Randomized Candidate Voting Methods for Preventing Manipulation

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Abstract

This paper investigates the problem of voting manipulation under complete information when manipulators know a full preference profile of other voters. Our main contribution includes analysis of a randomized candidate method as a potential solution to prevent voting manipulation and probabilistic measurement of the approximation of the randomized algorithm.

Introduction

The Gibbard-Satterthwaite theorem proves that any deterministic voting rule cannot simultaneously satisfy the following three properties.
i) Any alternative can be elected.
ii) Not dictatorial: no single player can determine the winner of the game.
iii) Strategy-proof: no player is better-off by misrepresenting his or her preference (Satterthwaite 1975).
Hence, we consider a randomized voting method as a potential solution to the voting manipulation.

Related Work

Bentert and Skowron provide a randomized voting method, described in The Randomized Candidate Voting Method section. While their research motivation is to approximate deterministic voting methods with less information to be efficient, this paper analyzes the randomized algorithm as a potential solution to the voting manipulation. Their results show that even $l = 2$, with hundreds of voters, we can approximate minimal variance (Benter and Skowron 2020).
Procaccia analyzes the approximation of the lottery extension, selecting a winner based on the Borda count distribution. While the lottery extension is strategy-proof, it has an upper bound of approximation $\frac{1}{2} + \Omega\frac{1}{\sqrt{m}}$ (Procaccia 2010).
Veselova introduces the concept of a manipulator having an incentive to misrepresent his preference under deterministic voting methods. Veselova defines that if there exists at least one possible situation when manipulation makes him better off, and nothing changes in all other situations, then the manipulator has an incentive to play strategically (Veselova 2020). We extend the concept of having an incentive to randomized voting methods defined in the Preliminaries section.
Ayadi et al. demonstrate a comparison of scoring voting methods Their research asks each voter to rank only top-k candidates. Their empirical result shows that harmonic scoring approximates better than Borda count (Ayadi, Amor, and Lang 2020). Since Bentert and Skowron prove that the randomized algorithm approximates well having large n with any decreasing order of the scoring method, we do not specify a scoring method in this paper. However, it will be beneficial to investigate the best scoring method in the randomized candidate method as a future study.

Our contribution

Our main contribution is the analysis of the randomized candidate method. We investigate followings:
i) The randomized candidate algorithm reduces the chances that a manipulator has an incentive to misrepresent their preference.
ii) Consequences of manipulation.
iii) Noise of the randomized candidate method.

The Randomized Candidate Voting Method

A brief description of the randomized candidate method introduced by Bentert and Skowron is the following. Given a set of candidates C with size m, with n number of voters, we first fix a candidate subset size, $l \leq m$, to be assigned to each voter. Each voter gives the linear order of preference. We assume no voter knows this candidate assignment of other voters, as well as score results of each candidate. In the end, we compute the total score the candidate received, divided by the number of times each candidate is ranked. The final score is n times the weighted score, and a candidate with the highest score will be elected.

Preliminaries

We first suppose a manipulator knows the preference profile of other voters. Secondly, the manipulator schemes to do constructive manipulation, which he wants to make his more favorite candidate elected.
Since each voter v_i is equipped with a linear order of preference over the candidates, $pos_v(c)$ means the position

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of c in v_i's preference ranking.
Based on the randomized candidate method, Bentert and Skowron, denote random variable X_c as the score that candidate c receives from the algorithm. Since we assume a manipulator has complete information of other voters, he can compute E[X_c] of all candidates, which implies that he can construct an aggregated ranking of all candidates based on E[X_c]. To analyze the algorithm let us denote C_{higher} as a set of candidates having higher E[X_c] than that of the manipulator's target candidate A. Similarly, denote C_{lower} as a set of candidates having lower E[X_c] than that of the manipulator's target candidate A or A itself. Bentert and Skowron show that the expected score from the randomized candidate algorithm can be expressed as follows:

$$E[X_c] = \frac{1}{\binom{m-1}{l-1}} \sum_{\text{all voter}} \sum_{i=1}^{l} \alpha_i \binom{pos_v(c)-1}{i-1}\binom{m-pos_v(c)}{l-i} \tag{1}$$

where α is a score vector, a decreasing order of points to be given to each candidate. A common scoring method is Borda rule, defined by $B(c) = m - pos(c)\ \forall c \in C$. Adding to those terminologies, we define that a manipulator has an incentive to misrepresent his preference under randomized candidate method when the manipulator can achieve the following inequality:

$$E[X_{higher} \mid \text{misrepresent preference}] \leq E[X_{higher} \mid truthful] \tag{2}$$
$$\forall higher \in C_{higher}.$$

For example, given a set of candidates = (A, B, C, D, E), suppose a manipulator's true preference is ($A \succ B \succ C \succ D \succ E$). We suppose aggregated ranking of E[X_c] is ($B \succ C \succ A \succ D \succ E$) where B has the highest E[X_c]. For $l = 2$, if candidate assignment is (C, D), then the manipulator has an incentive to misrepresent his preference. Since truthful voting is $C \succ D$, and misrepresented vote is $D \succ C$, the inequality (1) is satisfied.

Analysis 1: Chances of losing an incentive to manipulate

A manipulator needs aggregated ranking information to manipulate and increase the chances that his favorite candidate wins. Under deterministic voting methods, aggregated ranking is the total score of each candidate. Hence we consider how a manipulator would obtain an aggregated ranking under the randomized method. First, when n is small, a manipulator can compute each candidate's conditional probability of being elected under the manipulator's truthful voting. Then aggregated ranking can be produced based on the likelihood. It can be computed by counting the number of ways $X_c > X_i\ \forall i \in C$ divided by the total ways of candidate assignment for all voters, $\binom{m}{l}^n$. To count the number of ways X_c is the highest, it is necessary to verify $\binom{m}{l}^n$ cases. This implies that since $\binom{m}{l}$ is always an integer, the runtime of computing probabilities of each candidate being elected is exponential time. Thus, a manipulator being able to compute the exact probabilities of each candidate with a conventional computer is only possible when n is small.
However, this does not mean a manipulator stops misrepresenting his preference when n is large. As Bentert and Skowron prove, the probability that the score computed by the randomized candidate method differs from its expected score is upper bounded by $2 \cdot \exp -\delta^2 E[X_c]/3$ for $\delta \in [0, 1]$. Hence, if the manipulator can compute $E[X_c]$, then the actual score from the randomized candidate method would be close to the $E[X_c]$ if n is large. Hence, the manipulator schemes to satisfy the inequality (2). Hence, we have the following result.
Result:

$$Pr(\text{a manipulator loses an incentive to manipulate}) = \frac{\binom{pos(A)-1}{l} + \binom{m-pos(A)+1}{l}}{\binom{m}{l}}. \tag{3}$$

$\binom{pos(A)-1}{l}$ refers to the number of ways that candidates rank higher than A occupy all the assignment. In this situation, no matter how the manipulator order the assigned candidates, he cannot satisfy the inequality (2) without violating the inequality for other candidates in C_{higher}. $\binom{m-pos(A)+1}{l}$ refers to the number of ways that candidates rank lower than A or A occupy all the assignment. In this situation, the manipulator has no control to change the $E[X_c]\ \forall c \in X_{higher}$. Intuitively speaking, in this case, the manipulator cannot reduce the gap of score between his favorite candidate and candidates in the higher position. These cases are mutually exclusive. Hence, we sum these cases and divide by the total ways choosing l subset of candidates from m candidates. This explains the randomized candidate method can reduce the manipulator's incentive to manipulate with certain probability.

Analysis 2: Manipulators have a price to pay

Another virtue of the randomized candidate algorithm is that a manipulation increases the expected score of a candidate c $\in C_{lower}$ and not his target candidate A. This is due to the dependency of $E[X_c]$ on $pos(c)$, clearly from the equation (1). Let $pos(c')$ be a position of c in $C_{lower} \backslash A$ by manipulation and $pos(c)$ be by truthful preference such that $pos(c') < pos(c)$.
The equation (1) shows that $\frac{\binom{pos_v(c)-1}{i-1}\binom{m-pos_v(c)}{l-i}}{\binom{m-1}{l-1}}$ is the probability that the candidate c is ranked at position i. Since manipulation causes $pos(c') < pos(c)$, the probability that the candidate c is ranked at higher ranking i will be increased for c'. Moreover, since $\sum_{i=1}^{l} Pr(\text{c is ranked at i}) = 1$, the probability that the candidate c is ranked at lower ranking i will be decreased accordingly.
From (1), consider $\binom{pos(c')-1}{i-1}$ and $\binom{pos(c)-1}{i-1}$. These binomial expressions become 0 $\forall i > pos(c')$, pos(c) respectively. So for each $i = 1, 2, ..., pos(c')$, the expression is a non-zero integer. Since $pos(c') < pos(c)$, the number of terms in $\sum_{i=1}^{pos(c')} \binom{pos(c')-1}{i-1}$ is less than that of $\sum_{i=1}^{pos(c)} \binom{pos(c)-1}{i-1}$ ...(a). Similarly, consider $\binom{m-pos(c)}{l-i}$. The binomial expression becomes 0 $\forall i$ such

that $l-i < m-pos(c)$ where l, m are fixed. Since $pos(c') < pos(c)$, $m-pos(c') > m-pos(c) \Rightarrow \binom{m-pos(c)}{l-i} > \binom{m-pos(c')}{l-i}$ for the last index of i...(b).

From (a) and (b), the range of the score vector $\alpha = (a_1, a_2, ..., 1_{l-1})$ multiplied to the non-zero $\frac{\binom{pos(c')-1}{i-1}\binom{m-pos(c')}{l-i}}{\binom{m-1}{l-1}}$ is higher than that of $\frac{\binom{pos(c)-1}{i-1}\binom{m-pos(c)}{l-i}}{\binom{m-1}{l-1}}$. Since $\sum_{i=1}^{l} \frac{\binom{pos(c')-1}{i-1}\binom{m-pos(c')}{l-i}}{\binom{m-1}{l-1}} = 1$, the range of the score vector multiplied to $\sum_{i=1}^{l} \frac{\binom{pos(c')-1}{i-1}\binom{m-pos(c')}{l-i}}{\binom{m-1}{l-1}}$ is higher than $\sum_{i=1}^{l} \frac{\binom{pos(c)-1}{i-1}\binom{m-pos(c)}{l-i}}{\binom{m-1}{l-1}}$.

Analysis 3: Noise of the randomized method

While the randomized candidate method can reduce chances that manipulator has an incentive to manipulate, it has a cost of adding noise. Dubhashi and Ranjan prove that when random variables are negatively associated, we can still apply chernoff bounds (Dubhashi and Ranjan 1996). A voter v_i ranks a candidate in $pos_i(c)$ is the model of balls and bins, and hence X_c is negatively associated. Bentert and Skowron demonstrate that $\forall\ \delta \in$ [0,1], $Pr(|X_c - E[X_c]| \geq \delta \cdot E[X_c]) \leq 2 \cdot \exp -\delta^2 E[X_c]/3$ (Benter and Skowron 2020). This imply that if n is small, noise can be certain probability.

For example, for $\delta = 0.1$, suppose candidate A ranks the top for all voters, then $pos(A) = 1$ for all voters. Using Borda method, $E[X_A] = n \cdot ((l-1) \cdot \frac{\binom{m-1}{l-1}}{\binom{m-1}{l-1}}) = n \cdot (l-1)$. Then $Pr(|X_c - E[X_c]| \geq 0.1 \cdot E[X_c]) \leq 2 \cdot \exp -0.01n(l-1)/3$. With $n = 200$ voter and $l = 4$, $Pr(|X_c - E[X_c]| \geq 0.1 \cdot E[X_c]) \leq 2 \cdot \exp -6/3 = 27\%$. This analysis implies that if smaller number of voters increases the upper bound of probability that actual randomized score deviates from the expected score, which eventually leads to the increment of $E[X_c]$ for c in C_{lower}. Since the increment $E[X_c]$ depends on $pos(c)$, $E[X_A]$ does not change. This implies a manipulation increases the chances of winning for his less favorite candidates.

Conclusion and Future Direction

As we show already, the randomized candidate method can reduce the chances of manipulator having an incentive to misrepresent his preference. This result means the randomized candidate method could produce strategy-proof voting with some probability. However, if the number of voters is small, the algorithm adds the cost of noise.

A possible direction of this study includes three things. First, it would be beneficial to find ways to reduce noise that a randomized algorithm creates, actual score deviating from the expected score with some probability. Cloning voters and increase the size of n might be a potential solution. Secondly, there are multiple positional scoring methods, such as Borda and Harmonic weighting. Although chances that a manipulator has an incentive to misrepresent his preference does not depend on scoring methods, one scoring method could be better than the other in terms of noise. Lastly, exploring the destructive manipulation is also a possible future study.

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