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A Practitioner's Guide to Discrete-Time Yield Curve Modelling

With empirical illustrations and MATLAB examples

June 30, 2019

First draft

To Noa, Clara, Laura, and Emilia.

Preface

This *Element* was written during the fall of 2018, and it is intended for students and practitioners as a gentle and intuitive introduction to the field of discrete-time yield curve modelling. I strive to be as comprehensive as possible in the coverage of the included materials, while adhering to the over all premise that the *Element* should have a strong focus on the practical application of term structure models. Some (myself included) experience difficulties when embarking on the vast field of yield curve modelling approaches. It is my hope that the materials covered on the following pages can easy the entry into this field.

To emphasise the applied nature of the lecture notes, I have included MATLAB transcripts when relevant, and *Element* is accompanied by a set of newly programmed MATLAB object oriented classes that facilitates estimation the yield curve models used. Almost all the empirical examples and results shown can be replicated using these MATLAB materials. Of course, no warranty is provided for the code, and bugs are very likely still lurking around; if you find any then please do not hesitate to report them to me.

An overview of the MATLAB classes that I have programmed to help digest the content of this course is provided below. In addition to these generic functionalities I provide MATLAB scripts at the end of each empirically tilted chapter. To provide an overview, a list of these script files is also provided below. Note that all the provided codes can be inspected in MATLAB by typing edit and then the name of the code you want to see. It is recommended that the attached zip file is unpacked in a separate directory, and that the path (with sub-folders) is added to the MATLAB path.

The data that are used throughout these lecture notes are contained in the MATLAB files:
Data_TSM_Course_2018.mat and Data_GSW_factors_Course_2018.mat.

To illustrate how shadow-short rate models work, I have created a small graphical MATLAB add-in. This add-in can be installed by double clicking on the file name: ShadowRateExample.mlappinstall. More information on this is provided in chapter 3.5.

GSW.m is a class-file that can be used to convert Gurkaynak, Sack, and Wright (2006) yield curve factors, and in general Svensson and Söderlind (1997) factors, into yields at a set of pre-specified maturity points. The help file for this class is shown below.

GSW

```
Usage:  .getYields   ->  converts factors and lambdas into zero yieds
        .getForwards ->  converts factors and lambdas into forwards

GSW: converts Gurkaynak, Sack and Wright yield curve factors into
     yield and forward curves for a user-selected set of maturities.

     The conversion is based on the Svensson-Soderlind model as
     applied by GSW in their Fed working paper:
            "The U.S. Treasury Yield Curve: 1961 to the Present"
             Refet S. Gurkaynak, Brian Sack, and Jonathan H. Wright
             2006-28

     Gurkaynak, Sack and Wright data can be downloaded from the
     Fed homepage (http://www.federalreserve.gov/pubs/feds/2006/200628/200628abs.html).

Input:
   beta     :   GSW yield curve factors         (nObs-by-4)
   lambda   :   holds the time-decay parameters (nObs-by-2) [called tau by GSW]
   tau      :   vector of maturities (in months)(nTau-by-1)
                e.g. 24 is provided for a 2year maturity
Output:
   yields   :   zero-coupon yield curves
   forwards :   forward curves

   Activate calculations:
                          getYields   -> obtains zero coupon curves
                          getForwards -> obtains instantaneous forward curves
```

TSM.m is a class-file that allows for the estimation of various term structure models. The help file for this class is reproduced below.

TSM

```
Usage:   .getDNS        -> Dynamic Nelson-Siegel model and related metrics
         .getDSS        -> Dynamic Svensson-Soderlind model and related metrics
         .getSRB3       -> Short-Rate based 3-factor model and related metrics (following Nyholm 2018)
         .getSRB4       -> Short-Rate based 4-factor model and related metrics (following Nyholm 2018)
         .getJSZ        -> Joslin, Singleton, Zhu (2011)
         .getAFSRB      -> Arbitrage-free SRB model with 2,3, or 4 factors
         .getSRTPC1C2   -> Model with Short rate, 10-year termpremium, and 2 additional empirical factors
         .getSSR3       -> Shadow-short rate model with 3 factors (following Coche, Nyholm, Sahakyan 2018)

TSM : Dynamic term structure models of the Nelson-Siegel family.
      Arbitrage constraints are not imposed in this modelling
      framework. See Nelson-Siegel(1987), Diebold-Li(2011), Nyholm(2018)
      for references.

      Small sample bias correction using Pope's closed form solution
      can be invoked via the flagg "biasCorrect=1".

      The basic model setup is the following:

          Y_t(tau) = B(lambda) * beta_t + e_t
          beta_t   = mP + PhiP * (beta_{t-1}-mP) + v_t

      The model is estimated using two-step OLS.

      B(lambda) specifies the applied model. The following
      alternatives are valid:


 Input:
 ------
      yields   :  panel of yields      ( nObs-by-nTau )
      nF       :  number of factors    ( used only for the arbitrage-free models )
      tau      :  vector of maturities ( nTau-by-1 )
      mP_pre   :  pre-specified mean for the betas
      DataFreq :  Data frequency of the yield curve data. This is
                  relevant for the calculation of Term Premia.
                  The frequency is provided as the
                  "number of observations per year" to allow for
                  flexibility, so, e.g.:
                                    DataFreq      Interpretation
                                    ----------------------------
                                        360    =   daily
                                         52    =   weekly
                                         12    =   Monthly

      Z        :  panel of exogenous variables that can affect the
                  yield curve factors via the VAR model. They are
                  treated as unspanned factors as they do not impact
                  the pricing of bonds
 Output:
 -------
      beta   : panel of extracted factors
      lambda : optimal lambda values
      Yfit   : fitted yields
      RMSE   : Root mean squared error of yield residuals (in Basis points)
      e_t    : residuals from the yield equation
      v_t    : residuals from the beta equation
      PhiP   : matrix of autoregressive parameters
      mP     : estimated mean of the betas
      rho1   : vector that defines the short rate as a function of
                  the yield curve factors
      TP     : term premia calculated at the maturities tau
      Er     : expectations component (risk free term structure)
```

TSM2SSM.m is a class that translates an estimated TSM model into MATLAB's state-space format. This is for example relevant if we want to use MATLAB's built-in kalman-filter routines to generate conditional projections for the estimated yield curve factors. Once a TSM model has been estimated, the TSM2SSM class can be used to translate the model into SSM format. The help file for this class file is shown below.

TSM2SSM

```
Requirement: (1) an estimated TSM instance
             (2) access to MATLAB's Econometric Toolbox
                   the TSM2SSM class uses the state-space modelling
                   capabilities of this toolbox

Usage   :
             .getMdl  -> converts the supplied TSM into the variables
                           required for using the SSM toolbox


TSM2SSM : converts an estimated TSM model into MATLAB's state-space
             format to allow easier unconditional and conditional
             forecasting and scenario generation of the estimated
             TSM.

Input   :
---------
          .TSM       <-    estimated TSM class

Output  :
---------
          TSM2SSM.Mdl contains a MATLAB state-space model ready to
             be used by simulate, filter, smooth, and other
             functionalities embedded in the SSM module

             The following setup is dictated by the SSM module:

             state eqn:  X_{t} = A * X_{t-1} + B * e_{t}
             obs eqn  :  y_{t} = C * X_{t}   + D * u_{t}
```

EX_Script_Classes.m is a script filed that provides information on how class files are run. Many more examples are given in the end-chapter codes listed below:

1. Empirical_Investigation_of_Observed_Yields.m
2. P_and_Q_Measure_Vasicek_State_Space.m
3. P_and_Q_Measure_Vasicek_2_step_approach.m
4. Basic_yield_curve_setup.m
5. Modelling_yields_under_Q.m
6. P_and_Q_Measure_1.m[1]
7. Scenario_and_forecasting.m

[1] Used just for illustration, not shown as end-chapter code.

Contents

1

Empirical analysis of term structure data

1.1 Introduction

Before looking at the empirical behaviour of yields, we need to introduce some notation. Let y_t^τ denote a set of yields that together form a yield curve, i.e. a vector that stacks individual annual yields, with the same dating, t, but that are observed at different maturities, τ. In the practical examples included in this booklet, we will typically use $\tau = \{3, 12, 24, \ldots, 120\}$ months, but τ can naturally take any value, at which yields are observed. When referring to a panel of yield observations (of dimension number of dates by number of maturities), i.e. a collection of yield curves observed at different dates, we will either write y, $y(\tau)$, or Y.

In a factor model, X, will denote the extracted factors, and H, G, or B, will typically denote the corresponding loading matrix. Vector autoregressive models will be written as $z_t = m + \Phi \cdot (z_{t-1} - m) + e_t$, when written in mean-adjusted form, and sometimes as $z_t = c + \Phi \cdot z_{t-1} + e_t$, when written in constant form, i.e. $m = [I - \Phi]^{-1} \cdot c$.

At this point it may also be worth recalling that the yield curve is a by-product of the financial market trading process. Agents trade bonds that are quoted in prices, $p_t(\tau)$. A risk free bond, the ones we primarily deal with here, guarantee to pay Eur 1 (in reality some scaling of 1, most often Eur 100) at the maturity of the bond. The price today is therefore, as always in finance, the discounted value of the future promise: $P_t(\tau) = 1 \cdot (1 + y_t(\tau))^{-\tau} \Leftrightarrow y_t(\tau) = (P_t(\tau))^{1/-\tau} - 1$, in discrete time, and $P_t(\tau) = 1 \cdot e^{-y_t^\tau \cdot \tau} \Leftrightarrow y_t(\tau) = -\frac{1}{\tau} \cdot log\,(P_t(\tau))$, in continuous time.

We will exclusively be modelling Zero coupon bonds. Such data are important because they form the basis for fixed income pricing: since all coupon paying bonds can be expressed as portfolios of zero coupon bonds (of relevant maturities), once we know the prices of zero coupon bonds, we can also find the market-equilibrium price all existing coupon paying bonds. Most often, however, we do not work with prices, but instead focus on rates/yields, i.e. on the annualised percentage return the bonds gives, if we hold it to maturity. As implied by its name, a zero coupon bond does not pay any coupons during its life, and its cashflow stream is therefore simple, as illustrated in figure 1.1 for zero coupon bonds of 1, 2, and 10-year maturities.

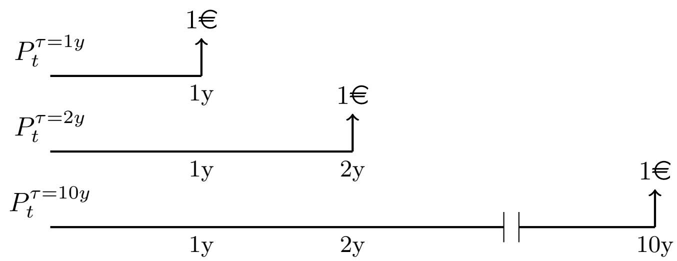


Fig. 1.1. Zero coupon cashflows

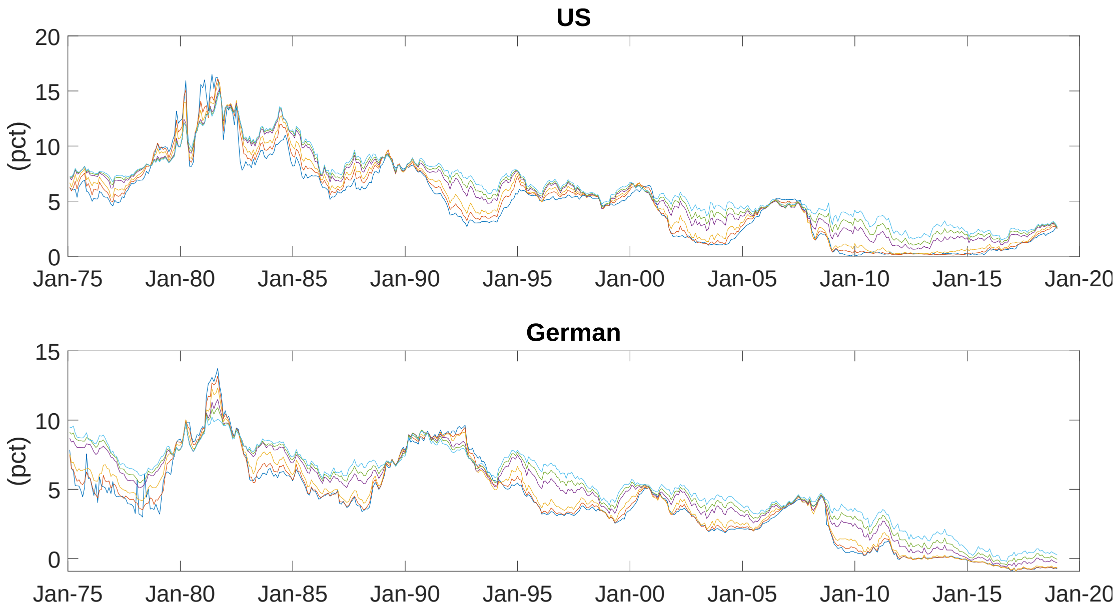
Typically, we get zero coupon data from Bloomberg, Reuters, and other data providers. These data are available at daily, weekly, and monthly observation frequencies, and at predefined target maturities, for example at $\{0.25, 1, 2, \ldots, 10, 15, 20, 30\}$ years.

1.2 Exploring yield curve data

The example data used in this section are stored in the MATLAB workspace file named "Data_YCM_2018". Data are obtained from public sources. The US data are downloaded from the Federal Reserve Board homepage.[1] These are the well-know and often used Gurkaynak, Sack, and Wright (2006) Data. German yield curve data are obtained from the homepage of the German Bundesbank.[2]

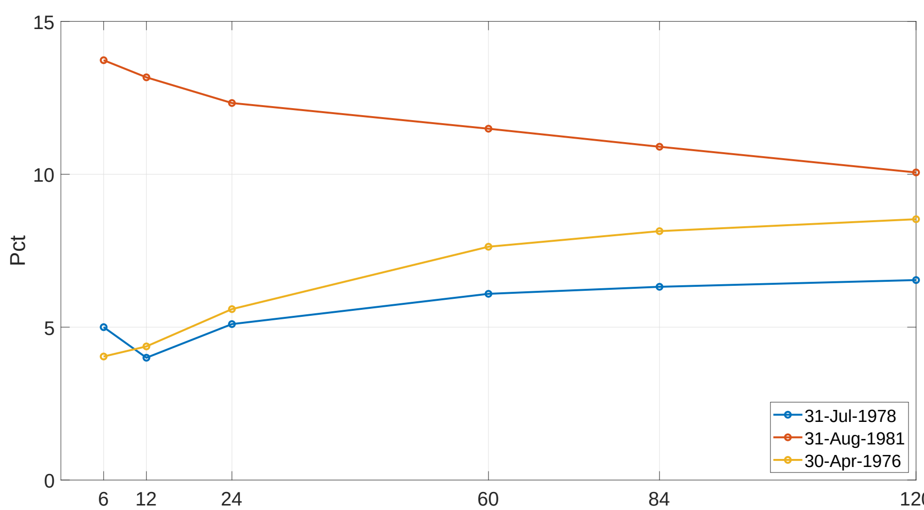
For each segment we have yields in percent per annum across maturities, as well as model based estimates for the expectations component and the term premium, both estimated at 10 year maturity point. We will return to these latter two variables later on, and for now only focus on the yield curve data. Let's load and plot these data: each data set contains monthly observations for the following variables: date, and yields, and spans the period from January 1975 to December 2018, i.e. a total of 528 time series observations for each of the 6 included maturities per yield curve segment. In addition to the time series evolution of yields shown in Figure 1.2 it is also informative to see what the yield curve looks like in the cross sectional dimension. For example, what does the average yield curve look like? And, what are some of the most extreme shapes and locations that yield have displayed historically? These questions are explored below.

[1] https://www.federalreserve.gov/pubs/feds/2006/200628/200628abs.html
[2] https://www.bundesbank.de/en/statistics/time-series-database.



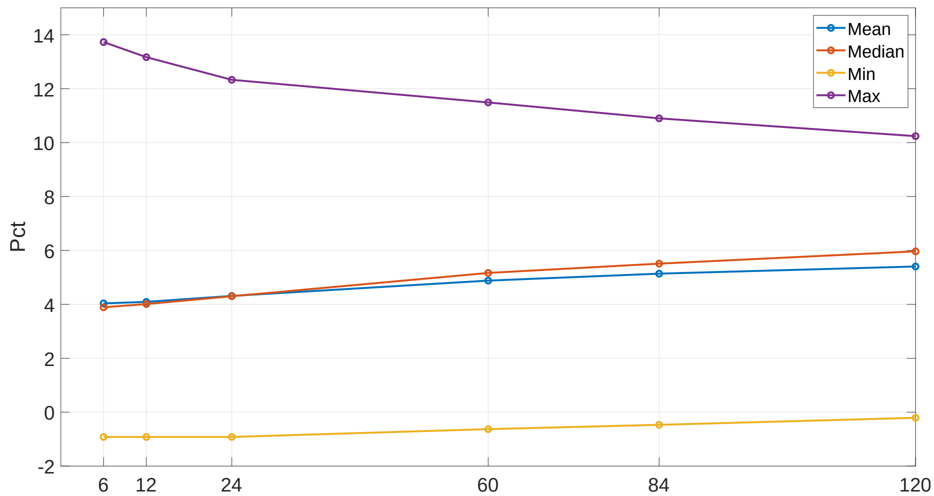
The figure shows the time series of yields, observed monthly and covering the period from 1975 to end-2018, for maturities of 6-months, 1-year, 2-year, 5-year, 7-year, and 10-year for Germany, and for the US market the following maturities are shown 3-months, 1-year, 2-year, 5-year, 7-year, and 10-year. Yields for the US, Germany, and the euro area are included in the plot. It is noted that the shortest maturity in the German market is 6-months (that is what is available from the German Bundesbank home page) while the shortest maturity available for the US data is 3-months.

Fig. 1.2. Yield curve data



The figure shows German yield curves on the days when the slope ($y^{\tau=10y}$-$y^{\tau=3m}$) reached its minimum, maximum and average value, for the period from January 1975 to December 2018.

Fig. 1.4. German yields with varying slopes



The figure shows the mean, median, min, and max of the German yields observed at a monthly frequency and covering the period from January 1975 to December 2018. The statistics are calculated across maturities.

Fig. 1.3. Summary of German yield curve data

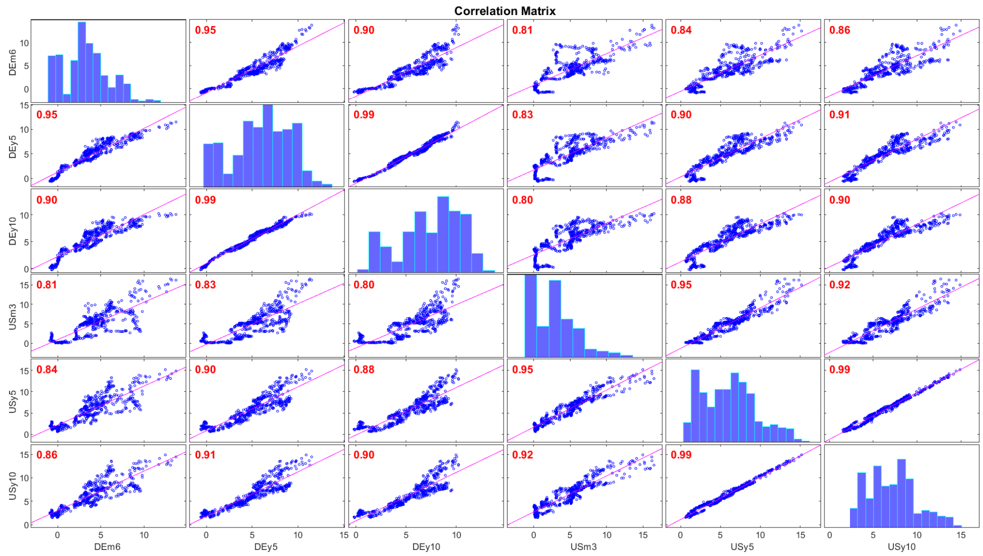
None of the curves shown in Figures 1.3 and 1.4 may actually have materialised historically, since the calculations are done for each of the maturity points separately.

Going back to the time series plots of the yields observed for the US and German market segments, it is also interesting to observe that there is a very high degree of correlation among yields within a given market segment, and that a similarly high degree of correlation exists between market segments. It almost seems as if every little up- and down-ward movement in one maturity is mirrored by the other maturities in that market segments, with more pronounced movements the higher the maturity. Similarly, the secular swings that yields display over the 20 years of data are equally well visible across market segments.

A more structured view on the within and between segment correlation is illustrated below. For presentational purposes, correlations are shown only for a subset of the included maturities.

Figure 1.5 provides a visual representation of the correlation between German and US yields. If we had included other or additional yield curve segments, in addition to the 3months, 5year, and 10year maturities, we would get qualitatively identical results. As expected based on the visual inspection of the time series plots, the cross correlations confirm our suspicion: yields within and across yield segments are very highly correlated. Note that a red number in the above correlation matrix indicates that the correlation is statistically significant from zero at a 1% significance level.

We could repeat the above correlation analysis for the first differences of the yield series - this would for example make sense, if yields were believed to be I(1) processes (i.e. integrated of order one). And, if we did this, we would obtain a correlation picture that is qualitatively identical to the one above.



The figure shows the pair-wise correlation between US and German yields observed at a monthly frequency and covering the period from January 1975 to December 2018. Correlations are calculated between the 3-month, 5-year, and 10-year maturity points. In each sub-element of the figure, the red number indicates the correlation coefficient, and the red line shows the fitted regression line. On the diagonal, histograms of the series are plotted.

Fig. 1.5. Correlations

Now, looking at the time series plots of the yield curve segments above, the conclusion that one may reach, based on a preliminary and casual visual inspection, is that the behaviour displayed by yields is somewhat different from what most people have in the back of their mind, when they think about the trajectory of a stationary I(0) process. While this is a relevant thought, the discussion of stationarity will be taken up later on, when we discuss the eigenvalues of estimated vector autoregressive processes (VAR models - not to be confused with VaR, i.e. value-at -risk). For now, we treat observed yields as coming from a stationary data generating process.

How can the overwhelming degree of correlation between yields be exploited? The answer is: by using Principal Component Analysis (PCA)/ factor models. At this stage, it is worth noting that virtually all term structure models, as well as many other important financial models, e.g. ATP and CAPM for equity return modelling, rely heavily on PCA modelling principles. In fact, this econometric technique is quite possibly the single mostly important modelling idea, in the field of quantitative time-series finance - to my mind, it is as important as PDEs (partial differential equations) are to the branch of finance that deals with derivative pricing. It is therefore fairly important to master this technique. The good news is, that it is not difficult at all.

Before embarking on the factor modelling principle, it is worth spending a few minutes on realising that modelling multiple yields directly is generally not a good idea. This would mean to apply the following modelling set-up, where Y is a vector of yields, c is a constant, F is a matrix of autoregressive coefficients, and e is a vector of residuals:

$$Y_t = c + F \cdot Y_{t-1} + e_t \tag{1.1}$$

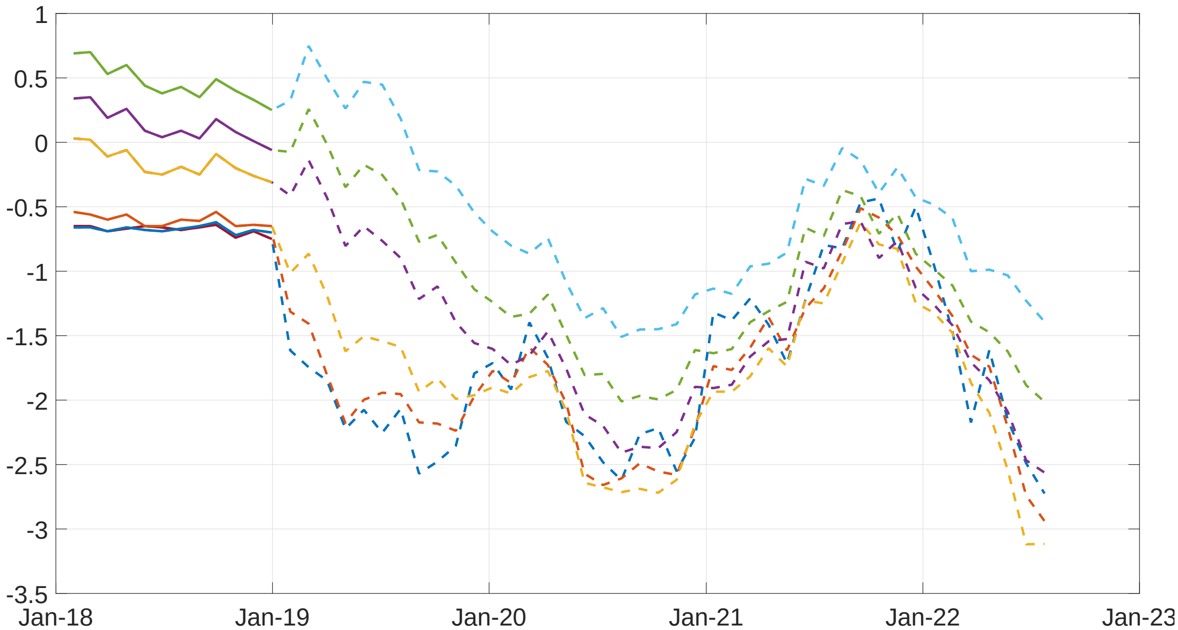
Arguments against this modelling strategy are, amongst others:

- The number of yields modelled may vary from market to market and over time. It is therefore not clear which maturities that should be included in the model.
- One may need to adapt the dimension of the model, depending on which market that is modelled. This is inconvenient as well as model results may not be comparable.
- Since correlation between yields is so high, we may run into the problem of multicollinearity
- Projected yield curves and yield curve forecasts may turn out to violate standard regularities, e.g. individual yield curve points may be out of sync with the rest of the curve.
- The econometrician has very little control over the simulations, for example, it is difficult to steer the projections in a certain direction, if that is desired. Likewise, it is difficult to avoid certain (unrealistic) yield curve shapes and developments.

This last point is illustrated in Figure 1.6, using the German data. It is dangerous these days to make statements about whether a given simulated yield curve has a realistic shape or not - and the future may prove me wrong - but despite what we have seen over the past years, I believe that the depicted simulated curves in Figure 1.7 are too oddly shaped to be considered for financial analysis (unless for some wild economic scenario): this applies to their shape and location, and to the overall simulated trajectory (Figure 1.6) for the yields over the coming 42 months. One may of course have a rule-of-thumb and program a routine that kicks out too oddly looking yield constellations and trajectories, but why bother? Why not simply follow the mainstream and well proven approach, i.e. to rely on factor models? This is what we will do next.

1.3 A first look at Principal Component models

Dimension reduction is one of the great feats of PCA / factors models: the core idea is that the majority of the variability of a given data set derives from a few underlying (sometimes not directly observable) factors. This concept is familiar, for example, the well-known CAPM prescribes that a single market factor is responsible for the expected return on all equities traded in the economy. Recall that the security market line is written as: $E[r_i] = r_f + \beta_i \cdot (r_m - r_f)$, where investors are rewarded only for taking market risk in excess of the risk free rate. r_m is the return on the market portfolio, i.e. the underlying factor in this model, r_f is the observable risk free rate, and β_i is the sensitivity of the i'th security's return, r_i. In factor model language, r_f is the constant, r_m is the underlying factor, and β_i is the factor sensitivity that translates



The figure shows how one can do naive forecasts of the yield curve, and what problems this may bring. A VAR model is fitted to individual maturity points using the full historical sample (from 1975 to end-2018) of German yields. Each maturity is then projected 42 monthly periods ahead using the VAR. These projections are started at the last observation covered by the data sample.

Fig. 1.6. Naive yield curve forecasts

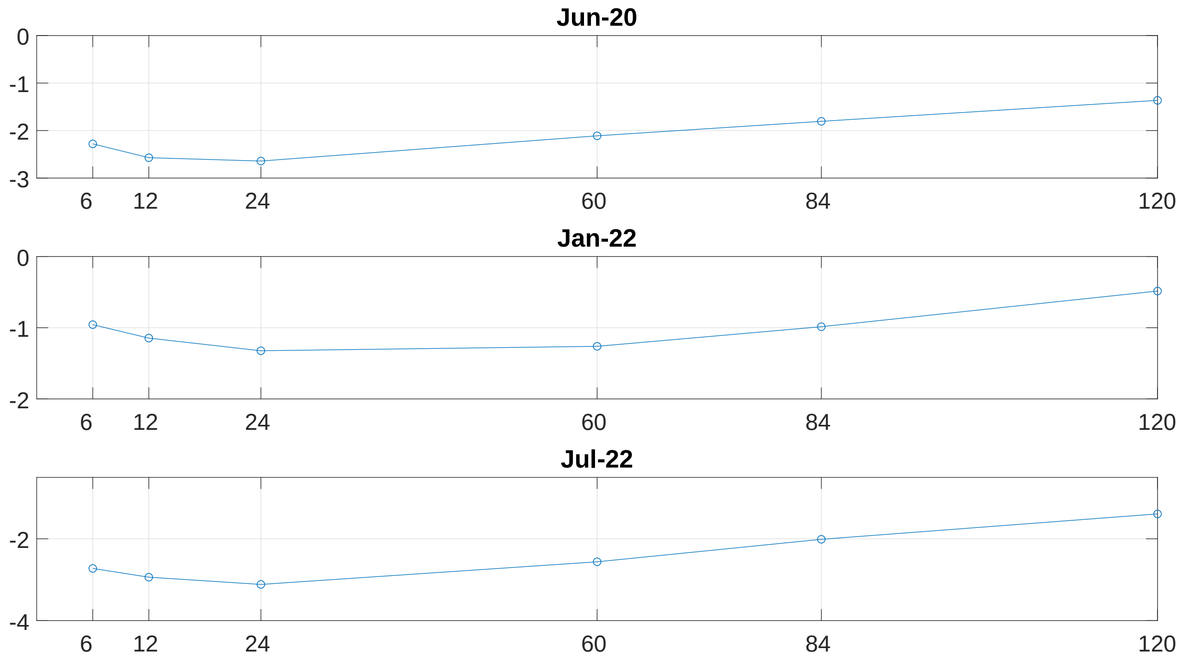
the factor observation into something that is applicable to the i'th security. We can naturally operate with more than one factor. Typically, term structure models include between 1 and 5 factors.

In general terms, and using matrix notation, we can write a factor model for the yield data in the following way:

$$\underbrace{Y_t}_{(\#\tau \times 1)} = \underbrace{G}_{(\#\tau \times \#F)} \cdot \underbrace{X_t}_{(\#F \times 1)} + \underbrace{\Sigma}_{(\#\tau \times \#\tau)} \cdot \underbrace{e_t}_{(\#\tau \times 1)} \quad e_t \sim N(0, I) \tag{1.2}$$

The dimensions of the variables are recorded below each entry, with $n\tau$ being the number of maturities that together form the yield curve, and nF being the number of included factors. So, our first job when using factor models is to settle on an appropriate number of factor to extract, i.e. to choose nF. But before getting to that point lets first get more familiar with the factor model concept.

Looking at the expression for Y_t in (1.2) indicates that if we know the factor loadings G, then we can find the factors X_t using linear regression, or by inversion. Underline the previous sentence! - we will use this 'trick' extensively when dealing with Nelson-Siegel type yield curve models later on. To preview a bit, let's quickly see how to back out the factors X using the full set of data - as mentioned, we will return to this issue in greater detail later on. First we write the above expression in terms of the full data set:



The figure shows randomly selected sample curves picked among the 42 projected curves.

Fig. 1.7. Randomly chosen projected yield curves

$$\underbrace{E\left[Y\right]}_{(\#\tau \times \#Obs)} = \underbrace{G}_{(\#\tau \times \#F)} \cdot \underbrace{X}_{(\#F \times \#Obs)}$$

where $nObs$ is the number of dates the data spans. Assume G is known, then, in the context of an OLS regression, G represents the explanatory variables and X the parameters to be estimated. We can therefore find X in the following ways:

$$\hat{X} = G^{-1} \cdot Y \tag{1.3}$$

or

$$\hat{X} = (G' \cdot G)^{-1} \cdot G' \cdot Y \tag{1.4}$$

where the first equation in (1.3) represents a pure inversion, and the second is the standard OLS formula. Returning to the main topic of this section, i.e. factor models, let's see if the DE and US data hide some interesting underlying patterns (i.e. factors), and let's try to construct a completely data-driven joint model for these to yield curve segments on the basis of such underlying factors.

The intention here is only to show how factor models can be useful for modelling term structure data, without infusing any term structure modelling knowledge - in other words, the illustrated strategy may be what an econometrician would choose to do, if she had not received any term structure schooling. Later on in the course, it will become clear, that such an econometrician can actually be quite successful at modelling term structure data!

A clarification about the term “factor models” is warranted here. When I refer to ”factor models” and ”factors” I do in fact mean ”Principal Components”, i.e. the outcome of applying the PCA function in MATLAB. So, through-out, it assume that yield curve factors can be formed as weighted averages of observed yields. Alternatively, if a true factor modelling approach was applied, the starting point would be some underlying latent factors that were causing the evolution observed in the yield curve, and we would try to extract these factors. As we shall see, we will typically revert to factors that are directly interpretable in terms of yield curve observables, e.g. the level, slope and curvature of the yields curve, or actual maturity points on the yield curve - we will not, however, include unobservable quantities, such as e.g. the effective stance of monetary policy, or the natural long-term rate, as factors in the models that we work with in this booklet.

Individual eigenvalues express how much of the overall variability in the data set, the respective eigenvector explains. To help decide how many factors that we need to include in our model, we can therefore link the number of factors to the overall variance that we want our model to capture. Table 1.3 shows the cumulative fraction explained by the first six extracted principal components/factors explain of the US and German data. So, 4 factors capture 100% of the historical variability of both US and German

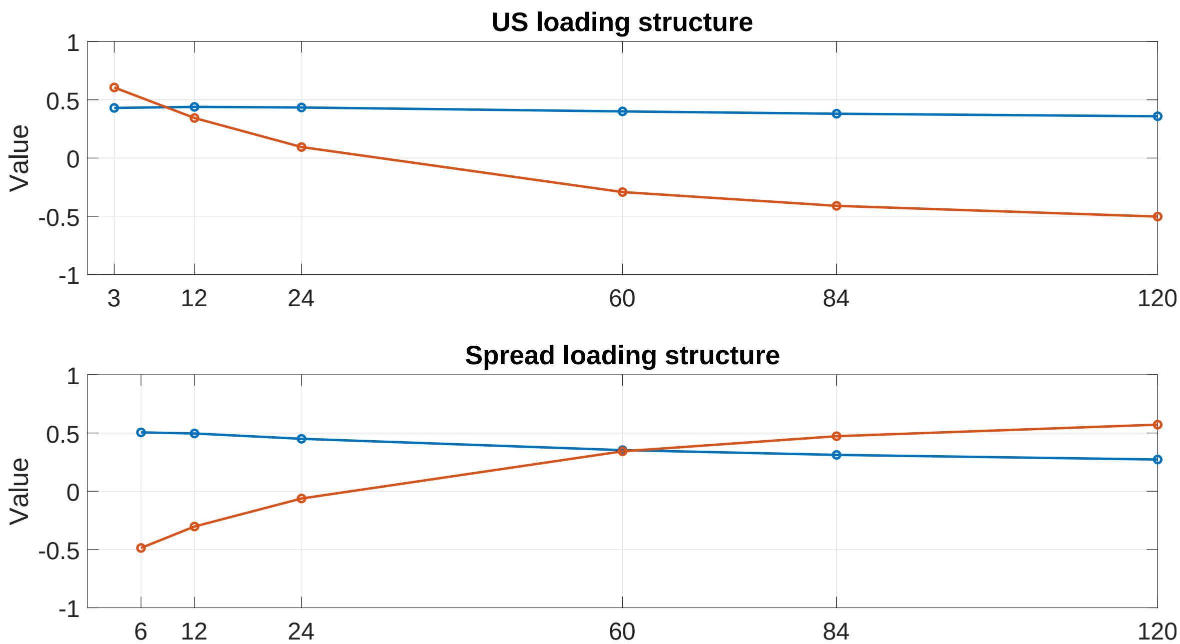
	US	DE
1st	0.9784	0.9713
2nd	0.9982	0.9983
3rd	0.9998	0.9996
4th	1.0000	0.9999
5th	1.0000	1.0000
6th	1.0000	1.0000

The table shows the cumulative fraction of variability explained by the principal components extracted from US and euro area yield curve data. The data covers the period from January 1975 to December 2018 and are observed monthly. The following yearly maturity points are included in the data sets: $0.25, 1, 2, 5, 7, 10$.

Table 1.1. Cumulative variability explained by the extracted yield curve factors

yields. That such a low number of factors explain all the variability underscores the high degree of cross-sectional correlation that we also documented above. If we believe that some of the variability in the observed data is due to noise, we should chose to model less than 4 factors: we don't want a model that propagates idiosyncratic noise from the past into the future. 3 factors also look to be on the high side, so a sensible choice may be to include 2 factors. In fact, the explained variability may suggest that only 1 factor is needed, since the most important factor explains 95% of the variability in the US data, and 98% of the variability in the German data. But, a model with just one factor is quite boring: in terms of e.g. scenario dynamics, it can only generate parallel shifts of the yield curve (i.e. duration effects), so also with a view to the type of yield curve perturbations a model can generate, it may be advisable to include a minimum of 2 factors.

But wait. If we want to construct a joint model for the two yield curve segments, perhaps it makes sense to include only one base segment and model the other segment as a spread curve against the chosen base-curve. What does the loading structure of the spread between the German and the US yield curves look like in comparison to the loading structure of the yield curves?



The figure shows the empirical loading structure for the US data (upper panel), the German data, and for the spread between the US and German data. The loading structure are obtained using principal component analysis.

Fig. 1.8. Yield curve data

Figure 1.8 shows the empirical loadings for the two estimated factors for the US and German data. It is interesting to see how similar the loadings are, both between the two yield curve segments, and between yield and spread loadings. This may even spark some curiosity, and desire to look at these phenomena in more detail: could it be that there are financially interpretable forces behind the observed patterns? And, going further, is it conceivable that such financial forces could be connected to developments in the broader economy, perhaps to expectations about growth, inflation, and agents perception about the risk? That would be something! We will save this issue for a later chapter, once we are armed with better yield curve modelling skills - remember: in this chapter we act as pure econometricians.

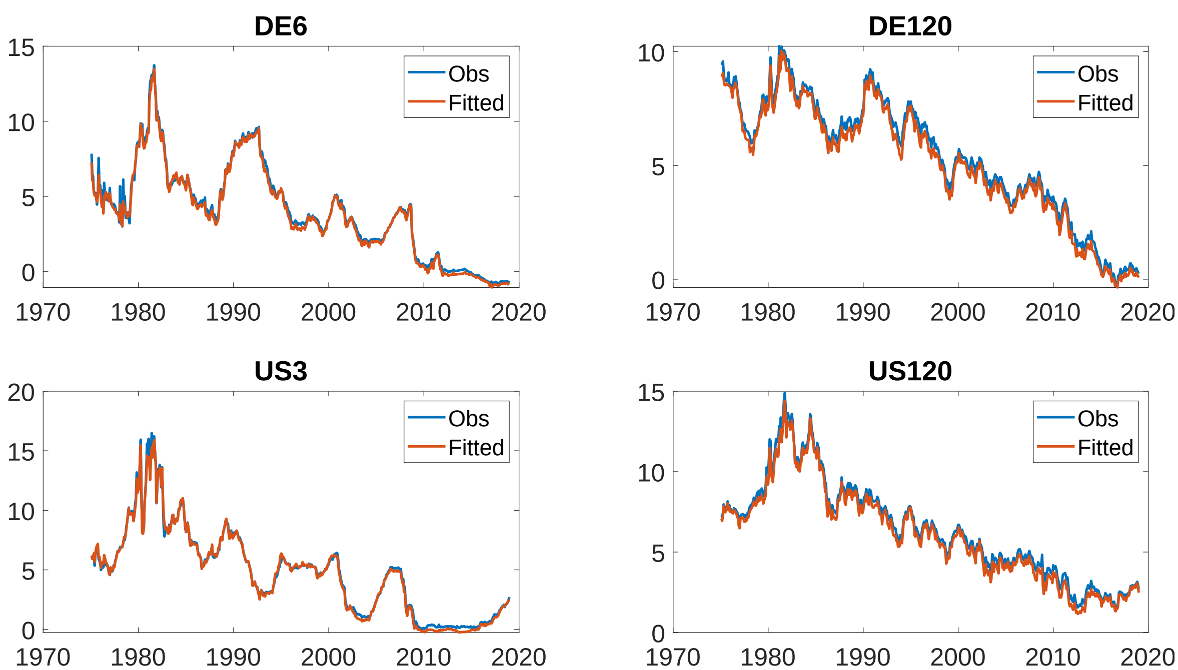
Given the similarity between the extracted loading structures, it seems reasonable to model the German term structure as a base element and the US curve as a spread element. Naturally, we could equally well do it the other way around, but granted the geographical location of my workplace, our choice is made.

	DE						US					
	6m	1y	2y	5y	7y	10y	3m	1y	2y	5y	7y	10y
RMSE	24	11	26	27	6	33	27	16	26	30	7	37

The table displays the fit of the joined yield curve model, comprising US and euro area German data, to the used data. The degree of fit is assessed via the root mean squared error (RMSE) denominated in basis points.

Table 1.2. Root Mean Squared Errors

The RMSEs delivered by this model are ok, without being overly impressive. To provide a visual comparison between observed and model fitted yields, below we plot the 3month and 10y segments of the German and US yield curve segments.



The figure shows factor loadings for the US and German data, and for the spread between the US and the euro area data.

Fig. 1.9. Empirical loading structures

It is now natural to add dynamics to our empirical model, such that we can use it as a projection and scenario-generation tool. To do this, we assume that a VAR(p) model is an appropriate devise to capture the dynamic behaviour of the yield curve factors. First, we want to identify the log-order p. Following the BIC criterion, a VAR(1) model is applied as an adequate description of the law of motion for the 4 yield curve factors. Our purely empirically derived joint German and US yield curve model is then ready to be put to work. The model can be summarised in the following way:

$$\begin{bmatrix} y_{DE} \\ y_{US} \end{bmatrix}_t = \left[\begin{array}{c:c} G_{DE} & 0 \\ \hdashline G_{DE} & G_{sprd} \end{array}\right] \cdot \left[\begin{array}{c} X_{DE} \\ \hdashline X_{sprd} \end{array}\right]_t + \Sigma e_t \tag{1.5}$$

$$\left[\begin{array}{c} X_{DE} \\ \hdashline X_{sprd} \end{array}\right]_t = \left[\begin{array}{c} c_{DE} \\ \hdashline c_{sprd} \end{array}\right] + \left[\begin{array}{c:c} \Phi_{DE,DE} & \Phi_{DE,sprd} \\ \hdashline \Phi_{sprd,DE} & \Phi_{sprd,sprd} \end{array}\right] \cdot \left[\begin{array}{c} X_{DE} \\ \hdashline X_{sprd} \end{array}\right]_{t-1} + \Sigma v_t \tag{1.6}$$

Eigenvalues of Φ is $[0.9569, 0.9569, 0.9720, 0.9921]$. Given that the maximum eigenvalue of the auroregressive matrix is less than one, the estimated VAR is stationary. So, lets see what kind of yield and return projections we can generate using this model. But, before we embark on this exercise, lets first backtest the model using a pseudo out-of-sample forecasting experiment. Our data sample covers 468 monthly observations from January 1975 to April 2018, and last five years of the sample is used for backtesting purposes. Naturally, this choice is somewhat arbitrary, since other equally appropriate combinations of the amount of data available for the first estimation of the model, and number of datapoints available for backtesting, naturally exists. The backtesting exercise is therefore structured in the following way:

- the model is estimated using data from January 1975 to May 2013
- Factor projections are generated using the Dynamic Model for the yield curve factors, shown above. Projections are generated for months 1 to 6 ahead, i.e. for April, May,...,September 2013
- The factor projections are converted to yields using the Yield Equation, shown above
- Projected yields are compared to observed yields at the appropriate horizon
- As a comparison, random walk projections are also generated and compared to the relevant observed yields
- One month is added to the dataset used to estimate the model and above steps are repeated until the end of the dataset is reached

Ok, the backtesting exercise is completed! More can naturally be done, but this is left to the reader, should he/she have the urge to go more into details at this stage. It is also left to the reader to evaluate the outcomes shown above, and to reach a conclusion on whether the model is useful for any practical purposes - apart from illustrative ones.

	DE						US					
	3m	1y	2y	5y	7y	10y	3m	1y	2y	5y	7y	10y
	Model											
Fitted	17	4	19	24	5	28	27	7	28	18	2	27
Forecast 1m ahead	18	8	21	30	17	28	25	11	30	23	19	35
Forecast 2m ahead	19	10	24	34	22	28	23	14	31	26	24	39
Forecast 3m ahead	20	12	27	39	27	28	22	17	33	28	28	42
Forecast 4m ahead	21	16	30	43	32	29	22	21	35	30	31	44
Forecast 5m ahead	22	18	34	48	37	30	23	24	36	33	34	47
Forecast 6m ahead	23	21	37	52	42	33	24	26	38	35	37	49
	Random-Walk											
Fitted	17	7	21	28	16	31	30	9	28	23	20	36
Forecast 1m ahead	17	7	22	31	21	33	33	13	28	27	27	42
Forecast 2m ahead	17	8	23	34	24	34	36	17	28	29	31	45
Forecast 3m ahead	17	10	25	37	29	36	40	21	29	33	36	48
Forecast 4m ahead	17	11	26	39	32	37	43	26	31	35	39	51
Forecast 5m ahead	17	12	28	41	34	38	47	31	33	37	41	52
Forecast 6m ahead	18	14	30	44	38	40	52	35	36	40	44	55

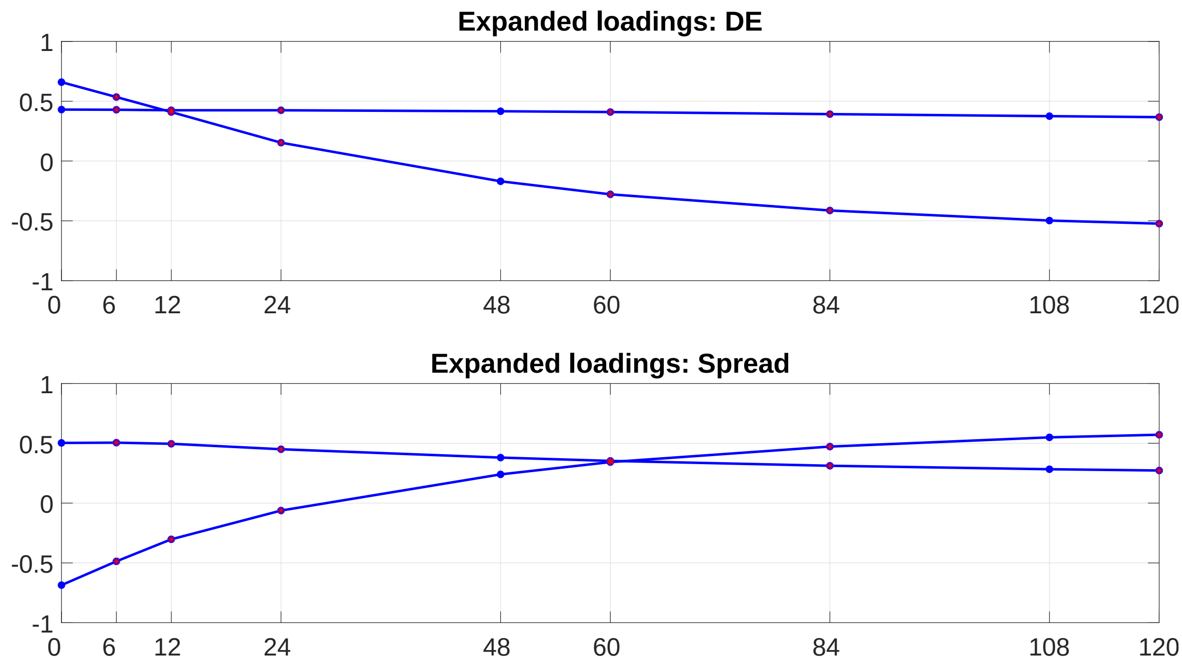
The table shows the s-step ahead prediction RMSEs in basis points for the joint model and the Random Walk model.

Table 1.3. Back-testing the joint model

With this out of the way, let's now see what kind of forward looking return distributions the model can generate. Assuming that we are working with continuously compounded yields, which we are, the holding period return on a τ-maturity bond over the period from, t to $t+j$ is $r^{\tau}_{t,t+j} = p^{\tau-j}_{t+j} - p^{\tau}_{t}$, where p is the log bond price. The intuition here is that we buy a bond at time t with maturity τ, (p^{τ}_{t}), and sell it j periods later, at time $t+j$, where the bond is j periods closer to redemption, its maturity is therefore $\tau - j$. Since $p^{\tau}_{t} = -\tau \cdot y^{\tau}_{t}$, we can rewrite the return in terms of yields as: $r^{\tau}_{t,t+j} = \tau \cdot y^{\tau}_{t} - (\tau - j) \cdot y^{\tau-j}_{t+j}$.

As an example we will use our model to simulate return distributions for the 12, 60, and 120 months segments of the curve, using the last observation in our data sample as a starting point. With this application in mind, it is clear that we cannot use the model directly. A bit of adjustment is needed since the empirical factor loadings only are available at the maturities, at which data are observed, i.e. for maturities $\{3, 12, 24, 60, 84, 120\}$ months, and since we also need yield observations at maturities $\{0, 48, 108\}$ months to calculate the desired returns. We therefore need somehow to enlarge our loading matrix such that it also comprises loadings for these additional maturity points. We will see later on that this is an easy operation, if we have a parametric description of the loading matrix (such as e.g. in the Nelson-Siegel model) - however for now, we have to come up with a solution applicable to the empirical problem at hand. And, that is simply to inter-/extra-polate:

It is probably a good idea to check visually whether the expanded loading matrix is ok. The expanded



The figure shows the model yields for the 3-months and 10-year segments of the curve compared to the corresponding observed yield curve segments.

Fig. 1.10. Model and observed yields

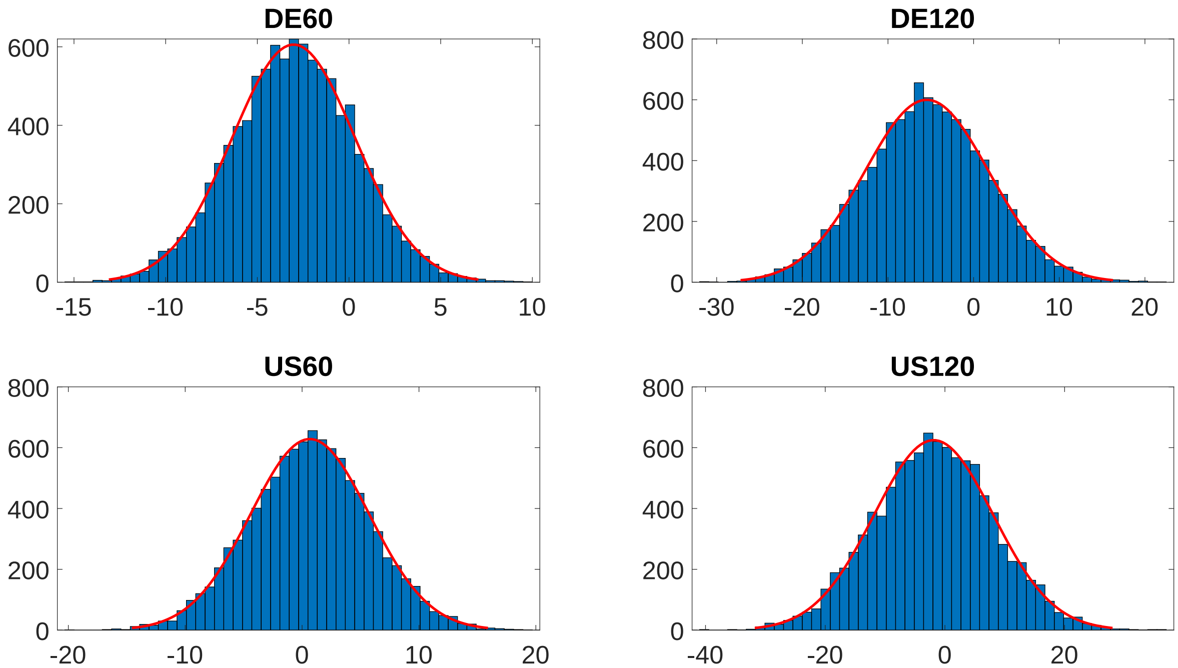
loadings are shown as blue lines while the original observations are indicated using red stars. It looks good, so it can be concluded that the chosen expansions-methodology did a good job. We can now proceed with the generation of yield simulations and return calculations. Figure 1.11 shows the simulated return

	DE			US		
	1y	5y	10y	1y	5y	10y
Mean	-0.74	-3.02	-5.47	2.65	0.67	-1.88
Std.	0.00	3.36	7.25	0.00	5.08	9.96

The table shows distribution statistics for the simulated return distributions for the 5-year and 10-year segments of the curve.

Table 1.4. Simulated return statistics

distributions for the 5 and 10 year maturity points. I have not shown the plots of the 1 year maturity points - why not? To illustrate the distributional properties of the simulated returns, the red-lines in the plots show superimposed normal distributions. These distributions fit the returns quite well, as expected, since the estimated model for the yields relies on the normal distribution. We will see later on how we can escape the world of normality and how distributions can be generated that match assumptions about the expected future trajectory of the economy.



The figure shows empirical return distributions evaluated again the normal distribution for the 5-year and 10-year segments of the curve.

Fig. 1.11. Return Distributions

The above example illustrates that the current low yield environment and a model that embeds mean-reversion to historically observed yield levels will predict negative mean-returns for both the US and the euro area markets.

1.4 MATLAB code

filename: Empirical_Investigation_of_Observed_Yields.m

```matlab
%%  CH 1: Empirical exploration of yield curve data
%
clear all;                             % clear all variables
close all;                             % close all figures
clc;                                   % clear command window
load('Data_YCM_2018.mat');

start_ = datenum('31-Jan-1975');       % defines the start date of the data samples.
                                       %    can be changed to test whether the results
                                       %    below are robust to other starting points.

indx_s   = find(US_data(:,1)==start_,1,'first');
indx_tau = [1 2 3 6 8 11];                    % selected maturities
tauDE    = [6 12 24 60 84 120]';              % defines the maturities
tauUS    = [3 12 24 60 84 120]';
Y_US     = US_data(indx_s:end,indx_tau+1);    % ... first column holds the date ...
Y_DE     = DE_data(indx_s:end,indx_tau+1);    % contains the yield curve
dates    = US_data(indx_s:end,1);             %   observations nObs-by-nTau
[nObs,nTau] = size(Y_US);                     % number of time series observations and
                                              %    number of maturities.

figure('units','normalized','outerposition',[0 0 1 1])
    subplot(2,1,1), plot( dates, Y_US )
    date_ticks = datenum(1975:5:2020,1,1);
    set(gca, 'xtick', date_ticks), ylabel('(pct)')
    datetick('x','mmm-yy','keepticks'), title('US')
    set(gca, 'FontSize', 20)

   subplot(2,1,2), plot(dates, Y_DE),
   date_ticks = datenum(1975:5:2020,1,1);
   set(gca, 'xtick', date_ticks), ylabel('(pct)')
   datetick('x','mmm-yy','keepticks'), title('German')
   set(gca, 'FontSize', 20)

print -depsc Empirical_YieldCurves_US_DE_EA

%% Cross sectional plots
figure('units','normalized','outerposition',[0 0 1 1])
    plot(tauDE,[mean(Y_DE)' median(Y_DE)' min(Y_DE)' max(Y_DE)' ], ...
        'o-','LineWidth',2), ...
    xticks(tauDE'), grid, 'on';
    xticklabels(tauDE' ), ...
    ylabel('Pct'), legend('Mean', 'Median', 'Min', ...
    'Max', 'Location','northeast')
    ylim([-2 15])
    set(gca, 'FontSize', 20)
print -depsc AverageYieldsDE

diff_S = Y_DE(:,end)-Y_DE(:,1);       % difference between the 10y and
```

```
                                        % 3m yields (a measure for the slope)

[~, indxS_med] = min(abs(diff_S-median(diff_S)));  % finds the index of
[~, indxS_min] = min(abs(diff_S-min(diff_S)));     % the curve having the
[~, indxS_max] = min(abs(diff_S-max(diff_S)));     % median, min, and max
                                                   % slope in the sample

figure('units','normalized','outerposition',[0 0 1 1])
    plot(tauDE,[Y_DE(indxS_med,:)' Y_DE(indxS_min,:)'...
              Y_DE(indxS_max,:)'],'o-', 'LineWidth',2 ), ...
    %title('Generic Slope-Based Shapes of the Yield Curve - Germany'), ...
    legend( datestr(dates(indxS_med,1)), datestr(dates(indxS_min,1)), ...
            datestr(dates(indxS_max,1)), 'Location', 'SouthEast' ), ...
    xticks(tauDE), xticklabels(tauDE'), ...
    ylabel('Pct'), grid, 'on'; ...
    ylim([0 15])
    set(gca, 'FontSize', 20)
print -depsc GenericYieldCurveShapesDE


%% Correlation analysis
subData = array2table([Y_DE(:,1), Y_DE(:,4), Y_DE(:,6), Y_US(:,1), Y_US(:,4), Y_US(:,6)]);
subData.Properties.VariableNames = { 'DEm6', 'DEy5', 'DEy10', ...
                                      'USm3', 'USy5', 'USy10' };
corrplot(subData,'type','Pearson','testR','on','alpha',0.01)

print -depsc YieldCorrPlot

%%
rng(42+42+42);  % fixing the starting point for the random number generator
                %    to ensure replicability
nHist    = 12;  % number of historical observations to inlude in the plot
nSim     = 42;  % number of periods to be simulated
VAR_y    = varm(nTau, 1);  % sets up a VAR1 model: 11 variables and 1 lag
est_DE   = estimate(VAR_y, Y_DE);  % estimate VAR1 model on all obs.
sim_DE   = simulate(est_DE, nSim, 'Y0', Y_DE(end,:)); % simulate the model
                                                      %  star at last obs
simDates = [ dates(end-11:end,1); ...
                dates(end,1)+(31:31:nSim*31)' ];
              % concatenating the dates for the last 12 data observations
              %    with the dates spanning the forecasts
data2plot = [ Y_DE(end-nHist+1:end,:); sim_DE];  %  hist. + sim. data

figure('units','normalized','outerposition',[0 0 1 1])
    plot(simDates, data2plot, '--', 'LineWidth',2), ...
    hold on, grid, 'on';
    plot(simDates(1:nHist,1), Y_DE(end-nHist+1:end,:), '-', ...
                                         'LineWidth',2)
    %title('Forecasting German Yields the Incorrect way'), ...
    set(gca, 'FontSize', 20)
    datetick('x','mmm-yy')
    print -depsc WrongProjections

figure('units','normalized','outerposition',[0 0 1 1])
```

```
    subplot(3,1,1), plot(tauDE,sim_DE(17,:),'o-'), ...
        ylim([-3 0]), grid,'on'; ...
                           title(datestr(simDates(17+nHist),'mmm-yy'))
    xticks(tauDE), xticklabels({tauDE}),
    set(gca, 'FontSize', 18)
    subplot(3,1,2), plot(tauDE,sim_DE(36,:),'o-'), ...
         ylim([-2.0 0.0]), grid,'on'; ...
                           title(datestr(simDates(36+nHist),'mmm-yy'))
    xticks(tauDE), xticklabels({tauDE}),
    set(gca, 'FontSize', 18)
    subplot(3,1,3), plot(tauDE,sim_DE(42,:),'o-'), ...
         ylim([-4 -0.5]), grid,'on'; ...
                           title(datestr(simDates(42+nHist),'mmm-yy'))
    xticks(tauDE), xticklabels({tauDE})
    set(gca, 'FontSize', 18)
    print -depsc FunnySimYields


%% A first look at factor models
%
[G_US, F_US, eig_US]    = pca(Y_US);      % run factor analysis on US data
[G_DE, F_DE, eig_DE]    = pca(Y_DE)       % run factor analysis on US data

[ cumsum(eig_US./sum(eig_US)) cumsum(eig_DE./sum(eig_DE)) ]

nF     = 2;
Spread = Y_US-Y_DE;                       % the pure spread in percentage
[G_Sprd, F_Sprd, eig_Sprd] = pca(Spread); % run factor analysis on US data

figure('units','normalized','outerposition',[0 0 1 1])
    subplot(2,1,1), plot(tauUS,G_US(:,1:nF),'o-', ...
        'LineWidth',2),  ylim([-1 1]),  title('US loading structure'),
    xticks(tauUS),xticklabels(tauUS'),
    ylabel('Value'), grid, 'on'; set(gca, 'FontSize', 20)
    subplot(2,1,2), plot(tauDE,G_Sprd(:,1:nF),'o-', ...
        'LineWidth',2), ylim([-1 1]), title('Spread loading structure'),
    xticks(tauDE), xticklabels(tauDE'),
    ylabel('Value'), grid, 'on'; set(gca, 'FontSize', 20)
    print -depsc EmpiricalLoadingStructures

%% Joint model for DE and US yields
%
Y2     = [ Y_DE Y_US ];                   % collecting the relevant yield segments
G_mdl  = [ G_DE(:,1:nF) zeros(nTau,nF);  % loading structure joint model
           G_DE(:,1:nF) G_Sprd(:,1:nF)] ;
F_mdl  = G_mdl\Y2';
Y2_hat = (G_mdl*F_mdl)';                  % fitted yield curves
err    = Y2-Y2_hat;                       % fitting errors

RMSE_bps = 100*(mean(err.^2)).^(1/2);    % RMSE in basis points

Tab_rmse = array2table(round(RMSE_bps)); % just for the display of output
for (j=1:nTau)
    Tab_rmse.Properties.VariableNames(1,j)      = {strcat(['DE',num2str(tauDE(j,1))])};
```

```
    Tab_rmse.Properties.VariableNames(1,j+nTau) = {strcat(['US',num2str(tauUS(j,1))])};
end
disp(Tab_rmse)
disp(round([ min(RMSE_bps) max(RMSE_bps) ]))
%% Comparing observed and fitted yields
figure('units','normalized','outerposition',[0 0 1 1])
subplot(2,2,1), plot(dates,[Y2(:,1) Y2_hat(:,1)], 'LineWidth',2), ...
                set(gca, 'FontSize', 20)
                title(Tab_rmse.Properties.VariableNames(1,1)), ...
                datetick('x','yyyy'),
                legend('Obs','Fitted','Location','northeast')

subplot(2,2,2), plot(dates,[Y2(:,6) Y2_hat(:,6)], 'LineWidth',2), ...
                set(gca, 'FontSize', 20)
                title(Tab_rmse.Properties.VariableNames(1,6)), ...
                datetick('x','yyyy'),
                legend('Obs','Fitted','Location','northeast')

subplot(2,2,3), plot(dates,[Y2(:,7) Y2_hat(:,7)], 'LineWidth',2), ...
                set(gca, 'FontSize', 20)
                title(Tab_rmse.Properties.VariableNames(1,7)), ...
                datetick('x','yyyy'),
                legend('Obs','Fitted','Location','northeast')

subplot(2,2,4), plot(dates,[Y2(:,12) Y2_hat(:,12)], 'LineWidth',2), ...
                set(gca, 'FontSize', 20)
                title(Tab_rmse.Properties.VariableNames(1,12)), ...
                datetick('x','yyyy'),
                legend('Obs','Fitted','Location','northeast')
                print -depsc EvaluatingJointModel


%% Adding dynamics to the model
%
maxLags = 6;
aic_bic = zeros(maxLags,2);
for ( j=1:maxLags )
    Mdl_    = varm(nF*2,j);
    Mdl_est = estimate(Mdl_,F_mdl');
    Info_   = summarize(Mdl_est);
    aic_bic(j,:) = [ Info_.AIC Info_.BIC ];
end
disp('Optimal lag-order according to:')
disp('    AIC   BIC ')
disp( [find(min(aic_bic(:,1))==aic_bic(:,1)) ...
       find(min(aic_bic(:,2))==aic_bic(:,2))])

Mdl_dynamics          = varm(nF*2,1);
Est_dynamics          = estimate(Mdl_dynamics, F_mdl');
sort(real(eig(Est_dynamics.AR{:,:})))

end_     = datenum('31-May-2013');   % end-date of the est data sample
indx_e   = find(dates==end_,1,'first')-1;
horizon_ = 6;                        % projection horizon
```

```
nCast     = nObs-indx_e-horizon_;     % number of times to re-estimate
err_mdl   = NaN(horizon_+1,nTau*2,nCast);  % container for the output
err_rw    = NaN(horizon_+1,nTau*2,nCast);  % container for the random-walk
Mdl_cast = varm(nF*2,1);
figure
for ( j=1:nCast )
    % estimate on expanding data window
    est_tmp   = estimate(Mdl_cast, F_mdl(:,1:indx_e+j)');
    % forecast VAR model
    F_cast    = forecast(est_tmp,horizon_,F_mdl(:,1:indx_e+j)');
    % forecast random-walk
    F_rw      = repmat(F_mdl(:,indx_e+j-1)',horizon_+1,1);
    Y_obs     = Y2(indx_e+j:indx_e+j+horizon_,:);
    F_cast    = [ F_mdl(:,indx_e+j)'; F_cast ];
    Y_cast    = (G_mdl*F_cast')';  % convert forecasted factors into yields
    Y_rw      = (G_mdl*F_rw')';    % convert random projections into yields
    err_mdl(:,:,j) = (Y_obs-Y_cast)*100;
    err_rw(:,:,j)  = (Y_obs-Y_rw)*100;
end
tab_Fcast_mdl_rmse = array2table( round( (mean(err_mdl.^2,3)).^(1/2)) );
tab_Fcast_mdl_rmse.Properties.VariableNames = Tab_rmse.Properties.VariableNames;
tab_Fcast_mdl_rmse.Properties.RowNames      = {'Fitted', 'Forecast 1m ahead', ...
                                               'Forecast 2m ahead', 'Forecast 3m ahead', ...
                                               'Forecast 4m ahead', 'Forecast 5m ahead', ...
                                               'Forecast 6m ahead'};

tab_Fcast_rw_rmse  = array2table( round( (mean(err_rw.^2,3)).^(1/2)) );
tab_Fcast_rw_rmse.Properties.VariableNames = tab_Fcast_mdl_rmse.Properties.VariableNames;
tab_Fcast_rw_rmse.Properties.RowNames      = tab_Fcast_mdl_rmse.Properties.RowNames;
disp(tab_Fcast_mdl_rmse)
disp(tab_Fcast_rw_rmse)

%% Creating the expanded loading matrix
%
tau_new     = sort([tauDE;[0;48;108]]);
nTau_new    = length(tau_new);
% inter- and extra-polation of the DE loadings
G_DE_ext   = interp1(tauDE,G_mdl(1:nTau,1:2),tau_new,'pchip');
% inter- and extra-polation of the Spread loadings
G_DE_Sprd = interp1(tauDE,G_mdl(nTau+1:end,3:4),tau_new,'pchip');
% expanded loading matrix
G_sim       = [ G_DE_ext zeros(nTau_new,2); G_DE_ext G_DE_Sprd ];

figure('units','normalized','outerposition',[0 0 1 1])
    subplot(2,1,1), plot(tau_new, G_sim(1:nTau_new,1:2),'b*-', ...
        'LineWidth',2)
    hold on, ylim([-1 1])
    subplot(2,1,1), plot(tauDE, G_mdl(1:nTau,1:2),'r*'),
    title('Expanded loadings: DE')
    set(gca, 'FontSize', 20), xticks(tau_new), xticklabels({tau_new})
    grid, 'on';
    subplot(2,1,2), plot(tau_new, G_sim(nTau_new+1:end,3:4),'b*-', ...
        'LineWidth',2)
    hold on, ylim([-1 1])
```

```
    subplot(2,1,2), plot(tauDE, G_mdl(nTau+1:end,3:4),'r*'),
    title('Expanded loadings: Spread')
    set(gca, 'FontSize', 20), xticks(tau_new), xticklabels({tau_new})
    grid, 'on';
    print -depsc ExpandedLoadingMatrix

%% Calculates return distributions
% defines the end-date of the first data sample
end_       = datenum('31-Dec-2018');
indx_e     = find(dates==end_,1,'first');
horizon_   = 12;                            % simulation horizon
nSim       = 1e4;                           % number of simulation paths
nAssets    = length(tau_new)-length(tauDE);  % number of points on the curve
                                        %    for which returns are generated
Sim_Ret    = NaN(nSim, nAssets);        % container for the simulated returns
Mdl_       = varm(nF*2,1);
% estimate the VAR model on the selected data
est_Mdl    = estimate(Mdl_, F_mdl(:,1:indx_e)');
Y0         = repmat(F_mdl(:,indx_e)',nSim,1);
F_sim1     = repmat(F_mdl(:,indx_e)',1,1,nSim);
% Simulated paths for the factors
F_sim2     = simulate(est_Mdl, horizon_, 'Y0', Y0, 'NumPaths', nSim);
% combining obs and simulated data
F_sim3     = cat(1,F_sim1,F_sim2);
% transposing first two dimensions
F_sim      = permute(F_sim3,[2 1 3]);
% container for simulated yields
Y_sim      = NaN(2*nTau_new,horizon_+1,nSim);% dim: Tau x horizon x sim_path
% container for the simulated annual returns
R_sim      = NaN(nSim,nAssets*2);
e1         = [3;6;9;12;15;18];     % indicator for relevant maturity points
e2         = [1;5;8;10;14;17];     %    at time t, and t+1
tau_ret    = [tau_new;tau_new]./12;

for ( j=1:nSim )                                   % calculating returns
    Y_sim(:,:,j) = G_sim*squeeze(F_sim(:,:,j));
    R_sim(j,:)   = (tau_ret(e1,1).*squeeze(Y_sim(e1,1,j)) - ...
                    tau_ret(e2,1).*squeeze(Y_sim(e2,horizon_+1,j)))';
end
ret_Tab = array2table([ round(mean(R_sim).*100)./100; ...
                    round(std(R_sim)*100)./100 ]);  % organising results
ret_Tab.Properties.VariableNames = ...
                          Tab_rmse.Properties.VariableNames(1,(2:2:12));
ret_Tab.Properties.RowNames = [{'Mean'};{'Std.'}];
disp('Summary of the simulated return distributions')
disp(ret_Tab)

figure('units','normalized','outerposition',[0 0 1 1])
    subplot(2,2,1), histfit(R_sim(:,2),50,'Normal'), ...
       set(gca, 'FontSize', 20), title(ret_Tab.Properties.VariableNames(2))
    subplot(2,2,2), histfit(R_sim(:,3),50,'Normal'), ...
       set(gca, 'FontSize', 20), title(ret_Tab.Properties.VariableNames(3))
    subplot(2,2,3), histfit(R_sim(:,5),50,'Normal'), ...
       set(gca, 'FontSize', 20), title(ret_Tab.Properties.VariableNames(5))
```

```
  subplot(2,2,4), histfit(R_sim(:,6),50,'Normal'), ...
     set(gca, 'FontSize', 20), title(ret_Tab.Properties.VariableNames(6))
   print -depsc ReturnDistributions
```

2

The $\mathbb{P}$ and $\mathbb{Q}$ measures

2.1 Introduction

It is impossible to escape a treatment of the $\mathbb{P}$ and $\mathbb{Q}$ measures. Even if we choose only to rely on models that do not impose arbitrage restrictions, such as e.g. the Nelson-Siegel family (among others, Nelson and Siegel (1987), Diebold and Li (2006)), and Diebold and Rudebusch (2013)) we need as a minimum to appreciate what we are missing (and gaining), such that our modelling choice is made in full consciousness. The main here is to bring into sharper focus the elements that are necessary for gaining an intuitive and practical understanding of the difference between the $\mathbb{P}$ and $\mathbb{Q}$ measures. In my opinion, this is sufficient for "blue-collar" yield-curve implementation work, i.e. the work that ensures the correct implementation of existing models in the context of financial decision support frameworks.[1]

2.2 Switching between measures

One of the central principles of financial theory is that asset prices (of equities, bonds, business projects, and so on) can be found as the sum of the discounted expected future cashflow stream, where the discount rate is set to match the riskiness of the cashflows being discounted. The risk adjustment is done by adding an appropriate risk premium to the discount rate, i.e. the discounting is done using $1 + r_t + \theta$, where r_t is the risk-free rate and θ is the market-determined equilibrium risk-premium, scaled by the risk of the cashflows in question. Another key insight is that financial option pricing does not fit immediately into this framework.[2] The main reason for this is that these assets have asymmetric pay-off schedules, and our traditional pricing tool-kits can only risk-adjust assets that have symmetric pay-off distributions.[3]

[1] In-depth treatments of the topics touched upon in this section can be found in e.g. Karatzas and Shreve (1996) and Mikosch (1998).

[2] You may wonder why I am bringing financial option pricing into play here, when the focus of attention is purely on fixed income pricing and yield curve modelling. But, please bear with me, I hope it will become clear.

[3] Think of how you would find the appropriately risk-adjusted discount rate, using the CAPM or APT, for pricing a call-option on the SP500 index. To determine the β of the call option in the CAPM world, we would need the covariance between the call-option's return (pay-offs) and the return on the market portfolio: how do we calculate

To solve this dilemma, Black, Scholes, and Merton, came up with a clever scheme where the cashflows, as opposed to the discount rate, undergo a risk adjustment. This adjustment is achieved by weighting the state-contingent cashflows by a new set of probabilities, drawn from a new probability distribution, such that the *expected* value of the cashflows can be discounted using the risk free rate (term structure). Since the risk free rate is used for discounting, the distribution and the accompanying probability measure, can be called *risk neutral*. This probability measure is also referred to as the *pricing measure*, because observed/theoretical prices are obtained using the adjusted probability distribution. After all, the correct pricing of financial options was the primary motivation behind the ideas developed by Black, Scholes, and Merton, so *pricing measure* seems like a very appropriate name. In the financial option pricing literature, as well as in the term structure literature, it has become common practice to associate the risk-neutral pricing measure with the letter $\mathbb{Q}$, and the historical/empirical measure by the letter $\mathbb{P}$.

The idea of adjusting the size of the cashflows to reflect the euro amount a risk averse investor would accept, instead of taking on a risky bet, is also known from introductory investment science text books, as the "certain-equivalent cashflow method". Often tucked away in an appendix, this method is presented as a way to determine a reference value for new products, or the premium companies should offer to entice new investors and make them participate in new equity or bond offerings. So, one way to see the $\mathbb{Q}$-measure is as an equilibrium solution to the certain-equivalent cashflow adjustment process: a $\mathbb{Q}$ distribution assigns risk-adjusted probabilities to each possible cashflow outcome-combination for the assets that exist in the economy, such that all assets are priced correctly. This means that any asset that is priced in the economy, can be written in the following way:

$$P_t = e^{-r_t} \cdot \mathbb{E}_t^{\mathbb{Q}}\left[P_{t+1}\right] = e^{-r_t} \int_S c_{t+1}(s+1) \cdot f_t^{\mathbb{Q}}(s+1) ds(t+1), \tag{2.1}$$

where P is the price, r is the risk free rate, and $c(s)$ is the cashflow in the possible (continuous) states$=s, \ldots, S$ of the world, e.g. $c_t \sim N(\mu, \sigma^2)$, and $f^{\mathbb{Q}}$ gives the accompanying (pricing) probability density function. Since we are dealing with risk-free bonds, it is also known that $P_0 = 1$, i.e. that all bonds repay their principal at the maturity date.

Since risk averse investor pay extra attention to outcomes of the world that they see as being undesirable (risky), the $\mathbb{Q}$-distribution is effectively a shifted/skewed version of the $\mathbb{P}$-distribution, where more probability mass is allocated to negative states of the world. We can write the relationship between the distributions in the following way:

$$f_t^{\mathbb{Q}}(s_{t+1}) = f_t^{\mathbb{P}}(s_{t+1}) \cdot \mathcal{R}_t(s_{t+1}), \tag{2.2}$$

the covariance between a variable that has a pay-off of the form $max(0, S - X)$ (the option) and the market portfolio that can assumed to be normally distributed? Pursuing this question is not necessarily a meaningful endeavour.

where $\mathcal{R}$ is the risk-adjustment function that financial market participants agree on, and which therefore becomes embedded in observed prices.[4]

The more pessimistic (risk averse) the financial market participants are, at a given point in time, the more attention (weight) is given to bad states of the world. But, what are these bad, or undesirable, outcomes, that demand a risk premium? The general answer is: states where the prices turn out to be low. For equities we would therefore expect the mean of the $\mathbb{Q}$ distribution to be lower than that of the $\mathbb{P}$ distribution. Conversely, if we look at fixed income markets, and our focus is on the yield curve, we would expect the mean of the $\mathbb{Q}$ distribution to be higher than that of the $\mathbb{P}$ distribution, given the inverse relation ship between bond prices and yields. This type of reasoning is of course only valid, when the risk premium is positive. If investors, for example, regard government bonds as a safe-heaven asset, then they are willing to pay a premium to acquire such securities, and the risk premium will, consequently, turn negative.

Participants in fixed-income markets market will require compensations for risk factors that may lead to yield increases. And, the higher the risk that yields increase, the higher the premium. So, if we first consider the shape of the term-structure of term premia, it is natural to expect that it is upward sloping in the maturity dimension: The higher the duration of the bond, the more exposed it is to yield developments, compared to a bond with lower maturity, over the same holding period. Secondly, it is reasonable to consider the economic factors that impact the yield curve, and which therefore demand a risk premium. For default-free bonds, the relevant factors are: the rate of economic growth, and the inflation rate. Uncertainty surrounding the future evolution of these macro gauges will therefore impact fixed-income term premia. Investors may also require compensation for holding illiquid bonds, that is bonds that may take longer time to sell than the investor would like to spend on this activity - here the compensation is of course not for the time spend, but for the adverse price movement that may materialise during the time it takes to find a buyer for the bond.

Some bonds are also exposed to credit risk. The issuer of the bond may be subjected to a credit-downgrade, whereby the bonds will trade at lower prices, because they are now priced off a new and higher yield curve. A down-grade action by rating agencies will typically be expected by market participants so the down-ward drift in market prices will to some extent happen before the rating agencies' official announcement. Rating down-grades are not the only possible credit event. It is also possible that the issuer defaults. In this case, the bond holders will receive a certain recover percentage, depending on the prices, at which the available assets can be sold.

In summary, investors require compensations for having exposure to the following systematic risk factors:

- the economic growth rate
- the inflation rate
- credit migration risk

[4] The function $\mathcal{R}$ is also called the Radon-Nikodym derivative, and it is assumed that $\mathcal{R}$ obey the conditions necessary such that $f^{\mathbb{Q}}$ behaves like and can be interpreted as a probability density function.

- default risk
- liquidity risk

However, in the remaining part of the booklet we will deal exclusively with credit and liquidity risk free bonds.

2.3 A simplified empirical example

Later on we will introduce the commonly used parametrisation of the market price of risk in the context of yield curve modelling, and go more into detail. For now, a simplified example is used to illustrate the idea.[5] Assume that fixed-income prices are governed by a single factor, the short rate, and that an AR(1) model gives a good characterisation of the dynamic behaviour of this factor:

$$r_t = c^{\mathbb{P}} + \alpha^{\mathbb{P}} \cdot r_{t-1} + \sigma \cdot e_t, \tag{2.3}$$

where r is the annualised three-month short rate, $c^{\mathbb{P}}$ is a constant, $\alpha^{\mathbb{P}}$ is the autoregressive coefficient, σ is the volatility of the process, and $e \sim N(0,1)$. As is evident, the model is written under the empirical $\mathbb{P}$-measure, and in passing, it is noticed that this set-up is similar to a discrete-time version of Vasicek (1977):

$$\Delta r_t = a \cdot (b - r_{t-1}) + \sigma \cdot e_t, \tag{2.4}$$

$$\Updownarrow$$

$$r_t = c^{\mathbb{P}} + \alpha^{\mathbb{P}} \cdot r_{t-1} + \sigma \cdot e_t \tag{2.5}$$

with the parameter-mapping, $\alpha^{\mathbb{P}} = 1 - a$, and $c^{\mathbb{P}} = a \cdot b$. We will return to the Vasicek (1977) model, below in Section 2.4.

If we use (2.3) together with (2.1), we can obtain $\mathbb{P}$- and $\mathbb{Q}$-measure price expressions for a τ maturity bond. First, the recursive structure of (2.1) is used to obtain:

$$\begin{aligned}
\tilde{P}_t^{\tau} &= \mathbb{E}_t^{\mathbb{P}}\left[e^{-r_t \Delta t} \cdot \tilde{P}_{t+1}^{\tau-1}\right] \\
&= \mathbb{E}_t^{\mathbb{P}}\left[e^{-r_t \Delta t} \cdot e^{-r_{t+1} \Delta t} \cdot \tilde{P}_{t+2}^{\tau-2}\right] \\
&= \mathbb{E}_t^{\mathbb{P}}\left[e^{-r_t \Delta t} \cdot e^{-r_{t+1} \Delta t} \cdot e^{-r_{t+2} \Delta t} \cdot \tilde{P}_{t+3}^{\tau-3}\right] \\
&= \dots
\end{aligned}$$

and because $\tilde{P}_T^0 = 1$, i.e. the bond repays its principal at maturity, this expression generalises to:

[5] For the more traditional exposition using a binomial tree and the portfolio-replication strategy to derive the risk neutral probabilities, see e.g. Hull (2006), Rebonato (2018), and Luenberger (1998)

$$\tilde{P}_t^{\tau} = \mathbb{E}_t^{\mathbb{P}} \left[e^{-\sum_t^{\tau} r_t \Delta t} \right], \tag{2.6}$$

and by similarity, we can write:

$$P_t^{\tau} = \mathbb{E}_t^{\mathbb{Q}} \left[e^{-\sum_t^{\tau} r_t \Delta t} \right]. \tag{2.7}$$

Using monthly observations for the three-month maturity point on the US risk-free zero-coupon term structure, covering the period from 1961 to 2018, the following $\mathbb{P}$-measure parameter estimates are obtained:[6]

	Estimate
$c^{\mathbb{P}}$	0.0763
$\alpha^{\mathbb{P}}$	0.9943
$\sigma^{\mathbb{P}}$	0.5886

Table 2.1. $\mathbb{P}$-measure estimates

Based on (2.3) the comparable $\mathbb{P}$-measure prices, $\tilde{P}_t^{\tau}$ can be calculated, with $\Delta t = 1/12$ (because we use a monthly observation frequency), and using the parameter estimates in Table 2.1. The good thing is that with the above set-up, i.e. using the assumption of an AR(1) model for the short rate, there is a closed-form solutions to the sum of the short rate that enters in equation (2.6):

$$\sum_t^{\tau} r_t = r_t \cdot \frac{1 - \alpha^{\tau}}{1 - \alpha} + \frac{c \cdot (\alpha^{\tau} - \alpha \cdot \tau + \tau - 1)}{(\alpha - 1)^2} \tag{2.8}$$

Note that the superscript on the parameters are omitted in the above expression because it is valid for any AR(1) model following the general notation used in equation (2.3).

With this, it is now possible to calculate $\mathbb{P}$-prices and compare them to observed $\mathbb{Q}$-prices, in order to gauge the size of the risk premium. When we use term structure models in practise, and apply them to observed market yields, it is easy to forget that yields are a by-product of the trading process: Investors observe market prices, and the trading commences until prices reach equilibrium, i.e. until all investors agree that the price is right (even if this moment is only a micro-second). However, we model yields and not prices, and we are therefore used to thinking about the risk premium in yield-space (and we will continue doing so below), but, in fact, the risk adjustment enters the stage through the pricing process, and is therefore originally a pricing concept, as also outlined above. Before reverting to our normal yield-thinking-mode, it may still be illustrative to see the risk premium as it materialises in price-space - even if this is only done using example prices.

[6] here we are using the data contained in the MATLAB file: Data_GSW_factors_Course_2018.mat.

On a randomly selected day, zero-coupon bond prices are sampled from the US market, see the row labelled $P^{\mathbb{Q}}$ in Table 2.2. Prices are sampled across the maturity spectrum, covering three- to 120 months. The next row in the table gives the corresponding $\mathbb{P}$-prices, i.e. the prices that would prevail if equation (2.6) together with the parameter estimates shown in Table 2.1, were used to price the bonds . The difference between the two price rows is the risk premium, i.e. the compensation that investors require to hold bonds at different maturities, here given in price-space.

τ in months	3	12	24	36	48	60	72	84	96	108	120
$P^{\mathbb{Q}}$ (Eur)	99.12	96.22	92.29	88.56	84.99	81.53	78.11	74.71	71.35	68.05	64.82
$P^{\mathbb{P}}$ (Eur)	99.48	97.76	95.14	92.24	89.17	85.98	82.73	79.47	76.22	73.03	69.90
Price of risk (Eur)	0.36	1.54	2.84	3.69	4.17	4.45	4.62	4.75	4.87	4.98	5.08

Table 2.2. $\mathbb{P}$- and $\mathbb{Q}$ prices and the price of risk, on a randomly selected day

Once the price of risk has been calculated in euro terms, we can fiddle with the parameters of the dynamic evolution of the yield curve factor in (2.3), such that we match the observed market prices as closely as possible. That is, we aim to find appropriate values for $c^{\mathbb{Q}}$ and $\alpha^{\mathbb{Q}}$, from this equation:

$$r_t = c^{\mathbb{Q}} + \alpha^{\mathbb{Q}} \cdot r_{t-1} + \sigma \cdot e_t, \tag{2.9}$$

This "appropriate adjustment" constitutes the risk-adjustment in yield space, and we will see later on, how exactly to map parameters between the two measures - for now this link is (intentionally) left to be vague.

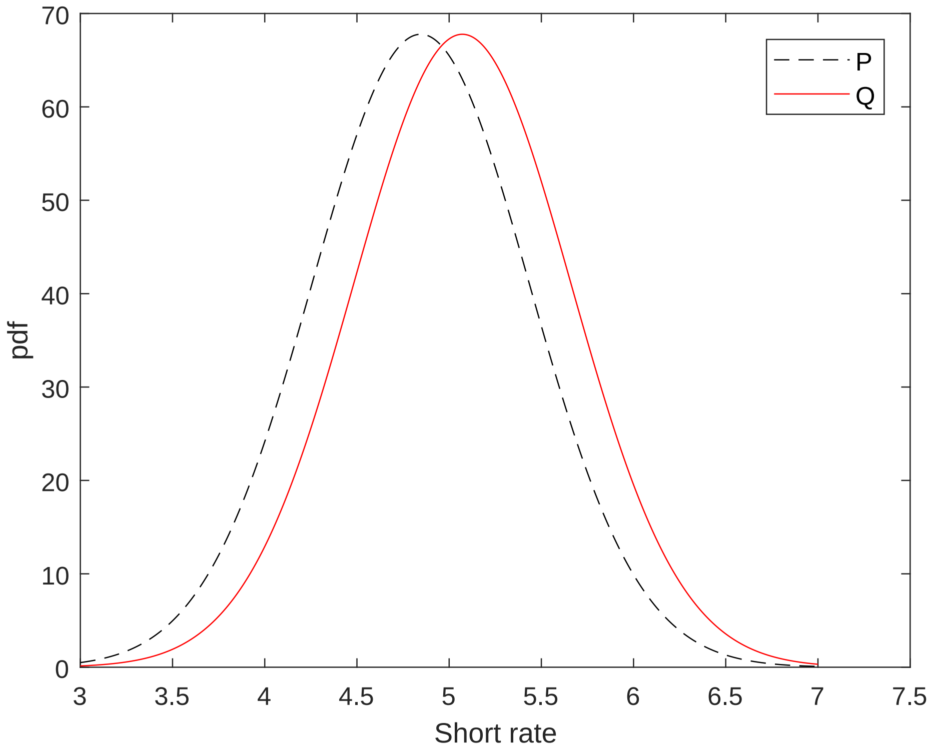
In our example, when the parameter-tinkering is done, we can draw the resulting distributions for the short rate, see Figure 2.1. Since the $\mathbb{Q}$-distribution falls to the right of the $\mathbb{P}$-distribution, it appears that a positive risk premium is present in the sampled data.

It is worth emphasising again, that the above is just an example. In general, we would not calibrate models using more observations than what was used above; in fact, models would typically be fitted to match a whole panel of yields covering no less than ten years of monthly time series observations, where each monthly observation would cover several maturity points.

2.4 A generic discrete-time one-factor model

A discrete-time one-factor model is presented here, as a prelude to the multi-factor models that we will concentrate on for the most part of the remainder of this booklet. The model below can be seen as the discrete-time counterpart of Vasicek (1977).

As above, we assume that the underlying factor driving the yield curve is the short rate, and that the short rate is governed by a stationary AR(1) process:



The figure shows an example of the relationship between the $\mathbb{P}$- and $\mathbb{Q}$ measure distributions. Only the mean differs between the two measures in this example.

Fig. 2.1. Example $\mathbb{P}$- and $\mathbb{Q}$-distributions on a randomly selected day

$$r_t = c^{\mathbb{P}} + \alpha^{\mathbb{P}} \cdot r_{t-1} + \sigma \cdot e_t^{\mathbb{P}}. \tag{2.10}$$

The bond price is an exponential affine function of the short rate:

$$P_t^\tau = exp\left(A_\tau + B_\tau \cdot r_t\right), \tag{2.11}$$

and we can therefore write the yield at maturity τ as:

$$y_t^\tau = -\frac{1}{\tau} \cdot log\left(P_t^\tau\right) = -\frac{A_\tau}{\tau} - \frac{B_\tau}{\tau} \cdot r_t. \tag{2.12}$$

In order for bond prices to exclude arbitrage opportunities, a single stochastic discount factor (SDF, also called the pricing kernel) is assumed to exist, and to price all bonds (and other asset in the economy):

$$P_t^\tau = E_t^{\mathbb{P}}\left[M_{t+1} \cdot P_{t+1}^{\tau-1}\right], \tag{2.13}$$

it is typically assumed that the SDF is parameterised in the following way:

$$M_{t+1} = exp\left(-r_t - \frac{1}{2}\lambda_t^2 - \lambda_t e_{t+1}^{\mathbb{P}}\right), \tag{2.14}$$

and that:

$$\lambda_t = \lambda_0 + \lambda_1 r_t, \tag{2.15}$$

Armed with these prerequisites, the fun can begin. By inserting (2.14) and (2.11) into (2.13), we get:

$$\begin{aligned} P_t^{\tau} &= E_t^{\mathbb{P}}\left[exp\Big(-r_t - \frac{1}{2}\lambda_t^2 - \lambda_t e_{t+1}^{\mathbb{P}}\Big) \cdot exp\Big(A_{\tau-1} + B_{\tau-1} \cdot r_{t+1}\Big)\right] \\ &= E_t^{\mathbb{P}}\left[exp\Big(-r_t - \frac{1}{2}\lambda_t^2 - \lambda_t e_{t+1}^{\mathbb{P}} + A_{\tau-1} + B_{\tau-1} \cdot r_{t+1}\Big)\right] \end{aligned} \tag{2.16}$$

into which we substitute (2.10):

$$P_t^{\tau} = E_t^{\mathbb{P}}\left[exp\Big(-r_t - \frac{1}{2}\lambda_t^2 - \lambda_t e_{t+1}^{\mathbb{P}} + A_{\tau-1} + B_{\tau-1} \cdot \Big(c^{\mathbb{P}} + \alpha^{\mathbb{P}} \cdot r_t + \sigma e_{t+1}^{\mathbb{P}}\Big)\right]. \tag{2.17}$$

We now separate terms into two groups, one related to the future, i.e. $t+1$, where the expectations operator is needed, and another group, which are know at time t, and where the expectations operator is therefore not needed:

$$\begin{aligned} P_t^{\tau} &= E_t^{\mathbb{P}}\left[exp\Big(-r_t - \frac{1}{2}\lambda_t^2 - \lambda_t e_{t+1}^{\mathbb{P}} + A_{\tau-1} + B_{\tau-1} \cdot \Big(c^{\mathbb{P}} + \alpha^{\mathbb{P}} \cdot r_t + \sigma e_{t+1}^{\mathbb{P}}\Big)\right] \\ &= E_t^{\mathbb{P}}\left[exp\Big(-r_t - \frac{1}{2}\lambda_t^2 - \lambda_t e_{t+1}^{\mathbb{P}} + A_{\tau-1} + B_{\tau-1} \cdot c^{\mathbb{P}} + B_{\tau-1} \cdot \alpha^{\mathbb{P}} \cdot r_t + B_{\tau-1} \cdot \sigma e_{t+1}^{\mathbb{P}}\Big)\right] \\ &= exp\Big(-r_t - \frac{1}{2}\lambda_t^2 + A_{\tau-1} + B_{\tau-1} \cdot c^{\mathbb{P}} + B_{\tau-1} \cdot \alpha^{\mathbb{P}} \cdot r_t\Big) \\ &\qquad \cdot E_t^{\mathbb{P}}\left[exp\Big(-\lambda_t \cdot e_{t+1}^{\mathbb{P}} + B_{\tau-1} \cdot \sigma \cdot e_{t+1}^{\mathbb{P}}\Big)\right] \\ &= exp\Big(-r_t - \frac{1}{2}\lambda_t^2 + A_{\tau-1} + B_{\tau-1} \cdot c^{\mathbb{P}} + B_{\tau-1} \cdot \alpha^{\mathbb{P}} \cdot r_t\Big) \\ &\qquad \cdot E_t^{\mathbb{P}}\left[exp\Big(\big(-\lambda_t + B_{\tau-1} \cdot \sigma\big) \cdot e_{t+1}^{\mathbb{P}}\Big)\right]. \end{aligned} \tag{2.18}$$

Since $\mathbb{E}\left[exp(aX)\right] = exp\left(\frac{1}{2}a^2\right)$ when $X \sim N(0,1)$, the expectations part of (2.18) can be written as:

$$\begin{aligned} E_t^{\mathbb{P}}\left[exp\Big(\big(-\lambda_t + B_{\tau-1} \cdot \sigma\big) \cdot e_{t+1}^{\mathbb{P}}\Big)\right] &= exp\left[\frac{1}{2}\left(-\lambda_t + B_{\tau-1} \cdot \sigma\right)^2\right] \\ &= exp\left[\frac{1}{2}B_{\tau-1}^2\sigma^2 - B_{\tau-1}\lambda_t\sigma + \frac{1}{2}\lambda_t^2\right]. \end{aligned} \tag{2.19}$$

The derived expression for the expectation part (2.19) can now be reinserted into (2.18)

$$
\begin{aligned}
P_t^\tau &= exp\Big(- r_t - \frac{1}{2}\lambda_t^2 + A_{\tau-1} + B_{\tau-1} \cdot c^{\mathbb{P}} + B_{\tau-1} \cdot \alpha^{\mathbb{P}} \cdot r_t \\
&\qquad\qquad + \frac{1}{2}B_{\tau-1}^2\sigma^2 - B_{\tau-1}\lambda_t\sigma + \frac{1}{2}\lambda_t^2\Big) \\
&= exp\Big(- r_t + A_{\tau-1} + B_{\tau-1} \cdot c^{\mathbb{P}} + B_{\tau-1} \cdot \alpha^{\mathbb{P}} \cdot r_t \\
&\qquad\qquad + \frac{1}{2}B_{\tau-1}^2\sigma^2 - B_{\tau-1}\lambda_t\sigma\Big).
\end{aligned} \tag{2.20}
$$

Recall the expression for the market price of risk, shown in equation (2.15). Insert it in (2.20), and collect terms related to r_t:

$$
\begin{aligned}
P_t^\tau &= exp\Big(- r_t + A_{\tau-1} + B_{\tau-1} \cdot c^{\mathbb{P}} + B_{\tau-1} \cdot \alpha^{\mathbb{P}} \cdot r_t + \frac{1}{2}B_{\tau-1}^2\sigma^2 - B_{\tau-1}\lambda_t\sigma\Big) \\
&= exp\Big(- r_t + A_{\tau-1} + B_{\tau-1} \cdot c^{\mathbb{P}} + B_{\tau-1} \cdot \alpha^{\mathbb{P}} \cdot r_t + \frac{1}{2}B_{\tau-1}^2\sigma^2 - B_{\tau-1}\sigma(\lambda_0 + \lambda_1 \cdot r_t)\Big) \\
&= exp\Big(A_{\tau-1} + B_{\tau-1} \cdot c^{\mathbb{P}} + B_{\tau-1} \cdot \alpha^{\mathbb{P}} \cdot r_t + \frac{1}{2}B_{\tau-1}^2\sigma^2 - B_{\tau-1}\sigma(\lambda_0 + \lambda_1 \cdot r_t) - r_t\Big) \\
&= exp\Big(A_{\tau-1} + B_{\tau-1} \cdot c^{\mathbb{P}} + \frac{1}{2}B_{\tau-1}^2\sigma^2 - B_{t-1}\sigma\lambda_0 \\
&\qquad\qquad + B_{\tau-1} \cdot \alpha^{\mathbb{P}} \cdot r_t - B_{\tau-1}\sigma\lambda_1 \cdot r_t - r_t\Big) \\
&= exp\Big(A_{\tau-1} + B_{\tau-1} \cdot c^{\mathbb{P}} - B_{t-1}\sigma\lambda_0 + \frac{1}{2}B_{\tau-1}^2\sigma^2 \\
&\qquad\qquad + \left(B_{\tau-1} \cdot \alpha^{\mathbb{P}} - B_{\tau-1}\sigma\lambda_1 - 1\right) \cdot r_t\Big) \\
&= exp\Big(A_{\tau-1} + B_{\tau-1}\left(c^{\mathbb{P}} - \sigma\lambda_0\right) + \frac{1}{2}B_{\tau-1}^2\sigma^2 \\
&\qquad\qquad + \left[B_{\tau-1}\left(\alpha^{\mathbb{P}} - \sigma\lambda_1\right) - 1\right] \cdot r_t\Big).
\end{aligned} \tag{2.21}
$$

It is obviously unnecessary to include all the intermediate steps in the above derivation, but for completeness, it is done anyway.

Matching coefficients between equations (2.11) and (2.21), it is seen that:

$$
\begin{aligned}
A_\tau &= A_{\tau-1} + B_{\tau-1}\left(c^{\mathbb{P}} - \sigma\lambda_0\right) + \frac{1}{2}B_{\tau-1}^2\sigma^2 \\
&= A_{\tau-1} + B_{\tau-1}c^{\mathbb{Q}} + \frac{1}{2}B_{\tau-1}^2\sigma^2
\end{aligned} \tag{2.22}
$$

$$
\begin{aligned}
B_\tau &= B_{\tau-1}\left(\alpha^{\mathbb{P}} - \sigma\lambda_1\right) - 1 \\
&= B_{\tau-1}\alpha^{\mathbb{Q}} - 1
\end{aligned} \tag{2.23}
$$

First, notice the nice interpretation of the constant and the autoregressive coefficient. When excluding arbitrage opportunities, by imposing a common risk-adjusted pricing equation for all assets that trade in the economy, see equation (2.13), the coefficients that determine the dynamics of the yield curve factor, r_t, under the market-pricing measure $\mathbb{Q}$, are being risk adjusted. We see that: $c^{\mathbb{Q}} = c^{\mathbb{P}} - \sigma\lambda_0$, and $\alpha^{\mathbb{Q}} = \alpha^{\mathbb{P}} - \sigma\lambda_1$, appear as the $\mathbb{Q}$-measure parameters, where σ has an interpretation as the amount of risk, and $\lambda_{0,1}$ can be interpreted as the price of risk. Second, the expressions for A_τ and B_τ have iterative structures, such that A_τ depends on $A_{\tau-1}$, and B_τ depends on $B_{\tau-1}$. This structure is no coincidence. It emerges as a natural consequence of the imposed sequential nature of the above pricing equation. With this structure, it is now possible to derive closed-form expressions for these parameters.

Starting with the general expression for B_τ in (2.23), gives:

$$
\begin{aligned}
B_1 &= B_0\alpha^{\mathbb{Q}} - 1 \\
B_2 &= B_1\alpha^{\mathbb{Q}} - 1 = \left(B_0\alpha^{\mathbb{Q}} - 1\right)\alpha^{\mathbb{Q}} - 1 = B_0\left(\alpha^{\mathbb{Q}}\right)^2 - \alpha^{\mathbb{Q}} - 1 \\
B_3 &= B_2\alpha^{\mathbb{Q}} - 1 = \left(B_0\left(\alpha^{\mathbb{Q}}\right)^2 - \alpha^{\mathbb{Q}} - 1\right)\alpha^{\mathbb{Q}} - 1 = B_0\left(\alpha^{\mathbb{Q}}\right)^3 - \left(\alpha^{\mathbb{Q}}\right)^2 - \alpha^{\mathbb{Q}} - 1 \\
B_4 &= B_0\left(\alpha^{\mathbb{Q}}\right)^4 - \left(\alpha^{\mathbb{Q}}\right)^3 - \left(\alpha^{\mathbb{Q}}\right)^2 - \alpha^{\mathbb{Q}} - 1.
\end{aligned} \tag{2.24}
$$

When the bond matures its price is $P_t^0 = exp\left(A_0 + B_0 \cdot r_t\right) = 1$, which implies that $A_0 = 0$ and $B_0 = 0$. The above expression therefore generalises in the following way:

$$
\begin{aligned}
B_\tau &= -\sum_{j=0}^{\tau-1}\left(\alpha^{\mathbb{Q}}\right)^j \\
&= -\frac{1-\left(\alpha^{\mathbb{Q}}\right)^\tau}{1-\alpha^{\mathbb{Q}}}
\end{aligned} \tag{2.25}
$$

where the last line results from the closed-form expression of the summed power-series. Doing the same exercise for the A_τ term, now that B_τ is known, gives:

$$
\begin{aligned}
A_\tau = &-\frac{c^{\mathbb{Q}}}{1-\alpha^{\mathbb{Q}}} \cdot \left[\tau - \frac{1-\left(\alpha^{\mathbb{Q}}\right)^\tau}{1-\alpha^{\mathbb{Q}}}\right] \\
&+ \frac{\sigma^2}{2\left(1-\alpha^{\mathbb{Q}}\right)^2} \cdot \left[\tau + \frac{1-\left(\alpha^{\mathbb{Q}}\right)^{2\tau}}{1-\left(\alpha^{\mathbb{Q}}\right)^2} - 2 \cdot \frac{1-\left(\alpha^{\mathbb{Q}}\right)^\tau}{1-\alpha^{\mathbb{Q}}}\right]
\end{aligned} \tag{2.26}
$$

Given the relationship between bond prices and yields in (2.12), the resulting yield equation for the discrete-time one-factor model can be written as:

$$
\begin{aligned}
y_t^\tau &= -\frac{1}{\tau}A_\tau - \frac{1}{\tau}B_\tau r_t + \sigma_y u_t \\
&= a_\tau + b_\tau r_t + \sigma_y u_t
\end{aligned} \tag{2.27}
$$

with

$$
\begin{aligned}
a_\tau &= \frac{c^{\mathbb{Q}}}{\tau\left(1-\alpha^{\mathbb{Q}}\right)} \cdot \left[\tau - \frac{1-\left(\alpha^{\mathbb{Q}}\right)^{\tau}}{1-\alpha^{\mathbb{Q}}}\right] \\
&\quad - \frac{\sigma^2}{2\tau\left(1-\alpha^{\mathbb{Q}}\right)^2} \cdot \left[\tau + \frac{1-\left(\alpha^{\mathbb{Q}}\right)^{2\tau}}{1-\left(\alpha^{\mathbb{Q}}\right)^2} - 2 \cdot \frac{1-\left(\alpha^{\mathbb{Q}}\right)^{\tau}}{1-\alpha^{\mathbb{Q}}}\right] \qquad (2.28) \\
b_\tau &= \frac{1-\left(\alpha^{\mathbb{Q}}\right)^{\tau}}{\tau\left(1-\alpha^{\mathbb{Q}}\right)}. \qquad (2.29)
\end{aligned}
$$

2.4.1 Estimating the short-rate model

Using example data collected from the US market, we can estimate the above derived model. To this end MATLAB's state-space toolbox (SSM) is used. Since the model relies on the short rate to be the underlying factor, that drives the dynamics of the model, it is assumed that the three-month rate can play this role. And, the model is therefore parameterised such that the yield curve factor is observed.

The model looks like this:

$$
\underbrace{r_t}_{1\times 1} = \underbrace{c^{\mathbb{P}}}_{1\times 1} + \underbrace{\alpha^{\mathbb{P}}}_{1\times 1} \cdot \underbrace{r_{t-1}}_{1\times 1} + \underbrace{\sigma_r}_{1\times 1} \underbrace{e_t}_{1\times 1} \qquad (2.30)
$$

$$
\underbrace{Y_t}_{\#\tau\times 1} = \underbrace{a_\tau}_{\#\tau\times 1} + \underbrace{b_\tau}_{\#\tau\times 1} \cdot \underbrace{r_t}_{1\times 1} + \underbrace{\Sigma_y}_{\#\tau\times\#\tau} \underbrace{u_t}_{\#\tau\times 1}, \qquad (2.31)
$$

with (2.30) being the state equation, and (2.31) being the observation equation, and with the dimension of the variables and parameters provided in brackets under the respective entries. Here $\#\tau$ refers to the number of maturities, at which the yield curve is observed, at a give point in time. To set up the model in MATLAB's SSM toolbox requires a bit of reworking of the model, such that it fits into the required format. Indeed, it is required that the equations of the model match the following generic set-up:

$$
\begin{aligned}
&\texttt{state equation:} && X_t = R \cdot X_{t-1} + S \cdot e_t \\
&\texttt{observation equation:} && Y_t = T \cdot X_t + U \cdot u_t.
\end{aligned}
$$

To align the one-factor model with this, the following is done for the state equation:

$$
\underbrace{\begin{bmatrix} r_t \\ \hdashline 1_t \end{bmatrix}}_{X_t} = \underbrace{\left[\begin{array}{c:c} \alpha^{\mathbb{P}} & c^{\mathbb{P}} \\ \hdashline 0 & 1 \end{array}\right]}_{R} \cdot \underbrace{\begin{bmatrix} r_{t-1} \\ \hdashline 1_{t-1} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} \sigma_r \\ \hdashline 0 \end{bmatrix}}_{S} e_t,
$$

where $1_t = 1$ is a constant that is equal to 1 for all values of t. The observation equation takes the following form:

$$\underbrace{\begin{bmatrix} y_t \\ \hdashline r_t \\ \hdashline 1_t \end{bmatrix}}_{Y_t} = \underbrace{\left[\begin{array}{c:c} b_\tau & a_\tau \\ \hdashline 1 & 0 \\ \hdashline 0 & 1 \end{array}\right]}_{T} \cdot \underbrace{\begin{bmatrix} r_t \\ \hdashline 1_t \end{bmatrix}}_{X_t} + \underbrace{\begin{bmatrix} \Sigma_y \\ \hdashline 0 \\ \hdashline 0 \end{bmatrix}}_{U} u_t.$$

It is well-known that a one-factor model is not flexible enough to capture both the time- and cross sectional behaviour of yields. In fact, it appears that when one factor models are used in the industry, they are applied to fit the yield curve at a given point in time, and while the model parameters should be stable over time, they are in reality not, so models are frequently re-estimated. There is therefore not much hope for the practical usefulness of the above state-space model, however, as an example, it is useful to carry on.

As a complement to the state-space approach, we can also explore the possibility that the amount of risk in the economy, is time-varying. With the intention to be as practical as possible, a two-step estimation approach is pursued to identify the relevant parameters of this model. First, the $\mathbb{P}$ dynamics is estimated, and the resulting parameter estimates are kept constant during the second stage of the estimation procedure. Second, the $\mathbb{Q}$ parameters are estimated (subject to the estimates obtained in step one). For the sake of clarity, it is recalled that the first step takes care of the time-series dimension of the data, while the second step is concerned with the cross sectional fit of the model, i.e. with the maturity dimension.

`step 1:`

$$r_t = c^{\mathbb{P}} + \alpha^{\mathbb{P}} \cdot r_{t-1} + \sigma_t e_t \tag{2.32}$$

$$\sigma_t^2 = \omega + \kappa\sigma_{t-1}^2 + \gamma e_t^2 \tag{2.33}$$

Having obtained the parameter estimates: $\hat{c}^{\mathbb{P}}$, $\hat{\alpha}^{\mathbb{P}}$, and the time series of time-varying variances, $\hat{\sigma}_t^2\ \forall t$, the next step can be completed:

`step 2:`

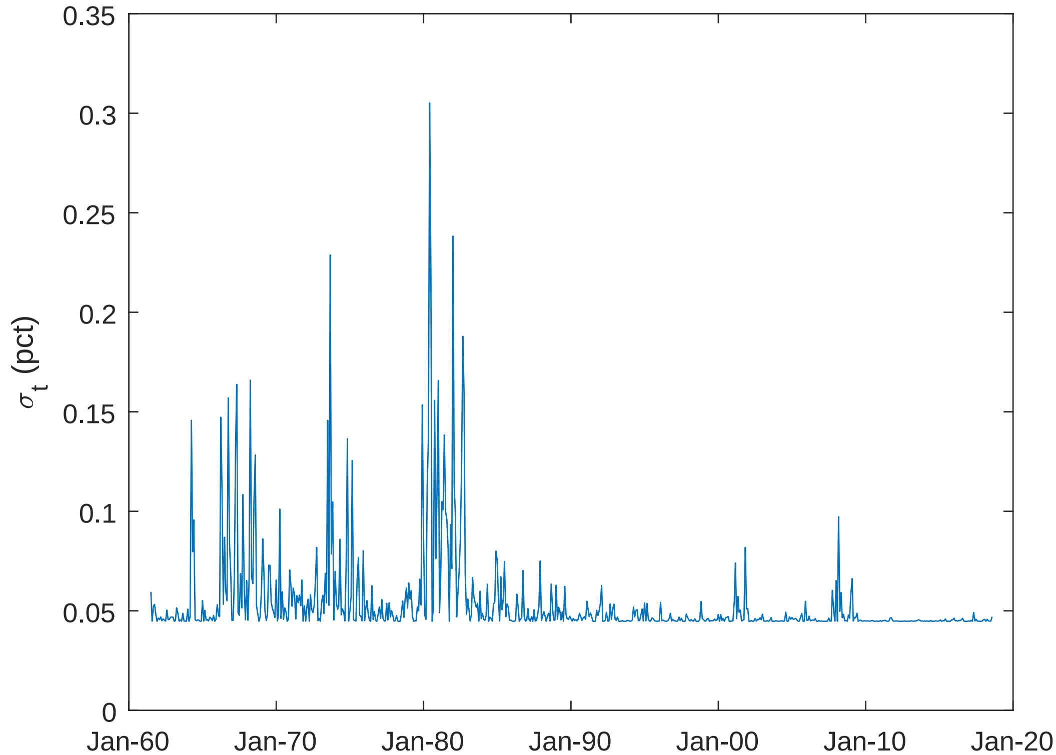
$$a_{\tau,t} = f\left(\hat{c}^{\mathbb{P}}, \hat{\alpha}^{\mathbb{P}}, \hat{\sigma}_t, \lambda_0, \lambda_1, \tau\right) \tag{2.34}$$

$$b_{\tau,t} = f\left(\hat{\alpha}^{\mathbb{P}}, \hat{\sigma}_t, \lambda_1, \tau\right) \tag{2.35}$$

which amounts to estimating the market price of risk parameters, λ_0 and λ_1. This can be done by minimising the sum of the squared errors between model and observed yields.

$$min_{\{\lambda_0,\lambda_1\}} = \sum_t \sum_\tau \left[Y - \hat{Y}\right]^2 = \sum_t \sum_\tau \left[Y - (a_\tau + b_\tau \cdot r_t)\right]^2. \tag{2.36}$$

The results from the above two estimation approaches and model specifications are sketched below.[7] The model fits are compared to that of a completely empirically determined one-factor model, where the factor is the observed short rate.[8]



The figure shows the estimated time-varying volatility of the short rate factor, obtained from am AR(1)-GARCH(1,1) model applied to monthly data.

Fig. 2.2. Time-varying volatility of the short rate factor

As anticipated, figure 2.3 shows that all three one-factor models have difficulties in matching the time series evolution of yields. In fact, the average root mean squared error of the models, across all maturities, is of a similar magnitude and far too large to qualify these models as applicable to capture both the time-series and cross-sections behaviour of yields. This observation underscores the usefulness of such one-factor

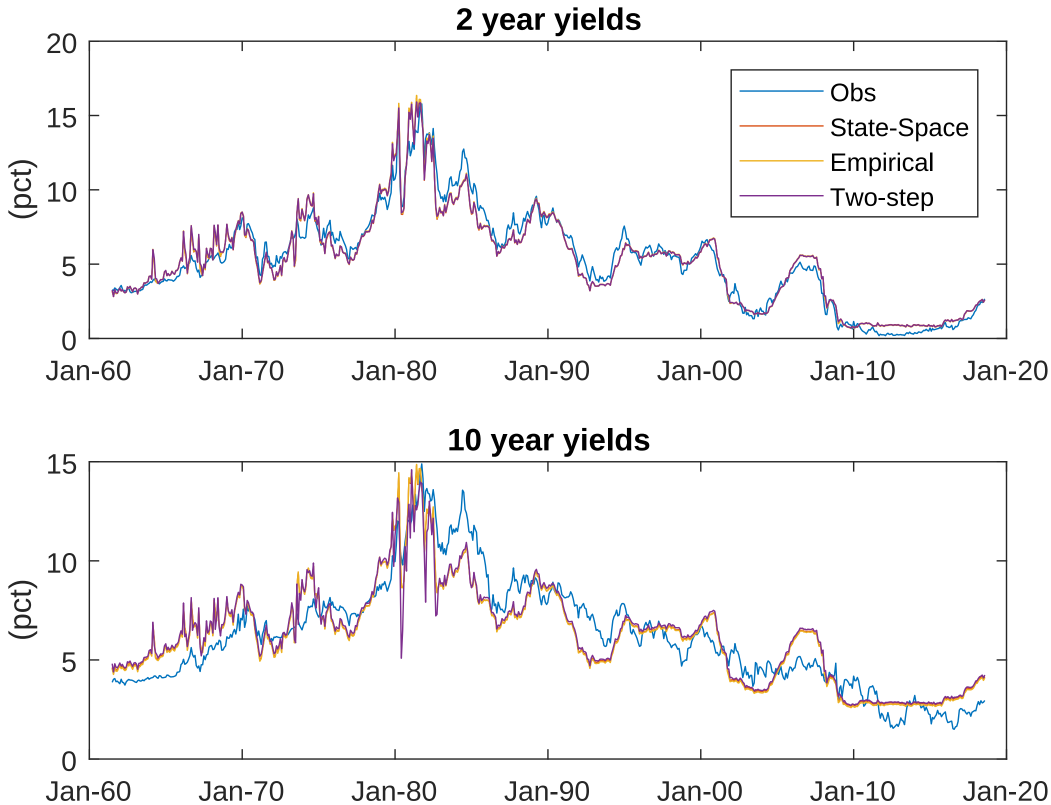
---

[7] The results are generated using the MATLAB scripts named: "P_and_Q_Measure_Vasicek_2_- step_approach.m" and "P_and_Q_Measure_Vasicek_State_Space.m", that accompany the booklet. See the Appendix MATLAB code, for a print of the code.

[8] This empirical model is estimated in MATLAB like this:

```
%% Empirical one-factor short-rate model
%
F     = [ones(nObs,1) RDNS.yields(:,1)];
H     = F\RDNS.yields;
Y_fit = F*H;
RMSE = (mean((RDNS.yields - Y_fit).^2)).^(0.5);
```

where Y contains the data for the 3-months to 10-year yields. Each column of Y contains the time-series observations for one maturity. Hence, the first column holds the short rate (the 3-months yield).



The figure shows the 2-year and 10-year observed and fitted yields. The models described in the text are used and comprise discrete-time Vasicek models estimated using a state-space approach, a simple empirical approach, and a two-step procedure allowing for time-varying volatility.

Fig. 2.3. Model fit

models as a means to fit the prevailing term-structure, on a day-to-day or intraday basis, e.g. for pricing purposes and for detecting rich and cheap bonds at a given time-point. For that purpose, such models are great.

2.5 Summary

The main objective of this section is to illustrate, in an intuitive a practical way, what we mean, when we refer to the $\mathbb{P}$ and $\mathbb{Q}$ measures in the context of yield curve models. This topic can be a stumbling block, and a source of confusion, when entering into this literature the first time. Sure, we can read, accept, and replicate what is written in text books and academic papers, but it may be difficult to discern from this a true sense of understanding. As mentioned in the introduction, my goal here is a modest one. But, I hope, after all, that the above may help to illustrate, and thereby further the understanding, of the $\mathbb{P}$ and $\mathbb{Q}$ measures, and the derived yield curve modelling frameworks.

(1) The arbitrage constraint amounts to assuming and imposing the existence of a unique pricing equation on the market being modelled. By pricing all assets that trade on this market, the unique pricing equation ensures that model prices for all assets are consistent with their exposure to the risk factors included in the model.

(2) Zero coupon bonds are priced under the $\mathbb{Q}$ measure as the discounted value of its terminal payment, i.e. the payment the bond makes when it matures, using the risk free rate as the discount rate. The recursive structure of the discounting approach, together with the unique pricing equation, implies that the loading structure, i.e. the matrix that converts yield curve factors into model yields, also can be found via a set of recursive equations.

(3) For many models it is possible to sole these recursive equations in closed-form, which makes model estimation faster.

(4) It is possible to find arbitrage-free counterparts to many of the yields curve models typically used by practitioners. For example, the Nelson-Siegel model and the Svensson-Soderlind models.

(5) When used appropriately, there is not much difference between arbitrage-free models, and models that do not impose arbitrage constraints. Still, the arbitrage-free models represent internally consistent frameworks and gives additional information about the market being modelled, for example, on the market prices of factor risks.

2.6 Appendix: MATLAB code

2.6.1 A discrete-time Vasicek model: state-space estimation

filename: P_and_Q_Measure_Vasicek_State_Space.m

```
%% State-space estimation of the Vasicek model
% Used in the section: the P and Q measures of the lecture notes

%% Load yield factors and construct yield curves
%
load('Data_GSW_factors_Course_2018.mat');
GSW_         = GSW;                  % creates an instance of the GSW class
GSW_.tau     = [3 12:12:120]';       % vector of maturities
GSW_.beta    = GSW_factors(:,2:5);   % yield curve factors
GSW_.lambda  = GSW_factors(:,6:7);   % lambdas
GSW_         = GSW_.getYields;       % getting yields
figure
    plot(GSW_factors(:,1),GSW_.yields(:,[1 11]));
    datetick('x','mmm-yy'), title('US yields'), legend('3m','10y')

RDNS             = TSM;              % creates an instance of the TSM class
RDNS.yields      = GSW_.yields;      % adds yields to the model
RDNS.tau         = GSW_.tau;         % adds maturities
RDNS.biasCorrect = 0;
RDNS.DataFreq    = 12;
RDNS.nF          = 3;
RDNS             = RDNS.getSRB3;     % estimates a 3 factor SRB model
figure
    plot(GSW_factors(:,1), RDNS.beta'),
    title('Extracted yield curve factors')
    datetick('x','mmm-yy'),
    legend('Short rate','Slope','Curvature')
figure
    plot(GSW_factors(:,1),[RDNS.beta(1,:)' RDNS.yields(:,1)]),
    title('Model and Observed short rate'),
    datetick('x','mmm-yy'), legend('Model','Observed')
figure
    plot(GSW_factors(:,1),[RDNS.TP(:,11) ACM(:,2)]),
         title('10Y Term Premium'),
         datetick('x','mmm-yy'), legend('SRB','ACM')

[nObs,nTau] = size(RDNS.yields);

%% Estimating the parameters of the discrete-time one-factor model
% Data are scaled to monthly decimals (percentage annual yields are
% converted to monthly decimal rates, because the formulas for the
% yield curve loadings are calculated for monthly step-sizes and thus
% for monthly rates.
%
scl_        = 1200;
Y           = [ RDNS.yields./scl_ ...
                RDNS.yields(:,1)./scl_ ...
```

```
                 ones(nObs,1) ];

cP   = 0.01;
aP   = 0.95;
s    = 1.15;
L0   = 0;
L1   = 0;
sY   = 1.25.*ones(nTau,1);

p0   = [ cP; aP; s; L0; L1; sY ];
lb_  = [ 0.00; 0.00; 0; -inf; -inf; zeros(nTau,1) ];
ub_  = [ 1.00; 1.00; 1;  inf; inf; 1000.*ones(nTau,1) ];

% constraints that ensure that all yield volatilities, ie. the
% entries of the variance-covariance matrix in the observation
% equation are equal for all maturities included in the analysis.
%
nP   = size(p0,1);
Aeq = zeros(nTau-1,nP);
Aeq(1,[6 7])   = [1 -1];Aeq(2,[7 8])  =[1 -1];Aeq(3,[8 9])  = [1 -1];
Aeq(4,[9 10])  = [1 -1];Aeq(5,[10 11])=[1 -1];Aeq(6,[11 12])= [1 -1];
Aeq(7,[12 13]) = [1 -1];Aeq(8,[13 14])=[1 -1];Aeq(9,[14 15])= [1 -1];
Aeq(10,[15 16])= [1 -1];
beq = zeros(size(Aeq,1),1);

Mdl_sr  = ssm(@(p) pMap(p, RDNS.tau));
options = optimoptions(@fmincon,'Algorithm','interior-point',...
                                'MaxIterations',1e6, ...
                                'MaxFunctionEvaluations',1e6, ...
                                'TolFun', 1e-6, 'TolX', 1e-6);

disp('... Estimating the model using the SSM module')
[ EstMdl_sr, pHat, pCov, logl, outFlags ] = ...
         estimate( Mdl_sr,Y,p0,'Display','iter','Aeq',Aeq,'beq',beq,...
                  'lb',lb_,'ub',ub_,'univariate',true,'options',options )

x_filter  = filter( EstMdl_sr, Y );  % extract filtered state variables
sr_filter = x_filter(:,1);           % filtered short rate

cP_  = pHat(1,1);
aP_  = pHat(2,1);
s_   = pHat(3,1);
L0_  = pHat(4,1);
L1_  = pHat(5,1);
sY_  = pHat(6:end,1);
mP   = (cP_/(1-aP_))*scl_;

a_tau_ = EstMdl_sr.C(1:nTau,2);
b_tau_ = EstMdl_sr.C(1:nTau,1);
Y_fit  = (a_tau_ + b_tau_*sr_filter')';

RMSE  = 100.*(mean((scl_.*Y(:,1:11)-scl_.*Y_fit).^2)).^(1/2)

figure
```

```
    plot(GSW_factors(:,1),[sr_filter Y(:,12)]),
    title('Yield curve factor')
    datetick('x','mmm-yy'), legend('obs','fit')
figure
    plot(GSW_factors(:,1),[Y_fit(:,1) Y(:,1)]),
    title('3M rate')
    datetick('x','mmm-yy'), legend('fit','obs')
figure
    plot(GSW_factors(:,1),[Y_fit(:,2) Y(:,2)]),
    title('1Y rate')
    datetick('x','mmm-yy'), legend('fit','obs')
figure
    plot(GSW_factors(:,1),[Y_fit(:,3) Y(:,3)]),
    title('2Y rate')
    datetick('x','mmm-yy'), legend('fit','obs')
figure
    plot(GSW_factors(:,1),[Y_fit(:,4) Y(:,4)]),
    title('3Y rate')
    datetick('x','mmm-yy'), legend('fit','obs')
figure
    plot(GSW_factors(:,1),[Y_fit(:,5) Y(:,5)]),
    title('4Y rate')
    datetick('x','mmm-yy'), legend('fit','obs')
figure
    plot(GSW_factors(:,1),[Y_fit(:,6) Y(:,6)]),
    title('5Y rate')
    datetick('x','mmm-yy'), legend('fit','obs')
figure
    plot(GSW_factors(:,1),[Y_fit(:,11) Y(:,11)]),
    title('10Y rate')
    datetick('x','mmm-yy'), legend('fit','obs')

%%
function [R,S,T,U,Mean0,Cov0,StateType] = pMap( p, tau )
%
% Setting up the matrices necessary to estimate the state-space model
%
nTau_1 = length(tau);
nTau   = max(tau);

cP  = p(1,1);
aP  = p(2,1);
s   = p(3,1);
L0  = p(4,1);
L1  = p(5,1);
sY  = p(6:end,1);

cQ  = cP - s*L0;
aQ  = aP - s*L1;

[ a_tau, b_tau ] = find_a_b(s,cQ,aQ,tau);

R = [ aP cP; 0 1 ];
S = [ s; 0 ];
```

```
T = [ b_tau a_tau; 1 0; 0 1 ];
U = [ diag(sY); zeros(2,nTau_1) ];

% ... other assignments
    Mean0     = [];
    Cov0      = [];
    StateType = [ 0; 1 ];
end

function [a_n, b_n] = find_a_b(s,cQ,aQ,tau)
% determines the loadings and the constant vector using the
% recursive equations and closed form expressions.
%
flagg = 1;    % 1-> closed form results, 0->iterative solution

a_nF = @(a_,n_,c_,s_) -c_/(1-a_)*( n_ - (1-a_.^n_)/(1-a_) )...
                      +(s_^2)/(2*((1-a_)^2)).*(n_ + ...
                      (1-a_.^(2*n_))./(1-a_^2) - 2*(1-a_.^n_)./(1-a_) );
b_nF = @(a_,n_) -(1-a_.^n_)./((1-a_));

if (flagg==0)
    nTau = max(tau(:));
    ttau = (1:1:nTau)';
    a_t  = zeros(nTau,1);
    b_t  = zeros(nTau,1);
    for (j=2:nTau+1)
        b_t(j,1) = b_t(j-1,1)*aQ - 1;
        a_t(j,1) = a_t(j-1,1) + b_t(j-1,1)*cQ - 0.5*s^2*(b_t(j-1,1))^2;
    end
    a_n = -a_t(tau+1,1)./tau;
    b_n = -b_t(tau+1,1)./tau;
else
    a_n = -a_nF(aQ,tau,cQ,s)./tau;
    b_n = -b_nF(aQ,tau)./tau;
end
end
```

2.6.2 A discrete-time Vasicek model: two-step estimation procedure

filename: P_and_Q_Measure_Vasicek_2_step_approach.m

```
%% Two-step estimation procedure for the discrete-time Vasicek model
% Used in the section: the P and Q measures of the lecture notes

%% Load yield factors and construct yield curves
%
load('Data_GSW_factors_Course_2018.mat');
GSW_        = GSW;                          % instance of the GSW class
GSW_.tau    = [3 12:12:120]';               % vector of maturities
GSW_.beta   = GSW_factors(:,2:5);           % yield curve factors
GSW_.lambda = GSW_factors(:,6:7);           % lambdas
GSW_        = GSW_.getYields;               % getting yields
figure
    plot(GSW_factors(:,1),GSW_.yields(:,[1 11]));
    datetick('x','mmm-yy'), title('US yields'), legend('3m','10y')

RDNS             = TSM;                     % instance of the TSM class
RDNS.yields      = GSW_.yields;             % adds yields to the model
RDNS.tau         = GSW_.tau;                % adds maturities
RDNS.biasCorrect = 0;
RDNS.DataFreq    = 12;
RDNS.nF          = 3;
RDNS             = RDNS.getSRB3;            % est. a 3 factor SRB model
figure
    plot(GSW_factors(:,1), RDNS.beta'),
    title('Extracted yield curve factors')
    datetick('x','mmm-yy'),
    legend('Short rate','Slope','Curvature')
figure
    plot(GSW_factors(:,1),[RDNS.beta(1,:)' RDNS.yields(:,1)]),
    title('Model and Observed short rate')
    datetick('x','mmm-yy'), legend('Model','Observed')
figure
    plot(GSW_factors(:,1),[RDNS.TP(:,11) ACM(:,2)]),...
    title('10Y Term Premium')
    datetick('x','mmm-yy'), legend('SRB','ACM')

[nObs,nTau] = size(RDNS.yields);
%% Time-varying volatility and the Vasicek model
% Below we implement a two-step approach to estimating the Vasicek model
% with time-varying volatility, as outlined in the lecture notes.
%
Y   = RDNS.yields./1200;   % US Yields in decimal form
tau = RDNS.tau;            % for maturities 3, 12:12:120 months
%
% ... Step 1
Sr            = Y(:,1);                          % 3-month rate = short rate
Mdl_AR_garch  = arima('ARLags',1,'Variance',garch(1,1), ...
                       'Distribution','Gaussian'); % AR(1)-GARCH(1,1) model
Est_AR_garch  = estimate(Mdl_AR_garch,Sr);         % estimate the model
```

```
[eps,s2]       = infer(Est_AR_garch,Sr);            % extract cond. variances
cP             = Est_AR_garch.Constant;
aP             = Est_AR_garch.AR{:};
s              = sqrt(s2);

%
% ... Step 2
p0  = [0;0];
lb_ = [-100;-100];
ub_ = [ 100; 100];
%
% minimise the squared residuals defined in the function
%    Est_Vasicek - see below
[pHat,fval,flagg,output,lamb_,G_,H_] = fmincon(@Est_Vasicek,p0,...
                                                    [],[],[],[],lb_,ub_,...
                                                    [],[],Y,s,cP,aP,tau,Sr)

[err2,Y_hat,a_tau,b_tau] = Est_Vasicek(pHat,Y,s,cP,aP,tau,Sr);
Y_hat = 12.*Y_hat;
RMSE  = 10000.*(mean((12.*Y-Y_hat).^2)).^(1/2);

figure
    plot(GSW_factors(:,1),sqrt(s2))
    datetick('x','mmm-yy')
    ylabel('\sigma_t')
%   print -depsc P_Q_distribution

function [err2,Y_hat,a_tau,b_tau] = Est_Vasicek(p,Y,s,cP,aP,tau,Sr)
% This function calculates the difference between model and observed
% yields that can be used to estimate the parameters $\lambda_0$ and
% $\lambda_1$
%
nObs = size(s,1);
nTau = max(tau);

a_nF = @(a_,n_,c_,s_) -c_/(1-a_)*( n_ - (1-a_.^n_)/(1-a_) )...
                      +(s_^2)/(2*((1-a_)^2)).*(n_ + ...
                      (1-a_.^(2*n_))./(1-a_^2) - 2*(1-a_.^n_)./(1-a_) );
b_nF = @(a_,n_) -(1-a_.^n_)./((1-a_));

L0 = p(1);
L1 = p(2);
cQ = cP - L0.*s;
aQ = aP - L1.*s;
a_tau = NaN(size(tau,1),nObs);
b_tau = NaN(size(tau,1),nObs);
Y_hat = NaN(nObs,size(tau,1));
for (j=1:nObs)
   a_tau(:,j) = -a_nF( aQ(j,1), tau, cQ(j,1), s(j,1) )./tau;
   b_tau(:,j) = -b_nF( aQ(j,1), tau )./tau;
   Y_hat(j,:) = (a_tau(:,j) + b_tau(:,j)*Sr(j,1))';
end
err2  = sum(sum((Y-Y_hat).^2));
end
```

3

The Basic Yield Curve Modelling Set-up

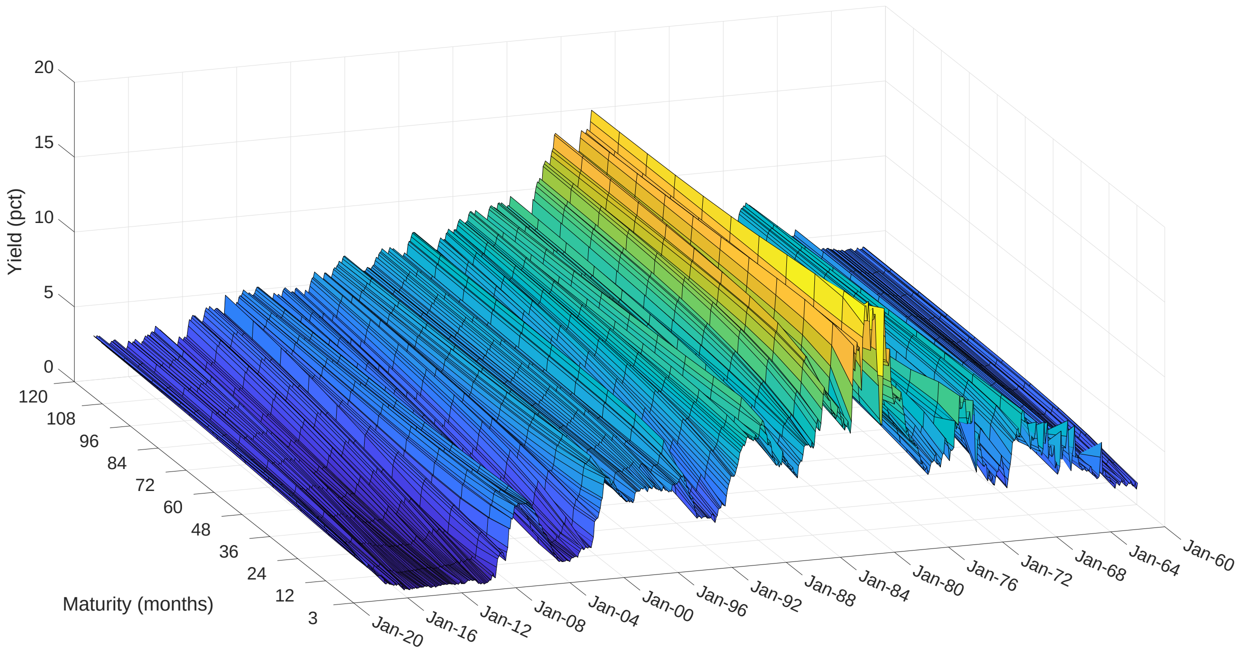
3.1 Introduction

Our staring point is the empirical observation that yields observed across the maturity spectrum are highly cross-correlated, and that their time series dynamics tend to exhibit some degree of autocorrelation. A good and practical modelling approach may therefore by to stack yields at different maturities, but observed at the same point in time, in a vector, and to collect all the vectors into a single panel of yield observations. The question is now, how do we model such a panel of correlated data points in a parsimonious way, while ensuring that as much of the information relevant to us is preserved? In coming up with an answer to this question, we will pursue a route that is purely empirically founded; the treatment of the arbitrage free pricing set-up will follow in later chapters. Here we will mainly follow the modelling ideas of Litterman and Scheinkman (1991), Nelson and Siegel (1987), Diebold and Li (2006), and in general, Diebold and Rudebusch (2013). In terms of estimation techniques, both state-space approaches and two-step OLS will be covered.

3.2 The factor structure of yields

Let Y be a data set of yield curve observations covering time and maturity dimensions. Figure 3.1 shows an example of what Y can look like. The shown data are US zero-coupon yields, observed at a monthly frequency for the period from June 1961 to July 2018, and covering maturities from 3 to 120-months. As in other parts of this booklet, these are the data we will work with.

To illustrate further, Y can be sliced in two dimensions (obviously!): a single slice of Y in the maturity dimension, Y_t, contains yield observations at different maturity points, at the date where the slice is carved out of the data set; in other words, Y_t constitutes a yield curve observed at time t. We can also slice the data in the date dimension, and then collect the time series observations of a given maturity point on the yield curve. These two ways of slicing Y are illustrated in Figure 3.2.



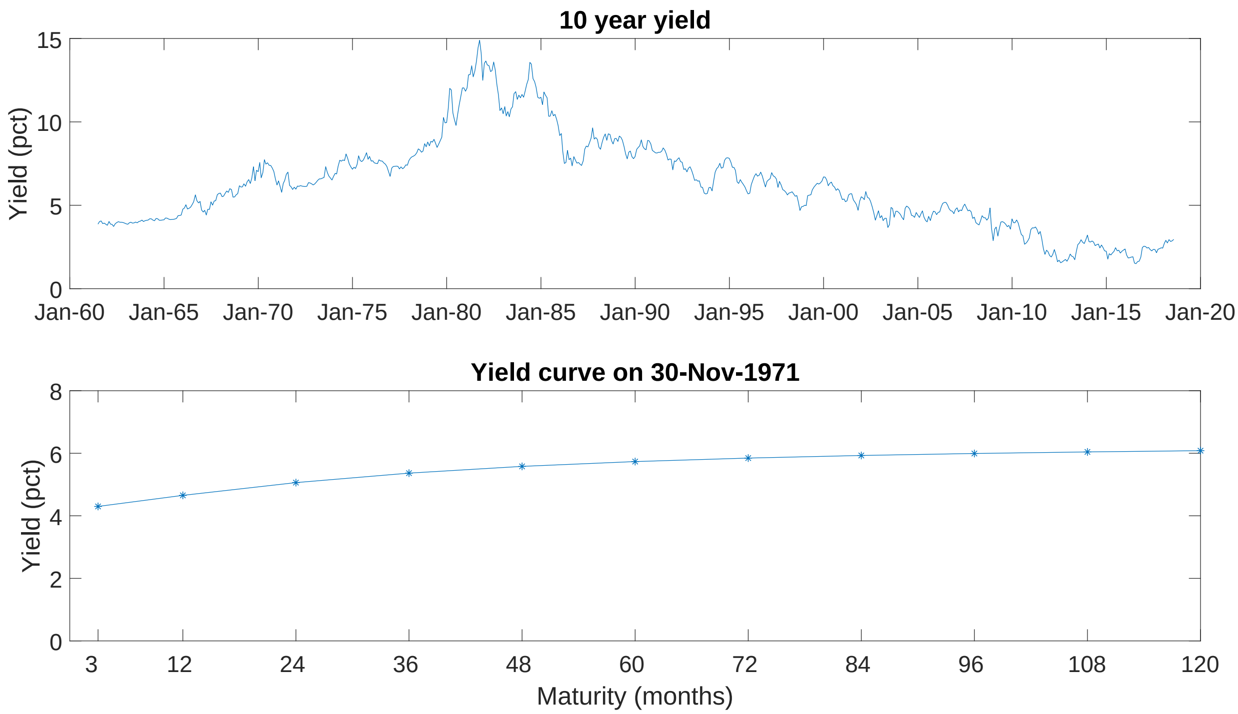
The figure shows the US example data used throughout the booklet. Monthly yield data are observed since 1961 to 2018, for maturities from 3-120 months. These data are from Gurkaynak, Sack, and Wright (2006) and made available and updated by the Federal Reserve Board.

Fig. 3.1. Yield curve data

Now, if we want to model the observations contained in Y, the natural starting point is to hypothesize, test, and estimate a time series model for Y_t. But, we have already seen empirically in chapter 1, that this may not be the best of ideas because of the strong cross sectional relationship that exist between yields observed at different maturities. We have also seen that a more viable strategy is to model a few yield curve factors, and to find out how these yield curve factors map into observed yields - as described in the papers referred in Section 3.1, and the related large body of related literature.

In its most general linear form, such an approach can be written as a two-equation dynamic system, which we typically refer to as a state-space model:[1]

[1] This does not necessary mean that we need to estimate the model using the Kalman filter. If, for example, we are working with observable yield curve factors, then an OLS estimation approach suffices. On the other hand, if factors enter non-linearly and are unobservable, then we need to use an appropriate filtering technique such as e.g. the unscented Kalman filter (see, e.g. Julier and Uhlmann (2004), Julier and Uhlmann (1997), and Wan and Merwe (2001)).



The figure shows the two dimensions of yield curve data. The upper panel displays the time series dimension, and the lower shown the maturity dimension.

Fig. 3.2. The maturity and time dimension of yield curve data

$$\texttt{state equation:} \qquad \underbrace{X_t}_{\#F\times 1} = \underbrace{k}_{\#F\times 1} + \underbrace{\Phi}_{\#F\times\#F} \cdot \underbrace{X_{t-1}}_{\#F\times 1} + \underbrace{\Sigma_X}_{\#F\times\#F} \cdot \underbrace{e_t}_{\#F\times 1}$$

$$\texttt{observation equation:} \qquad \underbrace{Y_t}_{\#\tau\times 1} = \underbrace{a}_{\#\tau\times 1} + \underbrace{b}_{\#\tau\times\#F} \cdot \underbrace{X_t}_{\#F\times 1} + \underbrace{\Sigma_Y}_{\#\tau\times\#\tau} \cdot \underbrace{u_t}_{\#\tau\times 1}.$$

where $(\#F)$ is the number of factors, and $(\#\tau)$ is the number of maturity points modelled.

The state equation governs the dynamic evolution of the yields curve factors, X, where k is a vector of constants, Φ is a matrix of autoregressive coefficients, Σ_X is the cholesky decomposition of the covariance matrix (i.e. it is a lower triangular matrix of covolatilities), and e_t is a vector of standard normal innovations, i.e. $e_t \sim N(0,1)$, so, $E\left[X_t|X_{t-1}\right] \sim N(\mu_X, \Sigma_X\Sigma_X')$. The observation equation translates the yield curve factors into yields, Y_t, as they are observed in the market place. In the state equation, a is a constant vector, b is the matrix that maps factor space into yield space, Σ_Y is a diagonal matrix of maturity specific yield volatilities, and $u_t \sim N(0,1)$.

It was shown in chapter 1 that a principal component analysis can cast light on the empirical factor structure underlying yields. What is hypothesised above, is addition that: (a) the factor structure can be parameterised in a parsimonious way (this idea was spearheaded by Nelson and Siegel (1987)), and that (b) the factors can be modelled by standard time series models e.g. as a VAR(1), as originally proposed by

Diebold and Li (2006). To test out these ideas, we employ MATLAB's state space modelling toolbox (SSM). Using MATLAB's build in toolboxes generally comes at the cost of having to conform with a required model set-up and so on. This is of course the same for the SSM module, although the barrier-of-entry with this toolbox may at first sight seem higher than with other toolboxes. Still, in my estimation, it is worth the effort (although one also have to forego the fun of implementing the Kalman-filter from scratch), because the added benefits far outweighs this initial investment of time.

To use the SSM toolbox it is required that the model to be estimated follows this generic set-up:

$$\begin{aligned}
&\texttt{state equation:} && X_t = R \cdot X_{t-1} + S \cdot e_t \\
&\texttt{observation equation:} && Y_t = T \cdot X_t + U \cdot u_t,
\end{aligned}$$

which means that the constants need to be integrated into the R and T matrices. This is done by including additional state variables that are preconditioned to be constant, and set equal to 1 at each observation point. Apart from this, it should be relatively straight forward to set up the model. The below set-up assumes that three factors are included in the model - but it naturally easy to accommodate any number of factors by appropriately adjusting the dimensions of the parameter matrices.

`State equation`

$$\begin{bmatrix} X(1) \\ X(2) \\ X(3) \\ \hdashline 1_{\#\tau} \end{bmatrix}_t = \left[\begin{array}{ccc:cccc} \Phi_{1,1} & \Phi_{1,2} & \Phi_{1,3} & k_1 & 0 & 0 & 0_{1,\#\tau-3} \\ \Phi_{2,1} & \Phi_{2,2} & \Phi_{2,3} & 0 & k_2 & 0 & 0_{1,\#\tau-3} \\ \Phi_{3,1} & \Phi_{3,2} & \Phi_{3,3} & 0 & 0 & k_3 & 0_{1,\#\tau-3} \\ \hdashline & 0_{\#\tau,3} & & & I_{\#\tau} & & \end{array}\right] \cdot \begin{bmatrix} X(1) \\ X(2) \\ X(3) \\ \hdashline 1_{\#\tau} \end{bmatrix}_{t-1}$$

$$+ \begin{bmatrix} \Sigma_{X(1,1)} & 0 & 0 \\ \Sigma_{X(2,1)} & \Sigma_{X(2,2)} & 0 \\ \Sigma_{X(3,1)} & \Sigma_{X(3,2)} & \Sigma_{X(3,3)} \\ \hdashline 0_{\#\tau,1} & 0_{\#\tau,1} & 0_{\#\tau,1} \end{bmatrix} e_t,$$

where $1_{\#\tau}$ is a constant unit vector of dimension $\#\tau$, and $I_{\#\tau}$ is the identity matrix of dimension $(\#\tau \times \#\tau)$. The rest of the dimension assignments follow the same principle.

The observation equation takes the following form:

`Observation equation`

$$
\left[\begin{array}{c} y^{3m} \\ y^{12m} \\ y^{24m} \\ \vdots \\ y^{120m} \\ \hdashline 1_{\#\tau} \end{array}\right]_t
=
\left[\begin{array}{ccc:cccccc}
b_{1,3m} & b_{2,3m} & b_{3,3m} & a_{3m} & 0 & 0 & 0 & \cdots & 0 \\
b_{1,12m} & b_{2,12m} & b_{3,12m} & 0 & a_{12m} & 0 & 0 & \cdots & 0 \\
b_{1,24m} & b_{2,24m} & b_{3,24m} & 0 & 0 & a_{24m} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & 0 \\
b_{1,120m} & b_{2,120m} & b_{3,120m} & 0 & 0 & 0 & 0 & 0 & a_{120m} \\
\hdashline
0 & 0 & 0 & \multicolumn{6}{c}{I_{(\#\tau \times \#\tau)}}
\end{array}\right]
\cdot
\left[\begin{array}{c} X(1) \\ X(2) \\ X(3) \\ \hdashline 1_{\#\tau} \end{array}\right]_t
$$

$$
+ \left[\begin{array}{ccccc}
\Sigma_{y,(1,1)} & 0 & 0 & \cdots & 0 \\
0 & \Sigma_{y,(2,2)} & 0 & \cdots & 0 \\
0 & 0 & \ddots & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \Sigma_{y,(\#\tau,\#\tau)} \\
\hdashline
\multicolumn{5}{c}{0_{\#\tau,\#\tau}}
\end{array}\right] u_t
$$

With the model now adhering to the notation used by MATLAB, it can be implemented and estimated using the SSM toolbox. This is done in the script included in the Annex part 4.3.1. Two model implementations are embedded in the code: one allows for the estimation of a fully empirical version of the model, i.e. where no prior structure is imposed on the constant vector, a, and the loading structure b in the observation equation; the other constrains a and b to follow the prescription by Nelson and Siegel (1987) using the parametrisation suggested by Diebold and Li (2006). This means that:

$$a = 0 \tag{3.1}$$

$$b_\tau = \left[1 \;\; \frac{1-e^{(-\gamma\cdot\tau)}}{\gamma\cdot\tau} \;\; \frac{1-e^{(-\gamma\cdot\tau)}}{\gamma\cdot\tau} - e^{(-\gamma\cdot\tau)}\right]. \tag{3.2}$$

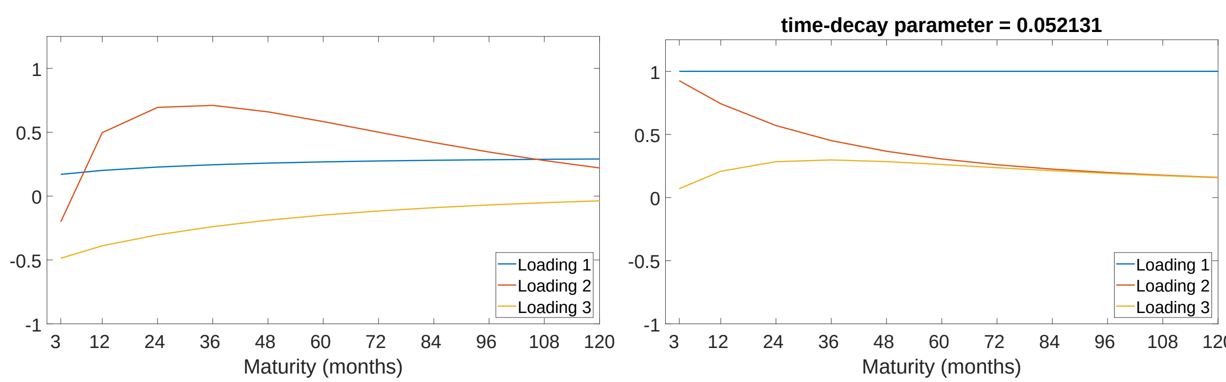
Note that the notation is changed slightly compared to what is traditionally used. We use γ to denote the time-decay parameter, which is most often denoted by λ is the literature. This is done to avoid notational confusion, since λ is elsewhere in this booklet used to denote the market price of risk.

When running the code shown in Annex part 4.3.1 (in its two guises by adjusting the input in line 52, choosing either 'Emp', for empirical, or 'NS', for the Nelson-Siegel model) we can compare the loadings and extracted factors. Since the empirical model is virtually unspecified in its generic form, it is not clear what to expect in terms of an outcome. In essence, this model is too flexibly specified, since there is nothing that locks down the scale and sign of the factors, and by the same token, there is noting ensuring that a reasonable and interpretable structure will emerge for the loadings contained in b. In principle, we have a linear regression model, $y = a + b \cdot x$, we only know y, and we try to determine a, b, and x, by using some clever estimation technique (i.e. the Kalman-filter). Clearly, there are many combinations of $a + b \cdot x$ that will fulfil the equation. And, we must therefore expect that, depending on the starting values, the iterative algorithm can converge to a multitude of maxima all providing exactly the same fit to data. If we would like to have a model that facilitates economic analysis, it is useful to attach a certain meaning to the factors, and that this meaning remains constant over time, i.e. across the multiple times the model will be re-estimated, as time progresses. We will look more carefully at this in the next section, for now we will push ahead, and see what we get when we run the code.

Figure 3.3 shows the loadings that are generated by the two models, and Figure 3.4 shows the extracted factors. Using these two estimates, which amounts to b and X in the above outlined model-notation, together with the constant, a, we can assess how well the estimated models fit the observed yields. This is done by means of the RMSE (root-mean-squared-error) expressed in basis points. Model predictions are denoted by $\hat{y}^{Emp}$ and $\hat{y}^{NS}$, respectively, and calculated in the following way:

$$\hat{y} = \hat{a} + \hat{b} \cdot \hat{X}. \tag{3.3}$$

Using the ˆ notation here underscores that estimates are used to produce the model predictions. This is an obvious fact, and this notation will therefore not be used throughout, unless the context is ambiguous. Another thing to note is that only the parameter estimates from the observation equation is used at the moment - but rest assured, we will return to the state equation, and use it extensively, in the section of the notes that looks at forecasting and scenario generation.



(a) The empirical model

(b) The Nelson-Siegel model

Panel (a) shows the loading structure of the empirical model specification, i.e. the parameter estimates contained in the b-matrix from the observation equation: $y = a + b \cdot X$. Panel (b) shows the same for the Nelson-Siegel model. The displayed loading structures provide a graphical representation of the loadings for the three estimated factors, i.e. the loadings for each factor across the modelled maturity dimension.

Fig. 3.3. Estimated loadings

While it may not be evident to the naked eye, there is quite some commonality between the loading structures shown in Figure 3.3. Taking the Nelson-Siegel loadings as the starting point, the first factor has an equal impact, on all yields, regardless of their maturity: that is, loading 1 equals unity across the maturity spectrum (see panel (b)). While the empirical model does not generate a constant value of one across the maturity spectrum for its first loading, the value is approximately constant. In the language of PCA analysis, where scale and signs of loadings and factors can switch around, this common feature, i.e. constancy across the maturity spectrum is enough to declare, that the fist factor has a similar interpretation for both model variants. And, by the virtue of its impact on the yield curve, shifting it upwards and downwards in a parallel fashion, this first factor can be seen as the duration risk factor. It is too early to say whether the duration factor is shifting the curve from the short, middle, or long end of the maturity dimension. To determine this, we need to look at the second factor. As is well-known, in the Nelson-Siegel model, the second factor constitutes the slope of the yield curve, or rather, the negative slope, i.e. the short-end yield minus the long-end yield. This is evident from the shape and location of Loading 2 in panel (b). At the short-end of the maturity spectrum, this factor records its maximum impact, and its impact falls to zero (beyond the maturities shown in Figure 3.3) as the maturity increases, following a convex trajectory. Thinking e.g. about how the three month yield is recovered from these two first factor loadings of the Nelson-Siegel model, implies that the first factor is defined as the long-end level of the yield curve (and it is from here that parallel shifts are induced on the yield curve), and the second factor is the negative slope:

$$
\begin{aligned}
\texttt{short rate} &= \texttt{loading 1} \cdot \texttt{Factor 1} + \texttt{loading 2} \cdot \texttt{Factor 2} \\
&= 1 \cdot \texttt{Factor 1} + 1 \cdot \texttt{Factor 2} \\
&= 1 \cdot \texttt{level} + 1 \cdot (-\texttt{level} + \texttt{short rate}) \\
&= \texttt{short rate}.
\end{aligned}
$$

Following this logic, it is established that the Nelson-Siegel model imposes factor interpretations for the first two factors that are equal to a yield curve level-factor, and to a negative (compared to the traditional definition) slope factor, respectively. In the above, we have also established that the first factor detected by the empirical model, is similar in shape to the level factor of the Nelson-Siegel model. The question is now, whether either of the two remaining loading structures in panel (a) of Figure 3.3 resembles the second Nelson-Siegel factor loading. It appears that Loading 3 exhibits a convex and increasing pattern, and if rotated around the x-axis, it compares well to the second Nelson-Siegel factor!

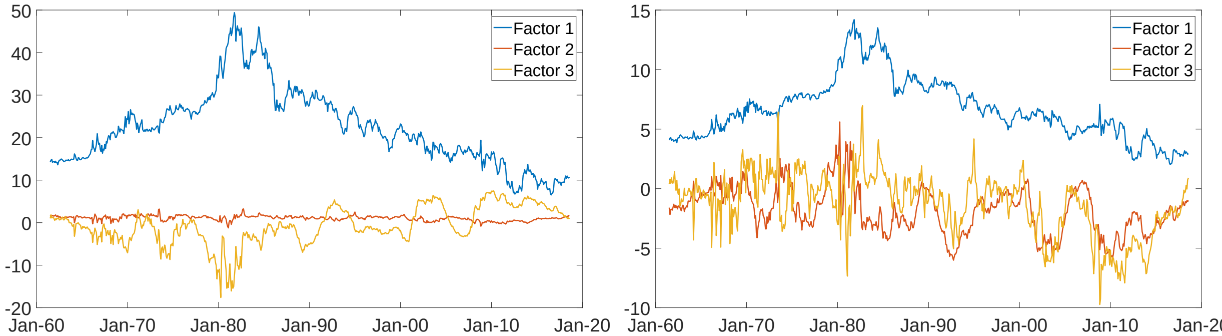
But, why does the empirical model swap the ordering of the factors around? Well, since we haven't given the model any information about how we want it to organise it does not know any better, and it orders the factors basically on the basis of the starting values given to the optimisation routine, and the path that the routine follows to reach the maximum. This is different when we do a PCA analysis, since most econometric packages order the extracted factors/principal components according to their eigenvalues, i.e. according to the amount of variance that they each explain. And, since PCA analysis in principle underpins the Nelson-Siegel model, the ordering of the NS-factors follows this principle - by the way, obviously, it also makes a lot of economic and intuitive sense to order the factors in this way, since it corresponds to placing the most risky factor first in the hierarchy and the least risky factor last.

Finally, the two remaining factors, factor two from the empirical model and factor three form the Nelson-Siegel model also match up in terms of patterns, with the weighting across maturities fitting that of a curvature impact, i.e. having little impact in the short and long ends of the curve and larger impact in the middle part of the maturity spectrum.

Sorting out the relationship between the empirical and the Nelson-Siegel model specifications by visual inspection of the loading structure shown in Figure 3.3, as we have done above, has hopefully helped to further our intuitive understanding of yield curve factor-models. At least, this was the main purpose of this exercise. In my experience, it is in general helpful to think about yield curve factor models as the the multiplication of loadings and factors (plus a constant), i.e. $y = a + b \cdot X$, in visual terms as the multiplication of the loadings, as e.g. shown in panel (a) of Figure3.3, with the time series of factors, shown in panel (a) of Figure 3.4. Having such a visual representation in the back of the mind helps to lock down the entities that enter the model and facilitates an immediate intuitive sense of the economic interpretation of the factors.

Instead of the somewhat lengthy deliberations above, there is an easier and (perhaps) more natural way to match up the factor interpretation across the two estimated models. Given that we have obtained the time series of the yield curve factors for each model, as shown in Figure 3.4, we can simply calculate the cross-correlations between the series. This is done in Table 3.1. And, luckily, the conclusions we drew above are confirmed: there is a correlation of 0.99 between factor 1 of the empirical and Nelson-Siegel models, a correlation of 0.96 between the second factor from the empirical model and the third Nelson-Siegel factor, and finally a correlation of -0.85 between the third empirically determined factor and the second Nelson-Siegel factor. The negative sign of the latter correlation coefficient match the fact that the loading structure for the third empirical factor had to be rotated around the x-axis (i.e. it had to be multiplied by -1) in order to obtain a pattern similar to that of the Nelson-Siegel model.

As the final issue in this section, we will have a look at how well the two models fit the data. To fix ideas, it is observed that the Nelson-Siegel model can be seen as a constrained version of the empirical model, since the Nelson-Siegel model imposes a certain functional structure on the loadings contained in the b-matrix. In fact, at first sight, it seems that Nelson and Siegel chose to impose some rather severe constraints: where the empirical model relies on $\#\tau \times 3 = 33$ parameters, all to be estimated, the Nelson-Siegel model uses only one single parameter, namely the time-decay parameter γ, together with functions of γ and τ. On the other hand, we also know that the Nelson-Siegel model is hugely popular, and one of the tools often used by central banks asset managers. If the model produced a poor fit to data, it probably



(a) The empirical model (b) The Nelson-Siegel model

Panel (a) shows the time series evolution of the extracted yield curve factors, i.e. the estimates contained in the X-matrix from the observation equation: $y = a + b \cdot X$. Panel (b) shows the same for the Nelson-Siegel model.

Fig. 3.4. Estimated factors

	Emp F1	Emp F2	Emp F3	NS F1	NS F2	NS F3
Emp F1	1.00					
Emp F2	0.32	1.00				
Emp F2	-0.68	-0.28	1.00			
NS F1	**0.99**	0.23	-0.59	1.00		
NS F2	0.19	0.14	**-0.85**	0.08	1.00	
NS F3	0.48	**0.96**	-0.54	0.38	0.38	1.00

The table shows the correlations between the extracted factors from the Empirically founded (Emp) model and from the Nelson-Siegel (NS) model.

Table 3.1. Factor correlations

would not be so widely used. So, it is no surprise that the two-parameter functional-forms utilised by the Nelson-Siegel model do not impose any devastating constraints. This is, of course, because the chosen functional forms match well the patterns that result from PCA analysis on yields, and that yields in most markets, and across time, are well captured by these patterns.

Using the US data, Table 3.2 shows the Root Mean Squared Error (RMSE) for both each model across maturities from three months to 10 years. Both models fit data very well, and they both have very low average RMSEs. While the empirical model fits slightly better, we see that the cost of the constraints imposed by the Nelson-Siegel model are very small, at most 1 to 3 basis points. And, these results are obtained on data covering the period from 1960 to 2018, so this results seems to have general validity, and it surely not an artefact of a carefully selected data sample.

τ in months	3	12	24	36	48	60	72	84	96	108	120
Empirical model (bps)	6	13	4	4	6	5	4	2	1	4	6
Nelson-Siegel (bps)	8	14	5	3	5	5	4	3	1	3	6
Difference	-2	-1	-1	1	-1	0	0	-1	0	1	0

Model fits are compared for the two estimated versions of the model: the empirical one and the Nelson-Siegel model. The table shows the root-mean-squared-error (RMSE) for each model. For each maturity point covered by the data, the RMSE is calculated as $\left[mean\left[\left(y_{\tau(i)} - \hat{y}^{j}_{\tau(i)}\right)^2\right]\right]^{\left(\frac{1}{2}\right)}$, where $y_{\tau(i)}$ and $\hat{y}^{j}_{\tau(i)}$ are time series for the i'th maturity point, $j \in \{Emp, NS\}$.

Table 3.2. RMSE (basis points)

3.3 Rotating the yield curve factors

As practitioners we may, at times, be interested in imposing a certain economic meaning on one or more of the yield curve factors, while still staying within the comforting remit of the Nelson and Siegel (1987) and Diebold and Li (2006) modelling frameworks.[2] For a given task, we may find that it is convenient to work directly with the short rate. For example, if yield curve scenarios need to be generated for risk assessment purposes, where a set of predefined scenarios are defined in terms of the future path of the monetary policy rate. In this case, it seems reasonable to model the short rate directly, rather than to backward engineer how the Nelson-Siegel level and slope factors would need to evolve, to match the predefined scenario paths for the short rate. It could may also be the case that a certain relationship between the short rate, the slope and some macroeconomic variables, are believed to exist. For example, we may believe that a Taylor-rule (Taylor (1993)) inspired relationship holds between macroeconomic variables, and that slope is related to the perceived risk in the fixed income markets, and that scenarios need to be generated against this set-up. Again, it seems more fruitful to rely on a Nelson-Siegel type model that rely on a short rate factor, rather than the level factor. Other example are: the evaluation of trading strategies and return decompositions. To the extent that trading positions are specified in terms of actual yield curve points, for example long/short the 2y-10y spread positions, curvature positions, e.g. as combinations of the 2Y-5Y-10Y, and so on, it may be relevant to model directly yield curve points, rather than the level, slope, and curvature factors.

Under the requirement that the desired alternative factor interpretation can be expressed as a linear combination of the existing factors, it is possible to find a rotation matrix $\mathcal{A}$, where $I = \mathcal{A}^{-1} \cdot \mathcal{A}$, such that the desired factor structure emerges.

Consider the observation equation from the standard dynamic model. We can naturally expand this expression by I, without changing it in any way. This is done below:

[2] Factor rotation is a well-known concept in statistical analysis, see e.g. Johnson and Wichern (1992)[ch. 9.4].

$$
\begin{aligned}
y_t &= a + b \cdot X_t + \Sigma_y \cdot u_t \\
&= a + b \cdot I \cdot X_t + \Sigma_y \cdot u_t \\
&= a + b \cdot \mathcal{A}^{-1} \cdot \mathcal{A} \cdot X_t + \Sigma_y \cdot u_t. \qquad (3.4)
\end{aligned}
$$

By doing this, we have obtained new interpretations of the factors and the factor loadings, that are in accordance with the chosen $\mathcal{A}$ matrix.

$$
\tilde{b} = b \cdot \mathcal{A}^{-1} \qquad (3.5)
$$

$$
\tilde{X}_t = \mathcal{A} \cdot X_t. \qquad (3.6)
$$

Later on we will see how to choose $\mathcal{A}$, for now the objective is to see how the state equation changes:

$$
\begin{aligned}
\tilde{X}_t &= \mathcal{A} \cdot X_t = \mathcal{A}\,(k + \Phi \cdot X_{t-1} + \Sigma_X \cdot e_t) \qquad (3.7) \\
&= \mathcal{A} \cdot k + \mathcal{A} \cdot \Phi \cdot X_{t-1} + \mathcal{A} \cdot \Sigma_X \cdot e_t \qquad (3.8) \\
&= \mathcal{A} \cdot k + \mathcal{A} \cdot \Phi \cdot \mathcal{A}^{-1} \cdot \tilde{X}_{t-1} + \mathcal{A} \cdot \Sigma_X \cdot e_t, \qquad (3.9) \\
&= \tilde{k} + \tilde{\Phi} \cdot \tilde{X}_{t-1} + \tilde{\Sigma}_X \qquad (3.10)
\end{aligned}
$$

where the second to last line follows from (3.6). The parameters of the rotated model can be read from equation (3.9), and are:

$$
\tilde{k} = \mathcal{A} \cdot k \qquad (3.11)
$$

$$
\tilde{\Phi} = \mathcal{A} \cdot \Phi \cdot \mathcal{A}^{-1} \qquad (3.12)
$$

$$
\tilde{\Sigma}_X = \mathcal{A} \cdot \Sigma_X. \qquad (3.13)
$$

In practical applications of rotated models, it is naturally enough to rotate the loading matrix, b, in the observation equation, and then to proceed with the estimation as usual. Doing this will result in the extraction of rotated factors as well. The above equations (3.11)-(3.13) are only needed, if a standard model has been estimated, and it subsequently needs to be rotated, or, if a rotated model has been estimated, and it needs to be un-rotated.

How is $\mathcal{A}$ determined? This naturally depends on the desired factor interpretation. Below I present two simple cases where the factors have interpretations as: [short rate, slope, curvature], and as the [2Y yield, 5Y yield, 10Y yield].

3.3.1 A short rate based model

As we have seen many times, the original Nelson-Siegel factors are level, slope and curvature. To obtain a factor structure that equals {`short rate, slope, curvature`} we see that the following $\mathcal{A}$-matrix will do the trick ("SRB stands for short rate based"):

$$\mathcal{A}^{SRB} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Lets insert it and see if it rotates that factors as desired:

$$\begin{bmatrix} \texttt{short rate} \\ \texttt{slope} \\ \texttt{curvature} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \texttt{level} \\ \texttt{-slope} \\ \texttt{curvature} \end{bmatrix} = \begin{bmatrix} \texttt{level-slope} \\ \texttt{slope} \\ \texttt{curvature} \end{bmatrix}.$$

It is important to recall that nothing is gained in terms of model fit, or improved forecasting performance, after a rotation is performed. This is clear since $I = \mathcal{A} \cdot \mathcal{A}^{-1}$. Only the factor interpretation is changed.

3.3.2 Using yields as factors

Another rotation that may be relevant for practical work, is a rotation towards an interpretation of the factors as yield curve maturity points. In the example below, the Nelson-Siegel factors are rotated to have interpretations as the [2Y, 5Y, 10Y] yields. To achieve this, we need to establish a link between the Nelson-Siegel factors and the desired factor interpretation. Often, it is mentioned in the literature that the following relationships hold: [level = 10Y yield], [-slope = 3m yield - 10Y yield], and [curvature = 2× 2Y yield - 10Y yield - 3m yield]. From this we could, in principle, obtain a rotation matrix, that approximately would give us the factor interpretation that we are looking for. But, it may be better to devise a general methodology that also would work, should we want to implement other types of factor interpretations/rotations. This can be done in a (perhaps) surprisingly simple way, by using linear regression. Of course, a pre-requisite for this methodology to work is that the factors we rotate towards are observable or can be estimated.

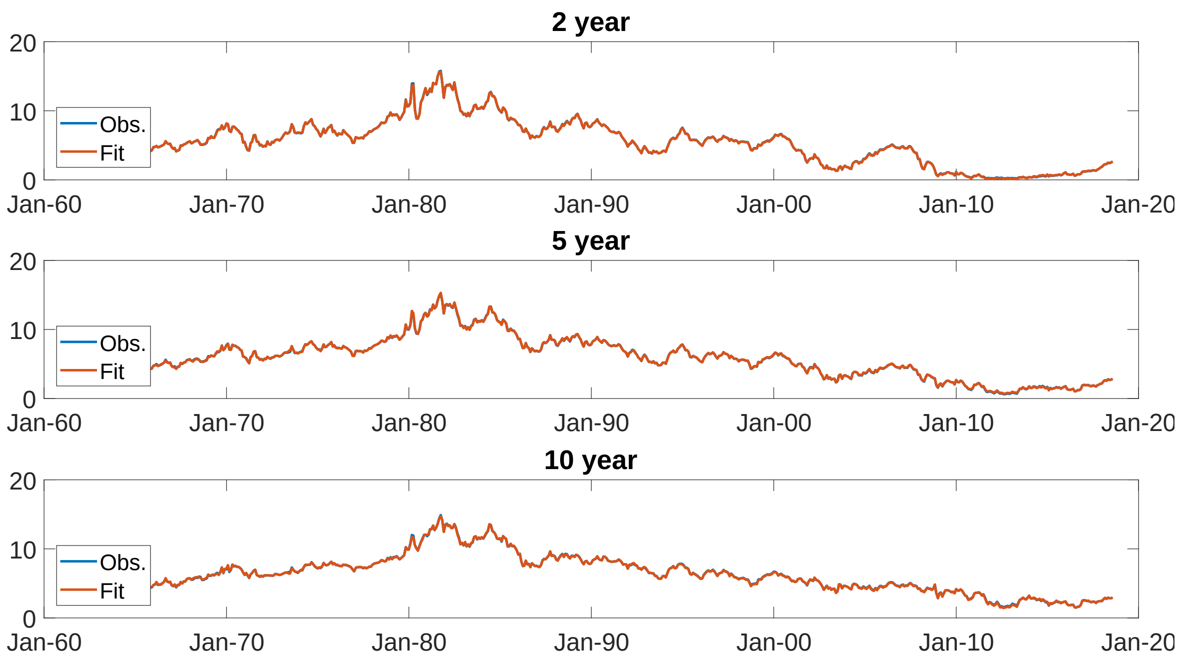
To fulfil our current factor appetite, we fill the entries of the rotation matrix via the three OLS regressions (see appendix with the MATLAB code to see how this can be implemented in practise):

$$\begin{aligned} y_{2Y} &= \mathcal{A}^{y}_{(1,:)} \cdot \hat{X}^{NS} \\ y_{5Y} &= \mathcal{A}^{y}_{(2,:)} \cdot \hat{X}^{NS} \\ y_{10Y} &= \mathcal{A}^{y}_{(3,:)} \cdot \hat{X}^{NS}. \end{aligned} \tag{3.14}$$

The regressions are run with the normalisation constraint that the impact of the level factor is equal to 1 for all rotated factor. Doing this gives:

$$\mathcal{A}^y = \begin{bmatrix} 1.00 & 0.56 & 0.28 \\ 1.00 & 0.31 & 0.26 \\ 1.00 & 0.15 & 0.16 \end{bmatrix}. \tag{3.15}$$

Even if we firmly belief that this approach will work, it may be a good idea to perform a double-check. First, we can visually inspect how well the rotated factors match the observed counterparts. This is done in Figure 3.5, with convincing results: the lines are indistinguishable. Second, we see whether the rotated



The figure compares the yield curve factors of a rotated dynamic Nelson-Siegel model, where the factors are the 2Y, 5Y, and 10Y yield curve points, with the actually observed 2Y, 5Y, and 10Y yield curve points. The lines fall on top of each other, therefore only one line is visible in the panels.

Fig. 3.5. 2Y, 5Y, and 10Y yields, and rotated NS factors

model fits the observed yields as well as the Nelson-Siegel model does. This assessment can be made on the basis of the RMSE across maturities. Table 3.3 shows the obtained numbers. As expected, the RMSEs of the rotated model are exactly the same as those obtained from the original Nelson-Siegel model.[3]

[3] It is worth noting that the RMSEs reported in Table 3.3 deviate slightly from the ones reported in Table 3.2. This is due to the difference in estimation methods applied: the results obtained in Table 3.2 rely on a state-space implementation, where as the results in Table 3.3 are based on a step-wise OLS implementation (via the TSM (Term Structure Model) object-oriented class), where the time-decay parameter is determined to a precision of three decimals.

τ in months	3	12	24	36	48	60	72	84	96	108	120
NS (OLS) (bps)	8.42	14.03	4.89	3.57	5.31	5.60	4.54	2.59	1.16	3.48	6.45
Rotated (bps)	8.42	14.03	4.89	3.57	5.31	5.60	4.54	2.59	1.16	3.48	6.45

RMSEs are calculated for the dynamic Nelson-Siegel model estimated using a two-step OLS methodology, as implemented in the object-oriented Term Structure Model (TSM) class, and for the model presented in the text, where the Nelson-Siegel factors are rotated to be the time series of the 2Y, 5Y, and 10Y yield curve points.

Table 3.3. RMSE (basis points)

3.4 The building blocks that shape the yield curve

Even if we resist the strong temptation to impose arbitrage constraints on our model, we will still be able to extract and analyse the fundamental building blocs that shape the location and dynamics of the yield curve. These are: (a) the term structure of term premia, and (b) the term structure of rate expectations. To assess the current economic environment in terms of risks and expectations to future economic growth, it is important to have reliable estimates of the term premium and the expected risk-free term structure. As also mentioned in the section describing the $\mathbb{P}$ and $\mathbb{Q}$ measures, the term premium, $\theta_{t,\tau}$, at time t for some maturity τ, is a summary measure for the risks that financial agents deem to face, when trading fixed income securities. The items displayed in italics font are typically not included in the list, when we deal with risk-free sovereign bonds, as we do here. However, they are included just to remind us of these additional systematic risk factors, when we start working with corporate bonds, and possibly less liquid market segments:

- uncertainty about the economic growth rate
- uncertainty about the inflation rate
- *credit migration risk*
- *default risk*
- *liquidity risk*

The risk-free term structure is constructed mechanically as the average of the short rate ($\mathbb{P}$)-expectation over future periods. It is risk free, because the one period (short) rate is risk-free, period after period. Another way to realise this, is to consider that a model needs to be fitted to historically observed short rate data, that represent past realisations of the risk-free rate.

Following Gürkaynak and Wright (2012), the risk-free term structure can be calculated as:

$$y^{rf}_{t,\tau} = \frac{1}{\tau} \cdot \mathbb{E}_t \sum_{j=0}^{\tau-1} r_{t,t+j}, \tag{3.16}$$

and, the term premium as the difference between the model fitted yield, $\hat{y}_{t,\tau}$ and the risk-free yield, $y^{rf}_{t,\tau}$:

$$\theta_{t,\tau} = \hat{y}_{t,\tau} - y_{t,\tau}^{rf}. \tag{3.17}$$

To illustrate how we can obtain rate expectations and term premia also from models that do not exclude arbitrage by construction, and to see to what extent they differ from model to model, we perform the following case studies, comparing the:

(A) 3-factor SRB model with the standard Nelson-Siegel parametrisation.
(B) 3- and 4-factor SRB models.
(C) 3-factor SRB model with and without bias correction (what bias correction entails will be outlined below).
(D) 3-factor SRB model with different assumptions on the mean of the short rate.
(E) 3-factor SRB model with published term premium and rate expectations from the Adrian, Crump, and Mönch (2013) and Kim and Wright (2005) models.

Before embarking on this task, we need to introduce the 3- and 4-factor versions of the short rate based (SRB) model. The state and transition equations of the 4-factor model will be presented, since the 3-factor model is simply a constrained version of the 4-factor model; the constraint being that the fourth factor is deleted in the 3-factor version of the model. We will use a discrete-time version of the model, and we will derive it formally in a later chapter. For now, only the relevant equations are presented, like in the case of the Nelson-Siegel model, as shown in equations (3.1) and (3.2).

The 4-factor model is shown in equations (3.18) and (3.19).

$$X_t = \mu + \Phi \cdot (X_{t-1} - \mu) + \Sigma_X \cdot e_t \tag{3.18}$$

$$\begin{aligned} y_{t,\tau} &= b_\tau \cdot X_t + \Sigma_Y \cdot u_t \\ &= \left[1, \quad 1 - \tfrac{1-\gamma^\tau}{(1-\gamma)\cdot\tau}, \quad \tfrac{1-\gamma^\tau}{(1-\gamma)\cdot\tau} - \gamma^{\tau-1}, \quad -\tfrac{1}{2}(\tau-1)(\gamma-1)\gamma^{(n-2)}\right] \cdot X_t + \Sigma_Y \cdot u_t \end{aligned} \tag{3.19}$$

Note that we have written the VAR(1) model in (3.18) in mean-adjusted form, so μ is the mean of the included factors. Since we know that the the first factor in X_t is the short rate, we can work on equation (3.16), and turn it into a closed-from expression, in the following way:

$$
\begin{aligned}
y^{rf}_{t,\tau} &= \frac{1}{\tau} \cdot \mathbb{E}_t \sum_{j=0}^{\tau-1} r_{t,t+j} \\
&= \frac{1}{\tau} \cdot \left[r_t + \mathbb{E}_t \sum_{j=1}^{\tau-1} r_{t,t+j} \right] \\
&= \frac{1}{\tau} \left\{ X_t' \cdot \iota + \left[\mathbb{E}_t \sum_{j=0}^{\tau-2} \mu + \Phi \cdot (X_{t+j} - \mu) \right]' \cdot \iota \right\} \\
&= \frac{1}{\tau} \left\{ X_t' \cdot \iota + \left[(\tau - 1) \cdot \mu + \frac{\Phi^\tau - \Phi}{\Phi - I} \cdot (X_t - \mu) \right]' \cdot \iota \right\}
\end{aligned}
\tag{3.20}
$$

where ι is a vector of appropriate dimension that selects the first element of the vector generated inside the brackets, i.e. $\iota = [1, 0, 0, 0]'$ when we work with a 4-factor model, and $\iota = [1, 0, 0]'$, when we work with a 3-factor model. The reason for separating out the $(X_t' \cdot \iota)$ part is to highlight that this equals r_t, and the factor, $(\Phi^\tau - \Phi)(\Phi - I)^{(-1)}$, originates as the limit of the sum of the power series implied by line three of (3.20).

It may be worth highlighting that it is much simpler to use the close-form expression in (3.20) compared to calculating the rate expectation using the summation over all τ, as implied by (3.16). The closed form expression is both faster and less computational intensive. While this is not a big deal when using monthly data, it may be an issue, if we estimate the model on daily data. Imagine we want to calculate the 10-year rate expectation, and we have estimated our model on daily data, and we that have data for the period from January 1961 to July 2018, a total of around $14,800$days. When using (3.20) we would then need to roll-forward the state equation (3.18) for $3,650$ observation points, for each of the $14,800$ days covered by our sample; implying that we would need to calculate Φ^k for $k \in \{1, 2, \ldots, 3650\}$ at each observation point, i.e. a total of 54mill calculations. Compared to this, it is easier to use (3.20), because only $14,800$ calculations are needed, i.e. one for each day. A second thing that is worth mentioning is that any of the two outlined calculation methods can be efficiently completed using the eigenvalue decomposition[4] of Φ^k:

$$
\Phi^k = V \cdot D^k \cdot V^{-1}, \tag{3.21}
$$

where V is the matrix of eigenvectors, and D is a diagonal matrix of eigenvalues. Recall that it requires much less computational effort to calculate the power of a diagonal matrix, because one just needs to raise each diagonal element of the matrix to the desired power.

Case (C) in the above list aims to investigate the relevance of bias-correcting the VAR(1) model, included in the state equation. When estimating a VAR model using OLS, on relatively few time-series data points as it is often the case in yield curve applications, the parameters of the VAR can be biased downwards. This implies that the estimated factors can exhibit a lower degree of persistency, compared to

[4] Can be obtained in MATLAB using the `eig` command.

their true process parameters. I think Bauer, Rudebusch, and Wu (2012) were the first to highlight this issue. Our bias correction method, which is implemented in MATLAB via the Term Structure Class (TSM) that accompany this booklet, is based on the description in Engsted and Pedersen (2014), of the analytical approach suggested by Pope (1990).

The persistency of the factors is naturally important for the decomposition of the yield curve into rate expectations and term premium components. Imagine, for example, that the short rate factor exhibits a very low degree of persistency, in fact so low that the process converges to its sample mean within three to seven years, for any of the short rate levels observed in the sample. Consequently, if we focus on the 10-year rate expectation, it will equal the sample mean for all dates covered by the sample, and the resulting time series of 10-year rate expectations is just a constant flat line equal to the sample mean, lets say, for example, 4.88%. Then we will obtain a time series of 10-year term premium estimates that is equal to $\hat{y}_{t,\tau=10years} - 4.88$, and assuming that the model fits data well, then this is very close to being equal to the observed series of 10-year yields minus 4.88%. Not a very believable result. At the other end of the absurdity-scale is a super-highly persistent process. Imagine a process where the short rate hardly moves from its starting point, and it e.g. takes $10,000$ years for the process to converge to its sample mean. Calculating the term premium in this case amounts approximately to calculating the slope of the yield curve (again assuming a good fit of the model), at any date covered covered by the sample. An equally unbelievable outcome. So, the persistency of the estimated VAR model typically has a large impact on the model derived expectations/term premium decomposition - we will check this empirically below.

All estimations done in the context of the five case studies are performed used the TSM class. To learn more about this object-oriented class, you can type `help TSM` at the command prompt. The basics of it is:

(1) To create an instance of the TSM class. An instance is, so to say, your private copy of the class, that you can work with. To create an instance, type: `<name>=TSM`, for example: `SRB_1 = TSM`.

(2) The created instance can now be populated with data: `SRB_1.yields=Y` (assuming that the data to be used for the estimation of the model are stored in the matrix `Y`), `SRB_1.tau=tau`, (assuming that the vector of maturities is stored in `tau`), and `SRB_1.DataFreq=12`, if data are sampled at a monthly time-interval. And so on. It should be noted that the variable names used in the TSM class are not optional, so the name that appears after the dot, i.e. in the example above `.yields`, `.tau`, and `.DataFreq`, has to be used as shown in the help file - the TSM class does not understand if yields, for example, are assigned to a container called `SRB_1.YieldsForTheModel`, or any other user-defined name. On the other hand, the name of the class instance, i.e. `SRB_1` can be chosen freely.

(3) Given that all data have been passed successfully to the created instance, any of the models covered by the class can be estimated. Four models are covered at the moment, but this number will increase over time. The following are covered: the Dynamic Nelson-Siegel model, the dynamic Svensson-Soderlind model, the 3-factor SRB model, and the 4-factor SRB model.

(4) Any of the models can be estimated using the command `.get<model name>`. For example, to estimate the 3-factor SRB model we would write: $SRB_1 = SRB\_1.getSRB3$. This estimates the desired model, and stores the results in the created class instance called `SRB_1`. The other models are estimated using the commands: `.getDNS`, `.getDSS`, and `.getSRB4`, respectively.

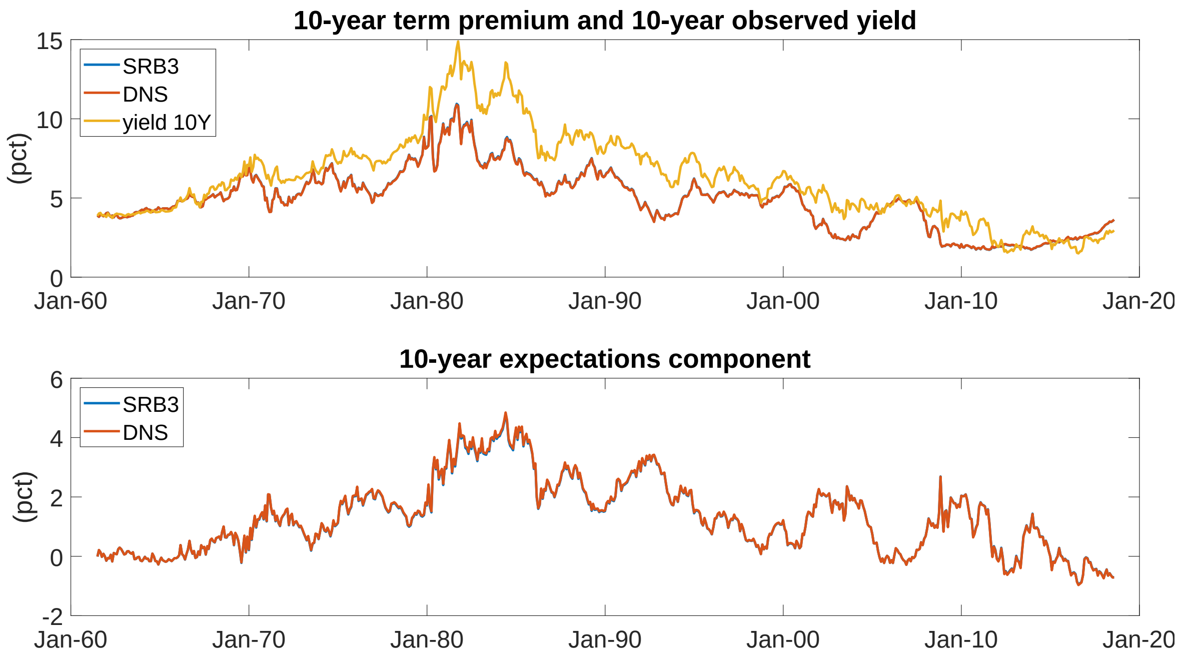
(5) The output covers, among other, model parameters, time series of extracted yield curve factors, RMSE, the term structure of term premia (at the provided maturities), and the term structure of rate expectations (at the provided maturities).

To complete the scenarios outlined above, we will implement the five steps from the above list. More details on the coding can be found in MATLAB appendix in section 4.3. Comparisons will be drawn in terms of rate expectations and term premia, typically at the 10-year maturity point; model fit in terms of RMSEs, and persistency of the VAR(1) model featuring in the state equation and judged on the basis of the eigenvalues of $\hat{\Phi}$.

Case (A): Comparing the 3-factor SRB model with the standard Dynamic Nelson-Siegel model

Figure 3.6 draws a comparison between the 3-factor SRB model, and the Dynamic Nelson-Siegel(DNS) model. While these models have different factor interpretations - the SRB-model explicitly includes the short rate, and the DNS model explicitly includes the long-term rate (i.e. the yield curve level) - they are very similar, and intimately linked via the rotation matrix $\mathcal{A}$. However, since the SRB model is derived in discrete-time (as we will see later on), and the DNS model is derived in a continuous time, the link between the two models, via a rotation-matrix, does not produce mathematically identical models.[5] It is actually not possible to rotate the DNS model into the SRB model, as it is used here, because of the mentioned difference between the models. But, it is possible to rotate the DNS model into a continuous-time SRB model, and the difference between this rotated model, and the SRB model (in discrete-time) is very, very small. So, this is the degree of intimately between the two models used in this section.

[5] The practical difference is that the SRB model's loading structure is defined in terms of power functions, while the DNS model relies on exponentials. The continuous-time limit of a power function is the exponential function: recall, for example, the link between discretely and continuously compounded interest rates: $lim_{n\rightarrow\infty}\left(1+\frac{r\cdot T}{n}\right)^n = e^{rT}$.



The figure shows the 10-year term premium estimates from the 3- and 4-factor SRB models. In the figure, the upper panel shows the 10-year term premium, and the lower panel shows the 10-year expectations component. Estimates from the standard $SRB3$ model is shown in blue, and the mean adjusted version is shown in (red). For comparison, the observed 10-year yield is plotted in yellow, in the upper panel.

Fig. 3.6. the SRB3 and DNS models

	RMSE (basis points)										
τ in months	3	12	24	36	48	60	72	84	96	108	120
SRB3	8.50	14.08	4.89	3.54	5.28	5.56	4.51	2.57	1.15	3.44	6.38
SRB3, bias corrected	8.42	14.03	4.89	3.57	5.31	5.60	4.54	2.59	1.16	3.48	6.45
	Eigenvalues of $\hat{\Phi}$										
	1	2	3								
SRB3	0.990	0.957	0.808								
DNS	0.990	0.957	0.807								

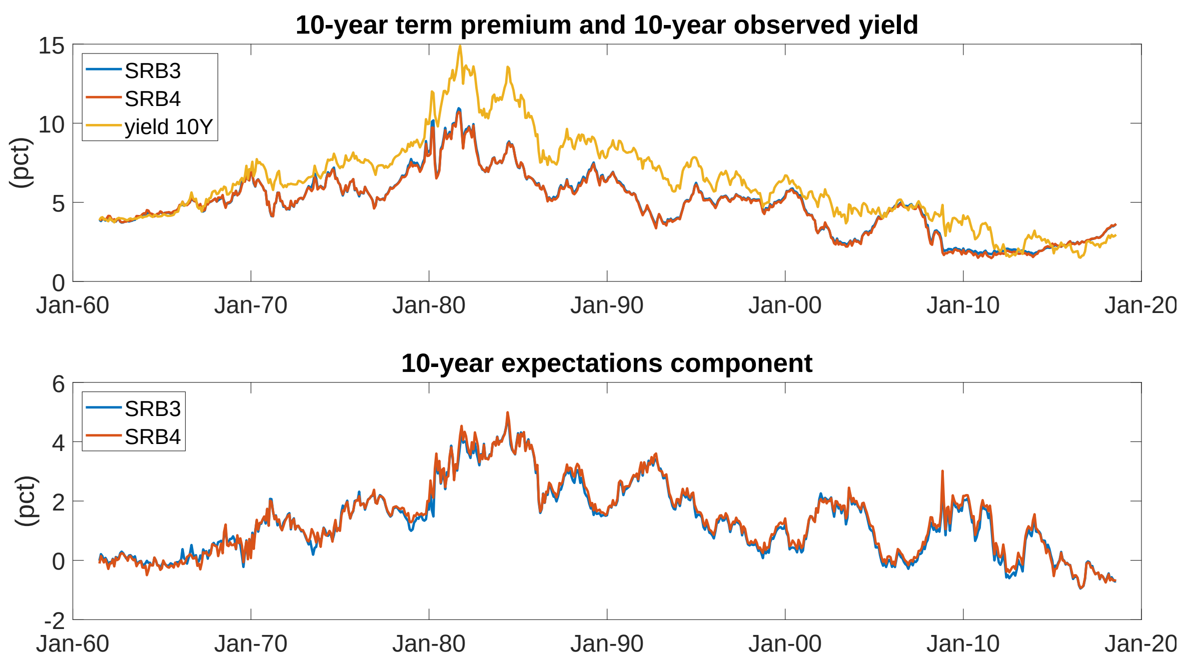
RMSEs are calculated and shown in basis points for the two models under investigation, together with the eigenvalues of $\hat{\Phi}$, sorted in descending order.

Table 3.4. Case A: RMSE and eigenvalues

The similarity between the models is confirmed in Figure 3.6, with the plots of the 10-year rate expectation and term premia being indistinguishable from one model to the next. A minor difference is observed on the third eigenvalue, where the SRB model is insignificantly more persistent than the DNS model; likewise, minor and non-significant differences are seen in terms of in-sample fits. For practical purposes in the area of extracting past information from the 3-factor SRB and DNS model, they are identical. In a later section of the booklet, we will see whether this conclusion also carries over to the forecasting performance of the models.

Case (B): Comparing 3- and 4-factor SRB models

Including an additional factor into the SRB model greatly improves the in sample fit, as seen in Table 3.5, but dispite of this, there is hardly any difference to detect between the models' output in terms of rate expectations and term premia, as seen in Figure 3.7. This result echoes the mantra that the potential merits of a model should never be judged only on its in-sample performance. Clearly, as more yield curve factors are added, the in-sample fit will, by definition, improve. One can think of a good in sample fit, as being a minimum requirement for including a given model into the toolbox of models that one relies on: as long as a model provides a reasonably good in sample fit, say below 10-20 basis points per maturity bucket, then it is worthwhile to consider whether other features makes it worthwhile to start using the model.



The figure shows the 10-year term premium estimates from the 3- and 4-factor SRB models. In the figure, the upper panel shows the 10-year term premium, and the lower panel shows the 10-year expectations component. Estimates from the standard $SRB3$ model is shown in blue, and the mean adjusted version is shown in (red). For comparison, the observed 10-year yield is plotted in yellow, in the upper panel.

Fig. 3.7. the SRB3-model with bias correction

	RMSE (basis points)										
τ in months	3	12	24	36	48	60	72	84	96	108	120
SRB3	8.50	14.08	4.89	3.54	5.28	5.56	4.51	2.57	1.15	3.44	6.38
SRB3, bias corrected	1.48	4.76	3.92	2.77	1.01	1.94	2.52	2.12	0.98	1.02	3.06
	Eigenvalues of $\hat{\Phi}$										
	1	2	3	4							
SRB3	0.990	0.957	0.808								
SRB4	0.990	0.957	0.893	0.640							

RMSEs are calculated and shown in basis points for the two models under investigation, together with the eigenvalues of $\hat{\Phi}$, sorted in descending order.

Table 3.5. Case B:RMSE and eigenvalues

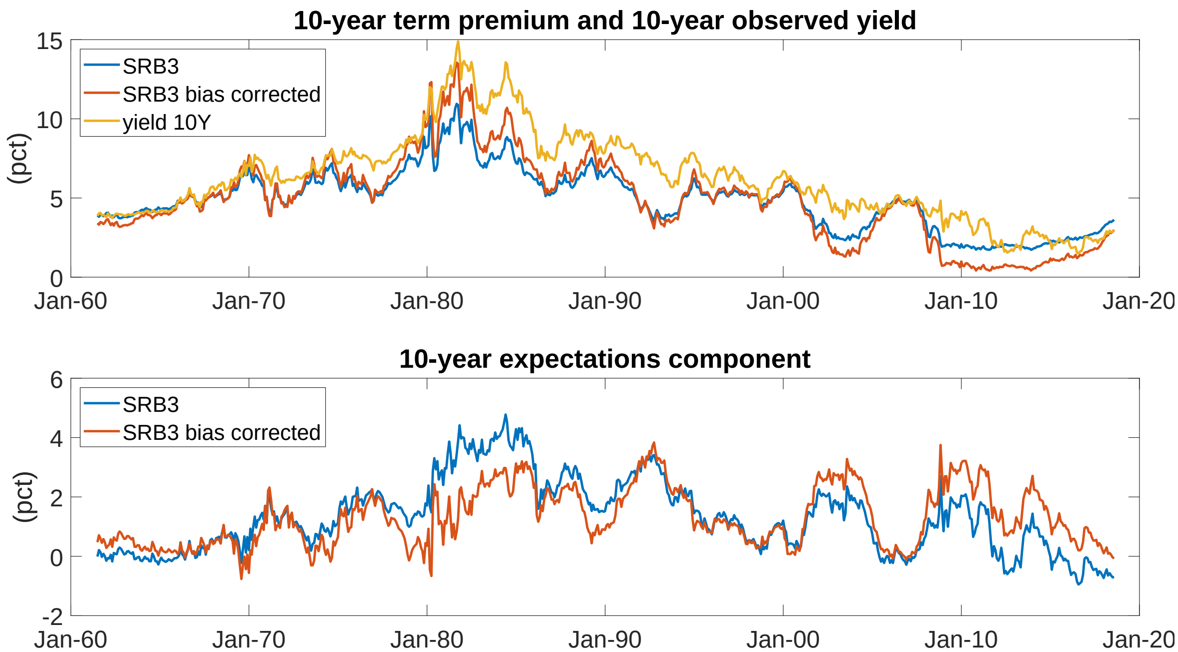
Case (C): Comparing 3-factor SRB models with and without bias correction

Bauer, Rudebusch, and Wu (2012) remind us that Φ, in the VAR model: $X_t = \mu + \Phi \cdot (X_{t-1} - \mu) + \Sigma_X e_t$, most likely will be biased downwards in term structure applications, because lagged endogenous variables are included, and the VAR is estimated using OLS. Too low persistency in the yield curve factors, and hence also in the short rate process, may severely impact measure derived from the term structure dynamics, such as rate expectations and term premia estimates. On the other hand, it is worth noting that the potential bias is reduced, as the number of time series observations is increased. For instance, Engsted and Pedersen (2014) show that the bias nearly disappears when the sample comprises 500 observations in the time series dimension. However, the simulation study they conduct is based on a VAR that at the outset exhibits somewhat less persistency compared to what is typically encountered in term structure models. So, it is not clear that their result can be directly transferred to a term structure context.

Using the closed form bias-correction methodology of Pope (1990), we compare the impact of bias correction on the 3-factor SRB model. Results are shown in Figure 3.8 and Table 3.6.

A higher degree of persistency implies that the short rate process reverts in a more sluggish manner towards its sample mean. The impact of this on derived rate expectations and term premia estimates, is that the time series evolution of the 10-year rate expectation (we use 10-years here because this is what is shown in the figure, but the conclusion holds for any maturity point) is that the rate expectation becomes more volatile, assuming that one or more rate cycles are contained in the data sample. The mirror image of this is, of course, that the term premia will evolve more smoothly. And, this is exactly what we observe in Figure 3.8.

Table 3.6 shows that the bias correction has absolutely no bearing on the in sample fit. It is interesting to note that accounting for potential biases in $\hat{\Phi}$ only affects the relative weighting of the rate expectation and term premia components (that together make up the model fitted yield), and not of the overall fit of the model.



The figure shows the 10-year term premium estimates from the 3-factor SRB model and a version of the model where Φ in the transition equation, $X_t = \mu + \Phi\left(\cdot X_{t-1} - \mu\right) + \Sigma_X e_t$, is bias corrected according to Pope (1990). In the figure, the upper panel shows the 10-year term premium, and the lower panel shows the 10-year expectations component. Estimates from the standard $SRB3$ model is shown in blue, and the mean adjusted version is shown in (red). For comparison, the observed 10-year yield is plotted in yellow, in the upper panel.

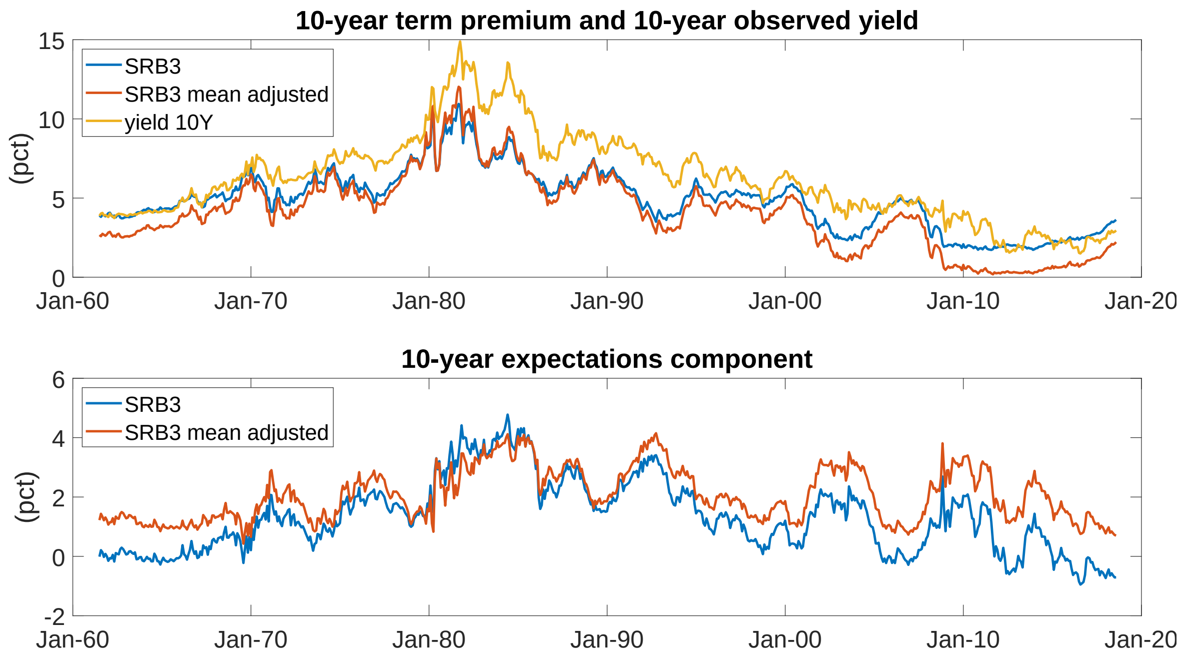
Fig. 3.8. the SRB3-model with bias correction

	RMSE (basis points)										
τ in months	3	12	24	36	48	60	72	84	96	108	120
SRB3	8.50	14.08	4.89	3.54	5.28	5.56	4.51	2.57	1.15	3.44	6.38
SRB3, bias corrected	8.50	14.08	4.89	3.54	5.28	5.56	4.51	2.57	1.15	3.44	6.38
	Eigenvalues of $\hat{\Phi}$										
	1	2	3								
SRB3	0.991	0.957	0.808								
SRB3, bias corrected	0.998	0.962	0.816								

RMSEs are calculated and shown in basis points for the two models under investigation, together with the eigenvalues of $\hat{\Phi}$, sorted in descending order.

Table 3.6. Case C: RMSE and eigenvalues

Case (D): The SRB model with a constraint on the mean of the short rate



The figure shows the 10-year term premium estimate from the 3-factor SRB model, and a version where the mean vector, μ, in $X_t = \mu + \Phi\left(\cdot X_{t-1} - \mu\right) + \Sigma_X e_t$, is altered. A constraint is imposed such that the mean of the short rate factor equals 2.00% (changed from 4.88%, which is its sample mean). The the slope and the curvature parameters are left at their sample means. In the figure, the upper panel shows the 10-year term premium, and the lower panel shows the 10-year expectations component. Estimates from the standard $SRB3$ model is shown in blue, and the mean adjusted version is shown in (red). For comparison, the observed 10-year yield is plotted in yellow in the upper panel.

Fig. 3.9. Mean adjusting the SRB3-model

For scenario analysis, or because the sample mean is judged to poorly reflect the true mean of one or more of the underlying yield curve factors, it may be relevant to impose constraints on the mean vector, μ, in the transition equation. Figure 3.9 shows the impact of doing this. For illustrative purposes, it is assumed that the true mean of the short rate is 2.00%, and this is imposed on the optimisation algorithm estimating the VAR parameters; the in sample mean is 4.88%, so changing it to 2.00% is somewhat of a moderate to substantial change. In Figure 3.9 a comparison between the standard 3-factor SRB model, i.e. where sample means are used for μ, and the constrained version of the model. It is clear that this constraint has a significant influence on the 10-year rate expectation, and the 10-year term premium. In fact, the time series evolution of the gauges shown in Figure 3.9 bears a lot of resemblance to the ones produced in Case C, i.e. where the bias correction is active, as shown in Figure 3.8.

Having a look at the eigenvalues in Table 3.7, confirms that not only is the mean of the short rate changed, also the persistence of the process has changed: the eigenvalue for the short rate process (the first factor) in the plain $SRB3$ model is 0.9909, when constraining the mean, this eigenvalue increases to 0.9956, and finally, when bias correction is introduced the eigenvalue equals 0.9976.

Why does the persistence of the VAR model change, when constraints are imposed on μ? One part of the system has to change such that the constrained set of means can be achieved, and the only part left in the equation is the Φ matrix, since the fit of the model, as seen in Table 3.7, is virtually unchanged. Let's consider the univariate case (which clearly generalises to the multivariate case), and assume that the yield curve factor is the short rate and it follows the AR(1) process:

	RMSE (basis points)										
τ in months	3	12	24	36	48	60	72	84	96	108	120
SRB3	8.50	14.08	4.89	3.54	5.28	5.65	4.51	2.57	1.15	3.44	6.38
SRB3, mean adjusted	8.42	14.03	4.89	3.56	5.31	5.60	4.54	2.59	1.16	3.47	6.45
	Eigenvalues of $\hat{\Phi}$										
	1	2	3								
SRB3	0.991	0.957	0.808								
SRB3, mean adjusted	0.996	0.960	0.827								

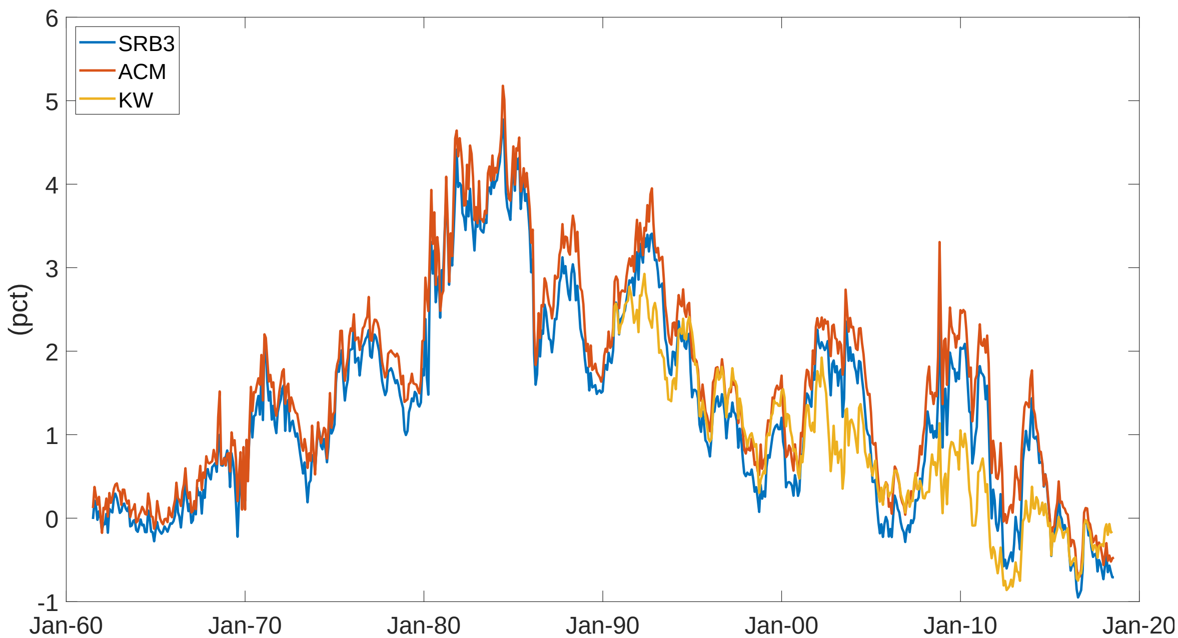
RMSEs are calculated and shown in basis points for the two models under investigation, together with the eigenvalues of $\hat{\Phi}$, sorted in descending order.

Table 3.7. Case D: RMSE and eigenvalues

Case (E): Comparing the 3-factor SRB model with published term premium from the ACM and KW models

It is seen that Figure 3.10 confirms the notion highlighted in above figures, that the main variation in term premia do not come from the applied model, but from the data sample used, and the thereby implied persistency of the underlying yield curve values and the convergence level for the factors, i.e. their sample mean. The KW premium estimate deviates most from the other two, and this is probably due to the different estimation window used. KW estimates only spans the period from 1990 and on-wards, and the persistency and the sample mean of the factors is therefore likely different from the parameter estimates used in ACM and SRB3. We seen that the $SRB3$ and ACM $10-$year term primia are very similar, both in terms of dynamic behaviour and levels.

A relevant question to ask, with respect to published model estimates, is whether the parameter estimates are updated regularly, or whether they are kept constant over time. One would think that it would be better to update parameters such that the derived metrics make use of as much information as possible. However, updating parameters means that the newly produced estimates are not backward-comparable, since earlier estimates were based on another set of parameter estimates. This then opens the gate to potential confusion, since different vintages of term premia estimates would have to be published, one vintage for each parameter update, and it does not take much imagination to envisage the problems that can transpire from such a setting, especially when the metrics are used to support policy decisions on e.g. strategic asset allocation issues. Another issue is the standard choice to be made in terms of the length of the data history to include in the estimation of any model, i.e. the trade-off between additional parameter accuracy against the possibility of covering distinct economic regimes. In the end, this choice is not so trivial.



The figure shows the 10-year term premium estimates from the 3-factor SRB mode, the ACM model (Adrian, Crump, and Mönch (2013)) and the KW model (Kim and Wright (2005)). The KW estimate is available from 1990 on-wards. Both the ACM and KW estimates are downloaded from Bloomberg.

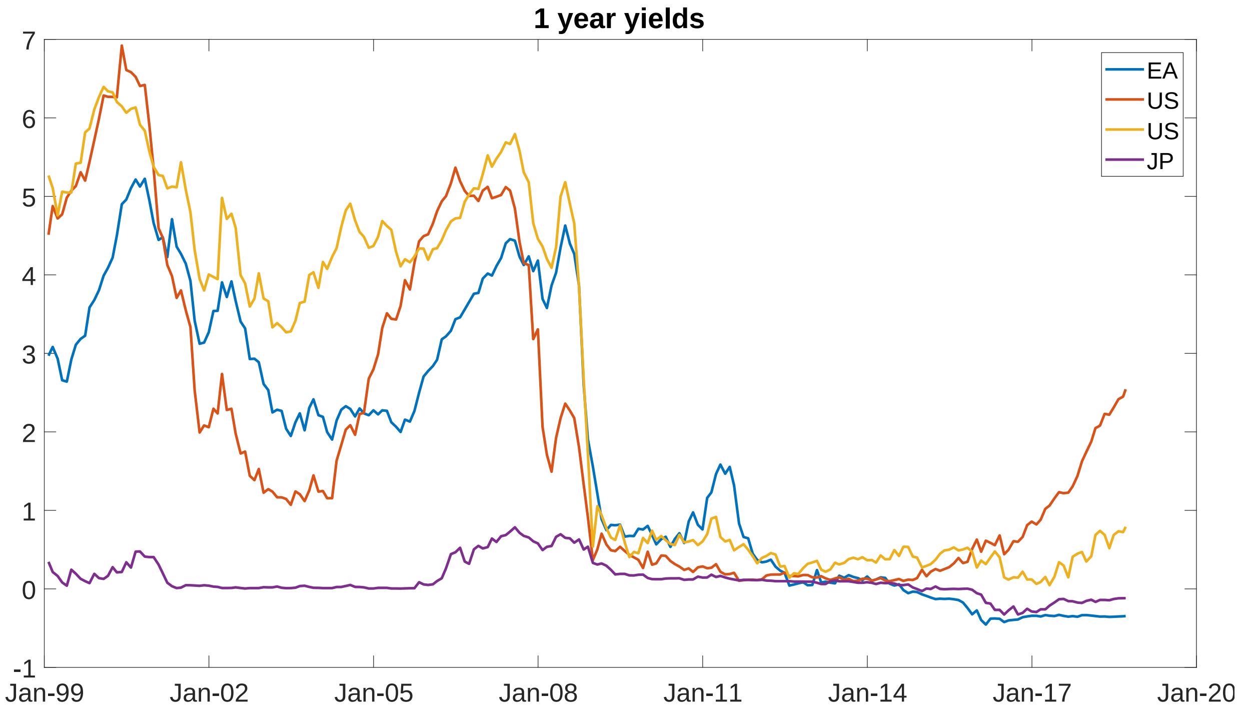
Fig. 3.10. SRB3, ACM, and KW 10-year permia

3.5 Modelling yields at the lower bound

A non-negligible part of the term structure literature deals with the modelling of the yield curve and its dynamics, when the level of yields approaches zero, or hovers around some low level.[6] Such approaches have become increasingly popular, as the monetary policy rates have decreased steadily in Japan, the US

[6] For a representative sample of the literature, see, Black (1995), Lemke and Vladu (2017), Kim and Priebsch (2013), Christensen and Rudebusch (2013), Wu and Xia (2015), Bauer and Rudebusch (2014), and Krippner (2015b).

and UK, as well as the euro zone, at least since around 2008/2009. To illustrate, Figure 3.11 shows the evolution of the short end of the term structure in the mentioned economics. The majority of this literature



The figure shows the evolution of the 1 year yields in the euro area, the US, UK and Japan, from January 1999 to August 2018. The data used are obtained from Bloomberg, and the following series are used: EUSWE1 Curncy (euro area OIS rate), I11101Y Index (on-the-run US curve), GUKG1 Index (UK generic yield curve), GTJPY1Y Govt (JP generic yield curve). Data are observed monthly.

Fig. 3.11. The maturity and time dimension of yield curve data

falls within the arbitrage-free framework, and it builds on Black (1995). Black suggests that the observed nominal rate, r_t, cannot be negative, because agents in the economy can hold cash at zero cost. We know now, that this is not necessarily completely true, since we have seen negative yields to a great extent, over the last years: just have another look at Figure 3.11 and observe that the 1 year yields is Japan and the euro area has been in negative territory since 2015. And, today (18 September 2018), according to Bloomberg, the German sovereign curve displays negative yields from the 3-months to the 6-years maturity points. So, there are storage costs, and the possibility of being robbed. For these, and possibly other reasons, it is possible to observe negative rates in the economy.[7] But, this does of course not invalidate the modelling

[7] A negative rate can be interpreted as the storage cost of money, and/or the insurance premium to be paid to avoid running the possibility of being robbed while have large amounts of cash tucked away in the mattress at home - just imagine, for example, how many mattresses Goldman Sachs would need.

idea proposed by Black (1995); rather than having a zero-lower bound, we can simply work with a lower bound, set at some reasonably low level.

Following Black (1995) the observed short rate, r_t, is modelled like a call-option, where the underlying asset is the unconstrained shadow short rate, s_t:

$$r_t = max(0, s_t) \tag{3.22}$$

Here, we will also use Black (1995) as a starting point, but then we will deviate from the main-stream approach, and build a Nelson-Siegel inspired shadow short rate model. This is done mainly for illustrative purposes, but also in the hope that it possibly could be useful from a yield-curve practitioners view-point.

One of the arguments for using a shadow short rate model is that traditional dynamic yield curve models have difficulties in matching the persistence displayed by yields when they evolve around a lower boundary, as e.g. seen in Figure 3.11 since 2009. As we have seen, yield curve factors are modelled using a stationary VAR model framework, and, consequently, yield curve factors, and thereby yields, will naturally converge back to their historical means, when projected forward. To circumvent this “problem”, the shadow short rate idea allows for the evolution of an unobserved short rate process, which is then truncated at some lower level, if the process at some point passes this threshold. In this way, if the underlying process, i.e. s_t, displays some level of persistence and stays in the truncation zone for an extended period of time, we will be able to replicate the observed dynamics of r_t, and then also the hovering dynamics of yields at longer maturities. There is one other potential benefit of shadow rate models, and that is if the short rate is modelled together with macroeconomic variables. It is econometrically challenging to model the joint dynamic evolution of the short rate and macroeconomic variables, if the short rate appears to be truncated, i.e. if it stays around the lower bound for years. Allowing the shadow short rate to move freely makes it an ideal candidate to enter into a model where the evolution of the yield curve and macroeconomic variables are modelled jointly.

Ok, enough introductory talk: lets get to work. First thing first: we need to agree on an appropriate functional form for the truncation function in (3.22). One short-hand approach is presented by Coche, Nyholm, and Sahakyan (2017), and this is what we will rely on here:

$$\tilde{y}_\tau = \begin{bmatrix} 1 & 1 - \frac{1-\gamma^\tau}{(1-\gamma)\cdot\tau} & -\gamma^{\tau-1} + \frac{1-\gamma^\tau}{(1-\gamma)\cdot\tau} \end{bmatrix} \cdot \tilde{X}_t \tag{3.23}$$

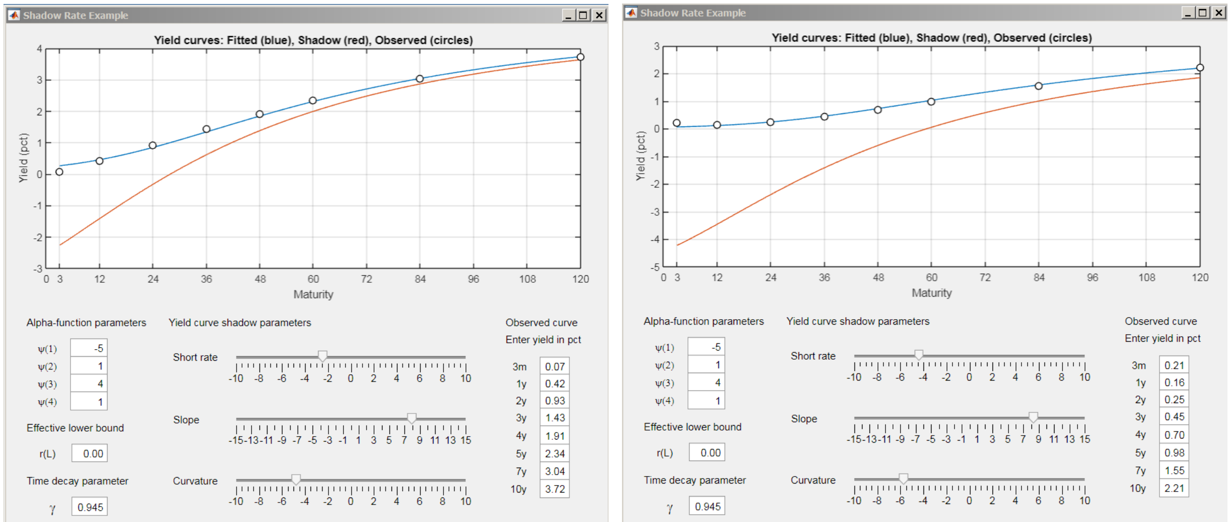
$$\alpha(X) = \frac{\mathtt{tanh}\left(\psi_1 \cdot X_{(2,1)} + \psi_2\right) + 3}{2} \cdot \frac{\mathtt{tanh}\left(\psi_3 \cdot X_{(3,1)} + \psi_4\right) + 3}{2} \in [1, 4] \tag{3.24}$$

$$y_\tau = r_L + \frac{\tilde{y}_\tau - r_L}{1 - e^{\left[-\alpha\left(\tilde{X}\right)\cdot(\tilde{y}_\tau - r_L)\right]}} \tag{3.25}$$

Lets have a look to better understand what the components of this function are about. It is important to know that (3.24) and (3.25) work together with the short-rate-based (SRB) version of the Nelson-Siegel model, i.e. where the factors have interpretations as [short rate, slope, curvature], as shown in (3.23). The variables $\tilde{y}$ and $\tilde{X}$, refer to the shadow yield curve and the shadow factors, respectively. The shadow factors are the shadow short rate, the shadow slope, and the shadow curvature.[8] Equation (3.25) indicates that the shadow yield curve, $\tilde{y}_t$, is transformed into the observed yield curve, y_t. This is similar to the traditional shadow-short-rate set-up, where the dynamics of the shadow rate impacts the shape and location of the whole yield curve. The α-function in (3.24) generates a scalar-weight that is applied to the shadow yield curve, depending on the values at time t, of the shadow-slope and the shadow-curvature factors, contained in $\tilde{X}$. ψ_1-ψ_4 are scaling constants applied to the shadow slope and the shadow curvature. There are quite a number of moving parts - the impact of parameter constellations of ψ_1 to ψ_4, together with the value of the shadow factors contained in $\tilde{X}$ can be explored using the interactive MATLAB app called "PG2TSM_SSR_original" contained in the MATLAB library accompanying this booklet.

Two screen-shots generated using the app is shown in Figure 3.12. This is done to give a brief view on how certain parameter settings affect the shape and location of the curves.[9]

[8] For practical reasons, when the model is estimated we constrain the shadow curvature to be equal to the curvature obtained from the short rate based version of the Nelson-Siegel model.
[9] The app is flexible and allows for analysing other curves, and all parameters can be selected by the user.



(a) September 2009 (b) November 2011

Panel (a) shows an example using data from 30 September 2009. The black circles in the figure shows the observed curve on this day. The red curve shows the shadow short rate, generated using the app. And, the blue line shows the corresponding fitted yield curve, i.e. the transformed shadow short rate using equations (3.23)-(3.24). Similarly, Panel (b) shows a fitting example using data observed on 30 November 2011.

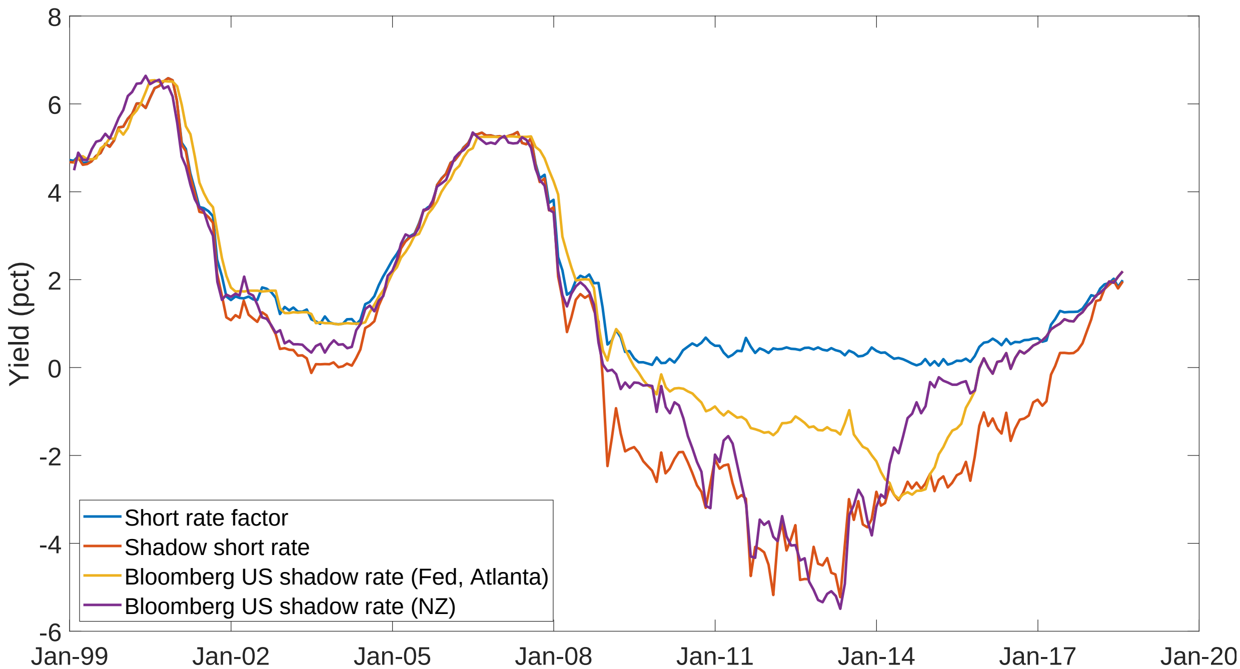
Fig. 3.12. Examples using the MATLAB Shadow Short Rate App

One thing is to use the above mentioned Shadow Short Rate App to fit the shadow short rate model to a single yield curve. Although it would be entertaining (at least for a while), it would be too consuming to fit shadow short rates to the whole set of monthly US yields cover the period from 1961 to 2018. Instead, we apply equations (3.23)-(3.24) to the whole data set at once, and minimise the overall sum of squared residuals to find the shadow short rate and the shadow slope; we impose the constraint that the shadow curvature is equal to the SRB curvature estimated on the observed yields. To obtain the estimates, the following steps are applied:

1. Estimate the SRB model via the TSM class, to obtain $\hat{X}_t$ $\forall t$ using (3.23)
2. Guess values for $\hat{\hat{X}}_t$ $\forall t$, a handy way to make these guesses is to use $\hat{X}_t$
3. calculate $\hat{\alpha}_t | \{\hat{\hat{X}}_t, \psi_1, \psi_2, \psi_3, \psi_4\}$, i.e. one value for $\hat{\alpha}$ per observation point included in the data set, conditional on the generated shadow factors and the fixed parameters ψ_1 to ψ_4. This step is done via (3.24)
4. calculate $\hat{y}_t$ $\forall t$ using (3.25)
5. calculate the sum of squared residuals between the observed yield curve data, y, and the fitted yield curves,

6. ask MATLAB to minimise this quantity: $\sum_t \sum_{tau} (y - \hat{y})^2$, calculated in step 5, by repeating steps 2-5, until optimal parameter values for the shadow short rate and the shadow slope are obtained

Following these steps produces a time series of shadow short rates as shown in Figure 3.13.



The figure shows the estimated shadow short rate using the estimation framework outlined in the text (red line), based on Coche, Nyholm, and Sahakyan (2017). This estimate is compared with two officially published estimates downloaded from Bloomberg. One is produced by Fed Atlanta using the model presented in Wu and Xia (2015) (yellow line), and the other is produced by Leo Krippner (Reserve Bank of New Zealand) following Krippner (2015b) (purple line). Both of these series are available via Bloomberg as "wuxiffrt index" and "nzssus index", respectively. The last (blue) line shows the short rate factor estimated by the SRB model.

Fig. 3.13. Shadow short rate estimates

It is observed that the time series behaviour of our measure[10] (red line) is similar to the estimate produced by Krippner (2015b) (purple line). The minimum value for both estimates occur on the same date (April 2013), and the dynamics towards and from this minimum point is roughly the same, with our measure falling a bit faster, and raising towards normalisation-levels a bit slower, compared to Krippner's measure. In contrast to this, Fed Atlanta's shadow short rate measure, based on Wu and Xia (2015) (yellow line), behaves in a distinctly different manner. It decreases very slowly, almost linearly until October 2013, after which the decline picks up speed, and it reaches its minimum on May 2014.

[10] By "our measure" I mean the estimate obtained from the methodology proposed by Coche, Nyholm, and Sahakyan (2017). And, "our" is used as an inclusive term here, that also comprises you, the reader: because, the code is available in the annex, and it can be used freely (at your own risk, of course).

A detailed chronology of the Fed's QE actions is provided in Krippner (2015a), and he further shows that the Krippner measure better reflects these actions, than the Wu and Xia short rate measure does. His conclusion is therefore, that as a gauge of the effective monetary policy stance, when unconventional policies are enacted, the Krippner measure is more precise. The reason for this is probably, as also highlighted by Krippner (2015a), that Krippner's model includes two-factors, and the model of Wu and Xia (2015) includes three factors. We typically model the yield curve using three factors, for example, the level, the slope, and the curvature. However, when yields are close to the effective lower bound, one of these dimensions will be redundant, because the short rate is fixed, and the level and the slope will consequently measure the same thing, namely the difference between a constant and the long end of the yield curve.[11] A two-factor model is therefore more appropriate to use in such cases. But, what happens then when the economy exits the lower bound period, and it again becomes relevant to use three factors? This is where the approach of Coche, Nyholm, and Sahakyan (2017) comes into play. The derived shadow short rate measure is based on three-factors, the short rate, the slope, and the curvature - but one of the factors (the curvature) is frozen, and left unchanged, when the economy enters the effective lower bound period. And, when exit is observed (well, rather, judged to have occurred), the factor is again unfrozen. With the chosen factor structure, it is enough to reduce the dimensionality in the direction of the curvature, because the short short rate and the slope will not produce redundancies. This would have been the case, if the traditional factor interpretation as level, slope, and curvature had been chosen. To some extent, the Coche, Nyholm, and Sahakyan (2017) methodology resembles that of a regime-switching model, where the regime is imposed exogenously on the curvature factor.

3.6 Summary

Honestly, the materials covered in this chapter went a bit beyond what I had planned at the outset. But, I hope that I have still managed to convey the main messages, at an acceptable and practical level, without messing things up too much. The main takeaways from the above materials can be summarised in the following way:

(1) Term structure models are best thought of in terms of a state-space model, where the state equation evolves the yield curve factors over time using a VAR(1) model, and the observation equation translates the yield curve factors into fitted yields using a loading matrix and possibly a constant as well.

(2) Even if a state-space model is used to characterise the dynamics and cross-sectional dimensions of yield curve data, it is not always necessary to estimate the model using the Kalman-filer. A two-step OLS procedure is often faster and yields the same results.

[11] Let l, s, c, be the level, the negative of the slope, and curvature, respectively. If $r = 0$, or some other fixed lower bound, then we have that $r = 0 = l + s \Leftrightarrow -s = l$.

(3) Assigning a certain economic and/or financial meaning to the yield curve factors is done via the choice of the loading matrix (in the observation equation).

(4) Yield curve factor models can be rotated, such that their factor interpretation changes, without affecting the model fit.

(5) Yield curves comprise information about future rate expectations and term premia. The distribution between these two important gauges is model and parameter dependant. Most important for the dissection of yields along these two dimensions is the model implied mean for the short rate, and the persistence of the VAR(1) in the state equation.

(6) The persistence of the VAR can be changed via bias correcting techniques applied to the (autoregressive) VAR parameters, and by imposing constraints on the mean vector in the VAR model.

(7) To better capture the behaviour of yields that evolve around some lower effective bound, it is possible to apply the concept of shadow short rate models. These models rely on a truncation function that maps unrestricted factors into restricted factors that match observed yields. In essence, when such a non-linear truncation function is included into the modelling set-up, the state-space model becomes non-linear in the underlying factors and the Kalman-filter is no longer usable. Instead, non-linear Kalman-filters must be applied, or alternative methodologies, as presented in Section 3.5.

3.7 Appendix: MATLAB code

3.7.1 Yield curve model estimation via the SSM toolbox

filename: Basic_yield_curve_setup.m

```
%% Script for: the basic yield curve modelling setup
% Access to the MATLAB class GSW is required.
%
%% Loading and plotting data
%
load('Data_GSW_factors_Course_2018.mat');
GSW_         = GSW;                    % creates an instance of the GSW class
GSW_.tau     = [3 12:12:120]';         % vector of maturities
GSW_.beta    = GSW_factors(:,2:5);     % yield curve factors
GSW_.lambda  = GSW_factors(:,6:7);     % lambdas
GSW_         = GSW_.getYields;         % getting yields

dates = GSW_factors(:,1);
Y     = GSW_.yields;
tau   = GSW_.tau;
nTau  = size(tau,1);

figure('units','normalized','outerposition',[0 0 1 1])
    surf(tau./12,dates,Y)
    date_ticks = datenum(1960:4:2020,1,1);
    set(gca, 'ytick', date_ticks);
    datetick('y','mmm-yy','keepticks')
    xticks(0:1:11), xticklabels({tau}),
    xlabel('Maturity (months)'), zlabel('Yield (pct)'),
    view([-109 38]),
    ytickangle(-25),
    set(gca, 'FontSize', 18)
    %print -depsc Y3D

nn = 3*42;
figure('units','normalized','outerposition',[0 0 1 1])
    subplot(2,1,1), plot(dates,Y(:,11)), datetick('x','mmm-yy'),
                    title('10 year yield'),
                    ylabel('Yield (pct)')
                    set(gca, 'FontSize', 18)
                    subplot(2,1,2), plot(tau,Y(nn,:)','*-'),
                    xlabel('Maturity (months)'),
                    title(['Yield curve on ' datestr(dates(nn))]),
                    ylabel('Yield (pct)'),
                    ylim([0,ceil(max(Y(nn,:))+1)])
                    xticks(tau), xticklabels({tau}),
                    set(gca, 'FontSize', 18)
    %print -depsc Yslices

%% ..................................................................
%  ... Empirical factor structure and the Nelson-Siegel parameterisation
%  ... This section uses the pMap function that appears at the end
```

```
%  ... of this script.
%  .................................................................

% ... Model selection
%
flagg   = 'Emp';                        % choose:  'NS'  -> Nelson-Siegel
                                       % or    :  'Emp' -> Empirical model
[nObs,nTau] = size(Y);
Y_dat       = [Y ones(nObs,nTau)];

% ... assigning starting values ...
%
Phi0 = [ 0.99 0.00 0.00;
         0.00 0.99 0.00;
         0.00 0.00 0.99];
k0   = [ 0; 0; 0 ];
Sx0  = [ 1.00;
         0.00; 1.00;
         0.00; 0.00; 1.00 ];
b0   = [ ones(nTau,1) linspace(1,0,nTau)' zeros(nTau,1) ];
a0   = zeros(nTau,1);
Sy0  = 1.00*ones(nTau,1);

p0  = [ Phi0(:); k0(:); Sx0(:); b0(:); a0(:); Sy0(:) ];
nP  = size(p0,1);

% ... defining upper and lower parameter bounds
%
lb_=-inf(nP,1); lb_(1:9,1)=-1; lb_(10:12,1)=0;
lb_([13;15;18])=0; lb_(19:51,1)=-1;
lb_(52:62,1)=-1; lb_(63:73,1)=0;

ub_ = inf(nP,1); ub_(1:9,1)= 1.1; ub_(10:12,1)=1;
ub_([13;15;18]) = 1; ub_(19:51,1)=1;
ub_(52:62,1)=1; ub_(63:73,1)=1;

% ... parameter constraits ...
%
nP  = size(p0,1);
% ... equal yield vols across all maturities
Aeq = zeros(nTau-1,nP);
Aeq(1,[63 64])=[1 -1];Aeq(2,[64 65])=[1 -1];Aeq(3,[65 66])=[1 -1];
Aeq(4,[66 67])=[1 -1];Aeq(5,[67 68])=[1 -1];Aeq(6,[68 69])=[1 -1];
Aeq(7,[69 70])=[1 -1];Aeq(8,[70 71])=[1 -1];Aeq(9,[71 72])=[1 -1];
Aeq(10,[72 73])=[1 -1];
% ... no constants in the observation equation (i.e. a=0)
if (strcmp(flagg,'NS'))
    Aeq(11:21,52:62) = eye(11);
end
% ... value of the constraints
beq = zeros(size(Aeq,1),1);

Mdl_    = ssm(@(p) pMap(p,flagg,tau));
options = optimoptions(@fmincon,'Algorithm','interior-point',...
```

```
                                'MaxIterations',1e6, ...
                                'MaxFunctionEvaluations',1e6, ...
                                'TolFun', 1e-6, 'TolX', 1e-6);

[ EstMdl_, p_hat ] = ...
         estimate( Mdl_,Y_dat,p0,'Display','iter','Aeq',Aeq,'beq',beq,...
                  'lb',lb_,'ub',ub_,'univariate',true,'options',options );

x_filter  = filter( EstMdl_, Y_dat );  % extract filtered state variables

% ... plotting the results
%
X_hat   = x_filter(:,1:3);
Phi_hat = EstMdl_.A(1:3,1:3);
k_hat   = diag(EstMdl_.A(1:3,4:6));
b_hat   = EstMdl_.C(1:11,1:3);
a_hat   = diag(EstMdl_.C(1:11,4:14));
if (strcmp(flagg,'NS'))
    L_hat   = p_hat(19);
end
Y_hat     = (a_hat + b_hat*X_hat')';
RMSE_bps  = 100.*(mean((Y-Y_hat).^2)).^(0.5);

figure('units','normalized','outerposition',[0 0 1 1])
    plot(dates,X_hat,'Linewidth',2), legend('Factor 1', 'Factor 2', 'Factor 3'),
   date_ticks = datenum(1960:10:2020,1,1);
    set(gca, 'xtick', date_ticks);
    datetick('x','mmm-yy','keepticks')
    set(gca, 'FontSize', 30)
%     if (strcmp(flagg,'NS'))
%         print -depsc EstFactors_NS
%     else
%         print -depsc EstFactors_Emp
%     end

figure('units','normalized','outerposition',[0 0 1 1])
    plot(tau,b_hat,'Linewidth',2),
    legend('Loading 1', 'Loading 2', 'Loading 3','Location','SE'),
    xlabel('Maturity (months)'), ylim([-1 1.25]),
    xticks([3 12:12:120]'), xticklabels({tau})
    if (strcmp(flagg,'NS'))
        title(['time-decay parameter = ', num2str(L_hat)])
    end
    set(gca, 'FontSize', 30)
%     if (strcmp(flagg,'NS'))
%         print -depsc EstLoadings_NS
%     else
%         print -depsc EstLoadings_Emp
%     end

disp(RMSE_bps)

%% ..............................
%  ...  Rotation matrices
```

```
%  ..............................
% ................................................
% ... rotating toward 2Y, 5Y and 10Y yields
% ................................................
NS_          = TSM;                  % create an instance of the TSM class
NS_.yields   = Y;                    % populating the model with input data
NS_.tau      = tau;
%NS_.mP_pre   = [0;0;0];
NS_.DataFreq = 12;

NS_ = NS_.getDNS;                    % estimate Dynamic Nelson-Siegel model
                                     %   using OLS

FunErr2 = @(p,dat_) sum(sum((dat_(:,1)-dat_(:,2:end)*p).^2));  % calc SSR

p0   = [1;0;0];  % starting values - no constant, only slope coefficients
Aeq  = [1 0 0];  % constraining the coefficient of the level factor to be =1
beq  = 1;
lb   = [0.99;0;0];  % just to help fmincon a bit
ub   = [1.01;1;1];

lst  = [3;6;11];
A_rotate = zeros(size(p0,1),size(p0,1));
for ( z=1:3 )
    dat      = [NS_.yields(:,lst(z,1)) NS_.beta'];
    [pHat]   = fmincon(FunErr2,p0,[],[],Aeq,beq,lb,ub,[],[],dat);
    A_rotate(z,:) = pHat';
end

% ... double checking if the objective is achieved
X_rotate = A_rotate*NS_.beta;
b_rotate = NS_.B*inv(A_rotate);

figure('units','normalized','outerposition',[0 0 1 1])
    subplot(3,1,1), plot(dates,[NS_.yields(:,3) X_rotate(1,:)'], ...
                                                  'LineWidth',2),
    date_ticks = datenum(1960:10:2020,1,1);
    set(gca, 'xtick', date_ticks), title('2 year');
    datetick('x','mmm-yy','keepticks'), legend('Obs.','Fit','Location','SW')
    set(gca, 'FontSize', 20)
    subplot(3,1,2), plot(dates,[NS_.yields(:,6) X_rotate(2,:)'], ...
                                                  'LineWidth',2),
    date_ticks = datenum(1960:10:2020,1,1);
    set(gca, 'xtick', date_ticks), title('5 year'),
    datetick('x','mmm-yy','keepticks'), legend('Obs.','Fit','Location','SW')
    set(gca, 'FontSize', 20)
    subplot(3,1,3), plot(dates,[NS_.yields(:,11) X_rotate(3,:)'], ...
                                                  'LineWidth',2),
    date_ticks = datenum(1960:10:2020,1,1);
    set(gca, 'xtick', date_ticks), title('10 year'),
    datetick('x','mmm-yy','keepticks'), legend('Obs.','Fit','Location','SW')
    set(gca, 'FontSize', 20)
    %print -depsc RotatedFactors2_5_10
```

```
RMSE_rotate = 100*(mean((Y - (b_rotate*X_rotate)').^2)).^(1/2)
[NS_.RMSE;RMSE_rotate]

%% ..........................................................
%  ... Building blocks of the yield curve
%  ..........................................................
%
% Case A: 3-factor SRB model and the DNS model
%
SRB3_ = TSM;
DNS_  = TSM;      % creating class instances

SRB3_.yields=Y; SRB3_.tau=tau; SRB3_.DataFreq=12;
DNS_.yields=Y;  DNS_.tau=tau;  DNS_.DataFreq=12;  % allocating data

SRB3_ = SRB3_.getSRB3;
DNS_  = DNS_.getDNS;                             % estimate the model

RMSE_A = [ SRB3_.RMSE; DNS_.RMSE ];               % generating output
EIG_A  = [ sort(real(eig(SRB3_.PhiP))); ...
           sort(real(eig(DNS_.PhiP))) ];

figure('units','normalized','outerposition',[0 0 1 1])
    subplot(2,1,1), plot(dates,  [ SRB3_.Er(:,11) DNS_.Er(:,11) ...
                                            Y(:,11)], 'LineWidth',2),
    date_ticks = datenum(1960:10:2020,1,1);
    set(gca, 'xtick', date_ticks), ylabel('(pct)')
    datetick('x','mmm-yy','keepticks'), legend('SRB3','DNS','yield 10Y',...
                                                    'Location','NW')
    set(gca, 'FontSize', 20)
    title('10-year term premium and 10-year observed yield')

    subplot(2,1,2), plot(dates, [SRB3_.TP(:,11) DNS_.TP(:,11) ], ...
                                                  'LineWidth',2),
    date_ticks = datenum(1960:10:2020,1,1);
    set(gca, 'xtick', date_ticks), ylabel('(pct)')
    datetick('x','mmm-yy','keepticks'), legend('SRB3','DNS',...
                                                     'Location','NW')
    set(gca, 'FontSize', 20)
    title('10-year expectations component')
    print -depsc Case_A_Er_TP

%
% Case B: 3- and 4-factor SRB models
%
SRB3_ = TSM;
SRB4_ = TSM;      % creating class instances


SRB3_.yields=Y; SRB3_.tau=tau; SRB3_.DataFreq=12;
SRB4_.yields=Y; SRB4_.tau=tau; SRB4_.DataFreq=12;  % allocating data

SRB3_ = SRB3_.getSRB3;
SRB4_ = SRB4_.getSRB4;                             % estimate the model
```

```


RMSE_B = [ SRB3_.RMSE; SRB4_.RMSE ];                % generating output
EIG_B  = [ sort(real(eig(SRB3_.PhiP))); ...
           sort(real(eig(SRB4_.PhiP))) ];

figure('units','normalized','outerposition',[0 0 1 1])
    subplot(2,1,1), plot(dates,  [ SRB3_.Er(:,11) SRB4_.Er(:,11) ...
                                                  Y(:,11)], 'LineWidth',2),
    date_ticks = datenum(1960:10:2020,1,1);
    set(gca, 'xtick', date_ticks), ylabel('(pct)')
    datetick('x','mmm-yy','keepticks'), legend('SRB3','SRB4', ...
                                             'yield 10Y','Location','NW')
    set(gca, 'FontSize', 20)
    title('10-year term premium and 10-year observed yield')

    subplot(2,1,2), plot(dates, [SRB3_.TP(:,11) SRB4_.TP(:,11) ], ...
                                                       'LineWidth',2),
    date_ticks = datenum(1960:10:2020,1,1);
    set(gca, 'xtick', date_ticks), ylabel('(pct)'),
    datetick('x','mmm-yy','keepticks'), legend('SRB3','SRB4', ...
                                                          'Location','NW')
    set(gca, 'FontSize', 20)
    title('10-year expectations component')
    %print -depsc Case_B_Er_TP

%
% Case C: 3-factor SRB model with and without bias correction
%
SRB3_    = TSM;
SRB3_BC = TSM;      % creating class instances

SRB3_.yields=Y; SRB3_.tau=tau; SRB3_.DataFreq=12;
SRB3_BC.yields=Y; SRB3_BC.tau=tau; SRB3_BC.DataFreq=12; % allocating data
SRB3_BC.biasCorrect = 1;                                % bias correction

SRB3_    = SRB3_.getSRB3;
SRB3_BC = SRB3_BC.getSRB3;                             % estimate the model

RMSE_C = [ SRB3_.RMSE; SRB3_BC.RMSE ];                 % generating output
EIG_C  = [ sort(real(eig(SRB3_.PhiP))); ...
           sort(real(eig(SRB3_BC.PhiP_bc))); ];

figure('units','normalized','outerposition',[0 0 1 1])
    subplot(2,1,1), plot(dates,  [ SRB3_.Er(:,11) SRB3_BC.Er(:,11) ...
                                                  Y(:,11)], 'LineWidth',2),
    date_ticks = datenum(1960:10:2020,1,1);
    set(gca, 'xtick', date_ticks), ylabel('(pct)')
    datetick('x','mmm-yy','keepticks'), legend('SRB3', ...
                                             'SRB3 bias corrected', ...
                                             'yield 10Y','Location','NW')
    set(gca, 'FontSize', 20)
    title('10-year term premium and 10-year observed yield')

```

```
    subplot(2,1,2), plot(dates, [ SRB3_.TP(:,11) SRB3_BC.TP(:,11)], ...
                                                 'LineWidth',2),
    date_ticks = datenum(1960:10:2020,1,1);
    set(gca, 'xtick', date_ticks), ylabel('(pct)')
    datetick('x','mmm-yy','keepticks'), legend('SRB3', ...
                                          'SRB3 bias corrected', ...
                                                    'Location','NW')
    set(gca, 'FontSize', 20)
    title('10-year expectations component')
    %print -depsc Case_C_Er_TP

%
% Case D: 3-factor SRB model with different assumptions on
%                  the mean of the short rate
%
SRB3_   = TSM;      % creating class instances
SRB3_ma = TSM;

SRB3_.yields=Y; SRB3_.tau=tau; SRB3_.DataFreq=12;  % allocating data
SRB3_ma.yields=Y; SRB3_ma.tau=tau; SRB3_ma.DataFreq=12;
SRB3_ma.mP_pre=[2.00;1.79;-1.19];

SRB3_   = SRB3_.getSRB3;                             % estimate the model
SRB3_ma = SRB3_ma.getSRB3;

RMSE_D = [ SRB3_.RMSE; SRB3_ma.RMSE ];               % generating output
EIG_D  = [ sort(real(eig(SRB3_.PhiP))); ...
           sort(real(eig(SRB3_ma.PhiP)))  ];

figure('units','normalized','outerposition',[0 0 1 1])
    subplot(2,1,1), plot(dates,  [ SRB3_.Er(:,11) SRB3_ma.Er(:,11) ...
                                            Y(:,11)], 'LineWidth',2),
    date_ticks = datenum(1960:10:2020,1,1);
    set(gca, 'xtick', date_ticks), ylabel('(pct)')
    datetick('x','mmm-yy','keepticks'), legend('SRB3',...
                                          'SRB3 mean adjusted', ...
                                        'yield 10Y','Location','NW')
    set(gca, 'FontSize', 20),
    title('10-year term premium and 10-year observed yield')

    subplot(2,1,2), plot(dates, [SRB3_.TP(:,11) SRB3_ma.TP(:,11)], ...
                                                 'LineWidth',2),
    date_ticks = datenum(1960:10:2020,1,1);
    set(gca, 'xtick', date_ticks), ylabel('(pct)')
    datetick('x','mmm-yy','keepticks'), legend('SRB3', ...
                                           'SRB3 mean adjusted', ...
                                           'Location','NW')
    set(gca, 'FontSize', 20),title('10-year expectations component')
    %print -depsc Case_D_Er_TP

%
% Case D: 3-factor SRB model against ACM and KW
%
SRB3   = TSM;      % creating class instances
```

```
SRB3.yields=Y;   SRB3.tau=tau;   SRB3.DataFreq=12;      % allocating data

SRB3   = SRB3.getSRB3;                            % estimate the model


figure('units','normalized','outerposition',[0 0 1 1])
    plot(dates, SRB3.TP(:,11), 'lineWidth',2)
    hold on
    plot(ACM(:,1), ACM(:,2), 'lineWidth',2)
    hold on
    plot(KW(:,1), KW(:,2), 'lineWidth',2)

    date_ticks = datenum(1960:10:2020,1,1);
    set(gca, 'xtick', date_ticks), ylabel('(pct)')
    datetick('x','mmm-yy','keepticks'), legend('SRB3','ACM','KW', ...
                                              'Location','NW')
    set(gca, 'FontSize', 20)
    %print -depsc Case_E_Er_TP

%% ..............................................
%  ... Modelling yields at the zero lower bound
%  ..............................................

% ... plot of 1Y rates in EU, US, UK, JP
figure('units','normalized','outerposition',[0 0 1 1])
    plot(Yield1Y(:,1), Yield1Y(:,2:end),'LineWidth',2),
    date_ticks = datenum(1999:3:2020,1,1);
    set(gca, 'xtick', date_ticks), title('1 year yields');
    datetick('x','mmm-yy','keepticks'),
    legend('EA','US','US','JP')
    set(gca, 'FontSize', 20)
    %print -depsc EU_US_UK_JP_1Y_yields

% ... Calling the TSM class to estimate a short rate based (SRB) model
SRB          = TSM;                  % create an instance of the TSM class
SRB.yields   = Y;                    % populating the model with input data
SRB.tau      = tau;
SRB.mP_pre   = [];
SRB.DataFreq = 12;

% ... step 0: fix the parameters that need to be fixed
%
rL = 0.00;               % preset effective lower bound

% ... step 1: estimate the 3-factors from the short rate based model
%
SRB = SRB.getSRB3;
X_  = SRB.beta;          % factors: short rate, slope, curvature
B_  = SRB.B;             % loading structure
Y_  = SRB.yields;        % observed yields (also contained in Y)

% ... step2: estimate the shadow short rate and the shadow slope
%
```

```
X_tmp = X_';
p0    = X_tmp(:);
Aeq   = zeros(nObs,3*nObs);
Aeq(1:nObs,2*nObs+1:end) = eye(nObs);
beq = X_(3,:)';

X_shadow = NaN(size(X_));
options_ = optimoptions(@fmincon,'Algorithm','sqp',...
                                 'MaxIterations',1e8, ...
                                 'MaxFunctionEvaluations',1e8, ...
                                 'TolFun', 1e-4, 'TolX', 1e-4, ...
                                 'display','iter');

FX_min        = @(p) Yshadow( p, Y_, B_, rL );
[ pHat_sr ]   = fmincon(FX_min, p0,[],[],Aeq,beq,[],[],[], options_);
alpha_        = pHat_sr(1:4,1);
X_shadow_hat  = reshape(pHat_sr,nObs,3);

figure('units','normalized','outerposition',[0 0 1 1])
    plot(dates, [ X_(1,:)' X_shadow_hat(:,1) ],'LineWidth',2)
    hold on
    plot(BB_US_shadow_rate(:,1),[BB_US_shadow_rate(:,3), ...
                                 BB_US_shadow_rate(:,2)], ...
                                               'LineWidth',2 )
%     yyaxis right
%     plot(BB_US_shadow_rate(:,1), -(BB_US_shadow_rate(:,4)),':g',...
%                                   'LineWidth',2)
    date_ticks = datenum(1999:3:2020,1,1);
    set(gca, 'xtick', date_ticks)
    datetick('x','mmm-yy','keepticks'),
    legend('Short rate factor','Shadow short rate', ...
           'Bloomberg US shadow rate (Fed, Atlanta)', ...
           'Bloomberg US shadow rate (NZ)', ...
           'Location','SW')
    set(gca, 'FontSize', 20)
    ylabel('Yield (pct)')
%     print -depsc Shadow_sr

figure('units','normalized','outerposition',[0 0 1 1])
    plot(dates, [ X_shadow_hat ],'LineWidth',2)
    date_ticks = datenum(1999:3:2020,1,1);
    set(gca, 'xtick', date_ticks), title('Short rates');
    datetick('x','mmm-yy','keepticks'),
    legend('Sr','Slope','curvature')

%% functions
%
function [R,S,T,U,Mean0,Cov0,StateType] = pMap( p, flagg, tau )
%
% Parameter mapping function for MATLAB's SSM mudule
%
    nTau = size(tau,1);

    Phi  = [p(1) p(4) p(7)  ;
```

```
            p(2) p(5) p(8)  ;
            p(3) p(6) p(9) ];

    k    = diag([p(10);p(11);p(12)]);

    Sx     = zeros(3,3);
    Sx(1,1)=p(13); Sx(2,1)=p(14); Sx(2,2)=p(15);
    Sx(3,1)=p(16); Sx(3,2)=p(17); Sx(3,3)=p(18);

    if (strcmp(flagg,'Emp'))
        b = [ p(19:29,1) p(30:40,1) p(41:51,1) ];
    elseif (strcmp(flagg,'NS'))
        L = p(19,1); p(20:51,1)=0;
        b = [ ones(nTau,1) ...
              (1-exp(-L.*tau))./(L.*tau) ...
              (1-exp(-L.*tau))./(L.*tau) - exp(-L.*tau)];
    else
        disp('The variable flagg must take on either of the following values')
        disp('NS (Nelson-Siegel)')
        disp('or, Emp (empirical model) ')
    end

    a  = diag(p(52:62,1));
    Sy = diag(p(63:73,1));

% ... Assigning the parameters following MATLAB's notation
%
    R = [ Phi k zeros(3,nTau-3); zeros(nTau,3) eye(nTau) ];
    S = [ Sx; zeros(nTau,3) ];

    T = [ b a; zeros(nTau,3) eye(nTau) ];
    U = [ Sy; zeros(nTau,nTau) ];

% ... other assignments
    Mean0     = [];
    Cov0      = [];
    StateType = [ 0 0 0 ones(1,nTau) ];
end

function [ err2, X_shadow, y_shadow, err ] = Yshadow( p0, Y_, B_, rL )
%
% calculating the sum of squared residuals from the static
%    shadow short rate model set-up
%
    nObs = size(Y_,1);

% ... Defining the shadow rate transformations
%
    alfa_  = @(Xshdw,zz) ( tanh(zz(1,1).*Xshdw(2,:)+zz(2,1)) ...
                  +3 )./2 .*( tanh( zz(3,1).*Xshdw(3,:)+zz(4,1) )+3 )./2;

    yFit_  = @(yS_,alpha_,rL_) rL_+(yS_-rL_)./(1-exp(-alpha_.*(yS_-rL_)));

% ... fixing some of the free parameters
```

```
%
    zz_ = [ -5.00; 1.00; 4.00; 1.00 ];

% ... calculating shadow yields
%
    X_shadow = reshape(p0,nObs,3)';
    y_shadow = B_*X_shadow;
    alpha    = alfa_(X_shadow,zz_);
    yFit     = yFit_(y_shadow,alpha,rL)';
    err      = Y_-yFit;
    err2     = sum(sum(err.^2));

end
```

4

Modelling Yields under the $\mathbb{Q}$-measure

4.1 Introduction

In this chapter we will look at yield curve models that exclude arbitrage by construction. The treatment is purposefully superficial, and it will focus on, and emphasise, the pure mechanics of the modelling frameworks. A deep-dive into the mathematical underpinnings of these models, and a full account of how the literature has evolved over time, is beyond the purpose of this booklet, as also indicated by its title.

As a gentle introduction, a 4-factor Short Rate Based (SRB) model will be derived, based on Nyholm (2018). Then we will move on to Joslin, Singleton, and Zhu (2011), a corner piece in the literature, and see how they cut to the bone of the inner workings of term structure models.

4.2 A discrete-time 4-factor SRB model

Our purpose here is to illustrate how the standard linear modelling set-up (see, e.g., Duffie and Kan (1996), Dai and Singleton (2000), and Ang and Piazzesi (2003)) can be used to derive a tailor-made discrete-time arbitrage-free model that has a loading structure similar to that of a dynamic Svensson and Söderlind (1997) model, but where the first factor can be interpreted as the short rate, rather than as the yield curve level. Since this will result in a 4-factor model, we get the 3-factor model for free, so to say, since we can reduce the factor space by simply omitting the forth factor and forth factor loading.

Within the continuous-time setting Christensen, Diebold, and Rudebusch (2011) have shown how to maintain the parametric loading structure of the Nelson and Siegel (1987), while ensuring that arbitrage constraints are fulfilled.[1] Discrete-time versions of the same model have been derived previously (Niu and Zeng (2012) and Li, Niu, and Zeng (2012)). Christensen, Diebold, and Rudebusch (2011) show that five factors are needed to generate an arbitrage-free term structure model, where the factor loadings match precisely those of Svensson and Söderlind (1997). Instead of providing an exact fit, here we derive a

[1] See also, Krippner (2013) and Diebold and Rudebusch (2013).

parsimonious four-factor model with a closed-form loading structure that maintain the characteristics of the Svensson and Söderlind (1997) model, where only one time-decay is used (recall that the original Svensson and Söderlind (1997) model relies on two time-decay parameters to define its loading structure).

As before, let X_t denote the vector of the modelled yield curve factors, at time t. Furthermore, let the dynamics of X_t be governed by vector autoregressive (VAR) processes of order one, under both the empirical measure, $\mathbb{P}$, and the pricing measure, $\mathbb{Q}$:

$$X_t = k^{\mathbb{P}} + \Phi^{\mathbb{P}} \cdot X_{t-1} + \Sigma^{\mathbb{P}} \epsilon_t^{\mathbb{P}}, \qquad \epsilon_t^{\mathbb{P}} \sim N(0,1) \tag{4.1}$$

$$X_t = k^{\mathbb{Q}} + \Phi^{\mathbb{Q}} \cdot X_{t-1} + \Sigma^{\mathbb{Q}} \epsilon_t^{\mathbb{Q}}, \qquad \epsilon_t^{\mathbb{Q}} \sim N(0,1). \tag{4.2}$$

with $\Sigma\Sigma' = \Omega$ being the variance of the residuals, and it is assumed that $\Sigma^{\mathbb{P}} = \Sigma^{\mathbb{Q}}$. It is noted that we do not use the mean-adjusted version of the VAR model here, the reason for this will become clear at the end of the section.

The risk free one-period short rate is assumed to be a function of X_t, such that:

$$r_t = \rho_0 + \rho_1' X_t. \tag{4.3}$$

In the model that we derive, we want our factors, X, to have interpretations as: the short rate, the slope, and curvature one and two. Given that the first factor is the short rate, we impose the following constraints on (4.3): $\rho_0 = 0$ and $\rho_1 = [1, 0, 0, 0]'$.

As in the derivation of the discrete-time version of the Vasicek (1977), in one of the previous sections of this booklet, we now impose absence of arbitrage on the model by introducing the unique pricing mechanism, that governs all traded assets:

$$P_{t,\tau} = \mathbb{E}_t \left[M_{t+1} \cdot P_{t+1,\tau-1} \right] \tag{4.4}$$

The idea here is that when the bond matures at time T, its value is know with certainty, since it is default-free: the bond pays its principal value on that day, so $P_{T,0} = 1$. At any time $t + j$ before maturity, the price of the bond can therefore be found as the one-period discounted-value of the price at time $t + j + 1$, all the way back to time t. Discounting is done using the stochastic discount factor (also called the pricing kernel), which is denoted by M_t, and this quantity is assumed to be given by:

$$M_{t+1} = \exp\left(-r_t - \frac{1}{2} \lambda_t' \lambda_t - \lambda_t' \epsilon_{t+1}^{\mathbb{P}} \right) \tag{4.5}$$

We recognise the uni-variate case of this expression, from when we derived the Vasicek (1977) model, but now we are dealing with a multi-factor model, since X contains four factors. So, we also bring the expression for the time-varying market price of risk into the multi-variate domain by specifying:

$$\lambda_t = \lambda_0 + \lambda_1 \cdot X_t, \tag{4.6}$$

with λ_t being of dimension (4×1) in our application, because we have four factors, λ_0 is of dimension (4×1), and λ_1 is a matrix of dimension (4×4).

It is recalled that:

$$y_{t,\tau} = -\frac{1}{\tau}\log(P_{t,\tau}), \tag{4.7}$$

and that we can write the yield curve expression as a linear (plus a constant, i.e. affine) function:

$$y_{t,\tau} = -\frac{A_\tau}{\tau} - \frac{B'_\tau}{\tau} X_t. \tag{4.8}$$

The bond price is therefore exponential affine in terms of A_τ and B_τ:

$$P_{t,\tau} = \exp\left(A_\tau + B'_\tau X_t\right). \tag{4.9}$$

To derive closed-form expressions for A_τ and B_τ, the fundamental pricing equation is invoked (4.5):

$$\begin{aligned} P_{t,\tau} &= \mathbb{E}_t\left[M_{t+1} \cdot P_{t+1,\tau-1}\right] && (4.10)\\ &= \mathbb{E}_t\left[\exp\left(-r_t - \frac{1}{2}\lambda'_t\lambda_t - \lambda'_t\epsilon^{\mathbb{P}}_{t+1}\right) \cdot \exp\left(A_{\tau-1} + B'_{\tau-1}X_{t+1}\right)\right]. && (4.11) \end{aligned}$$

The expression for X_{t+1} (see equation 4.1) is substituted:

$$P_{t,\tau} = \mathbb{E}_t\left[\exp\left(-r_t - \frac{1}{2}\lambda'_t\lambda_t - \lambda'_t\epsilon^{\mathbb{P}}_{t+1}\right) \cdot \exp\left(A_{\tau-1} + B'_{\tau-1}\left(k^{\mathbb{P}} + \Phi^{\mathbb{P}}X_t + \Sigma\epsilon^{\mathbb{P}}_{t+1}\right)\right)\right], \tag{4.12}$$

and, the terms are then separated into two groups: one to which the expectations operator should be applied, i.e. $t+1$ terms, and another group, which are known at time t:

$$\begin{aligned} P_{t,\tau} =& \exp\left(-r_t - \frac{1}{2}\lambda'_t\lambda_t + A_{\tau-1} + B'_{\tau-1}k^{\mathbb{P}} + B'_{\tau-1}\Phi^{\mathbb{P}}X_t\right)\\ &\cdot\mathbb{E}_t\left[\exp\left(-\lambda'_t\epsilon^{\mathbb{P}}_{t+1} + B'_{\tau-1}\Sigma\epsilon^{\mathbb{P}}_{t+1}\right)\right]. \end{aligned} \tag{4.13}$$

The question is then, how can we calculate the expectations part of (4.13):

$$\mathbb{E}_t\left[\exp\left(-\lambda'_t + B'_{\tau-1}\Sigma\right)\epsilon^{\mathbb{P}}_{t+1}\right]. \tag{4.14}$$

To this end, the moment generating function of the multivariate normal distribution is used. Since $\epsilon^{\mathbb{P}} \sim N(0, I)$, it is know that:

$$\mathbb{E}[\exp(a'\epsilon^{\mathbb{P}})] = \exp\left(\frac{1}{2}a' \cdot I \cdot a\right), \tag{4.15}$$

so, the expectation in (4.13) can be calculated, using $a' = (-\lambda_t' + B_{\tau-1}'\Sigma)$, as:

$$\begin{aligned}
&\exp\left[\frac{1}{2}(-\lambda_t' + B_{\tau-1}'\Sigma) \cdot I \cdot (-\lambda_t' + B_{\tau-1}'\Sigma)'\right] \\
=&\exp\left[\frac{1}{2}(-\lambda_t' + B_{\tau-1}'\Sigma) \cdot I \cdot (-\lambda_t + \Sigma' B_{\tau-1})\right] \\
=&\exp\left[\frac{1}{2}\left(\lambda_t'\lambda_t - \lambda_t'\Sigma' B_{\tau-1} - B_{\tau-1}'\Sigma\lambda_t + B_{\tau-1}'\Sigma\Sigma' B_{\tau-1}\right)\right],
\end{aligned} \tag{4.16}$$

and, since $B_{\tau-1}'\Sigma\lambda_t$ is a scalar, and for a scalar h, we know that $h = h'$, so $B_{\tau-1}'\Sigma\lambda_t = \lambda_t'\Sigma' B_{\tau-1}$. We can then write:

$$\begin{aligned}
&\mathbb{E}_t\left[\exp\left(-\lambda_t' + B_{\tau-1}'\Sigma\right)\epsilon_{t+1}^{\mathbb{P}}\right] \\
=&\exp\left[\left(\frac{1}{2}\lambda_t'\lambda_t - B_{\tau-1}'\Sigma\lambda_t + \frac{1}{2}B_{\tau-1}'\Sigma\Sigma' B_{\tau-1}'\right)\right].
\end{aligned} \tag{4.17}$$

This term is then reinserted into (4.13), giving:

$$P_{t,\tau} = \exp\left(-r_t + A_{\tau-1} + B_{\tau-1}'k^{\mathbb{P}} + B_{\tau-1}'\Phi^{\mathbb{P}} X_t - B_{\tau-1}'\Sigma\lambda_t + \frac{1}{2}B_{\tau-1}'\Sigma\Sigma' B_{\tau-1}'\right). \tag{4.18}$$

It is recalled that $r_t = \rho_1' X_t$, and that $\lambda_t = \lambda_0 + \lambda_1 X_t$. Inserting these expressions into (4.18), gives:

$$\begin{aligned}
P_{t,\tau} = \exp\Big(&-\rho_1' X_t + A_{\tau-1} + B_{\tau-1}'k^{\mathbb{P}} + B_{\tau-1}'\Phi^{\mathbb{P}} X_t \\
&-B_{\tau-1}'\Sigma\left(\lambda_0 + \lambda_1 X_t\right) + \frac{1}{2}B_{n-1}'\Sigma\Sigma' B_{\tau-1}'\Big).
\end{aligned} \tag{4.19}$$

Reorganising this expression into terms that load on X_t and terms that do not, help matching coefficients with respect to equation (4.9):

$$\begin{aligned}
P_{t,\tau} = \exp\Big(&A_{\tau-1} + B_{\tau-1}'\left(k^{\mathbb{P}} - \Sigma\lambda_0\right) + \frac{1}{2}B_{\tau-1}'\Sigma\Sigma' B_{\tau-1}' \\
&+B_{\tau-1}'\Phi^{\mathbb{P}} X_t - \rho_1' X_t - B_{\tau-1}'\Sigma\lambda_1 X_t\Big),
\end{aligned} \tag{4.20}$$

which is:

$$\begin{aligned}
P_{t,\tau} = \exp\Bigg(&A_{\tau-1} + B_{\tau-1}'\left(k^{\mathbb{P}} - \Sigma\lambda_0\right) + \frac{1}{2}B_{\tau-1}'\Sigma\Sigma' B_{\tau-1}' \\
&+\left[B_{\tau-1}'\left(\Phi^{\mathbb{P}} - \Sigma\lambda_1\right) - \rho_1'\right] X_t\Bigg).
\end{aligned} \tag{4.21}$$

Matching the coefficients of (4.21) with those of (4.9) establishes the recursive formulas for A_τ and B_τ:

$$A_\tau = A_{\tau-1} + B'_{\tau-1}k^{\mathbb{Q}} + \frac{1}{2}B'_{\tau-1}\Sigma\Sigma' B'_{\tau-1} \tag{4.22}$$

$$B'_\tau = B'_{\tau-1}\Phi^{\mathbb{Q}} - \rho'_1 \tag{4.23}$$

with $k^{\mathbb{Q}} = k^{\mathbb{P}} - \Sigma\lambda_0$, and $\Phi^{\mathbb{Q}} = \Phi^{\mathbb{P}} - \Sigma\lambda_1$. Recall that $\rho_0 = 0$ in our model setup. Using recursive substitution, we realise that the expression for B'_n also can be written in the following way:[2]

$$B_\tau = -\left[\sum_{k=0}^{\tau-1}\left(\Phi^{\mathbb{Q}}\right)^k\right]' \cdot \rho_1. \tag{4.24}$$

It is convenient to write the loading structure in this way, when we want to find a closed-form solution for B_τ, because the expression in (4.24) is the sum of a matrix power series, and we know that this can be solved if $\Phi^{\mathbb{Q}}$ comes from a stationary VAR model.

The last task remaining is then to find a $\Phi^{\mathbb{Q}}$ matrix that, when inserted in (4.24) gives us loadings that are as similar as possible to the ones appearing in the Svensson and Söderlind (1997), while still imposing the constraints mentioned above, that ensures that the first factor is the short rate. So, let's start the guessing game. What happens, for example, if we use the following matrix?

$$\Phi^{\mathbb{Q}} = \begin{bmatrix} 1 & 1-\gamma & 1-\gamma & 1-\gamma \\ 0 & \gamma & \gamma-1 & \gamma-1 \\ 0 & 0 & \gamma & \gamma-1 \\ 0 & 0 & 0 & \gamma \end{bmatrix}. \tag{4.25}$$

A closed-form expressions for B_τ can then be derived by first finding $\left(\Phi^{\mathbb{Q}}\right)^k$:

---

[2] We see this by the use of an example. For $\tau = 3$, we have:

$$\begin{aligned} B'_1 &= -\rho'_1 \\ B'_2 &= B'_1\Phi^{\mathbb{Q}} - \rho'_1 = -\rho'_1\Phi^{\mathbb{Q}} - \rho'_1 \\ B'_3 &= B'_2\Phi^{\mathbb{Q}} - \rho'_1 = (-\rho'_1\Phi^{\mathbb{Q}} - \rho'_1)\Phi^{\mathbb{Q}} - \rho'_1 \\ &= -\rho'_1\left(\Phi^{\mathbb{Q}}\right)^2 - \rho'_1\Phi^{\mathbb{Q}} - \rho'_1 \\ &= -\rho'_1\left(\left(\Phi^{\mathbb{Q}}\right)^2 + \left(\Phi^{\mathbb{Q}}\right)^1 + \left(\Phi^{\mathbb{Q}}\right)^0\right) \\ &= -\rho'_1\left[\sum_{k=0}^{2}\left(\Phi^{\mathbb{Q}}\right)^k\right] \end{aligned}$$

so,

$$B_3 = -\left[\sum_{k=0}^{2}\left(\Phi^{\mathbb{Q}}\right)^k\right]' \rho_1,$$

which generalises to equation (4.24).

$$\left(\Phi^{\mathbb{Q}}\right)^k = \begin{bmatrix} 1 & 1-\gamma^k & -k\gamma^{k-1}(\gamma-1) & -\frac{k}{2}\gamma^{k-2}\left((k+1)\gamma^2-2k\gamma+k-1\right) \\ 0 & \gamma^k & k\gamma^{k-1}(\gamma-1) & \frac{k}{2}\gamma^{k-2}\left((k+1)\gamma^2-2k\gamma+k-1\right) \\ 0 & 0 & \gamma^k & k\gamma^{k-1}(\gamma-1) \\ 0 & 0 & 0 & \gamma^k \end{bmatrix}, \tag{4.26}$$

and then by substituting (4.26) into (4.24), we get:

$$B_\tau = -\begin{bmatrix} \tau \\ \sum_{k=0}^{\tau-1} 1-\gamma^k \\ \sum_{k=0}^{\tau-1} -k\gamma^{k-1}(\gamma-1) \\ \sum_{k=0}^{\tau-1} -\frac{k}{2}\gamma^{k-2}\left((k+1)\gamma^2-2k\gamma+k-1\right) \end{bmatrix}. \tag{4.27}$$

Solving (4.27) gives:[3]:

$$B_\tau = -\begin{bmatrix} \tau \\ \tau - \frac{1-\gamma^\tau}{(1-\gamma)} \\ -\tau\gamma^{\tau-1} + \frac{1-\gamma^\tau}{(1-\gamma)} \\ -\frac{1}{2}\tau(\tau-1)(\gamma-1)\gamma^{\tau-2} \end{bmatrix}. \tag{4.28}$$

An expression for the yield curve at time t is then obtained if Y_t collects $y_{t,\tau}$ $\forall\tau$ by increasing maturity, and if $a = -A_\tau/\tau$ and $b = -B'_\tau/\tau$ are defined similarly. The expression for the yield curve observed at time t is then:

$$Y_t = a + bX_t + \Sigma_Y u_t. \tag{4.29}$$

with:

[3] The first entry of (4.27) follows immediately, the second entry uses $\sum_{k=0}^{\tau-1} x^k = \frac{1-x^\tau}{1-x}$, the third and fourth entries can be found by consecutive substitution. For example, for $\tau = 5$ the third entry of (4.27) is: $4\gamma^4 - \gamma^3 - \gamma^2 - \gamma^1 - \gamma^0$, which generalizes to $(\tau-1)\gamma^{\tau-1} - \sum_{k=0}^{\tau-2}\gamma^k$. Similarly, the fourth entry of (4.28) for $\tau = 5$ is: $-(0 + 1(\gamma-1)\gamma^0 + 3(\gamma-1)\gamma^1 + 6(\gamma-1)\gamma^2 + 10(\gamma-1)\gamma^3)$, which generalizes to $-\frac{1}{2}\tau(\tau-1)(\gamma-1)\gamma^{\tau-2}$.

$$b = \begin{bmatrix} 1 \\ 1 - \frac{1-\gamma^{\tau}}{(1-\gamma)\cdot\tau} \\ -\gamma^{\tau-1} + \frac{1-\gamma^{\tau}}{(1-\gamma)\cdot\tau} \\ -\frac{1}{2}(\tau-1)(\gamma-1)\gamma^{\tau-2} \end{bmatrix}. \tag{4.30}$$

4.2.1 The relationship between the SRB model and the Joslin, Singleton and Zhu (2011) framework

Here we will briefly look at Joslin, Singleton, and Zhu (2011) (JSZ). This is a core paper in the literature because it shows that many of the existing model parametrisations have the same core, and are in fact identical upto a rotation. They also present an algorithm to estimate arbitrage-free term structure models that is fast and that converges effortlessly. In addition, they provide insights on what to expect in terms of forecasting performance of certain term structure models, and why some models have equal forecasting performance (to the degree of uncertainty present in the data). It is a true treat for the reader - if only the paper was easier to understand, it would be perfect.

From my reading of Joslin, Singleton, and Zhu (2011), the key takeaways are:

1. Gaussian dynamic term structure models (GDTSM) can be parameterised such that the paranmeters that govern the $\mathbb{P}$ measure, and thus the $\mathbb{P}$-measure forecasts of the yield curve factors, X, do not appear in the measurement-error density. This means that the $\mathbb{P}$- and $\mathbb{Q}$-measure parameters can be estimated separately.
2. It also means that constraints imposed under $\mathbb{Q}$ do not affect the dynamics of the yield curve factors under $\mathbb{P}$, so no-arbitrage constraints cannot help in providing better model forecasts.
3. For a N-factor GDTSM the following parameters need to be specified:
   3.1. r_{∞}^{Q}, the long-run mean of the short rate under $\mathbb{Q}$
   3.2. $\gamma^{\mathbb{Q}}$, mean reversion speed (eigenvalues) of the factor dynamics under $\mathbb{Q}$. Other notation uses λ for this parameter, but we have reserved λ for the market price of risk.
   3.3. $\varSigma^{\mathbb{P}}$, the conditional covariance matrix of the yield factors from the VAR model governing their dynamics.
4. The JSZ model characterization framework is based on the idea 'similar' matrices, known from linear algebra, where similarity is defined on the basis of the Jordan form. JSZ apply this idea to GDTSMs: if a given model's $\mathbb{Q}-$dynamics can be re-written in Jordan form, with ordered eigenvalues, then the model is identical (up to a rotation) to the JSZ canonical form.

If a comparison is made to the notation used in Joslin, Singleton, and Zhu (2011), it may be relevant to note that they specify VAR models in difference form. Throughout the booklet, we have looked at VAR models in level form. Although it is not a big deal, I will continue using the level form here, and thus rewrite (and adapt their notation) to what we have been using so far.[4]

Let's start by looking at the issue from an intuitive angle on the basis of a general VAR model for the yield curve factors under the $\mathbb{Q}$-measure. As in Joslin, Singleton, and Zhu (2011) we choose a specific set of yield curve factors that are formed as linear combinations of yields. JSZ refer to these factors as being "portfolios of yields" i.e. implying that they can be obtained by applying a weighting matrix to the yield curve data in the following way[5]:

$$X_t = W \cdot y_t. \tag{4.31}$$

According to this definition, any linear combination of yields qualify, thus also principal components. We continue by writing a general VAR model for these factors:

$$X_t = k^{\mathbb{Q}} + \Phi^{Q} \cdot X_{t-1} + \Sigma^{Q} u_t, \tag{4.32}$$

We know that Φ^Q governs the dynamics of the VAR model under the $\mathbb{Q}$-measure, and thus the shape of the yield loadings b_τ and the constant, a_τ. But, what are the core components of this matrix? If we look at the eigenvalue decomposition of Φ^Q, the eigenvalues express the degree of persistency of the matrix, i.e. how fast (or slow) it converges to its steady-state (assuming, as always, that the VAR is stationarity). The eigenvectors of Φ (not of X, just to be clear) can be interpreted as the "direction" the matrix points in, or the space that it spans. This is a very vague statement, I know, and I apologies not to be able to explain it better, but you know, we typically do not use these eigenvectors for anything else than multiplying them on the diagonal matrix of eigenvalues, when we e.g. want to find the s-step ahead projection matrix.[6] But in the current context, it may be enough to be vague to get the overall intuitive understanding of the idea proposed in Joslin, Singleton, and Zhu (2011). Vague or not, let's continue. We can naturally imagine many different sets of yield curve factors (one for each occasion), and all sets spanning different directions, and all formed according to (4.31). Some of these sets could, for example, be (a) the three first principal components; (b) level, slope, and curvature; (c) short rate slope and curvature; and (d) the 3-month yield, the 3-year yield, and the 10-year yield. The crux of JSZ is that all these possible factors definitions, i.e. also our examples in (a)-(d), can be converted (or rotated) into a common single basis form. So, in fact, the various factors definitions, and their associated models, are all (just) variations over a single core model; all having identical properties, but appearing to be different.

---

[4] A VAR model written in difference form looks like this: $\Delta X_t = k + \tilde{\Phi} X_{t-1} + \Sigma e_t$, which can be written as, $X_t - X_{t-1} = k + \tilde{\Phi} X_{t-1} + \Sigma e_t$, and as $X_t = k + \tilde{\Phi} X_{t-1} + X_{t-1} + \Sigma e_t$. So, the difference between the difference and level forms is that $\Phi = \tilde{\Phi} + I$, where Φ refers to the autoregressive matrix in the level form.

[5] Joslin, Singleton, and Zhu (2011) denote the yield factors by $\mathcal{P}$, but we continue by using X to denote the factors.

[6] Recall from our discussion of the term premia in a previous chapter, that we can calculate the s-step ahead projection of the VAR as the s'th power of the diagonal matrix containing the eigenvalues, pre- and post-multiplied by the eigenvectors.

To express the GDTSM in its purest form, Joslin, Singleton, and Zhu (2011) rely on the Jordan decomposition of $\Phi^{\mathbb{Q}}$. The Jordan decomposition is a generalisation of the eigenvalue decomposition, see e.g. Hamilton (1994)[pp.730-31], in that it explicitly handles repeated eigenvalues. An eigenvalue decomposition can still be successfully completed, even if there are repeated eigenvalues, as long as the eigenvectors form a full-rank matrix. But this is not guaranteed to all ways be the case, hence the generalisation represented by the Jordan decomposition (formulas [A.4.26] and [A.4.27] from Hamilton (1994)):

$$J = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \cdots & J_n \end{bmatrix} \tag{4.33}$$

and

$$J_i = \begin{bmatrix} \gamma_i & 1 & 0 & \cdots & 0 \\ 0 & \gamma_i & 1 & \cdots & 0 \\ 0 & 0 & \gamma_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \gamma_i \end{bmatrix} \tag{4.34}$$

where the i'th eigenvalue is denoted by γ_i. Similarly to the eigenvalue decomposition, the Jordan decomposition is given by:

$$\Phi^{\mathbb{Q}} = V \cdot \Phi^J \cdot V^{-1}. \tag{4.35}$$

To facilitate the rotation from the Jordan-form to any other observational equivalent model, we start with the VAR model:

$$X_t^J = k^J + \Phi^J \cdot X_{t-1}^J + \Sigma^J e_t^J, \tag{4.36}$$

where J refers to the Jordan-form. A general rotation of this VAR model is implemented below:

$$
\begin{aligned}
X_t &= N + M \cdot X_t^j \\
&= N + M \cdot \left(k^J + \Phi^J \cdot X_{t-1}^J + \Sigma^J e_t\right) \\
&= N + M \cdot k^J + M \cdot \Phi^J \cdot X_{t-1}^J + M \cdot \Sigma^J e_t \\
&= N + M \cdot k^J + M \cdot \Phi^J \cdot M^{-1} \cdot (X_{t-1} - N) + M \cdot \Sigma^J e_t \\
&= N - M \cdot \Phi^J \cdot M^{-1} \cdot N + M \cdot k^J + M \cdot \Phi^J \cdot M^{-1} \cdot X_{t-1} + M \cdot \Sigma^J e_t \\
&= \underbrace{\left(I - M\Phi^J M^{-1}\right) N + M k^J}_{k^{\mathbb{Q}}} + \underbrace{M\Phi^J M^{-1}}_{\Phi^Q} \cdot X_{t-1} + \underbrace{M\Sigma^J}_{\Sigma^Q} e_t \\
&= k^{\mathbb{Q}} + \Phi^{\mathbb{Q}} \cdot X_{t-1} + \Sigma^{\mathbb{Q}} e_t.
\end{aligned}
\tag{4.37}
$$

Line 4 follows from line 1, since $X_t = N + M \cdot X_t^j \Leftrightarrow X_t^j = M^{-1} \cdot (X_t - N)$.

We know from (4.3) that the short rate depends on the yield curve factors. In the SRB model this link is simply defined by the parameter constraints $\rho_0 = 0$ and $\rho_1 = [1, 0, \ldots, 0]'$. However, for other factors, ρ_0 and ρ_1 contain parameters that need to be estimated. So, we need also to show how the introduced rotation affects the short rate equation:

$$
\begin{aligned}
r_t &= \rho_0^J + \rho_1^J X_t^j \\
&= \rho_0^J + \rho_1^J M^{-1} \left(X_t - N\right) \\
&= \underbrace{\rho_0^J - \rho_1^J M^{-1} \cdot N}_{\rho_0} + \underbrace{\rho_1^J M^{-1}}_{\rho_1} \cdot X_t \\
&= \rho_0 + \rho_1 X_t.
\end{aligned}
\tag{4.38}
$$

With this behind us, we can now show that the SRB-model developed in section 4.2 is a constrained member of the general family of Gaussian dynamic term structure model derived by Joslin, Singleton, and Zhu (2011).[7]

Let J be the Jordan matrix, and V be a rotation matrix such that equation (4.25) can be reformulated as:

$$
\Phi^{\mathbb{Q}} = V \cdot J \cdot V^{-1}. \tag{4.39}
$$

Choosing V to be:

[7] This is not overly surprising since the SRB model is a generalization of the arbitrage-free Nelson-Siegel model suggested by Christensen, Diebold, and Rudebusch (2011) (CDR), and since Joslin, Singleton, and Zhu (2011) show that the CDR model is a constrained member of the JSZ family.

$$V = \begin{bmatrix} 1 & -(\gamma-1)^2 & \gamma-1 & -1 \\ 0 & \gamma^2 - 2*\gamma+1 & 1-\gamma & 1 \\ 0 & 0 & \gamma-1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.40}$$

implies that:

$$J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma & 1 & 0 \\ 0 & 0 & \gamma & 1 \\ 0 & 0 & 0 & \gamma \end{bmatrix}, \tag{4.41}$$

Since (4.41) is in Jordan form with repeated eigenvalues, there exists a mapping between the $\mathbb{Q}-$dynamics I propose above in (4.25) and the framework suggested by JSZ. The proposed SRB model is therefore a constrained member of the JSZ family of models.[8]

4.2.2 The relationship between the 4-factor SRB model and the Svensson-Söderlind model

Since the 4-factor SRB model aims to replicate the Svensson and Söderlind (1997) loading structure, as closely as possible, but by the use of a single time-decay parameter, γ, it may be relevant to draw a comparison between the two models.

It is recalled that the Svensson-Soderlind loadings are given by:

$$H = \begin{bmatrix} 1 \\ \frac{1-e^{-\kappa_1 n}}{\kappa_1 n} \\ \frac{1-e^{-\kappa_1 n}}{\kappa_1 n} - e^{-\kappa_1 n} \\ \frac{1-e^{-\kappa_2 n}}{\kappa_2 n} - e^{-\kappa_2 n} \end{bmatrix}, \tag{4.42}$$

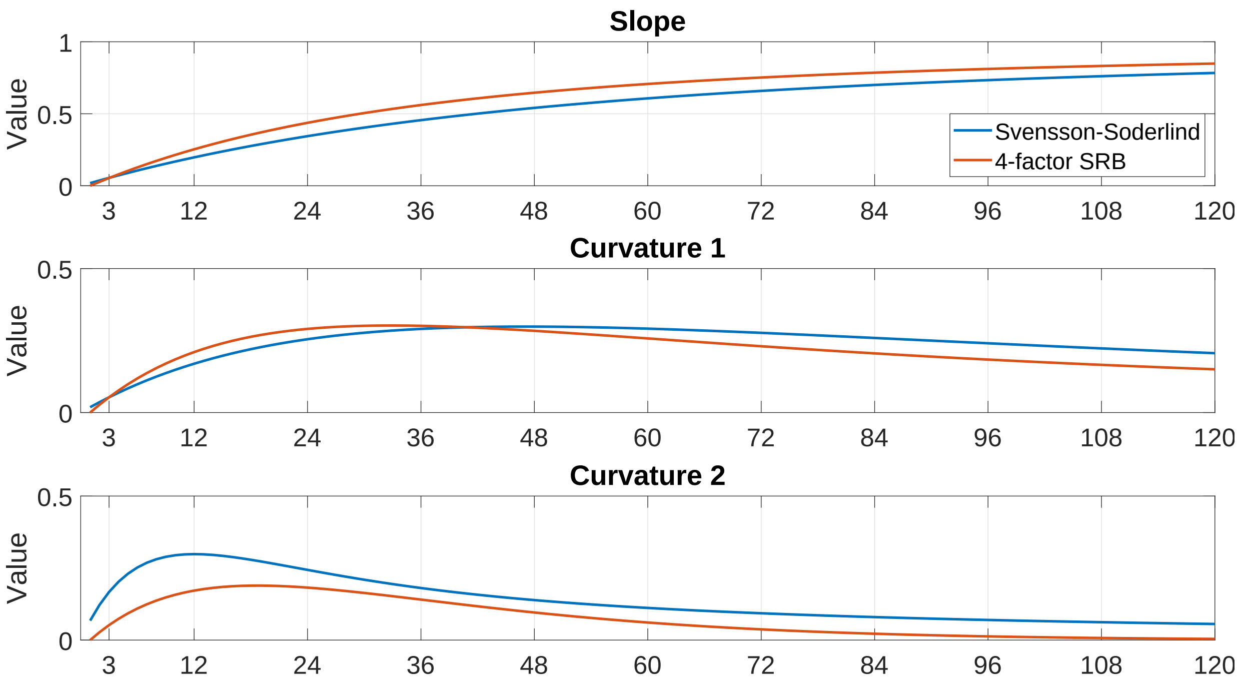
and that the loading structure of the SRB model is given by $B = B_n/n$, where B is given by equation (4.28):

[8] The restriction of repeated eigenvalues, compared to the canonical JSZ form, is not rejected by the data used in the paper at a 5% level, using a likelihood ratio test.

$$B = -\begin{bmatrix} 1 \\ 1 - \frac{1-\gamma^n}{(1-\gamma)\cdot n} \\ -\gamma^{n-1} + \frac{1-\gamma^n}{(1-\gamma)\cdot n} \\ -\frac{1}{2}(n-1)(\gamma-1)\gamma^{n-2} \end{bmatrix}. \tag{4.43}$$

Figure 4.2.2 compares the loading structures of the Svensson and Söderlind (1997) and the SRB models. The shape parameter of the SRB model is set to $\gamma = 0.945$, and the two Svensson-Söderlind shape parameters take on the values $\kappa_1 = 0.0381$ and $\kappa_2 = 0.1491$ (similar to what is found on US data). The loadings for the first factor are not shown in the figure, as they equal 1 for both models, across the included maturities. The first panel in the figure shows the loadings for the slope factor; and to facilitate easy comparison, the loading of the Svensson-Söderlind model is rotated to match that are the SRB model: let H_{slope} be the original slope loading for the Svensson-Söderlind model, panel 1 then plots $1 - H_{slope}$. The second and third panels compare the loadings for the first and second curvature loadings. Returning to the first panel. It shows that the loadings for the slope factor are quite similar across the two models, although the SRB loading assumes slightly higher values throughout the maturity spectrum, and also seems to arch upwards a bit more than the Svensson-Söderlind loading does. Level differences between the loading structures can naturally be subsumed by the corresponding factor values, so the shape attained by the loadings are of greater importance for the relative comparison between the models. Similarly, the second and the third panels show relatively good correspondence between the curvature loadings of the two models. Panel 2 indicates that the SRB model loading peaks around a maturity of 30 months, while the corresponding Svensson-Söderlind loading peaks around 40 month.

To test the impact of the detected differences in the loading structures, as documented above, the models are fitted to the US data used in other sections of this booklet. Table 4.2.2 documents that both models produce very low root mean squared errors and that the added flexibility of the Svensson-Söderlind model, via its reliance on two shape parameters, κ_1 and κ_2 (see equation (4.42)), as opposed to the one used by the SRB model (γ), gives it an economically insignificant edge of 1 basis points on average. The worst fitted maturity of the SRB model is the 1-year segment with a RMSE of 4.7 basis points, and the average RMSE across the eleven included maturities is 2.68 basis points. In comparison, the Svensson-Söderlind model produces the worst RMSE at the 2-year segment of 3.0 basis points, with the average RMSE of 1.58 basis points across the included maturities.



The figure compares the loading structures of the Svensson-Soderlind and the SRB models. The shape parameter of the SRB model is $\gamma = 0.945$, and the two Svensson-Soderlind shape parameters are estimated to be $\kappa_1 = 0.0381$ and $\kappa_2 = 0.1491$. The loadings for the first factor are not shown as they equal 1 for both models across the included maturities. The first panel shows the loadings for the slope factor, and to facilitate the comparison, the Svensson-Soderlind loading structure is rotated, to match that of the SRB model, and this is done in the following way: Let $H_{\texttt{slope}}$ be the original slope loading for the Svensson-Soderlind model, panel 1 then plots $1 - H_{\texttt{slope}}$. The second and third panels compare the loadings for the first and second curvature loadings.

Fig. 4.1. Loading Structures

	3m	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y	Average
SRB Model	2.8	4.7	4.1	2.5	1.6	2.3	2.3	2.1	2.0	1.5	3.4	2.6
Svensson-Söderlind	0.3	2.6	3.0	2.0	0.7	1.4	1.9	1.6	0.8	0.7	2.3	1.5

The table shows the root mean squared errors in basis points of the SRB and the Svensson and Söderlind (1997) models when estimated using monthly US yield curve data covering the period from January 1961 to November 2017. Data are observed at maturities spanning 3 months to 10 years. The shape parameter of the SRB model is $\gamma = 0.945$, and the two Svensson-Söderlind shape parameters are estimated to be $\kappa_1 = 0.0381$ and $\kappa_2 = 0.1491$.

Table 4.1. Root Mean Squared Errors (basis points)

4.3 Appendix: MATLAB code

4.3.1 Yield curve model estimation via the SSM toolbox

filename: Modelling_yields_under_Q.m

```
%% Modelling yields under Q
% All we do here is to plot the loading structures of the Svensson-
% Soderlind and the 4-factor SRB models
%

tau    = ( 1:1:120 )';
nTau   = size(tau,1);

Bfunc_SS = @(lambda_,tau_,nTau_) ...
[ ones(nTau_,1) (1-exp(-lambda_(1,1).*tau_))./ (lambda_(1,1).*tau_) ...
(1-exp(-lambda_(1,1).*tau_))./(lambda_(1,1).*tau_)-exp(-lambda_(1,1).*tau_) ...
(1-exp(-lambda_(2,1).*tau_))./(lambda_(2,1).*tau_)-exp(-lambda_(2,1).*tau_) ];


Bfunc_SRB4 = @(lambda_,tau_,nTau_) ...
    [ ones(nTau_,1) 1-(1-lambda_.^tau_)./((1-lambda_).*tau_) ...
     -(lambda_.^(tau_-1))+(1-lambda_.^tau_)./((1-lambda_).*tau_) ...
     -0.5.*(tau_-1).*(lambda_-1).*lambda_.^(tau_-2) ];

L_SS   = [ 0.0381; 0.1491 ];

L_SRB4 = 0.945;

B_SS   = Bfunc_SS( L_SS, tau, nTau );
B_SRB4 = Bfunc_SRB4( L_SRB4, tau, nTau );

tau_plot = [3 12:12:120]';
figure('units','normalized','outerposition',[0 0 1 1])
    subplot(3,1,1), plot( tau, 1-B_SS(:,2) , ...
        'LineWidth',2),  ylim([0 1]),  title('Slope'),
         ylabel('Value'), set(gca, 'FontSize', 20)
    hold on
    subplot(3,1,1), plot(tau,B_SRB4(:,2) , ...
         'LineWidth',2), grid 'on'
         xticks(tau_plot),xticklabels(tau_plot)
         legend('Svensson-Soderlind','4-factor SRB','Location','SE')

    subplot(3,1,2), plot( tau, B_SS(:,3) , ...
        'LineWidth',2),  ylim([0 0.5]),  title('Curvature 1'),
         ylabel('Value'), set(gca, 'FontSize', 20)
    hold on
    subplot(3,1,2), plot(tau,B_SRB4(:,3) , ...
         'LineWidth',2), grid 'on'
         xticks(tau_plot),xticklabels(tau_plot)

    subplot(3,1,3), plot( tau, B_SS(:,4) , ...
        'LineWidth',2),  ylim([0 0.5]),  title('Curvature 2'),
         ylabel('Value'), set(gca, 'FontSize', 20)
    hold on
    subplot(3,1,3), plot(tau,B_SRB4(:,4) , ...
         'LineWidth',2), grid 'on'
         xticks(tau_plot),xticklabels(tau_plot)
         print -depsc Loadings_SS_SRB4
```

5

Model implementation

5.1 Introduction

In addition to the modelling explanations provided so far in this booklet, it is also important to discuss how model implementation is achieved in practice. When looking at a model on the internet or in a paper (here I am of course referring to a term structure model), it is not always clear how the authors manage to apply the model to data, and how they obtain the relevant parameter estimates. The aim of the current section is therefore to discuss practical issues, supported by step-wise implementation guidelines.

5.2 A brief note on model implementation

Before getting started on outlining detailed implementation recipes and coding up the Joslin, Singleton, and Zhu (2011) model and the arbitrage-free version of the Dynamic Nelson-Siegel model, following Nyholm (2018), this section describes the central building blocks that arbitrage-free models consist of. As we have seen in chapter 4 these building blocks consists of the $\mathbb{P}$-measure dynamics, the $\mathbb{Q}$-measure dynamics, the parametrisation of the market price of risk, and the no-arbitrage pricing relationship. These elements, and how they interact, are illustrated in Figure 5.1.

$$\lambda_t = \lambda_0 + \lambda_1 \cdot X_t$$

$$r_t = \rho_0 + \rho_1' \cdot X_t$$

$$P_{t,\tau} = \mathbb{E}_t[M_{t+1} \cdot P_{t+1,\tau-1}]$$

$$M_{t+1}^{\mathbb{P}} = \texttt{exp}\left(-r_t - \tfrac{1}{2}\lambda_t' \cdot \lambda_t - \lambda_t' \cdot \epsilon_{t+1}^{\mathbb{P}}\right) \qquad M_{t+1}^{\mathbb{Q}} = \texttt{exp}\left(-r_t\right)$$

$$X_t = k^{\mathbb{P}} + \Phi^{\mathbb{P}} \cdot X_{t-1} + \Sigma^{\mathbb{P}} \cdot \epsilon_t^{\mathbb{P}} \qquad X_t = k^{\mathbb{Q}} + \Phi^{\mathbb{Q}} \cdot X_{t-1} + \Sigma^{\mathbb{Q}} \cdot \epsilon_t^{\mathbb{Q}}$$

$$\Sigma^{\mathbb{Q}} = \Sigma^{\mathbb{P}}$$

$$k^{\mathbb{Q}} = k^{\mathbb{P}} - \Sigma^{\mathbb{P}} \cdot \lambda_0$$

$$\Phi^{\mathbb{Q}} = \Phi^{\mathbb{P}} - \Sigma^{\mathbb{P}} \cdot \lambda_1$$

Fig. 5.1. Building blocks for arbitrage-free models

The triangle in the figure is meant to illustrate that there are three ways to parameterise an arbitrage-free model. A model is parameterised by selecting two of the three corners in the triangle, and by letting the third, un-selected, corner being implied by the parameter relationships recorded in the lower part of Figure 5.1. More specifically, you can choose to:

1. estimate the parameters governing the $\mathbb{P}$- and the $\mathbb{Q}$-dynamics, i.e. the two lower parts of the triangle, leaving the parameters of the market price of risk to be implied by the parameter relationships: $k^{\mathbb{Q}} = k^{\mathbb{P}} - \Sigma^{\mathbb{P}} \cdot \lambda_0$, and $\Phi^{\mathbb{Q}} = \Phi^{\mathbb{P}} - \Sigma^{\mathbb{P}} \cdot \lambda_1$.
2. estimate the parameters governing the $\mathbb{P}$-dynamics and the market price of risk, and letting the parameters of the $\mathbb{Q}$-dynamics being implied by the parameter relationships: $k^{\mathbb{Q}} = k^{\mathbb{P}} - \Sigma^{\mathbb{P}} \cdot \lambda_0$, and $\Phi^{\mathbb{Q}} = \Phi^{\mathbb{P}} - \Sigma^{\mathbb{P}} \cdot \lambda_1$.
3. estimate the parameters governing the $\mathbb{Q}$-dynamics and the market price of risk, and letting the parameters of the $\mathbb{P}$-dynamics being implied by the parameter relationships: $k^{\mathbb{Q}} = k^{\mathbb{P}} - \Sigma^{\mathbb{P}} \cdot \lambda_0$, and $\Phi^{\mathbb{Q}} = \Phi^{\mathbb{P}} - \Sigma^{\mathbb{P}} \cdot \lambda_1$.

A final issue to consider is what the main object of the estimation routine is going to be. In Figure 5.1, it is observed that the no-arbitrage relationship is specified in term of bond prices by $P_{t,\tau} = \mathbb{E}_t[M_{t+1} \cdot P_{t+1,\tau-1}$.

However, it is rarely the case that observed market prices are used to fit yield curve models.[1] The vast majority of models use yields as the primary object for writing up the objective function, i.e. most models minimise the the (squared) difference between observed and model-yields, in order to estimate the model parameters. This is also the approach that we rely on below. However, it should be mentioned that there is another strand of literature using fixed income returns, as the basis for fitting model parameters, see among others, Adrian, Crump, and Mönch (2013).

5.3 Implementing the Joslin, Singleton and Zhu (2011) model

Although the MATLAB code for estimating the Joslin, Singleton, and Zhu (2011) model is available on the internet[2] we still choose to implement the model here. Our implementation will most likely be less efficient, compared to the code made available by the authors. The JSZ code available on the net typically converges within seconds - this is hard to beat, and originates from the clever step-wise estimation approach suggested by Joslin, Singleton, and Zhu (2011). We will also implement the model in a step-wise fashion, but integrate the code into out TSM-class. While it may seem as poor judgement not simply to use what is already available on the net, in our context, where we may want to include exogenous variables and to do conditional projections, we would anyway have to adapt the JSZ code to our particular needs. So, in the end, it may be easier (and more fun) to implement the model our selves.

In section 4.2.1 we met the JSZ model, and saw how the SRB model is a constrained member of the of JSZ family. Here we will take a deep-dive and present the model parameters that need to be estimated and how we achieve model convergence.

Following the traditional linear yield curve modelling set-up the three factors included in the Joslin, Singleton, and Zhu (2011) model are governed by VAR(1) dynamics. In their paper JSZ write up the dynamics in difference form, we will however continue using the level-form as we have done throughout this booklet:

$$X_t = k^{\mathbb{P}} + \Phi^{\mathbb{P}} \cdot X_{t-1} + \Sigma e_t^{\mathbb{P}} \tag{5.1}$$

$$X_t = k^{\mathbb{Q}} + \Phi^{\mathbb{Q}} \cdot X_{t-1} + \Sigma e_t^{\mathbb{Q}} \tag{5.2}$$

$$r_t = \rho_0 + \rho_1 \cdot X_t \tag{5.3}$$

where r_t is a linear function of the factors, and the residual covariance is given by $\Sigma\Sigma'$. JSZ normalise their model by requiring that $\Phi^{\mathbb{Q}}$ is in Jordan form, i.e.

[1] One exception to this rule is the Smith and Wilson (2000) model, which use is prescribed in the context of European Solvency II calculations for Insurance companies, see CEIOPS (2010). This model is fitted directly to prices via discount functions.

[2] See, http://www-bcf.usc.edu/ sjoslin/.

$$\Phi^{\mathbb{Q}} = J(\gamma^{\mathbb{Q}}) = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & J_n \end{bmatrix} \tag{5.4}$$

and

$$J_i = \begin{bmatrix} \gamma_i^{\mathbb{Q}} & 1 & 0 & \cdots & 0 \\ 0 & \gamma_i^{\mathbb{Q}} & 1 & \cdots & 0 \\ 0 & 0 & \gamma_i^{\mathbb{Q}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \dots & \gamma_i^{\mathbb{Q}} \end{bmatrix} \tag{5.5}$$

where the i'th eigenvalue is denoted by $\gamma_i^{\mathbb{Q}}$. With this in place, it turns out that the likelihood function can be partitioned in a convenient way:

$$f(y_t|y_{t-1};\theta) = \underbrace{f(y_t|X_t, \Sigma_X^{\mathbb{P}}, \rho_0, \rho_1; \gamma^{\mathbb{Q}}, k_\infty^{\mathbb{Q}}, \Sigma_y)}_{\texttt{translates factors into yields}} \times \underbrace{f(X_t|X_{t-1}; k^{\mathbb{P}}, \Phi^{\mathbb{P}}, \Sigma_X^{\mathbb{P}})}_{\texttt{evolves factors over time}} \tag{5.6}$$

where the parameters to be estimated, $\theta = \left\{\gamma^{\mathbb{Q}}, k_\infty^{\mathbb{Q}}, \Sigma_y, \Sigma_X^{\mathbb{Q}}, k^{\mathbb{P}}, \Phi^{\mathbb{P}}, \Sigma_X^{\mathbb{P}}, \rho_0, \rho_1\right\}$, are neatly separated into one group that converts yield curve factors into yield, i.e. the $\mathbb{Q}$-measure parameters, and the parameters that cater for the time-series evolution of the factors under the empirical $\mathbb{P}$-measure. As a consequence, our traditional state-space set-up:

$$y_\tau = a_\tau + b_\tau \cdot X_t + \Sigma_y u_t \tag{5.7}$$

$$X_t = k^{\mathbb{P}} + \Phi^{\mathbb{P}} \cdot X_t + \Sigma_X^{\mathbb{P}} e_t^{\mathbb{P}} \tag{5.8}$$

can be broken down into two distinct operations where, first, the state equation is estimated, and second, the observation equation is estimated. From (4.22) and (4.24) we know that the no-arbitrage restriction imposes the following functional form on the parameters that enter the observation equation:

$$A_\tau = A_{\tau-1} + B'_{\tau-1} k^{\mathbb{Q}} + \frac{1}{2} B'_{\tau-1} \Sigma \Sigma' B'_{\tau-1} \tag{5.9}$$

$$B_\tau = -\left[\sum_{k=0}^{\tau-1} \left(\Phi^{\mathbb{Q}}\right)^k\right]' \cdot \rho_1, \tag{5.10}$$

and

$$a_\tau = -\frac{A_\tau}{\tau}$$

$$b_\tau = -\frac{B_\tau}{\tau}.$$

In addition, the JSZ normalisation implies the following parameter constraints:

$$r_t = \rho_0 + \rho_1' \cdot X_t,$$

$$\rho_0 = 0,$$

$$\rho_1 = \iota, \tag{5.11}$$

$$k^{\mathbb{Q}} = \begin{bmatrix} k_\infty^{\mathbb{Q}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{5.12}$$

With these constraints and (5.4) exact identification of the model is achieved. In practise this means that we in some sense uncover the structural parameters that govern the yield curve dynamics and cross sectional behaviour, at its most fundamental level. It is noted that $k_\infty^{\mathbb{Q}}$ is the long-run constant for the short rate under the $\mathbb{Q}$-measure, such that the long-run mean is $r_\infty^{\mathbb{Q}} = \frac{-k_\infty^{\mathbb{Q}}}{\gamma_1^{\mathbb{Q}}}$, when $\gamma_1^{\mathbb{Q}}$ is not a repeated root.

As we saw in section 4.2.1 JSZ use principal components as the staring point for setting up their model. So, together with (5.2) the dynamics of the PCA factors, $\mathcal{P}$, can be written in the following way, where W denote the PCA weights:

$$\mathcal{P}_t = W \cdot y_t. \tag{5.13}$$

To be clear about the notation used, we denote by $a_{J,\tau}$ and $b_{J,\tau}$ the constant and the factor loadings for the yield curve model where the underlying process for X_t is in its most fundamental form, i.e. where $\Phi^{\mathbb{Q}}$ is in Jordan form:

$$\mathbb{E}[y_t] = a_{J,\tau} + b_{J,\tau} \cdot X_t. \tag{5.14}$$

Using (5.13) the PCA based model can now be written in terms of the parameters that define the Jordan basis form:

$$
\begin{aligned}
\mathcal{P}_t &= W \cdot \mathbb{E}[y_t] \\
&= W \cdot (a_{J,\tau} + b_{J,\tau} \cdot X_t) \\
&= W \cdot a_{J,\tau} + W \cdot b_{J,\tau} \cdot X_t \\
&\Updownarrow \\
W \cdot b_{J,\tau} \cdot X_t &= \mathcal{P}_t - W \cdot a_{J,\tau} \\
X_t &= (W \cdot b_{J,\tau})^{-1} \cdot \mathcal{P}_t - (W \cdot b_{J,\tau})^{-1} \cdot W \cdot a_{J,\tau} \\
&= (W \cdot b_{J,\tau})^{-1} \cdot (\mathcal{P}_t - W \cdot a_{J,\tau}) \qquad (5.15)
\end{aligned}
$$

To complete the model, such that it can be estimated, the expression above is inserted into (5.14):

$$
\begin{aligned}
\mathbb{E}[y_t] &= a_{J,\tau} + b_{J,\tau} \cdot (W \cdot b_{J,\tau})^{-1} \cdot \mathcal{P}_t - b_{J,\tau} \cdot (W \cdot b_{J,\tau})^{-1} \cdot W \cdot a_{J,\tau} \\
&= a_{J,\tau} - b_{J,\tau} \cdot (W \cdot b_{J,\tau})^{-1} \cdot W \cdot a_{J,\tau} + b_{J,\tau} \cdot (W \cdot b_{J,\tau})^{-1} \cdot \mathcal{P}_t \\
&= \Big(I - b_{J,\tau} \cdot (W \cdot b_{J,\tau})^{-1} \cdot W\Big) \cdot a_{J,\tau} + b_{J,\tau} \cdot (W \cdot b_{J,\tau})^{-1} \cdot \mathcal{P}_t \\
&= a_{\mathcal{P},\tau} + b_{\mathcal{P},\tau} \cdot \mathcal{P}_t
\end{aligned}
$$

where

$$
a_{\mathcal{P},\tau} = \Big(I - b_{J,\tau} \cdot (W \cdot b_{J,\tau})^{-1} \cdot W\Big) \cdot a_{J,\tau} \qquad (5.16)
$$

$$
b_{\mathcal{P},\tau} = b_{J,\tau} \cdot (W \cdot b_{J,\tau})^{-1} \qquad (5.17)
$$

Via equations (5.16) and (5.17) JSZ create a link between the dynamics that characterise the time series dynamics and cross sectional behaviour of yields, as represented by the Jordan form, and the parameters that govern the yield curve, when using principal components as underlying factors.

As mentioned earlier, JSZ make their MATLAB code available on the net, and this suit of functions works extremely well and converges exceptionally fast. My implementation below is much less general than the JSZ code, and it does not converge as fast as their code does. However, the educational benefits of making our own implementation hopefully outweighs the programming deficiencies. To estimate a version of the Joslin, Singleton, and Zhu (2011) model the following steps are followed:

1. ρ_0 and ρ_1 are determined by the normalisation constraints, so no short rate regression is needed
2. use principal component analysis to identify the factors $\mathcal{P}$ and the weights W
3. find $k^{\mathbb{P}}$, $\Phi^{\mathbb{P}}$, and $\Sigma_{\mathcal{P}}^{\mathbb{P}}$ using linear regression, or maximum likelihood (if constraints are imposed on the parameters), from the time series evolution of $\mathcal{P}_t$
4. make a first guess on the eigenvalues contained in $\gamma^{\mathbb{Q}}$, and for $k_{\infty}^{\mathbb{Q}}$
5. calculate $a_{J,\tau}$ and $b_{J,\tau}$ using the recursive equations in (5.9) and (5.10)
6. use (5.16) and (5.17) to find $a_{\mathcal{P},\tau}$ and $b_{\mathcal{P},\tau}$
7. then find $\gamma^{\mathbb{Q}}$, and for $k_{\infty}^{\mathbb{Q}}$ as the solution to the minimisation problem below:

$$\left\{\hat{\gamma^{\mathbb{Q}}}, \hat{k}_{\infty}^{\mathbb{Q}}\right\} = \underset{\gamma^{\mathbb{Q}}, k_{\infty}^{\mathbb{Q}}}{\texttt{argmin}} \sum_t \sum_\tau \left(y - \left(a_{\mathcal{P}} + b_{\mathcal{P}} \cdot \mathcal{P}\right)\right)^2 \tag{5.18}$$

where y is the whole panel of yield curve observations spanning all dates and maturities.

After having obtained the parameter estimates of the Jordan form of the model, the remaining parameters of the PCA founded model can be determined. This is done here, starting with the short rate equation:

$$\begin{aligned} r_t &= \rho_{J,0} + \rho_{J,1} \cdot X_t \\ &= rho_{J,0} + \rho_{J,1} \cdot \left(W \cdot b_{J,\tau}\right)^{-1} \left(\mathcal{P}_t - W \cdot a_{J,\tau}\right) \\ &= \rho_{J,0} - \rho_{J,1} \cdot \left(W \cdot b_{J,\tau}\right)^{-1} \cdot W \cdot a_{J,\tau} + \rho_{J,1} \cdot \left(W \cdot b_{J,\tau}\right)^{-1} \cdot \mathcal{P}_t \\ &= \rho_{\mathcal{P},0} + \rho_{\mathcal{P},1} \cdot \mathcal{P}_t \end{aligned}$$

where

$$\rho_{\mathcal{P},0} = \rho_{J,0} - \rho_{J,1} \cdot \left(W \cdot b_{J,\tau}\right)^{-1} \cdot W \cdot a_{J,\tau} \tag{5.19}$$

$$\rho_{\mathcal{P},1} = \rho_{J,1} \cdot \left(W \cdot b_{J,\tau}\right)^{-1}. \tag{5.20}$$

And, then for the $\mathbb{Q}$-dynamics of the factors:

$$\begin{aligned} X_t &= m^J + \Phi^J \cdot \left(X_{t-1} - m^J\right) \\ &\Updownarrow \\ \left(W \cdot b_{J,\tau}\right)^{-1} \cdot \left(\mathcal{P}_t - W \cdot a_{J,\tau}\right) &= m^J + \Phi^J \cdot \left(W \cdot b_{J,\tau}\right)^{-1} \cdot \left(\mathcal{P}_{t-1} - W \cdot a_{J,\tau}\right) - \Phi^J \cdot m^J \\ &\Updownarrow \\ \left(W \cdot b_{J,\tau}\right)^{-1} \cdot \mathcal{P}_t &= \left(W \cdot b_{J,\tau}\right)^{-1} \cdot W \cdot a_{J,\tau} + m^J \\ &\quad + \Phi^J \cdot \left(W \cdot b_{J,\tau}\right)^{-1} \cdot \left(\mathcal{P}_{t-1} - W \cdot a_{J,\tau}\right) - \Phi^J \cdot m^J \\ &\Updownarrow \\ \left(W \cdot b_{J,\tau}\right)^{-1} \cdot \mathcal{P}_t &= \Phi^J \cdot \left(W \cdot b_{J,\tau}\right)^{-1} \mathcal{P}_{t-1} + \left(I - \phi^J\right) \cdot \left(m^J + \left(W \cdot b_{J,\tau}\right)^{-1} \cdot W \cdot a_{J,\tau}\right) \\ &\Updownarrow \\ \mathcal{P}_t &= \left(W \cdot b_{J,\tau}\right) \cdot \Phi^J \cdot \left(W \cdot b_{J,\tau}\right)^{-1} \mathcal{P}_{t-1} \\ &\quad + \left(W \cdot b_{J,\tau}\right) \left(I - \Phi^J\right) \cdot \left(m^J + \left(W \cdot b_{J,\tau}\right)^{-1} \cdot W \cdot a_{J,\tau}\right) \end{aligned} \tag{5.21}$$

This means that the parameters of the dynamic evolution of the principal components can be found in the following way:[3].

[3] Note that (5.21) is written in the constant form, and not in mean-adjusted form. Consequently, the mean is found via the generic expression: $m = (I - \Phi)^{-1} \cdot c$

$$\Phi^{\mathbb{Q}}_{\mathcal{P}} = (W \cdot b_{J,\tau}) \cdot \Phi^{J} \cdot (W \cdot b_{J,\tau})^{-1} . \tag{5.22}$$

$$m^{\mathbb{Q}}_{\mathcal{P}} = \left(I - (W \cdot b_{J,\tau}) \cdot \Phi^{J} \cdot (W \cdot b_{J,\tau})^{-1}\right)^{-1} (W \cdot b_{J,\tau}) \cdot \left(I - \Phi^{J}\right) \cdot \left(m^{\mathbb{Q}}_{\mathcal{J}} + (W \cdot b_{J,\tau})^{-1} \cdot W \cdot a_{J,\tau}\right) . \tag{5.23}$$

The notation may have gotten a bit out of hand in the above. I hope it is still roughly clear, what we have achieved (or rather what Joslin, Singleton, and Zhu (2011) have achieved), i.e. by specifying the term structure model in its most fundamental form, via the Jordan basis, helped clarify which of the variables that are central for its empirical implementation and, by afterwards rotating the model to one that relies on principal components as underlying factors, a link was established to data, and we are therefore able to estimate the model. In addition, it was shown in equation (5.6), that the $\mathbb{P}$ and $\mathbb{Q}$-measures can be separated - this is of course important from a model forecasting perspective: basically, this separation principle tell us that superior forecasting performance of a model is unrelated to whether or not it belongs to the family of arbitrage free models. And, it directs our attention to what may facilitate superior forecasting performance, namely the careful selection of exogenous variables to include, and to the number of factors that the model specification relies on.

As far as notation goes, the intention was that whenever a J appears, as a super- or subscript, it means that the parameter belongs to the Jordan form of the model, and whenever a $\mathcal{P}$ appears, it indicates that the parameter belongs to the model based on principal components. Hopefully, this is not too confusing after all.

5.4 Implementing the arbitrage-free SRB model

Similar to the above section, a very short implementation guideline is provided here for the arbitrage-free SRB model. This model is also integrated into the TSM class (for completeness). Two, three, four, factors models are supported.

A step-wise estimation algorithm is used (following Nyholm (2018)), which can be seen as a special case of Andreasen and Christensen (2015) and Rios (2015):

1. conditional on $\gamma^{\mathbb{Q}}$, the arbitrage-free yield loading, b_τ, is known in closed-form from (4.43)
2. using the yield equation, as in (5.7), the yield factors can be found as: $X = b_\tau^{-1} \cdot y'$, where b_τ^{-1} is the pseudo-inverse of b_τ. This is similar to JSZ's approach in equation (5.13), where PCA weights are used to construct the underlying yield curve factors, although here we use b_τ^{-1} as the weighting matrix (because we want to impose a certain economic interpretation onto the extracted factors)
3. the optimal value for γ is found via grid-search, as the γ that minimises $\sum_t \sum_\tau (y - b_\tau(\gamma) \cdot X)^2$
4. using the extracted factors and (5.8), the parameters governing the $\mathbb{P}$-measure dynamics can be found

5. recalling that $\Phi^{\mathbb{Q}}$ is a function of γ^{Q}, the last remaining parameter, $m^{\mathbb{Q}}$, is found as the solution to $\underset{m^{\mathbb{Q}}}{\texttt{argmin}} \sum_t \sum_\tau (y - (a_\tau + b_\tau \cdot X))^2$

5.5 Constructing a model with the short rate and the 10-year term premium as underlying factors

A specific factor structure can help communicating the results of the model easier to third parties (including decision makers), by better supporting a given narrative and communication style. For example, as we have seen, the underlying economic building blocks of the yield curve are the rate expectation and the term premium components. Often in economic analysis, yield curve levels and changes around important events, such as, among other things, Governing council meetings, major economic news release dates, and when some unexpected news hits the market, are typically broken down into these components to give a reading of how the financial market participants interpret the event. It is naturally important to know the degree to which market participants see an event as affecting the future economic environment (the rate expectations) and how it affects their perception of current and future risks (the term premium component). Such decompositions are typically done on the basis of term structure models that use principal components as underlying factors, and where the factors therefore have interpretations as the level, the slope and the curvature; and, most often such models that fall in the camp that excludes arbitrage by construction.

So, the aim of the current section is to build an empirical model that includes the short rate, the term premium and curvature factors, as underlying yield curve factors. This is, perhaps surprisingly, not done very often and the literature on models having this kind of factor structure is very scarce. Actually, to the best of my knowledge a notable exception from this generalisation is the seminal paper by Creal and Wu (2017).

As always, our approach is modest and it cuts a fair amount of corners. The first corner we cut is the one where the arbitrage-free models rest. By relying on a purely empirical model we are able to finalise its implementation quickly - and we can then look at how to build an arbitrage-free model, with the same factor structure, at a later stage. The following steps are applied:

1. Extraction of observable factors: We rely on the SRB3 model to generate the two factors that are directly attainable, i.e. the short rate, the 10-year term premium. Any reasonable term-structure model could in principle to used to this end. So, now we have the two first elements of X, denoted by $X_{(1:2)}$.
2. Finding the loading structure $b_{\tau,(1:2)}$: The yield loadings $b_{\tau,(1:2)}$ that match $X_{(1:2)}$ are obtained by inversion (or linear regression) $y = X_{(1:2)} \cdot b'_{\tau,(1:2)} \Leftrightarrow b'_{\tau,(1:2)} = X^{-1}_{(1:2)} \cdot y$, where the pseudo-inverse is used to obtain $X^{-1}_{(1:2)}$.
3. Obtaining the remaining factors: The third and forth factors are obtained via PCA performed on the residuals from the model using the two factors obtained in step 1, and the loading structure found in step 2. The residuals are found as: $e = y - X_{(1:2)} \cdot b'_{\tau,(1:2)}$, and the first two factors (i.e. the ones having

the largest eigenvalues) are sampled as $X_{(3:4)}$. That is, the PCA is performed on the covariance matrix, $\Omega_y = 1/T \cdot e'e$.

4. Finding the loading structure $b_{\tau,(1:4)}$: Similar to step 2, b_τ is found as: $b'_\tau = X^{-1} \cdot y$, where X hold the time series of all four factors.

Steps 1-4 completes the yield equation of the model, and since the factors contained in X are observable, the parameters that govern their dynamic evolution can be obtained by VAR analysis, as it is done in all the models that we have looked at so far. For convenience, we include this empirical model in our TSM class, that accompany this booklet.

6

Scenario generation with yield curve models

6.1 Introduction

In this chapter we bring together the elements from the other chapters, and show concrete examples of how yield curve models can be used in a risk management context. What we will look at is unconditional and conditional forecasting as well as scenario generation. To this end, three case studies will be solved:

(1) A horse-race between the models we have looked at so far. Starting in Januar 1994 the models are evaluated against each other in terms of how well they predict future yield developments.
(2) Macroeconomic variables are included, and a sub-set of the models are used to generate conditional forecasts.
(3) Scenarios are constructed where the future path of the yield curve is forced to pass through a set of exogenously determined future fixed points.

We are naturally using the same US yield curve data in this chapter that we have been using throughout this booklet.

6.1.1 The horse-race

Eighteen different model specifications are tested. Their differences fall along three dimensions (i) whether they impose arbitrage constraints, or not; (ii) whether bias correction is performed on the VAR dynamics, or not; and (iii) and whether they rely on 2, 3, or 4 factors.

The pseudo out-of-sample forecasting experiment is carried out in the following way:

1. sample data from June 1961 till January 1994
2. estimate the model under consideration
3. perform model forecasts for the horizon of 1-month to 12-months ahead, for each of the maturity points covered by the data sample i.e. for the $\{0.25, 1, 2, \ldots, 10\}$ year segments of the yield curve
4. calculate and store the differences between observed and forecasted yields
5. add one month to the sample, and repeat the above steps

6. repeat the process for the 282 data points covered by the evaluation sample, i.e. from February 1994 to June 2017 (we loose 1-year of data due to the 12-months forecasting horizon)
7. repeat the above for each of the eighteen models to be evaluated

Tables 6.1, 6.2, and 6.3, show the resulting forecast RMSE for each model at forecast horizons of 1, 2, 3, 6, and 12 ahead. Only a representative set of yield curve segments (maturities) are shown, and these are the 3-month, 5, and 10-year segments. For example, the first data row of Table 6.1 shows the ability of the dynamic Nelson-Siegel (DNS) model to forecast the 3-month maturity-segment of the yield curve: 1-month ahead the DNS model misses observed yields with an RMSE of 18 basis points, 2-months ahead the model misses with an RMSE of 27, and so on for the 3, 6 and 12-months projection horizons, with RMSEs of, 38, 67, and 122 basis points, respectively. For a model-free comparison, the last data line in each of the tables shows the RMSE of the Random walk model.

It is interesting to see that the forecasting performance reported here is very similar to the results of Diebold and Li (2006): both in terms of size and pattern across maturity segments and forecasting horizons - with one significant difference, which is addressed below. It is recalled that Diebold and Li (2006) conducted their analysis on US data covering the period from 1985 to 2000, and that the forecasting experiment they conducted was based on pseudo our-of-sample forecasts beginning in 1994. Our analysis extends the data sample to cover historical data going back to 1961 and our forecast also start in 1994, but extend to 2018. Still, results are quite well aligned. To illustrate the similarity of the produced forecasting performance, some representative figures are reported here. Table 5 in Diebold and Li (2006) show that the RMSE of the 3-months, 5-year and 10-year maturity points, forecasted 6-months ahead are 52, 78, and 72, respectively. In comparison, our study gives the following RMSEs 67, 74, 61. Differences of similar sizes are seen for the tested maturities and forecasting horizons. So, there is a much smaller difference between the numbers produced by Diebold and Li (2006) and our results, despite of the differences in the historical and forecast-evaluation periods.

There might be some who are of the opinion that the documented differences are large - roughly 10 basis points difference in term of standard deviation is not small, they may say. To assess whether the size of the difference is large or not, it may be illustrative to consider the standard error on the forecasts themselves. Inspecting the estimated models suggest that a comparable (i.e for the same maturity point and forecast horizon) forecast error is in the range of 20-30 basis points, so, to my reading the differences between the Diebold and Li (2006) results and our results are immaterial.

Now, as mentioned above, there is one important difference between our results and those of Diebold and Li (2006). And, this relates to the behaviour of the short end of the curve. Starting with the Fed's response to the 2007/2008 financial crises by lowering the policy rate, see the top panel of Figure 1.2, the dynamics of the short end has been under the control of the monetary authority and has remained at very low levels until around year 2017. In addition, the Fed's asset purchase programmes, implemented during this period, have impacted the term structure of term premia. However, our forecasting results imply that the dynamics of longer maturities of the curve can be well approximated by the same DGP as before 2007 - only the short end is materially impacted - as is clear from the performance of the 3-months segment of the curve shown in Table 6.1.

	Forecast horizon				
	1-month	2-months	3-months	6-months	12-months
DNS	18	27	38	67	122
DNS (bc)	17	27	38	66	111
DSS	18	28	39	68	122
DSS (bc)	18	28	39	66	122
SRB-3	18	27	38	67	122
SRB-3 (bc)	17	27	38	66	111
SRB-4	18	28	39	68	122
SRB-4 (bc)	18	28	39	67	122
JSZ	29	51	70	111	166
JSZ (bc)	26	44	60	91	133
AFSRB-2	30	31	39	67	122
AFSRB-2 (bc)	32	35	43	68	122
AFSRB-3	18	27	38	67	122
AFSRB-3 (bc)	17	28	39	67	122
AFSRB-4	30	45	63	122	200
AFSRB-4 (bc)	18	28	39	68	122
SRTPC1C2	44	47	53	74	122
SRTPC1C2 (bc)	38	42	50	74	122
Random walk	38	42	50	74	122

The RMSE of model forecasts calculated for the period covering January 1994 to July 2018 are shown. Each model is re-estimated each at each monthly observation point that is included in the evaluation period (using an expanding data sample), i.e. for each of the 282 months that falls in the period between January 1994 and July 2017. 12 months ahead forecasts are generated at each of the 282 observation points covered by the evaluation sample (this is why the last month estimations are performed is July 2017). The table shows the RMSE of the forecasts for 1,2,3,6, and, 12-months ahead, for the 3-month yield curve segment. Model names featuring a "(bc)" have been bias corrected using Pope (1990). The actual models that hides behind the shown abbreviations can be found in the MATLAB TSM class that accompany this booklet.

Table 6.1. Forecast RMSEs for the 3-month yield curve segment (basis points)

	Forecast horizon				
	1-month	2-months	3-months	6-months	12-months
DNS	30	44	53	74	100
DNS (bc)	29	42	51	70	92
DSS	27	41	50	71	97
DSS (bc)	27	41	49	69	91
SRB-3	30	44	53	74	100
SRB-3 (bc)	29	42	51	70	92
SRB-4	28	42	51	72	98
SRB-4 (bc)	27	41	50	69	91
JSZ	29	41	49	69	93
JSZ (bc)	29	42	50	67	85
AFSRB-2	34	47	55	77	100
AFSRB-2 (bc)	33	45	53	72	95
AFSRB-3	30	44	53	75	100
AFSRB-3 (bc)	30	43	51	71	93
AFSRB-4	67	77	84	100	133
AFSRB-4 (bc)	29	41	50	70	92
SRTPC1C2	44	53	60	78	100
SRTPC1C2 (bc)	44	53	61	79	100
Random walk	44	53	61	79	100

The RMSE of model forecasts calculated for the period covering January 1994 to July 2018 are shown. Each model is re-estimated each at each monthly observation point that is included in the evaluation period (using an expanding data sample), i.e. for each of the 282 months that falls in the period between January 1994 and July 2017. 12 months ahead forecasts are generated at each of the 282 observation points covered by the evaluation sample (this is why the last month estimations are performed is July 2017). The table shows the RMSE of the forecasts for 1,2,3,6, and, 12-months ahead, for the 5-year yield curve segment. Model names featuring a "(bc)" have been bias corrected using Pope (1990). The actual models that hides behind the shown abbreviations can be found in the MATLAB TSM class that accompany this booklet.

Table 6.2. Forecast RMSEs for the 5-year yield curve segment (basis points)

	Forecast horizon				
	1-month	2-months	3-months	6-months	12-months
DNS	28	40	45	61	75
DNS (bc)	29	40	46	61	73
DSS	27	39	45	61	76
DSS (bc)	28	40	46	62	74
SRB-3	28	40	45	61	75
SRB-3 (bc)	29	40	46	61	73
SRB-4	27	39	45	61	76
SRB-4 (bc)	27	40	45	62	74
JSZ	31	44	52	75	111
JSZ (bc)	30	42	48	67	86
AFSRB-2	38	47	52	65	74
AFSRB-2 (bc)	39	49	54	67	77
AFSRB-3	29	40	45	60	74
AFSRB-3 (bc)	29	41	46	61	73
AFSRB-4	49	49	51	61	79
AFSRB-4 (bc)	28	40	46	62	73
SRTPC1C2	45	52	57	70	86
SRTPC1C2 (bc)	46	53	59	74	87
Random walk	46	53	59	74	87

The RMSE of model forecasts calculated for the period covering January 1994 to July 2018 are shown. Each model is re-estimated each at each monthly observation point that is included in the evaluation period (using an expanding data sample), i.e. for each of the 282 months that falls in the period between January 1994 and July 2017. 12 months ahead forecasts are generated at each of the 282 observation points covered by the evaluation sample (this is why the last month estimations are performed is July 2017). The table shows the RMSE of the forecasts for 1,2,3,6, and, 12-months ahead, for the 10-year yield curve segment. Model names featuring a “(bc)” have been bias corrected using Pope (1990). The actual models that hides behind the shown abbreviations can be found in the MATLAB TSM class that accompany this booklet.

Table 6.3. Forecast RMSEs for the 10-year yield curve segment (basis points)

If we had been asked to make an unconditional guess, we would probably have said that the precision of the model forecasts would deteriorate as the forecast-horizon is increased. And, this is also what Tables 6.1, 6.2, and 6.2 confirm. Furthermore, being aware of the implemented policy measures since 2007/2008, we would also have said that the dynamics of the short end would be forecasted poorly, and that a shadow-short rate model may alleviate this problem (the reason why the SSR model is not included in the horse race is because it is very time-consuming to re-estimate this model 282 times). Finally, based on the separation of the likelihood function as derived by Joslin, Singleton, and Zhu (2011), see equation (5.6), we may also have reached the conclusion - since all tested models are based on a VAR(p) model for the factor dynamics, that their forecasting performance must be reasonably close; although differences may materialise due to a marginal better performance of e.g. a VAR(3) model, compared to VAR(2) and VAR(4) alternatives.

One of the practical conclusions that can be drawn from this horse race is that the choice of yield curve model has little impact on the precision of the unconditional forecasting performance. It does not seem feasible to choose the "best" model on the basis of its forecasting skills, when forecasts are made unconditionally. We have confirmed this empirically via the results of the performed horse race, as shown in Tables 6.1, 6.2, and 6.3. And, theoretical considerations also support this conclusion, as argued above.

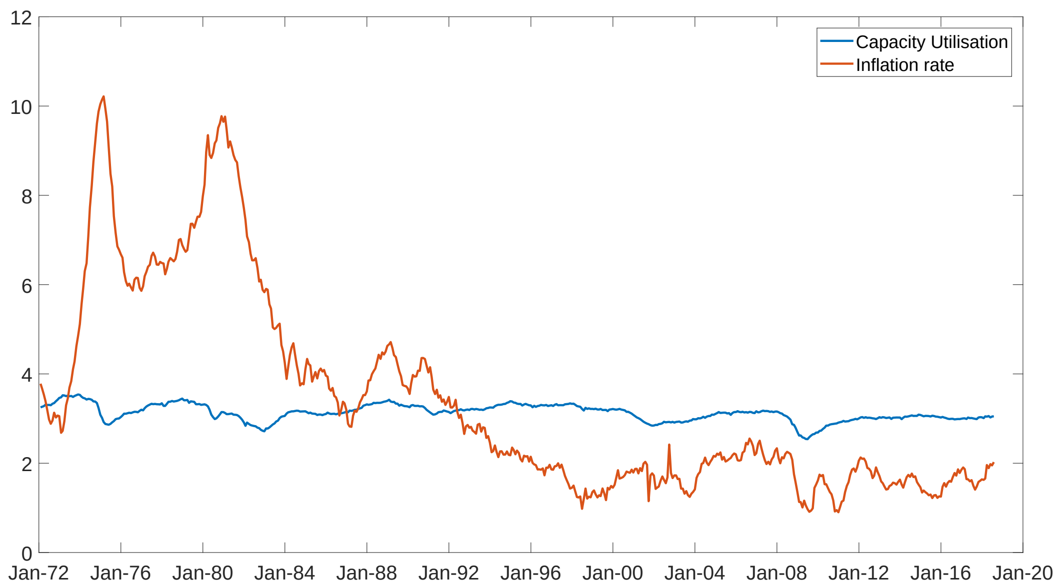
What should then guide our choice of model set-up? One element could be models' conditional forecasting performance; but we are not now going to conduct another horse race to explore different conditioning datasets - this would take too much time, as there are so many different constellations of macroeconomic variables that may be relevant. Another element is more subjective, yet important, and it relates to the factor structure that better fit the narratives we aim to support with our model. And, this of course depends on the business context in which the model is used. So, unfortunately, all the hard work that went in to producing the results shown in Tables 6.1, 6.2, and 6.3, while perhaps helpful from a general perspective, will not solve the model-selection problem for us.

6.1.2 Conditional projections

In many practical application we are interested in generating conditional yield curve scenarios. For example, in the context of strategic fixed income asset allocation, our objective is to assess risk and return characteristics of bond indices and portfolios in a setting where the investment horizon is long. For this purpose, unconditional forecasts are not overly useful, because the underlying VAR model that governs the dynamic evolution of the yield curve factors (and therefore also the yields) will typically converge to its sample mean before the end of the investment horizon. It is therefore limited what kind of relevant questions such model projections can answer. In stead, when deciding on long-term asset allocations, it is relevant to know what the yield curve will look like, if one or the other macroeconomic environment materialises, and how the yield curve converges to such scenario-based economic outcomes. In effect, the yield curve model becomes a tool that can help illustrate the consequence of various, more or less realistic, economic scenario-developments on expected returns and risks along the maturity dimension of the investable asset universe. So, rather then hoping for a yield curve model that will generate accurate point forecasts, we seek a model that links macroeconomic variables to yield curve developments, such that accurate conditional yield distributions can be generated (as opposed to accurate point estimates).

The reason why macroeconomic variables are pulled to the forefront here is that we may have a better grasp at what value such variables would take in different possible future scenarios, and it would be comparable harder for us to directly predict how the yield curve would evolve. This is why we build a bridge between the dynamic evolution of yield curves and macroeconomic (or other) variables, such that we can generate scenarios for these bridge-variables and subsequent extract the yield curve evolution, because this is what we are genuinely interested in.

As mentioned in the previous section, there are many macro variables to choose from. Out of convenience, we choose to use the same type of macroeconomic variables as used in Diebold, Rudebusch, and Aruoba (2006), namely manufacturing capacity utilisation (CU) and annual price inflation (INFL)[1]. These data are obtained from the FRED database, and are shown in Figure 6.1. These two variables gauge the level of real economic activity relative to potential, and the inflation rate. The macroeconomic variables are observable at a monthly frequency since January 1972, and our conditional forecasting experiment is therefore limited to using data from this date onwards.



US macroeconomic variables observed monthly since 1972. Manufacturing capacity utilisation (CU) and annual price inflation (INFL) are shown. Capacity utilisation (FRED code MCUMFN) is divided by 25 to align its scale. The inflation rate is calculated as the 12-month percent change in the Personal Consumption Expenditures Excluding Food and Energy (Chain-Type Price Index), (FRED code PCEPILFE).

Fig. 6.1. Capacity utilisation and inflation

We then form a set of possible macroeconomic scenarios. Although the conditional yield distributions were mentioned above as a main reason for this exercise, to keep things manageable in terms of visual

[1] The variable INFL is the 12-month percent change in the Personal Consumption Expenditures Excluding Food and Energy (Chain-Type Price Index), (FRED code PCEPILFE). The used CU variable has FRED code MCUMFN. Diebold, Rudebusch, and Aruoba (2006) also include the federal funds rate in their study, but since we will rely on a multi-factor short-rate based model (SRB3), we already have the short rate included among the yield curve factors that we model.

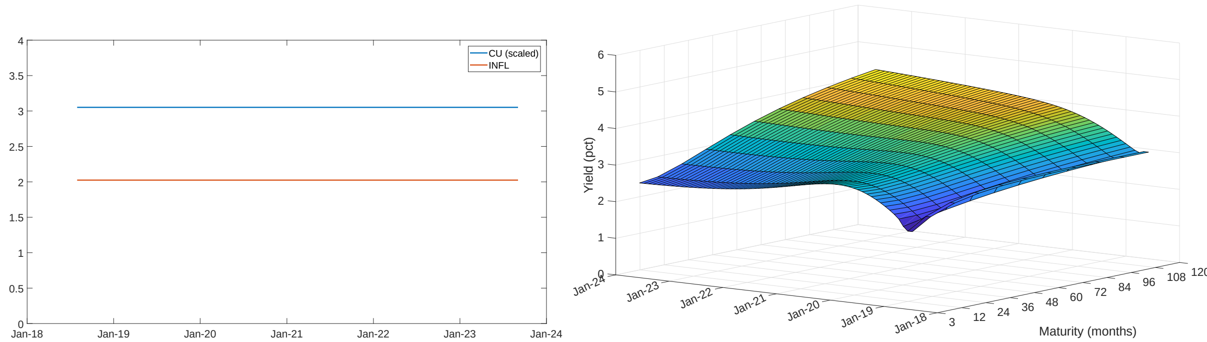
representations, we show only the mean paths of the yield curve. The presented framework can easily be used to also generate distributions.

Since the objective is to extract the model implied trajectories for the yield curve factors β^y conditional on the macroeconomic variables β^{macro}, we first need to estimate the parameters that govern the joint evolution of $\beta = [\beta^y, \beta^{macro}]$. For demonstration purposes alone we do this in the context of the SRB3 model (any of the term structure models included in these lecture note can naturally be used). After the model parameters have been identified, we then use the Kalman filter to extract the model implied conditional projections of interest. The game-plan is the following:

1. Estimate the SRB3 model by including the two macro variables as exogenous variables (using the TSM class).
2. Set-up the VAR(1)-part of the model (i.e. the part involving the dynamic evolution of the yield curve factors) in a MATLAB SSM model. This allows us to easily calculate the conditional projections for the yield curve factors using MATLAB's pre-programmed Kalman filter.
3. Make scenario-projections for the macroeconomic variables over a 5-year period. To exemplify, four scenario developments are examined: (1) Random walk (INLF and CU stays constant at their levels as observed at the end of the sample period); (2) exponential growth in inflation until a level of 4% is reached, after which the inflation rate normalises over a period of 12 months, and it then stays constant at 2.5% until the end of the projection horizon; (3) a linear drop in inflation to a level of 1.75% followed by a linear recovery; (4) a steady increase in economic growth over a period of four years, while the inflation rate is under control.
4. The conditional-scenario projections for the yield curve factors are converted into yields using the estimated SRB3-model's loading structure.

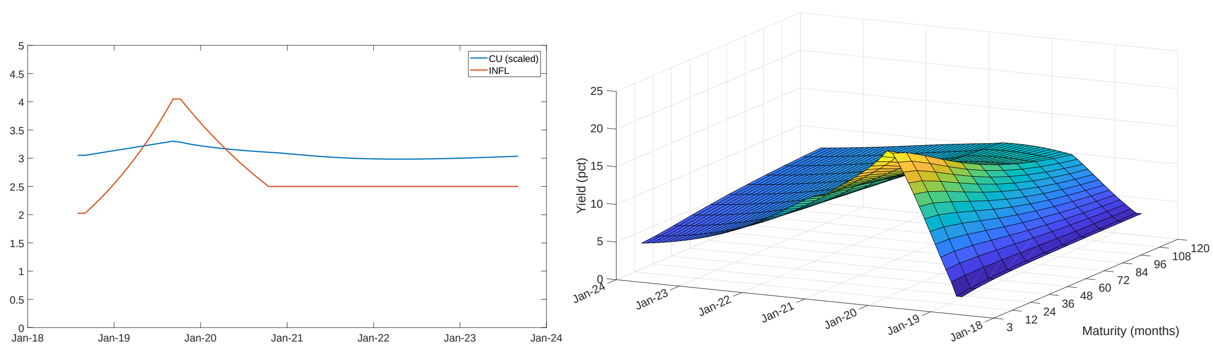
It is important to emphasise that the parametrisation of the model that is used here, has not been validated, nor has it undergone any testing/calibration to ensure that its economic narrative is sound: in other words, the model is used exactly as it comes, directly from the machine-room. The purpose here is, of course, not to construct a model that can enter directly into the SAA/policy process. The objective is simply to illustrate how the provided tool box can be used to make conditional yield curve projections.

It is left to the reader to investigate and inspect the outcome of each of the sketched scenarios using Figures 6.2, 6.3, 6.4, and 6.5.



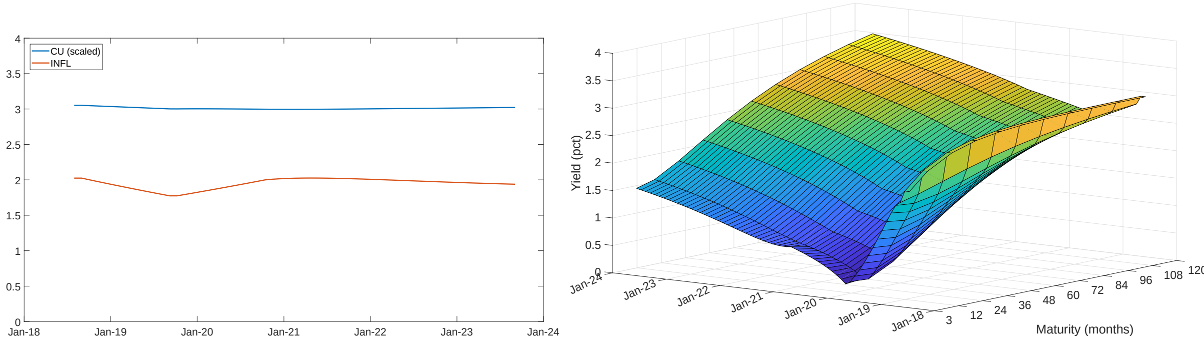
The LHS panel shows the macroeconomic scenario as illustrated by the assumed evolution of the macroeconomic variables. The macroeconomic developments are only dictated for the period of time that defines the scenario. After this period, the Kalman filter is used to find the relevant projections. The yield curve factors of the model are also obtained via the Kalman filter as projections that are calculated conditional on the macroeconomic developments. The RHS panel shows the corresponding development in the yield curve. The scenario spans a horizon of 5-years.

Fig. 6.2. Scenario 1: Random walk for inflation and economic growth



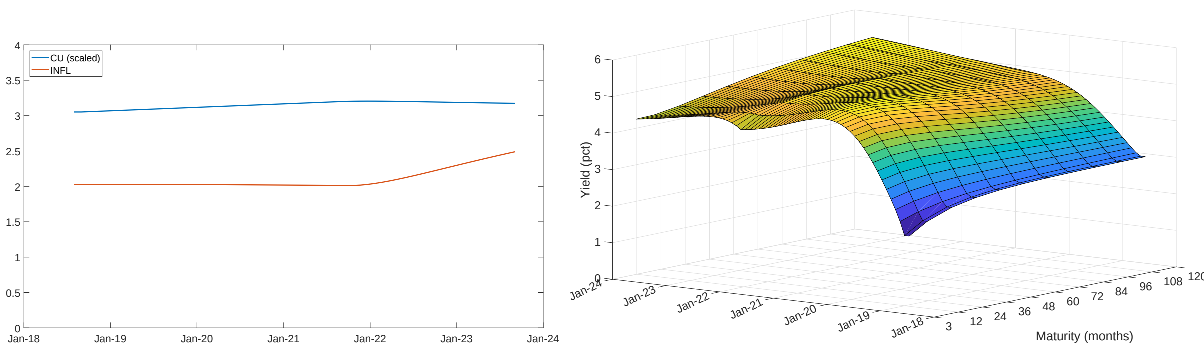
The LHS panel shows the macroeconomic scenario as illustrated by the assumed evolution of the macroeconomic variables. The macroeconomic developments are only dictated for the period of time that defines the scenario. After this period, the Kalman filter is used to find the relevant projections. The yield curve factors of the model are also obtained via the Kalman filter as projections that are calculated conditional on the macroeconomic developments. The RHS panel shows the corresponding development in the yield curve. The scenario spans a horizon of 5-years.

Fig. 6.3. Scenario 2: exponential growth in inflation



The LHS panel shows the macroeconomic scenario as illustrated by the assumed evolution of the macroeconomic variables. The macroeconomic developments are only dictated for the period of time that defines the scenario. After this period, the Kalman filter is used to find the relevant projections. The yield curve factors of the model are also obtained via the Kalman filter as projections that are calculated conditional on the macroeconomic developments. The RHS panel shows the corresponding development in the yield curve. The scenario spans a horizon of 5-years.

Fig. 6.4. scenario 3: a sudden drop in inflation



The LHS panel shows the macroeconomic scenario as illustrated by the assumed evolution of the macroeconomic variables. The macroeconomic developments are only dictated for the period of time that defines the scenario. After this period, the Kalman filter is used to find the relevant projections. The yield curve factors of the model are also obtained via the Kalman filter as projections that are calculated conditional on the macroeconomic developments. The RHS panel shows the corresponding development in the yield curve. The scenario spans a horizon of 5-years.

Fig. 6.5. scenario 4: steady economic growth

It is recalled that the mapping between the projected yield curve factors and the scenario yields is given by the observation equation of our well-know yield curve state-space model:

$$Y = a + b \cdot X_t + \Sigma_Y \cdot u_t \tag{6.1}$$

Above we have plotted only the mean scenarios, i.e. $E[Y]$. But, it is of course possible to use this frame work to generate distributions around the shown mean paths. This can be done by drawing innovations for the state and observation equations and by feeding these though Σ_X and Σ_Y, respectively. Another often used method is to block-bootstrap the historical residuals, possibly only sampled from historical periods that are judged to be similar to the one characterising the projection horizon.

6.1.3 Fix-point scenarios

Sometimes we may be able to construct scenarios directly using the yield curve factors; this is particularly the case when our chosen model is built on yield curve factors that have interpretations that we can relate to and that have straight-forward economic interpretations. For example, we may be interested in a scenario where the curve steepens by $x\%$ or to a certain level, or perhaps a scenario where the yield curve steepens to some pre-defined level. Such scenarios can be used e.g. to analyse the size of portfolio losses and gains given the materialisation of some future shape and location of the yield curve. The point is, contrary to the previous section where the yield curve and its dynamic evolution was tied to macroeconomic variables, that sometimes we are able to directly specify scenarios exclusively in terms of the future values that the yield curve factors are assumed to take on.

To cater for the generation of such scenarios, a neat little re-formulation of the VAR model is helpful. Let $X_{t+h}^{\texttt{target}}$ be the sequence of future fixed points that the factors are assumed to take on. $h = \{h_1, h_2, \ldots, h_n\}$ is a vector of future horizons, that define the scenario factor values (and thus naturally the scenario yield curves). In order to illustrate this process, we will focus on a single future horizon, but the process naturally genralises to multiple horizons, as we shall see in the empirical illustration below.

Starting with the VAR(1) model that govern the dynamic evolution of the yield curve factors (and suppressing the expectations operator for ease of notation), and making a projection for the horizon h_1 gives the following:

$$X_{t+h_1} = \mu + \Phi^{h_1} \left(X_t - \mu \right). \tag{6.2}$$

We now want to ensure that this projection exactly meets a given future set of yield curve factor values, such that $X_{t+h_1} = X_{t+h_1}^{target}$. We also want to retain the factor interpretation that is embedded in our chosen term structure model. In other words, we need to leave the eigenvectors of Φ unchanged during this exercise, because it is the eigenvectors that define the direction of the factor and thus their economic interpretation. This leaves us with the persistency parameters (the eigenvalues of Φ), and the mean m to be eligible for changing. Lets implement the changes via the presistency parameters, and write Φ using the eigenvalue decomposition:

$$X_t = \mu + V \cdot D \cdot V^{-1} \cdot \left(X_t - \mu \right), \tag{6.3}$$

where V contains the eigenvectors, and D holds eigenvalues on the diagonal. We can then write:

$$
\begin{aligned}
X_{t+h_1}^{\texttt{target}} - \mu = V \cdot D^{h_1} \cdot V^{-1} \cdot (X_t - \mu) \\
\Updownarrow \\
D^{h_1} \cdot V^{-1} \cdot (X_t - \mu) = V^{-1} \cdot \left(X_{t+h_1}^{\texttt{Target}} - \mu\right) \\
\Updownarrow \\
D^{h_1} = V^{-1} \cdot \left(X_{t+h_1}^{\texttt{Target}} - \mu\right) \odot \left(V^{-1} \cdot (X_t - \mu)\right)^{-1},
\end{aligned}
\tag{6.4}
$$

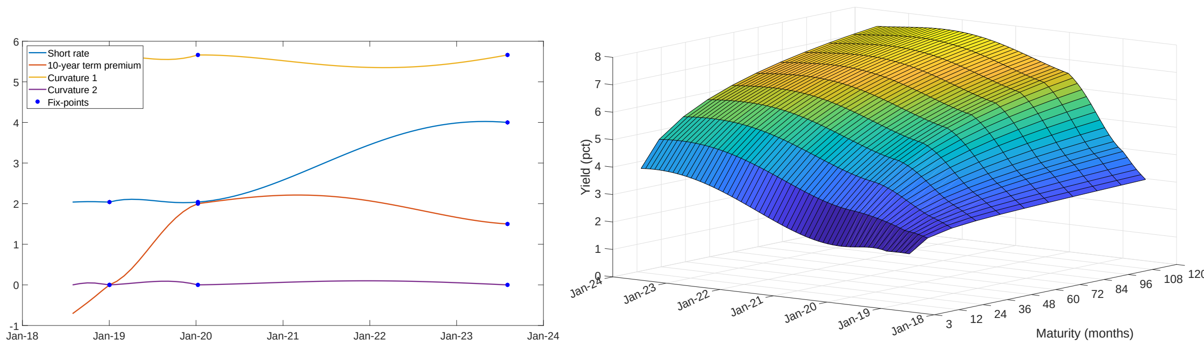
where $\odot$ represents the element-by-element multiplication. The last line works because D is diagonal.

Intuitively, it makes sense to change the persistency part of the system to force the yield curve factors to pass through a certain future fix point. The persistency of the model (the eigenvalues) is what makes the model converge to its sample mean: the higher the persistency (the higher the eigenvalues) the slower the convergence to historical means. So, by keeping the eigenvectors fixed, the economic interpretation of the factors remains unchanged, while the persistency is changed such that the desired fix-point values can be met at the desired projection horizon.

To illustrate this process we again generate a scenario over the coming five year, now using the SRTPC1C2-model, i.e. the one where we have included the short rate and the 10-year term premium as yield curve factors (together with two curvature factors). Having the short rate and the term term premium as underlying factors allows to us make scenarios where future fixed-point values are specified exogenously for the value that these factors may take on at future dates covered by the projection horizon.

Following a period of central bank intervention in the fixed income markets via bond purchases, as seen in the US from 2008 to 2018, where the purpose of such interventions is to compress the term premium - and thereby the yield curve - it is likely that we have in mind particular future trajectories for the term premium and the short end of the yield curve, and it is therefore handy to be able to model these factors explicitly in the context of a formal modelling-and-projection framework. But, the relevance of having a direct handle on these factors naturally extends beyond the quantitative easing example provided here, and is of a general interest in monetary policy modelling as well as strategic investment analysis.

In the current example, it is assumed that we have three fix-points: (1) after 6-months where the 10-year term premium equals 0.00% (at the start of the projection-horizon sample (July 2018) the 10-year term premium is estimated to be around -0.70%); (2) after an additional 18-months, the 10-year term premium equals 4.00%; and (3) at the end of the 60-months projection horizon the 10-year term premium equals 1.5% and the short rate equals 4.00%. The resulting factor trajectories and yield curve evolutions are displayed in Figure 6.6.



The LHS panel shows the development of the yield curve factors of the SRTPC1C2-model, i.e. the short rate, the 10-year term premium, and the two curvature factors. The projected scenario is defined by three fixed-point vectors that the factors are required to pass through, (1) after 6-months where the 10-year term premium equals 0.00% (at the start of the projection-horizon sample (July 2018) the 10-year term premium is estimated to be around -0.70%); (2) after an additional 18-months, the 10-year term premium equals 4.00%; and (3) at the end of the 60-months projection horizon the 10-year term premium equals 1.5% and the short rate equals 4.00%. The resulting factor trajectories and yield curve evolutions are displayed in. The RHS panel shown the corresponding yield curve projection.

Fig. 6.6. A yield curve scenario using fixed-point projections

The familiar relationship between factors and yields recalled in equation (6.1) is also used here to translate projected yield curve factors into yield evolutions.

In the LHS panel of Figure 6.6 it is observed that the projections for the short rate and the 10-year term premium pass through the scenario fixed-points at the pre-specified future dates, and that the RHS panel of Figure 6.6, traces out the scenario yields that follow as a logical consequence of the yield curve factor trajectories.

6.2 MATLAB code

filename: Scenario_and_forecasting.m

```
%% Scenario generation and forecasting
%
% preparing the data
%
warning('off','all')
load('Data_GSW_factors_Course_2018.mat');
GSW_         = GSW;                   % creates an instance of the GSW class
GSW_.tau     = [3 12:12:120]';        % vector of maturities
GSW_.beta    = GSW_factors(:,2:5);    % yield curve factors
GSW_.lambda  = GSW_factors(:,6:7);    % lambdas
GSW_         = GSW_.getYields;        % getting yields

dates = GSW_factors(:,1);
Y     = GSW_.yields;
tau   = GSW_.tau;
nTau  = size(tau,1);

figure
    plot(dates,Y(:,11))
    date_ticks = datenum(1960:4:2020,1,1);
    set(gca, 'xtick', date_ticks);
    datetick('x','mmm-yy','keepticks')

figure('units','normalized','outerposition',[0 0 1 1])
    plot(US_MacroVariables(:,1),[US_MacroVariables(:,2)./25 ...
                     US_MacroVariables(:,3) ],'LineWidth',2)
    date_ticks = datenum(1972:4:2020,1,1);
    set(gca, 'xtick', date_ticks);
    datetick('x','mmm-yy','keepticks')
    set(gca, 'FontSize', 18),
    legend('Capacity Utilisation','Inflation rate')
  %  print -depsc MacroVariables


%% The horse-race
% The following models are included in the horce-race
% ---------------------------------------------------
%  DNS         -> Dynamic Nelson-Siegel model
%  DNS_bc      -> Dynamic Nelson-Siegel model, bias corrected
%  DSS         -> Dynamic Svensson-Soderlind model
%  DSS_bc      -> Dynamic Svensson-Soderlind model, bias corrected
%  SRB3        -> Short-Rate based 3-factor model
%  SRB3_bc     -> Short-Rate based 3-factor model, bias corrected
%  SRB4        -> Short-Rate based 4-factor model
%  SRB4_bc     -> Short-Rate based 4-factor model, bias corrected
%  JSZ         -> Joslin, Singleton, Zhu (2011)
%  JSZ_bc      -> Joslin, Singleton, Zhu (2011), bias corrected
%  AFSRB       -> Arbitrage-free SRB model with 2,3, or 4 factors
%  AFSRB_bc    -> Arbitrage-free SRB model with 2,3, or 4 factors, bias c.
```

```
%  SRTPC1C2    -> Model with Short rate, 10-year term premium,
%                       and 2 additional empirical factors
%  SRTPC1C2_bc -> Model with Short rate, 10-year term premium,
%                       and 2 additional empirical factors, bias corrected
%
% Note that program execution could possibly be increased by combining
%      the pseudo out-of-sample forecasts, performed for each model,
%      inside one loop. However, with an eye to clarity of the code,
%      a slower model-by-model programming set-up is used.
%
% fDate     = datenum('31-Jan-1994'); % start date for the horse-race
% horizon   = 12;                     % forecast horizon
% startIndx = find(fDate==dates,1,'first');
% nIter     = GSW_.nObs - startIndx - horizon;
%
% %
% % ... DNS
% %
% DNS_fErr = NaN(horizon+1,GSW_.nTau,nIter);
% for ( j=1:nIter )
%     estYields      = Y(1:startIndx+j,:);
%     oYields        = Y(startIndx+j-1:startIndx+j-1+horizon,:);
%     A_TSM          = [];
%     A_TSM          = TSM;
%     A_TSM.yields   = estYields;
%     A_TSM.tau      = tau;
%     A_TSM.DataFreq = 12;
%     A_TSM.nF       = 3;
%     A_TSM.biasCorrect = 0;
%     A_TSM          = A_TSM.getDNS;
%     castY          = [];
%     A_SSM          = TSM2SSM;
%     A_SSM.TSM      = A_TSM;
%     A_SSM          = A_SSM.getMdl;
%     castY          = [ A_SSM.Data(startIndx+j-1,:); forecast(A_SSM.Mdl, ...
%                                  horizon, A_SSM.Data(startIndx+j-1,:) )];
%     DNS_fErr(:,:,j)    = oYields-castY(:,1:11);
% end
% DNS_fRMSE = 100.*sqrt(mean((DNS_fErr.^2),3));
%
% %
% % ... DNSbc
% %
% DNSbc_fErr = NaN(horizon+1,GSW_.nTau,nIter);
% for ( j=1:nIter )
%     estYields      = Y(1:startIndx+j,:);
%     oYields        = Y(startIndx+j-1:startIndx+j-1+horizon,:);
%     A_TSM          = [];
%     A_TSM          = TSM;
%     A_TSM.yields   = estYields;
%     A_TSM.tau      = tau;
%     A_TSM.DataFreq = 12;
%     A_TSM.nF       = 3;
%     A_TSM.biasCorrect = 1;
```

```
%     A_TSM           = A_TSM.getDNS;
%     castY           = [];
%     A_SSM           = TSM2SSM;
%     A_SSM.TSM       = A_TSM;
%     A_SSM           = A_SSM.getMdl;
%     castY           = [ A_SSM.Data(startIndx+j-1,:); forecast(A_SSM.Mdl, ...
%                                     horizon, A_SSM.Data(startIndx+j-1,:) )];
%     DNSbc_fErr(:,:,j)    = oYields-castY(:,1:11);
% end
% DNSbc_fRMSE = 100.*sqrt(mean((DNSbc_fErr.^2),3));
%
% %
% % ... DSS
% %
% DSS_fErr = NaN(horizon+1,GSW_.nTau,nIter);
% for ( j=1:nIter )
%     estYields       = Y(1:startIndx+j,:);
%     oYields         = Y(startIndx+j-1:startIndx+j-1+horizon,:);
%     A_TSM           = [];
%     A_TSM           = TSM;
%     A_TSM.yields    = estYields;
%     A_TSM.tau       = tau;
%     A_TSM.DataFreq = 12;
%     A_TSM.nF        = 3;
%     A_TSM.biasCorrect = 0;
%     A_TSM           = A_TSM.getDSS;
%     castY           = [];
%     A_SSM           = TSM2SSM;
%     A_SSM.TSM       = A_TSM;
%     A_SSM           = A_SSM.getMdl;
%     castY           = [ A_SSM.Data(startIndx+j-1,:); forecast(A_SSM.Mdl, ...
%                                     horizon, A_SSM.Data(startIndx+j-1,:) )];
%     DSS_fErr(:,:,j) = oYields-castY(:,1:11);
% end
% DSS_fRMSE = 100.*sqrt(mean((DSS_fErr.^2),3));
%
% %
% % ... DSSbc
% %
% DSSbc_fErr = NaN(horizon+1,GSW_.nTau,nIter);
% for ( j=1:nIter )
%     estYields       = Y(1:startIndx+j,:);
%     oYields         = Y(startIndx+j-1:startIndx+j-1+horizon,:);
%     A_TSM           = [];
%     A_TSM           = TSM;
%     A_TSM.yields    = estYields;
%     A_TSM.tau       = tau;
%     A_TSM.DataFreq = 12;
%     A_TSM.nF        = 3;
%     A_TSM.biasCorrect = 1;
%     A_TSM           = A_TSM.getDSS;
%     castY           = [];
%     A_SSM           = TSM2SSM;
%     A_SSM.TSM       = A_TSM;
```

```
%     A_SSM           = A_SSM.getMdl;
%     castY           = [ A_SSM.Data(startIndx+j-1,:); forecast(A_SSM.Mdl, ...
%                                     horizon, A_SSM.Data(startIndx+j-1,:) )];
%     DSSbc_fErr(:,:,j) = oYields-castY(:,1:11);
% end
% DSSbc_fRMSE = 100.*sqrt(mean((DSSbc_fErr.^2),3));
%
%
% %
% % ... SRB3
% %
% SRB3_fErr = NaN(horizon+1,GSW_.nTau,nIter);
% for ( j=1:nIter )
%     estYields       = Y(1:startIndx+j,:);
%     oYields         = Y(startIndx+j-1:startIndx+j-1+horizon,:);
%     A_TSM           = [];
%     A_TSM           = TSM;
%     A_TSM.yields    = estYields;
%     A_TSM.tau       = tau;
%     A_TSM.DataFreq  = 12;
%     A_TSM.nF        = 3;
%     A_TSM.biasCorrect = 0;
%     A_TSM           = A_TSM.getSRB3;
%     castY           = [];
%     A_SSM           = TSM2SSM;
%     A_SSM.TSM       = A_TSM;
%     A_SSM           = A_SSM.getMdl;
%     castY           = [ A_SSM.Data(startIndx+j-1,:); forecast(A_SSM.Mdl, ...
%                                     horizon, A_SSM.Data(startIndx+j-1,:) )];
%     SRB3_fErr(:,:,j) = oYields-castY(:,1:11);
% end
% SRB3_fRMSE = 100.*sqrt(mean((SRB3_fErr.^2),3));
%
%
% %
% % ... SRB3bc
% %
% SRB3bc_fErr = NaN(horizon+1,GSW_.nTau,nIter);
% for ( j=1:nIter )
%     estYields       = Y(1:startIndx+j,:);
%     oYields         = Y(startIndx+j-1:startIndx+j-1+horizon,:);
%     A_TSM           = [];
%     A_TSM           = TSM;
%     A_TSM.yields    = estYields;
%     A_TSM.tau       = tau;
%     A_TSM.DataFreq  = 12;
%     A_TSM.nF        = 3;
%     A_TSM.biasCorrect = 1;
%     A_TSM           = A_TSM.getSRB3;
%     castY           = [];
%     A_SSM           = TSM2SSM;
%     A_SSM.TSM       = A_TSM;
%     A_SSM           = A_SSM.getMdl;
%     castY           = [ A_SSM.Data(startIndx+j-1,:); forecast(A_SSM.Mdl, ...
```

```
%                                    horizon, A_SSM.Data(startIndx+j-1,:) )];
%     SRB3bc_fErr(:,:,j) = oYields-castY(:,1:11);
% end
% SRB3bc_fRMSE = 100.*sqrt(mean((SRB3bc_fErr.^2),3));
%
%
% %
% % ... SRB4
% %
% SRB4_fErr = NaN(horizon+1,GSW_.nTau,nIter);
% for ( j=1:nIter )
%     estYields      = Y(1:startIndx+j,:);
%     oYields        = Y(startIndx+j-1:startIndx+j-1+horizon,:);
%     A_TSM          = [];
%     A_TSM          = TSM;
%     A_TSM.yields   = estYields;
%     A_TSM.tau      = tau;
%     A_TSM.DataFreq = 12;
%     A_TSM.nF       = 4;
%     A_TSM.biasCorrect = 0;
%     A_TSM          = A_TSM.getSRB4;
%     castY          = [];
%     A_SSM          = TSM2SSM;
%     A_SSM.TSM      = A_TSM;
%     A_SSM          = A_SSM.getMdl;
%     castY          = [ A_SSM.Data(startIndx+j-1,:); forecast(A_SSM.Mdl, ...
%                                    horizon, A_SSM.Data(startIndx+j-1,:) )];
%     SRB4_fErr(:,:,j) = oYields-castY(:,1:11);
% end
% SRB4_fRMSE = 100.*sqrt(mean((SRB4_fErr.^2),3));
%
%
% %
% % ... SRB4bc
% %
% SRB4bc_fErr = NaN(horizon+1,GSW_.nTau,nIter);
% for ( j=1:nIter )
%     estYields      = Y(1:startIndx+j,:);
%     oYields        = Y(startIndx+j-1:startIndx+j-1+horizon,:);
%     A_TSM          = [];
%     A_TSM          = TSM;
%     A_TSM.yields   = estYields;
%     A_TSM.tau      = tau;
%     A_TSM.DataFreq = 12;
%     A_TSM.nF       = 4;
%     A_TSM.biasCorrect = 1;
%     A_TSM          = A_TSM.getSRB4;
%     castY          = [];
%     A_SSM          = TSM2SSM;
%     A_SSM.TSM      = A_TSM;
%     A_SSM          = A_SSM.getMdl;
%     castY          = [ A_SSM.Data(startIndx+j-1,:); forecast(A_SSM.Mdl, ...
%                                    horizon, A_SSM.Data(startIndx+j-1,:) )];
%     SRB4bc_fErr(:,:,j) = oYields-castY(:,1:11);
```

```
% end
% SRB4bc_fRMSE = 100.*sqrt(mean((SRB4bc_fErr.^2),3));
%
%
% %
% % ... JSZ
% %
% JSZ_fErr = NaN(horizon+1,GSW_.nTau,nIter);
% for ( j=1:nIter )
%     estYields      = Y(1:startIndx+j,:);
%     oYields        = Y(startIndx+j-1:startIndx+j-1+horizon,:);
%     A_TSM          = [];
%     A_TSM          = TSM;
%     A_TSM.yields   = estYields;
%     A_TSM.tau      = tau;
%     A_TSM.DataFreq = 12;
%     A_TSM.nF       = 3;
%     A_TSM.biasCorrect = 0;
%     A_TSM          = A_TSM.getJSZ;
%     castY          = [];
%     A_SSM          = TSM2SSM;
%     A_SSM.TSM      = A_TSM;
%     A_SSM          = A_SSM.getMdl;
%     castY          = [ A_SSM.Data(startIndx+j-1,:); forecast(A_SSM.Mdl, ...
%                                     horizon, A_SSM.Data(startIndx+j-1,:) )];
%     JSZ_fErr(:,:,j) = oYields-castY(:,1:11);
% end
% JSZ_fRMSE = 100.*sqrt(mean((JSZ_fErr.^2),3));
%
%
% %
% % ... JSZ_bc
% %
% JSZbc_fErr = NaN(horizon+1,GSW_.nTau,nIter);
% for ( j=1:nIter )
%     estYields      = Y(1:startIndx+j,:);
%     oYields        = Y(startIndx+j-1:startIndx+j-1+horizon,:);
%     A_TSM          = [];
%     A_TSM          = TSM;
%     A_TSM.yields   = estYields;
%     A_TSM.tau      = tau;
%     A_TSM.DataFreq = 12;
%     A_TSM.nF       = 3;
%     A_TSM.biasCorrect = 1;
%     A_TSM          = A_TSM.getJSZ;
%     castY          = [];
%     A_SSM          = TSM2SSM;
%     A_SSM.TSM      = A_TSM;
%     A_SSM          = A_SSM.getMdl;
%     castY          = [ A_SSM.Data(startIndx+j-1,:); forecast(A_SSM.Mdl, ...
%                                     horizon, A_SSM.Data(startIndx+j-1,:) )];
%     JSZbc_fErr(:,:,j) = oYields-castY(:,1:11);
% end
% JSZbc_fRMSE = 100.*sqrt(mean((JSZbc_fErr.^2),3));
```

```
%
%
% %
% % ... AFSRB2
% %
% AF2_fErr = NaN(horizon+1,GSW_.nTau,nIter);
% for ( j=1:nIter )
%     estYields      = Y(1:startIndx+j,:);
%     oYields        = Y(startIndx+j-1:startIndx+j-1+horizon,:);
%     A_TSM          = [];
%     A_TSM          = TSM;
%     A_TSM.yields   = estYields;
%     A_TSM.tau      = tau;
%     A_TSM.DataFreq = 12;
%     A_TSM.nF       = 2;
%     A_TSM.biasCorrect = 0;
%     A_TSM          = A_TSM.getAFSRB;
%     castY          = [];
%     A_SSM          = TSM2SSM;
%     A_SSM.TSM      = A_TSM;
%     A_SSM          = A_SSM.getMdl;
%     castY          = [ A_SSM.Data(startIndx+j-1,:); forecast(A_SSM.Mdl, ...
%                                   horizon, A_SSM.Data(startIndx+j-1,:) )];
%     AF2_fErr(:,:,j) = oYields-castY(:,1:11);
% end
% AF2_fRMSE = 100.*sqrt(mean((AF2_fErr.^2),3));
%
%
% %
% % ... AFSRB2
% %
% AF2bc_fErr = NaN(horizon+1,GSW_.nTau,nIter);
% for ( j=1:nIter )
%     estYields      = Y(1:startIndx+j,:);
%     oYields        = Y(startIndx+j-1:startIndx+j-1+horizon,:);
%     A_TSM          = [];
%     A_TSM          = TSM;
%     A_TSM.yields   = estYields;
%     A_TSM.tau      = tau;
%     A_TSM.DataFreq = 12;
%     A_TSM.nF       = 2;
%     A_TSM.biasCorrect = 1;
%     A_TSM          = A_TSM.getAFSRB;
%     castY          = [];
%     A_SSM          = TSM2SSM;
%     A_SSM.TSM      = A_TSM;
%     A_SSM          = A_SSM.getMdl;
%     castY          = [ A_SSM.Data(startIndx+j-1,:); forecast(A_SSM.Mdl, ...
%                                   horizon, A_SSM.Data(startIndx+j-1,:) )];
%     AF2bc_fErr(:,:,j) = oYields-castY(:,1:11);
% end
% AF2bc_fRMSE = 100.*sqrt(mean((AF2bc_fErr.^2),3));
%
% %
```

```
% % ... AFSRB3
% %
% AF3_fErr = NaN(horizon+1,GSW_.nTau,nIter);
% for ( j=1:nIter )
%     estYields      = Y(1:startIndx+j,:);
%     oYields        = Y(startIndx+j-1:startIndx+j-1+horizon,:);
%     A_TSM          = [];
%     A_TSM          = TSM;
%     A_TSM.yields   = estYields;
%     A_TSM.tau      = tau;
%     A_TSM.DataFreq = 12;
%     A_TSM.nF       = 3;
%     A_TSM.biasCorrect = 0;
%     A_TSM          = A_TSM.getAFSRB;
%     castY          = [];
%     A_SSM          = TSM2SSM;
%     A_SSM.TSM      = A_TSM;
%     A_SSM          = A_SSM.getMdl;
%     castY          = [ A_SSM.Data(startIndx+j-1,:); forecast(A_SSM.Mdl, ...
%                              horizon, A_SSM.Data(startIndx+j-1,:) )];
%     AF3_fErr(:,:,j) = oYields-castY(:,1:11);
% end
% AF3_fRMSE = 100.*sqrt(mean((AF3_fErr.^2),3));
%
%
% %
% % ... AFSRB3_bc
% %
% AF3bc_fErr = NaN(horizon+1,GSW_.nTau,nIter);
% for ( j=1:nIter )
%     estYields      = Y(1:startIndx+j,:);
%     oYields        = Y(startIndx+j-1:startIndx+j-1+horizon,:);
%     A_TSM          = [];
%     A_TSM          = TSM;
%     A_TSM.yields   = estYields;
%     A_TSM.tau      = tau;
%     A_TSM.DataFreq = 12;
%     A_TSM.nF       = 3;
%     A_TSM.biasCorrect = 1;
%     A_TSM          = A_TSM.getAFSRB;
%     castY          = [];
%     A_SSM          = TSM2SSM;
%     A_SSM.TSM      = A_TSM;
%     A_SSM          = A_SSM.getMdl;
%     castY          = [ A_SSM.Data(startIndx+j-1,:); forecast(A_SSM.Mdl, ...
%                              horizon, A_SSM.Data(startIndx+j-1,:) )];
%     AF3bc_fErr(:,:,j) = oYields-castY(:,1:11);
% end
% AF3bc_fRMSE = 100.*sqrt(mean((AF3bc_fErr.^2),3));
%
%
% %
% % ... AFSRB4
% %
```

```
% AF4_fErr = NaN(horizon+1,GSW_.nTau,nIter);
% for ( j=1:nIter )
%     estYields      = Y(1:startIndx+j,:);
%     oYields        = Y(startIndx+j-1:startIndx+j-1+horizon,:);
%     A_TSM          = [];
%     A_TSM          = TSM;
%     A_TSM.yields   = estYields;
%     A_TSM.tau      = tau;
%     A_TSM.DataFreq = 12;
%     A_TSM.nF       = 4;
%     A_TSM.biasCorrect = 0;
%     A_TSM          = A_TSM.getAFSRB;
%     castY          = [];
%     A_SSM          = TSM2SSM;
%     A_SSM.TSM      = A_TSM;
%     A_SSM          = A_SSM.getMdl;
%     castY          = [ A_SSM.Data(startIndx+j-1,:); forecast(A_SSM.Mdl, ...
%                               horizon, A_SSM.Data(startIndx+j-1,:) )];
%     AF4_fErr(:,:,j) = oYields-castY(:,1:11);
% end
% AF4_fRMSE = 100.*sqrt(mean((AF4_fErr.^2),3));
%
%
% %
% % ... AFSRB4_bc
% %
% AF4bc_fErr = NaN(horizon+1,GSW_.nTau,nIter);
% for ( j=1:nIter )
%     estYields      = Y(1:startIndx+j,:);
%     oYields        = Y(startIndx+j-1:startIndx+j-1+horizon,:);
%     A_TSM          = [];
%     A_TSM          = TSM;
%     A_TSM.yields   = estYields;
%     A_TSM.tau      = tau;
%     A_TSM.DataFreq = 12;
%     A_TSM.nF       = 4;
%     A_TSM.biasCorrect = 1;
%     A_TSM          = A_TSM.getAFSRB;
%     castY          = [];
%     A_SSM          = TSM2SSM;
%     A_SSM.TSM      = A_TSM;
%     A_SSM          = A_SSM.getMdl;
%     castY          = [ A_SSM.Data(startIndx+j-1,:); forecast(A_SSM.Mdl, ...
%                               horizon, A_SSM.Data(startIndx+j-1,:) )];
%     AF4bc_fErr(:,:,j) = oYields-castY(:,1:11);
% end
% AF4bc_fRMSE = 100.*sqrt(mean((AF4bc_fErr.^2),3));
%
%
% %
% % ... SRTPC1C2
% %
% SRTPC1C2_fErr = NaN(horizon+1,GSW_.nTau,nIter);
% for ( j=1:nIter )
```

```
%     estYields      = Y(1:startIndx+j,:);
%     oYields        = Y(startIndx+j-1:startIndx+j-1+horizon,:);
%     A_TSM          = [];
%     A_TSM          = TSM;
%     A_TSM.yields   = estYields;
%     A_TSM.tau      = tau;
%     A_TSM.DataFreq = 12;
%     A_TSM.nF       = 4;
%     A_TSM.biasCorrect = 0;
%     A_TSM          = A_TSM.getSRTPC1C2;
%     castY          = [];
%     A_SSM          = TSM2SSM;
%     A_SSM.TSM      = A_TSM;
%     A_SSM          = A_SSM.getMdl;
%     castY          = [ A_SSM.Data(startIndx+j-1,:); forecast(A_SSM.Mdl, ...
%                                 horizon, A_SSM.Data(startIndx+j-1,:) )];
%     SRTPC1C2_fErr(:,:,j) = oYields-castY(:,1:11);
% end
% SRTPC1C2_fRMSE = 100.*sqrt(mean((SRTPC1C2_fErr.^2),3));
%
%
% %
% % ... SRTPC1C2bc
% %
% SRTPC1C2bc_fErr = NaN(horizon+1,GSW_.nTau,nIter);
% for ( j=1:nIter )
%     estYields      = Y(1:startIndx+j,:);
%     oYields        = Y(startIndx+j-1:startIndx+j-1+horizon,:);
%     A_TSM          = [];
%     A_TSM          = TSM;
%     A_TSM.yields   = estYields;
%     A_TSM.tau      = tau;
%     A_TSM.DataFreq = 12;
%     A_TSM.nF       = 4;
%     A_TSM.biasCorrect = 1;
%     biasCorrect    = 1;
%     A_TSM          = A_TSM.getSRTPC1C2;
%     castY          = [];
%     A_SSM          = TSM2SSM;
%     A_SSM.TSM      = A_TSM;
%     A_SSM          = A_SSM.getMdl;
%     castY          = [ A_SSM.Data(startIndx+j-1,:); forecast(A_SSM.Mdl, ...
%                                 horizon, A_SSM.Data(startIndx+j-1,:) )];
%     SRTPC1C2bc_fErr(:,:,j) = oYields-castY(:,1:11);
% end
% SRTPC1C2bc_fRMSE = 100.*sqrt(mean((SRTPC1C2bc_fErr.^2),3));
%
% %% preparing output tables
% %
% % RMSE of all models
% %       For maturities      : 3m 1Y 5Y 10Y
% %       and forecasts adead : 1m 2m 3m 6m 12m
% %
%
```

```
% ahead = [2;3;4;7;13];
% %
% % ... for the 3m maturity segment
% %
% ZZmat_3m = [  DNS_fRMSE(ahead,1)';
%               DNSbc_fRMSE(ahead,1)';
%               DSS_fRMSE(ahead,1)';
%               DSSbc_fRMSE(ahead,1)';
%               SRB3_fRMSE(ahead,1)';
%               SRB3bc_fRMSE(ahead,1)';
%               SRB4_fRMSE(ahead,1)';
%               SRB4bc_fRMSE(ahead,1)';
%               JSZ_fRMSE(ahead,1)';
%               JSZbc_fRMSE(ahead,1)';
%               AF2_fRMSE(ahead,1)';
%               AF2bc_fRMSE(ahead,1)';
%               AF3_fRMSE(ahead,1)';
%               AF3bc_fRMSE(ahead,1)';
%               AF4_fRMSE(ahead,1)';
%               AF4bc_fRMSE(ahead,1)';
%               SRTPC1C2_fRMSE(ahead,1)';
%               SRTPC1C2bc_fRMSE(ahead,1)'];
%
% %
% % ... for the 1Y maturity segment
% %
% ZZmat_1Y = [  DNS_fRMSE(ahead,2)';
%               DNSbc_fRMSE(ahead,2)';
%               DSS_fRMSE(ahead,2)';
%               DSSbc_fRMSE(ahead,2)';
%               SRB3_fRMSE(ahead,2)';
%               SRB3bc_fRMSE(ahead,2)';
%               SRB4_fRMSE(ahead,2)';
%               SRB4bc_fRMSE(ahead,2)';
%               JSZ_fRMSE(ahead,2)';
%               JSZbc_fRMSE(ahead,2)';
%               AF2_fRMSE(ahead,2)';
%               AF2bc_fRMSE(ahead,2)';
%               AF3_fRMSE(ahead,2)';
%               AF3bc_fRMSE(ahead,2)';
%               AF4_fRMSE(ahead,2)';
%               AF4bc_fRMSE(ahead,2)';
%               SRTPC1C2_fRMSE(ahead,2)';
%               SRTPC1C2bc_fRMSE(ahead,2)'];
%
%
% %
% % ... for the 5Y maturity segment
% %
% ZZmat_5Y = [  DNS_fRMSE(ahead,6)';
%               DNSbc_fRMSE(ahead,6)';
%               DSS_fRMSE(ahead,6)';
%               DSSbc_fRMSE(ahead,6)';
%               SRB3_fRMSE(ahead,6)';
```

```
%                SRB3bc_fRMSE(ahead,6)';
%                SRB4_fRMSE(ahead,6)';
%                SRB4bc_fRMSE(ahead,6)';
%                JSZ_fRMSE(ahead,6)';
%                JSZbc_fRMSE(ahead,6)';
%                AF2_fRMSE(ahead,6)';
%                AF2bc_fRMSE(ahead,6)';
%                AF3_fRMSE(ahead,6)';
%                AF3bc_fRMSE(ahead,6)';
%                AF4_fRMSE(ahead,6)';
%                AF4bc_fRMSE(ahead,6)';
%                SRTPC1C2_fRMSE(ahead,6)';
%                SRTPC1C2bc_fRMSE(ahead,6)'];
%
% %
% % ... for the 10Y maturity segment
% %
% ZZmat_10Y = [ DNS_fRMSE(ahead,11)';
%                DNSbc_fRMSE(ahead,11)';
%                DSS_fRMSE(ahead,11)';
%                DSSbc_fRMSE(ahead,11)';
%                SRB3_fRMSE(ahead,11)';
%                SRB3bc_fRMSE(ahead,11)';
%                SRB4_fRMSE(ahead,11)';
%                SRB4bc_fRMSE(ahead,11)';
%                JSZ_fRMSE(ahead,11)';
%                JSZbc_fRMSE(ahead,11)';
%                AF2_fRMSE(ahead,11)';
%                AF2bc_fRMSE(ahead,11)';
%                AF3_fRMSE(ahead,11)';
%                AF3bc_fRMSE(ahead,11)';
%                AF4_fRMSE(ahead,11)';
%                AF4bc_fRMSE(ahead,11)';
%                SRTPC1C2_fRMSE(ahead,11)';
%                SRTPC1C2bc_fRMSE(ahead,11)'];
%
% ZZZ_tex_3msegment = latex(vpa(sym(ZZmat_3m),2));
% ZZZ_tex_1Ysegment = latex(vpa(sym(ZZmat_1Y),2));
% ZZZ_tex_5Ysegment = latex(vpa(sym(ZZmat_5Y),2));
% ZZZ_tex_10Ysegment = latex(vpa(sym(ZZmat_10Y),2));
%
% %% RW forecasts
% %
% RW_fErr = NaN(horizon+1,GSW_.nTau,nIter);
% for ( j=1:nIter )
%     RW_cast        = Y(startIndx+j-1,:);
%     oYields        = Y(startIndx+j-1:startIndx+j-1+horizon,:);
%     RW_fErr(:,:,j) = oYields-RW_cast;
% end
% RW_fRMSE = 100.*sqrt(mean((SRTPC1C2bc_fErr.^2),3));
%
% ZZ_RW = RW_fRMSE(ahead,[1 2 6 11])';
%
% ZZZ_tex_RW = latex(vpa(sym(ZZ_RW),2));
```

```
%
% %%  Plots
% %
%
% %
% % ... 1Y maturity, 12 months ahead
% %
% hori = 13;
% matu = 11;
% figure
%      subplot(2,1,1), plot(dates(startIndx+1:end-12,1),Y(startIndx+1:end-12,matu))
%      datetick('x','mmm-yy')
%      subplot(2,1,2), plot(dates(startIndx+1:end-12,1),squeeze(DNS_fErr(hori,matu,:).^2))
%      hold on
%      subplot(2,1,2), plot(dates(startIndx+1:end-12,1),squeeze(JSZbc_fErr(hori,matu,:).^2))
%      hold on
%      subplot(2,1,2), plot(dates(startIndx+1:end-12,1),squeeze(AF2_fErr(hori,matu,:).^2))
%      datetick('x','mmm-yy'), legend('DNS','JSZ','AFSRB2')

%% Conditional forecasting exercise
%
nCast          = 60;
indxStart      = find(dates==US_MacroVariables(1,1),1,'first');
                                     % index to match yield and macro data
datesX         = dates(indxStart:end,1);
datesCast      = (dates(end,1):31:dates(end,1)+(nCast)*31)';
SRB3           = TSM;
SRB3.yields    = Y(indxStart:end,:);
SRB3.tau       = tau;
SRB3.DataFreq = 12;
SRB3.nF        = 3;
SRB3.eXo       = [US_MacroVariables(:,2)./25 US_MacroVariables(:,3)];
SRB3           = SRB3.getSRB3;
%
% ... convert the VAR part of the model into SSM format
%
SRB3_SSM      = TSM2SSM;
SRB3_SSM.TSM = SRB3;
SRB3_SSM      = SRB3_SSM.getMdl;
nX            = SRB3_SSM.TSM.nF+SRB3_SSM.TSM.nVarExo;  % number of factors and exogenous variables
AA            = [ SRB3_SSM.Mdl.A(1:nX,1:nX*2);
                  zeros(nX,nX) eye(nX) ];
BB            = [ SRB3_SSM.Mdl.B(1:nX,1:nX); zeros(nX,nX)];
CC            = eye(nX*2);
stateType     = [ zeros(1,nX), ones(1,nX) ];
castMdl       = ssm(AA,BB,CC,'statetype',stateType);  % VAR model as SSM model

beta_Cast      = [ NaN( size(SRB3_SSM.Mdl.B,2), nCast); ones(nX,nCast)];
beta_Cast(1:nX,1) = SRB3_SSM.TSM.beta(:,end);
                                 % start projections at last obs of factors

% ......................................
% ... Conditional forecasting examples
% ......................................
```

```
%
% 0: unconditional forecast
%
beta_0    = beta_Cast;

filter_0 = [ [SRB3_SSM.TSM.beta(:,end)' ones(1,size(BB,2))] ; filter(castMdl,beta_0') ];
Y_0      = [SRB3_SSM.Mdl.C(1:nTau,1:nX*2)*filter_0']';

figure('units','normalized','outerposition',[0 0 1 1])
    surf(tau./12,datesCast,Y_0)
    date_ticks = datenum(2018:1:2024,1,1);
    set(gca, 'ytick', date_ticks);
    datetick('y','mmm-yy','keepticks')
    xticks(0:1:11), xticklabels({tau}),
    xlabel('Maturity (months)'), zlabel('Yield (pct)'),
    zlim([0 5])
    view([-53 16]),
    ytickangle(25),
    set(gca, 'FontSize', 18)
    %print -depsc Forecast_Y0

figure('units','normalized','outerposition',[0 0 1 1])
    plot(datesCast,filter_0(:,4:5),'LineWidth',2)
    date_ticks = datenum(2018:1:2024,1,1);
    set(gca, 'xtick', date_ticks);
    datetick('x','mmm-yy','keepticks')
    ylabel(' (pct)'), legend('CU (scaled)', 'INFL')
    ylim([0 4])
    set(gca, 'FontSize', 18)
    %print -depsc Forecast_X0

%
% 1: random walk assumption on macro variables
%
beta_1               = beta_Cast;
beta_1(4:5,2:nCast) = repmat(beta_1(4:5,1),1,nCast-1);

filter_1 = [ [SRB3_SSM.TSM.beta(:,end)' ones(1,size(BB,2))] ; filter(castMdl,beta_1') ];
Y_1      = [SRB3_SSM.Mdl.C(1:nTau,1:nX*2)*filter_1']';

figure('units','normalized','outerposition',[0 0 1 1])
    surf(tau./12,datesCast,Y_1)
    date_ticks = datenum(2018:1:2024,1,1);
    set(gca, 'ytick', date_ticks);
    datetick('y','mmm-yy','keepticks')
    xticks(0:1:11), xticklabels({tau}),
    xlabel('Maturity (months)'), zlabel('Yield (pct)'),
    zlim([0 6])
    view([-53 16]),
    ytickangle(25),
    set(gca, 'FontSize', 18)
    %print -depsc Forecast_Y1

```

```
figure('units','normalized','outerposition',[0 0 1 1])
    plot(datesCast,filter_1(:,4:5),'LineWidth',2)
    date_ticks = datenum(2018:1:2024,1,1);
    set(gca, 'xtick', date_ticks);
    datetick('x','mmm-yy','keepticks')
    legend('CU (scaled)', 'INFL')
    ylim([0 4])
    set(gca, 'FontSize', 18)
    %print -depsc Forecast_X1

%
% 2: Inflation overshooting, and increased CU
%
nn      = 12;
beta_2 = beta_Cast;
a       = beta_Cast(5,1);
b       = 2*a;
a1      = 2.5;
g1      = (b/a)^(1/nn);
g2      = (a1/b)^(1/nn);
infl_   = [a*g1.^(0:nn) b*g2.^(0:nn) ];
cu_     = linspace(beta_Cast(4,1),beta_Cast(4,1)+0.25,nn+1);

beta_2(4,1:length(cu_))   = cu_;
beta_2(5,1:length(infl_)) = infl_;
beta_2(5,length(infl_):end) = 2.5;

filter_2 = [ [SRB3_SSM.TSM.beta(:,end)' ones(1,size(BB,2))] ; filter(castMdl,beta_2') ];
Y_2      = [SRB3_SSM.Mdl.C(1:nTau,1:nX*2)*filter_2']';

figure('units','normalized','outerposition',[0 0 1 1])
    surf(tau./12,datesCast,Y_2)
    date_ticks = datenum(2018:1:2024,1,1);
    set(gca, 'ytick', date_ticks);
    datetick('y','mmm-yy','keepticks')
    xticks(0:1:11), xticklabels({tau}),
    xlabel('Maturity (months)'), zlabel('Yield (pct)'),
    zlim([0 25])
    view([-64 25]),
    ytickangle(25),
    set(gca, 'FontSize', 18)
    %print -depsc Forecast_Y2

figure('units','normalized','outerposition',[0 0 1 1])
    plot(datesCast,filter_2(:,4:5),'LineWidth',2)
    date_ticks = datenum(2018:1:2024,1,1);
    set(gca, 'xtick', date_ticks);
    datetick('x','mmm-yy','keepticks')
    legend('CU (scaled)', 'INFL')
    ylim([0 5])
    set(gca, 'FontSize', 18)
    %print -depsc Forecast_X2
```

```
%
% 3: New drop in inflation
%
nn      = 12;
beta_3 = beta_Cast;
a       = beta_Cast(5,1);
b       = a-0.25;
a1      = 2;
g1      = (b/a)^(1/nn);
g2      = (a1/b)^(1/nn);
infl_   = [a*g1.^(0:nn) b*g2.^(0:nn) ];
cu_     = linspace(beta_Cast(4,1),beta_Cast(4,1)-0.05,nn+1);

beta_3(4,1:length(cu_))   = cu_;
beta_3(5,1:length(infl_)) = infl_;
%beta_3(5,length(infl_):end) = 2.5;

filter_3 = [ [SRB3_SSM.TSM.beta(:,end)' ones(1,size(BB,2))] ; filter(castMdl,beta_3') ];
Y_3      = [SRB3_SSM.Mdl.C(1:nTau,1:nX*2)*filter_3']';

figure('units','normalized','outerposition',[0 0 1 1])
    surf(tau./12,datesCast,Y_3)
    date_ticks = datenum(2018:1:2024,1,1);
    set(gca, 'ytick', date_ticks);
    datetick('y','mmm-yy','keepticks')
    xticks(0:1:11), xticklabels({tau}),
    xlabel('Maturity (months)'), zlabel('Yield (pct)'),
    zlim([0 4])
    view([-53 16]),
    ytickangle(25),
    set(gca, 'FontSize', 18)
    %print -depsc Forecast_Y3

figure('units','normalized','outerposition',[0 0 1 1])
    plot(datesCast,filter_3(:,4:5),'LineWidth',2)
    date_ticks = datenum(2018:1:2024,1,1);
    set(gca, 'xtick', date_ticks);
    datetick('x','mmm-yy','keepticks')
    legend('CU (scaled)', 'INFL','location','NW')
    ylim([0 4])
    set(gca, 'FontSize', 18)
    %print -depsc Forecast_X3


%
% 4: high growth, inflation under control
%
nn      = 36;
beta_4 = beta_Cast;
a       = beta_Cast(5,1);
b       = a;
a1      = 2;
g1      = (b/a)^(1/nn);
g2      = (a1/b)^(1/nn);
```

```
infl_  = [a*g1.^(0:nn/2) b*g2.^(0:nn/2) ];
cu_    = linspace(beta_Cast(4,1),beta_Cast(4,1)+0.15,nn+1);

beta_4(4,1:length(cu_))   = cu_;
beta_4(5,1:length(infl_)) = infl_;
%beta_4(5,length(infl_):end) = 2.5;

filter_4 = [ [SRB3_SSM.TSM.beta(:,end)' ones(1,size(BB,2))] ; filter(castMdl,beta_4') ];
Y_4      = [SRB3_SSM.Mdl.C(1:nTau,1:nX*2)*filter_4']';

figure('units','normalized','outerposition',[0 0 1 1])
    surf(tau./12,datesCast,Y_4)
    date_ticks = datenum(2018:1:2024,1,1);
    set(gca, 'ytick', date_ticks);
    datetick('y','mmm-yy','keepticks')
    xticks(0:1:11), xticklabels({tau}),
    xlabel('Maturity (months)'), zlabel('Yield (pct)'),
    zlim([0 6])
    view([-53 16]),
    ytickangle(25),
    set(gca, 'FontSize', 18)
    %print -depsc Forecast_Y4

figure('units','normalized','outerposition',[0 0 1 1])
    plot(datesCast,filter_4(:,4:5),'LineWidth',2)
    date_ticks = datenum(2018:1:2024,1,1);
    set(gca, 'xtick', date_ticks);
    datetick('x','mmm-yy','keepticks')
    legend('CU (scaled)', 'INFL','location','NW')
    ylim([0 4])
    set(gca, 'FontSize', 18)
    %print -depsc Forecast_X4

%% Fix-point projections
%
% h_target:  is the number of periods ahead at which the target is met
% X_target:  is the fix-point forecast for the yield curve factor
% V       :  id the eigenvector of Phi

% % Function that calculate the adjusted mean
% m_target    = @(X_t, Phi, X_target,h_target) ...
%                 1/h_target*((eye(length(X_t))-Phi)^(-1)*(X_target-Phi*X_t));

% Function that calculates
D_target = @(X_t, V, m, X_target, h_target) ...
                    diag(((V^(-1)*(X_target-m))./(V\(X_t-m))).^(1/h_target));

% ... Using the model with factors equal to the: short rate, term premium,
%            and C1 and C2
%
nCast     = 60;
datesCast = (dates(end,1):31:dates(end,1)+(nCast-1)*31)';

SR_TP = TSM;
```

```
SR_TP.yields   = GSW_.yields;
SR_TP.tau      = GSW_.tau;
SR_TP.nF       = 3;
SR_TP.DataFreq = 12;
SR_TP          = SR_TP.getSRTPC1C2;  % est model with SR,TP,C1,C2

% ... Generating scenarios
%
% ... Scenario 1: (a) TP goes to 0% in 6 months,
%                 (b) Thereafter TP goes to 2% after additional 12 months
%                        while short rate stays low
%                 (c) At the end of the 60 months projection horizon,
%                        the short rate converges to 4% and the TP to 3%

X_t1           = SR_TP.beta(:,end);
X_t1(2,1)      = 0;
X_t2           = SR_TP.beta(:,end);
X_t2(2,1)      = 2;
X_t3           = SR_TP.beta(:,end);
X_t3(1,1)      = 4.00;
X_t3(2,1)      = 1.50;
h1             = 6;
h2             = 12;
h3             = 42;
beta_proj      = NaN(SR_TP.nF,h1+h2+h3);
beta_proj(:,1) = SR_TP.beta(:,end);

[V,D] = eig(SR_TP.PhiP);
D_1   = D_target( beta_proj(:,1), V, SR_TP.mP, X_t1, h1-1 );
for ( j=2:h1+1 )
    beta_proj(:,j) = SR_TP.mP + (V*(D_1)*V^(-1)) * ...
                                       (beta_proj(:,j-1) - SR_TP.mP);
end
D_2   = D_target( beta_proj(:,h1), V, SR_TP.mP, X_t2, h2 );
for ( j=h1+1:h1+h2+1 )
    beta_proj(:,j) = SR_TP.mP + (V*(D_2)*V^(-1)) * ...
                                       (beta_proj(:,j-1) - SR_TP.mP);
end
D_3   = D_target( beta_proj(:,h1+h2), V, SR_TP.mP, X_t3, h3 );
for ( j=h1+h2+1:h1+h2+h3 )
    beta_proj(:,j) = SR_TP.mP + (V*(D_3)*V^(-1)) * ...
                                       (beta_proj(:,j-1) - SR_TP.mP);
end
beta_proj = real(beta_proj);
Y_proj    = (SR_TP.B*real(beta_proj))';
fDates    = cumsum([h1;h2;h3]);

figure('units','normalized','outerposition',[0 0 1 1])
    surf(tau./12,datesCast,Y_proj)
    date_ticks = datenum(2018:1:2024,1,1);
    set(gca, 'ytick', date_ticks);
    datetick('y','mmm-yy','keepticks')
    xticks(0:1:11), xticklabels({tau}),
    xlabel('Maturity (months)'), zlabel('Yield (pct)'),
```

```
    zlim([0 8])
    view([-53 16]),
    ytickangle(25),
    set(gca, 'FontSize', 18)
    %print -depsc Y_fixed_point_1

figure('units','normalized','outerposition',[0 0 1 1])
    plot(datesCast,beta_proj,'LineWidth',2),
    hold on
    plot(datesCast(fDates,1),beta_proj(:,fDates')','*b','LineWidth',5), ...
         legend('Short rate','10-year term premium','Curvature 1','Curvature 2','Fix-points','Location'
             ,'NW')
    hold off
    date_ticks = datenum(2018:1:2024,1,1);
    set(gca, 'xtick', date_ticks);
    datetick('x','mmm-yy','keepticks')
    set(gca, 'FontSize', 18)
    %print -depsc beta_fixed_point_1
```

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