Randomized Candidate Voting Methods for Preventing Manipulation

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Abstract

This paper investigates the problem of voting manipulation under complete information when manipulators know a full preference profile of other voters. Our main contribution includes analysis of a randomized candidate method as a potential solution to prevent voting manipulation and probabilistic measurement of the approximation of the randomized algorithm.

Introduction

The Gibbard-Satterthwaite theorem proves that any deterministic voting rule cannot simultaneously satisfy the following three properties.
i) Any alternative can be elected.
ii) Not dictatorial: no single player can determine the winner of the game.
iii) Strategy-proof: no player is better-off by misrepresenting their preference (Satterthwaite 1975).
Hence, we consider a randomized voting method as a potential solution to the voting manipulation.
A brief description of the randomized candidate method introduced by Bentert and Skowron is the following. Given a set of candidates C with size m, with n number of voters, we first fix a candidate subset size, $l \leq m$, to be assigned to each voter. Each voter gives the linear order of preference. We assume no voter knows this candidate assignment of other voters, as well as score results of each candidate. In the end, we compute the total score the candidate received, divided by the number of times each candidate is ranked. The final score is n times the weighted score, and a candidate with the highest score will be elected. The score computed from the algorithm is under condition where all m candidates are ranked at least one time. However, since $Pr(c \in C \text{ never ranked} | \text{ n voters}) = (\frac{m-l}{m})^n$, this is exponentially small when n is large.
Given the randomized candidate method, our main contribution is the analysis of the algorithm to investigate followings:
i) The randomized candidate algorithm reduces the chances that a manipulator has an incentive to misrepresent their preference. ii) Consequences of manipulation. iii) Noise of the randomized candidate method.

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Related Work

While Bentert and Skowron's research motivation is to approximate deterministic voting methods with less information to be efficient, this paper analyzes the randomized algorithm as a potential solution to the voting manipulation. Their results show that even $l = 2$, with hundreds of voters, we can approximate minimal variance (Benter and Skowron 2020).
Procaccia analyzes the approximation of the lottery extension, selecting a winner based on the Borda count distribution. While the lottery extension is strategy-proof, it has an upper bound of approximation $\frac{1}{2} + \Omega\frac{1}{\sqrt{m}}$ (Procaccia 2010).
Veselova introduces the concept of a manipulator having an $incentive$ to misrepresent their preference under deterministic voting methods. Veselova defines that if there exists at least one possible situation when manipulation makes him better off, and nothing changes in all other situations, then the manipulator has an incentive to play strategically (Veselova 2020). We extend the concept of having an incentive to randomized voting methods defined in the Preliminaries section.
Ayadi et al. demonstrate a comparison of scoring voting methods Their research asks each voter to rank only top-k candidates. Their empirical result shows that harmonic scoring approximates better than Borda count (Ayadi, Amor, and Lang 2020). Since Bentert and Skowron prove that the randomized algorithm approximates well having large n with any decreasing order of the scoring method, we do not specify a scoring method in this paper. However, it will be beneficial to investigate the best scoring method in the randomized candidate method as a future study.

Preliminaries

We first suppose a manipulator knows the preference profile of other voters. Secondly, the manipulator schemes to do constructive manipulation, which they want to make their most favorite candidate, A, elected.
Since each voter v_i is equipped with a linear order of preference over the candidates, $pos_{v_i}(c)$ means the position of a candidate c in v_i's preference ranking.
Based on the randomized candidate method, Bentert and Skowron, denote random variable X_c as the score that candidate c receives from the algorithm. Since we assume a

manipulator has complete information of other voters, they can compute E[X_c] of all candidates, which implies that they can construct an aggregated ranking of all candidates based on E[X_c]. To analyze the algorithm let us denote C_{higher} as a set of candidates having higher E[X_c] than that of the manipulator's target candidate A. Similarly, denote C_{lower} as a set of candidates having lower E[X_c] than that of the manipulator's target candidate A.

Bentert and Skowron show that the expected score from the randomized candidate algorithm can be expressed as follows:

$$E[X_c] = \frac{1}{\binom{m-1}{l-1}} \sum_{\text{all voter}} \sum_{i=1}^{l} \alpha_i \binom{pos_v(c)-1}{i-1} \binom{m-pos_v(c)}{l-i} \tag{1}$$

where α is a score vector, a decreasing order of points to be given to each candidate. A common scoring method is Borda rule, defined by $B(c) = m - pos(c)\ \forall c \in C$.

Adding to those terminologies, we extend Veselova's definition of $incentive$ to our case that a manipulator has an incentive to misrepresent their preference under randomized candidate method when they can achieve the following inequality:

There exists at least one $higher \in C_{higher}$ such that

$$E[X_{higher} \mid \text{misrepresent preference}] < E[X_{higher} \mid truthful]$$

and

$$E[X_{higher} \mid \text{misrepresent preference}] \leq E[X_{higher} \mid truthful] \tag{2}$$

for all other $higher \in C_{higher}$. For example, given a set of candidates = (A, B, C, D, E), suppose a manipulator's true preference is ($A \succ B \succ C \succ D \succ E$). We suppose aggregated ranking of E[X_c] is ($B \succ C \succ A \succ D \succ E$) where B has the highest E[X_c]. For $l = 2$, if candidate assignment is (C, D), then the manipulator has an incentive to misrepresent their preference. Since truthful voting is $C \succ D$, and misrepresented vote is $D \succ C$, the inequality (2) is satisfied.

Chances of losing an incentive to manipulate

A manipulator needs aggregated ranking information to manipulate and increase the chances that their favorite candidate A wins. Under deterministic voting methods, aggregated ranking is produced based on the total score of each candidate. Hence we consider how a manipulator would obtain an aggregated ranking under the randomized method. Ideallly, they need the conditional probability of winning for each candidate under truthful voting. Although computing the probability is still work in progress, a manipulator can still obtain aggregated ranking based on $E[X_c]$ of each candidate $c \in C$, and they scheme to satisfy the inequality (2). Hence, we have the following result.

$$Pr(\text{a manipulator loses an incentive to manipulate}) = \frac{\binom{pos(A)}{l} + \binom{m-pos(A)+1}{l}}{\binom{m}{l}} \tag{3}$$

$\binom{pos(A)}{l}$ refers to the number of ways that candidates $\in C_{higher}$ and A occupy all the assignment. If A is in the subset, the manipulator always ranks A top. If $l = 2$, their votes is always $A \succ c \in C_{higher}$, which is thier truthful voting. If $l > 2$, no matter how the manipulator orders the assigned candidates, they cannot satisfy the inequality (2) without violating the inequality for other candidates in C_{higher}. This is same when A is not in the subset. $\binom{m-pos(A)+1}{l}$ refers to the number of ways that candidates rank lower than A and A occupy all the assignment. In this situation, the manipulator has no control to change the $E[X_c]\ \forall c \in X_{higher}$. Intuitively speaking, in this case, the manipulator cannot reduce the gap of score between their favorite candidate and candidates in the higher position. These cases are mutually exclusive. Hence, we sum these cases and divide by the total ways choosing l subset of candidates from m candidates. This explains the randomized candidate method can reduce the manipulator's incentive to manipulate with certain probability.

Manipulators have a price to pay

Another lemma of the randomized candidate algorithm is that a manipulation increases the expected score of a candidate c $\in C_{lower}$ and not their target candidate A. This is due to the dependency of $E[X_c]$ on $pos(c)$, clearly from the equation (1). Let $pos(c \mid \text{manipulation})$ be a position of c in $C_{lower} \setminus \{A\}$ by manipulation and $pos(c \mid \text{truthful})$ be the position by truthful preference. We know $pos(c \mid \text{manipulation}) < pos(c \mid \text{truthful})$.

The equation (1) shows that $\frac{\binom{pos_v(c)-1}{i-1}\binom{m-pos_v(c)}{l-i}}{\binom{m-1}{l-1}}$ is the probability that the candidate c is ranked at position i. Since manipulation causes $pos(c \mid \text{manipulation}) < pos(c \mid \text{truthful})$, the probability that the candidate c is ranked at lower i will be increased for $pos(c \mid \text{manipulation})$, which will increase $E[X_c]$ for $c \in C_{lower}$. Moreover, since $\sum_{i=1}^{l} Pr(\text{c is ranked at i}) = 1$, the probability that the candidate c is ranked at higher i will be decreased accordingly. As manipulation does not change $pos(A)$, the manipulator cannot increase $E[X_A]$.

Noise of the randomized method

While the randomized candidate method can reduce chances that manipulator has an incentive to manipulate, it has a cost of adding noise. Dubhashi and Ranjan prove that when random variables are negatively associated, we can still apply chernoff bounds (Dubhashi and Ranjan 1996). A voter v_i ranks a candidate in $pos_i(c)$ is the model of balls and bins, and hence X_c is negatively associated. Bentert and Skowron demonstrate that $\forall\ \delta \in$ [0,1], $Pr(|X_c - E[X_c]| \geq \delta \cdot E[X_c]) \leq 2 \cdot \exp -\delta^2 E[X_c]/3$ (Benter and Skowron 2020). This imply that if n is small, noise can be certain probability.

For example, for $\delta = 0.1$, suppose candidate A ranks the top for all voters, then $pos(A) = 1$ for all voters. Using Borda method, $E[X_A] = n \cdot ((l-1) \cdot \frac{\binom{m-1}{l-1}}{\binom{m-1}{l-1}}) = n \cdot (l-1)$. Then $Pr(|X_c - E[X_c]| \geq 0.1 \cdot E[X_c]) \leq 2 \cdot \exp -0.01n(l-1)/3$. With $n = 200$ voter and $l = 4$, $Pr(|X_c - E[X_c]| \geq 0.1 \cdot E[X_c]) \leq 2 \cdot \exp -6/3 = 27\%$.

Conclusion and Future Direction

As we show already, the randomized candidate method can reduce the chances of manipulator having an incentive to misrepresent their preference. This result means the randomized candidate method could produce strategy-proof voting with some probability. However, if the number of voters is small, the algorithm adds the cost of noise.

A possible direction of this study includes three things. First, this paper analyses the case when constructive manipulation with one target. Hence, it would be beneficial to explore more cases when a manipulator has multiple target of constructive manipulation. Analysis1 would be updated. Secondly, there are multiple positional scoring methods, such as Borda and Harmonic weighting. Although chances that a manipulator has an incentive to misrepresent their preference does not depend on scoring methods, one scoring method could be better than the other in terms of the size of noise. Lastly, exploring the destructive manipulation is also a possible future study.

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