normally distributed pseudo random numbers

The numbers you generate behave like samples with prob. dist. fcn.

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

randn (10) outputs a matrix in which each entry is one of these pseudo random numbers.

m = randn(2) ;

Now move to matlab.

1^{st} example : 2×2

raises the question: easiest way to get

$$\begin{pmatrix} \xi_{11} & \frac{\xi_{12} + i\eta_{12}}{\sqrt{2}} \\ \frac{\xi_{12} - i\eta_{12}}{\sqrt{2}} & \xi_{22} \end{pmatrix} :$$

m = randn(2),

then $L = \frac{m + m^{+}}{2}$.

$$\begin{pmatrix} r_{11} & \frac{a+b}{2} + i\,\frac{c+d}{2} \\ \frac{a+b}{2} - i\frac{c+d}{2} & r_{22} \end{pmatrix}$$

what is the dist. fcn. for

$\frac{a+b}{2}$ if a and b are indep.

with dist $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$?

answer: it is the same dist. as the dist. fcn. for $\frac{a}{\sqrt{2}}$ (or $\frac{b}{\sqrt{2}}$)

How do you see this?

You compute it. As a refresher,

HW: understand why $\mathbb{P}\left(\frac{a+b}{2} < \lambda\right) = \mathbb{P}\left(\frac{a}{\sqrt{2}} < \lambda\right)$

and be able to explain it, cold!

Now we begin experimenting!

1. Generate random matrices and find eigenvalues: 2x2, then 5x5, then 10x10.

2. How do we study the eigenvalues? Number statistics. Expected number of eigenvalues in sets, PARTICULARLY intervals, because the eigenvalues are real.

a. Observe and begin talking about density of eigenvalues as a function of 1 variable.

b. Of course you could consider more complicated things: talk about examples, such as the joint law of adjacent eigenvalues (i.e. try to respect the fact that eigenvalues are ordered).

in each interval...
what about the
average #?

It (avg) is an analytic
fcn. of the endpoints of the
interval in question.
to see this will require
some analysis.

In the lab, we can determine
some approx. of this:

$(x, x+\Delta)$ small interval

(a, x) ; $(a, x+\Delta)$

$$\frac{\#\text{ in} - \#\text{ in}}{\Delta} = \frac{\#\text{ in } (x, x+\Delta)}{\Delta}$$

$$\frac{\text{avg. \# in } (x, x+\Delta)}{\Delta}.$$

$\Delta = \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}$

$x = 0$

$\#$ experiments $= 10000$

$N = 2.$

$$C \int_{-2\sqrt{2}}^{2\sqrt{2}} \sqrt{8 - x^2}\, dx = 1$$

$$\frac{C \cdot 8\pi}{2} = 1 \quad C = \frac{1}{4\pi}$$

$$\frac{\sqrt{8}}{4\pi}$$

experiments suggest
$\mathbb{E}(\#\text{ evals in } (a, x))$
is diff'ble wrt. x.

$$\mathbb{E}(\#\text{ in } (a,x)) = \int_a^x \rho(s)\, ds$$

$\rho(s) \approx .$

Nice way to visualize ρ:

intervals $(-8, -8+2\Delta)$ $j = 0$
$(-8+2\Delta, -8+4\Delta)$
$(-8+2(j-1)\Delta, -8+2j\Delta)$
$\vdots$
$(8-2\Delta, 8)$

center
$-8+(2j-1)\Delta$

In lab find
Avg. in I_j

$$\int_{I_j} \rho\, ds \approx \rho(m_j) \cdot 2\Delta$$

Q: What is $\int_{-\infty}^{\infty} \rho\, ds$?

$0, 1, 2, \ldots N$?