# Discrete Probability

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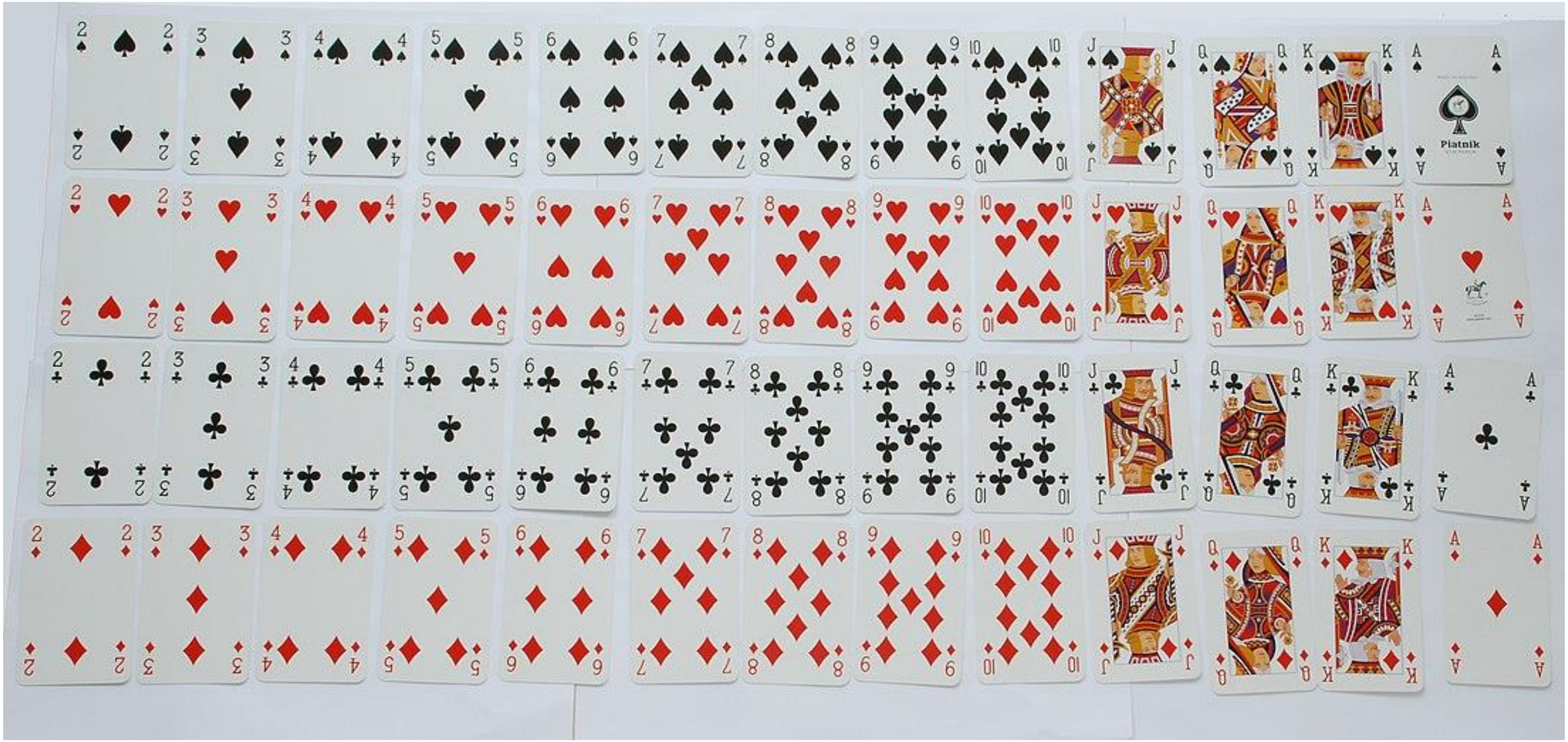
# Discrete Probability

1. An Introduction to Discrete Probability
2. Probability Theory
3. Bayes' Theorem
4. Expected Value and Variance

# Example from Card games

There are (13 x 4) = 52 cards in a pack


Piatnik

# Poker game: Royal flush

- What is the chance of getting a royal flush?
  - That's the cards 10, J, Q, K, and A of the same suit

# Probability

- **Experiment:**
  - a procedure that yields one of the possible outcomes
- **Sample space:** a set of possible outcomes
- **Event:** a subset of possible outcomes (E is a subset of S)
- **Assuming the outcomes are equally likely, the probability of an event E, defined by a subset of outcomes from the sample space S is**
  - **P(E)= |E| / |S|**
- The cardinality of the subset divided by the cardinality of the sample space.

# Probability

- Event E, Sample space S, all outcomes equally likely, then

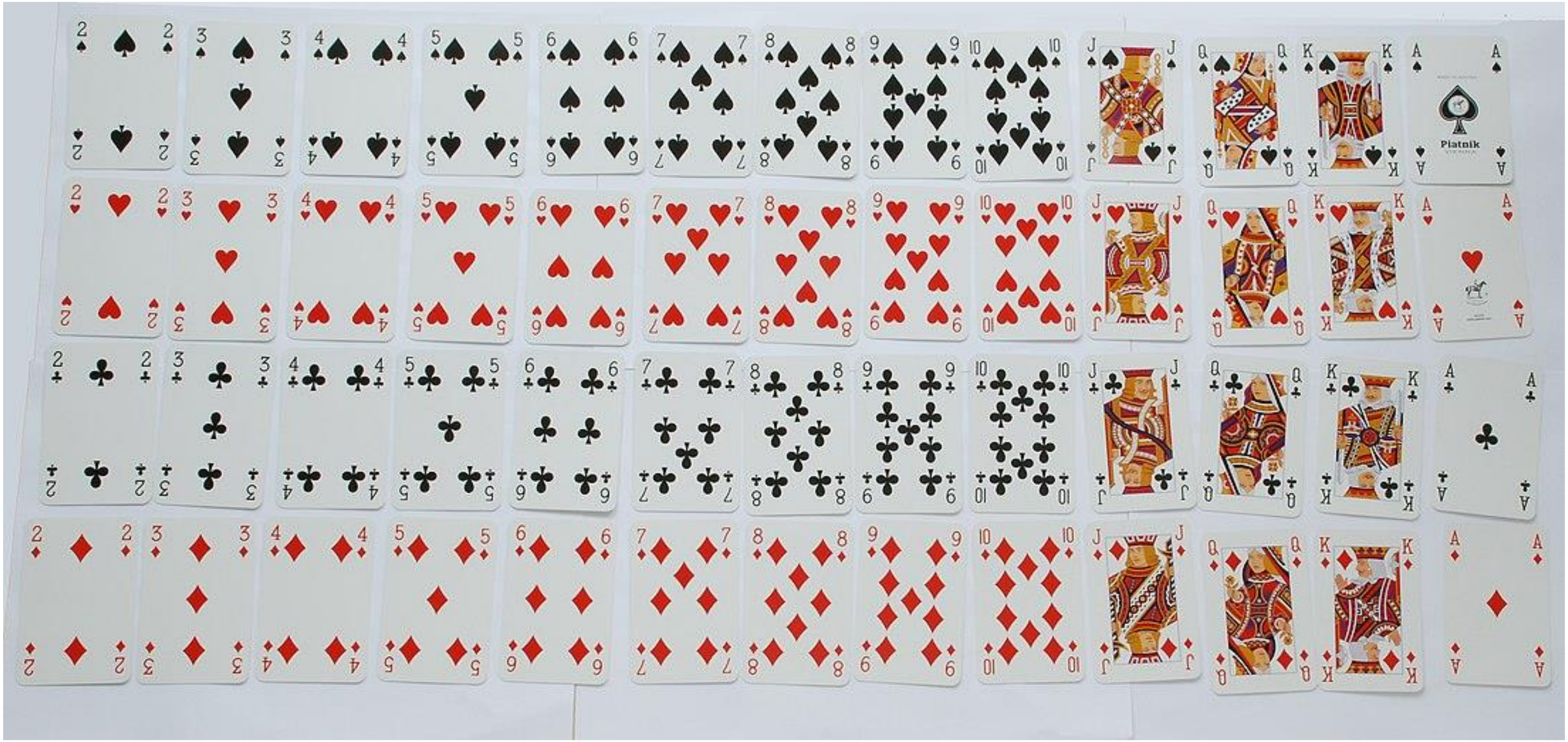
$$P(E) = |E| / |S|$$

**Example:**

- roll of two dices
- What is the probability that the outcome is 7.
- All possible outcomes (sample space S):
- (1,6 ) (2,6) …(6,1),…(6,6)  total: 36
- Outcomes leading to 7 (event E)
- (1,6 ) (2,5) …(6,1)  total: 6
- P(sum=7)= 6/36 = 1/6

# Example from Card games

There are (13 x 4) = 52 cards in a pack


Piatnik

# Poker game: Royal flush

- What is the chance of getting a royal flush?
  - That's the cards 10, J, Q, K, and A of the same suit
- There are only 4 possible royal flushes

# Probability

- Event E, Sample space S, all outcomes equally likely, then

$$P(E) = |E| \,/\, |S|$$

$\binom{40}{6}$

**Example:**

- Odd of winning a lottery: 6 numbers out of 40.
- Total number of outcomes (sample space S):
    - C(40,6) = 3,838,380
- Winning combination (event E): 1
- Probability of winning:
    - P(E) =1/C(40,6)= 34! 6! / 40! = 1/3,838,380

# Probability

- Event E, Sample space S, all outcomes equally likely, then

  **P(E)= |E| / |S|**

$\underbrace{\checkmark\ \checkmark\ \checkmark\ \checkmark\ \checkmark}\ \boxed{\phantom{x}}$

**Example (cont):**

- Odd of winning a second prize in lottery: hit 5 of 6 numbers selected from 40.
- Total number of outcomes (sample space S):
  - C(40,6) = 3,838,380
- Second prize (event E): C(6,5)*(40-6)= 6*34
- Probability of winning:
  - P(E) =6*34/C(40,6)= (6*34)/3,838,380

# Probability

- Event E, Sample space S, all outcomes equally likely, then

  **P(E)= |E| / |S|**

√ √ √ √ □ □

C(6,4) c(40-6,2)

**Example (cont):**

- Odd of winning a **third prize** in lottery: hit 4 of 6 numbers selected from 40.
- Total number of outcomes (sample space S):
  - C(40,6) = 3,838,380
- Third prize (event E): C(6,4)*C(40-6,2) = C(6,4)*C(34,2)
- Probability of winning:
  - P(E) = C(6,4)*C(34,2) /C(40,6)

# Probability

- Event E, Sample space S, all outcomes equally likely, then

  **$P(E) = |E| / |S|$**

**Another lottery:**

- 6 numbers (ordered) selected out of 40 numbers (with repetitions)
- Total number of outcomes:
  - Permutations with repetitions: $= 40^6$ $40^6$
- Number of winning configuration: 1
  - $P(\text{win}) = 1/40^6$

**And its modification:**

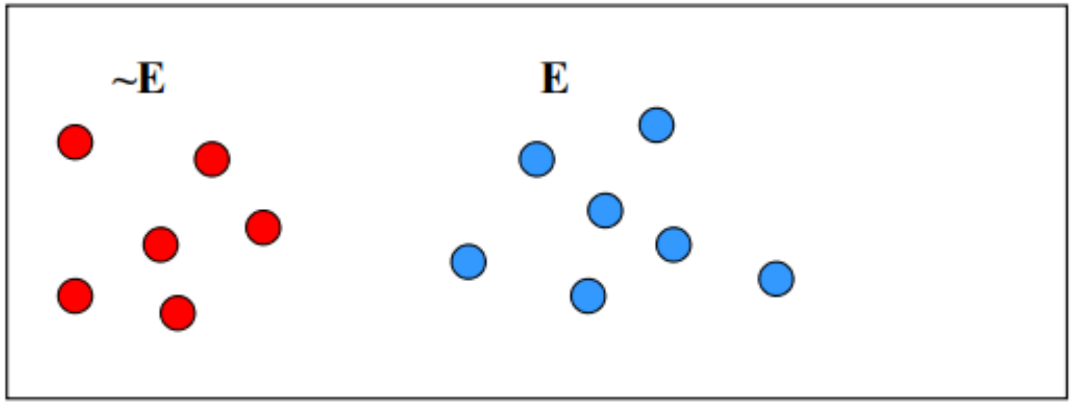
- If the winning combination is order independent:
  - E.g. (1,5,17,25,5,13) is equivalent to (5,17,5,1,25,13)
  - Number of winning permutations = number of permutations of 6 = 6!
  - $P(\text{win}) = 6! / 40^6$

# Probability

**Theorem:** Let E be an event and ~E its complement with regard to S. Then:

- P(~E) = 1 – P(E)

Sample space



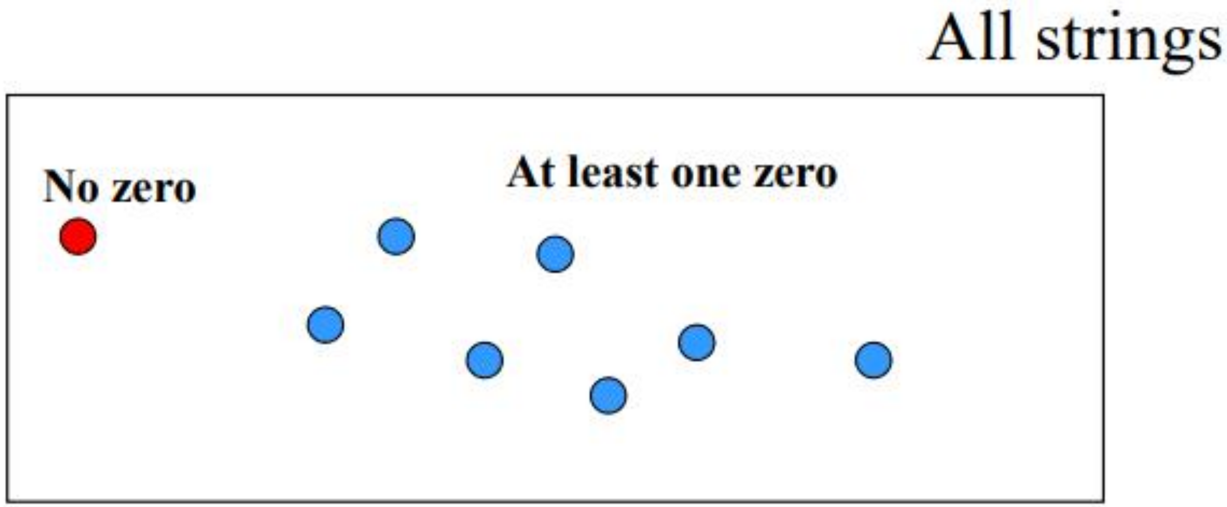

**Proof.**

**P(~E) = ( |S|- |E|) /| S | = 1- |E| / |S|**

# Probability

**Example:**

- 10 randomly generated bits. What is the probability that there is at least one zero in the string.



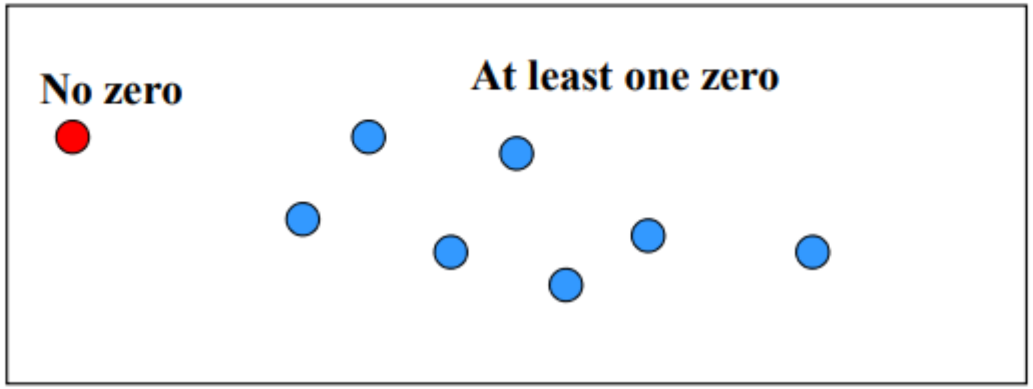

- Event: seeing no-zero string   P(E) =?
- ~Event: seeing at least one zero in the string

# Probability

**Example:**

- 10 randomly generated bits. What is the probability that there is at least one zero in the string.

All strings




- Event: seeing no-zero string $P(E) = 1/2^{10}$
- ~Event: seeing at least one zero in the string

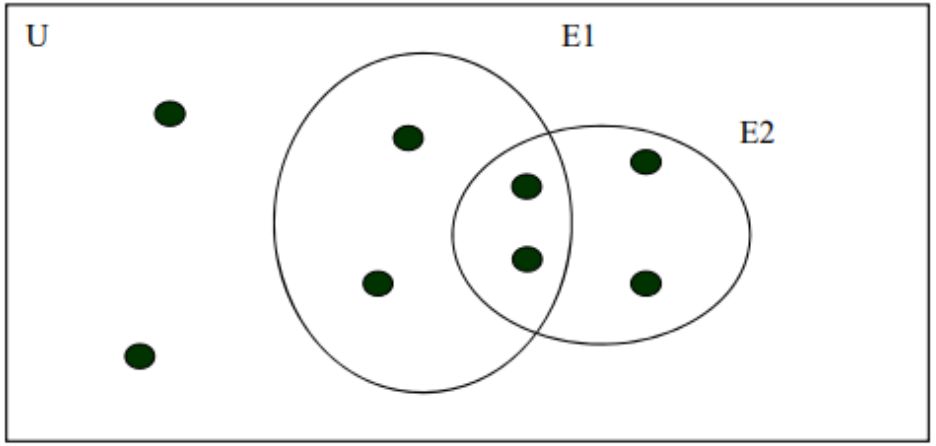
  $P(\sim E) = 1 - P(E) = 1 - 1/2^{10}$

# Probability

**Theorem.** Let E1 and E2 be two events in the sample space S. Then:

- $P(E1 \cup E2) = P(E1) + P(E2) - P(E1 \cap E2)$

- This is an example of the inclusion-exclusion principle



# Probability

**Theorem.** Let E1 and E2 be two events in the sample space S. Then:

- $P(E1 \cup E2) = P(E1) + P(E2) - P(E1 \cap E2)$

**Example:** Probability that a positive integer $<= 100$ is divisible either by 2 or 5.

# Probability

- Assumption applied so far:
  - **the probabilities of each outcome are equally likely.**
- However in many cases outcomes may not be equally likely.

**Example:** a biased coin or a biased dice.

- **Biased Coin:**
  - Probability of head 0.6,
  - probability of a tail 0.4.
- **Biased Dice:**
  - Probability of **6:** 0.4,
  - Probability of **1, 2, 3, 4, 5:** 0.12 each

# Probability

**Three axioms of the probability theory:**

(1) Probability of a discrete outcome is:

- $0 <= P(s) <= 1$

(2) Sum of probabilities of all (disjoint) outcomes is = 1

(3) For any two events E1 and E2 holds:

$$P(E1 \cup E2) = P(E1) + P(E2) - P(E1 \cap E2)$$

# Probability distribution

**Definition:** A function **p: S → [0,1]** satisfying the three conditions is called **a probability distribution**

**Example:** a biased coin

- Probability of head 0.6, probability of a tail 0.4
- **Probability distribution:**
  - Head → 0.6
  - Tail → 0.4

The sum of the probabilities sums to 1

# Probability of an Event

**Definition**: The probability of the event $E$ is the sum of the probabilities of the outcomes in $E$.

$$P(E) = \sum_{s \in E} P(s)$$

- Note that now no assumption is being made about the distribution.

# Example

**Probability of an event** $$P(E) = \sum_{s \in E} P(s)$$

**Example:** Suppose that a die is biased so that 3 appears twice as often as each other number, but that the other five outcomes are equally likely. What is the probability that an odd number appears when we roll this die?

**Solution:** We want the probability of the event $E = \{1,3,5\}$.
Probabilities of outcomes:

# Probabilities of Complements and Unions

- Complements still hold. Since each outcome is in either E or $\overline{E}$ but not both,

$$p(\overline{E}) = 1 - p(E)$$

- Unions: $\sum_{s \in S} p(s) = 1 = p(E) + p(\overline{E}).$

also still holds under the new definition.

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

# Example

- If you choose a number between 1 and 100, what is the probability that it is divisible by 2 or 5 or both?
- Let $n$ be the number chosen
  - $p(2|n) = 50/100$ (all the even numbers)
  - $p(5|n) = 20/100$
  - $p(2|n)$ and $p(5|n) = p(10|n) = 10/100$
  - $p(2|n)$ or $p(5|n) = p(2|n) + p(5|n) - p(10|n)$

$$= 50/100 + 20/100 - 10/100$$

$$= 3/5$$

# When is gambling worth?

**Disclaimer**. *This is a statistical analysis, not a moral or ethical discussion*

- What if you gamble \$1, and have a ½ probability to win \$10?
  - If you play 100 times, you will win (on average) 50 of those times

- What if you gamble \$1 and have a 1/100 probability to win \$10?
  - If you play 100 times, you will win (on average) 1 of those times

# Powerball lottery


**Disclaimer**. This is a statistical analysis, not a moral or ethical discussion

- Modern powerball lottery is a bit different
  - Source: http://en.wikipedia.org/wiki/Powerball
- You pick 5 numbers from 1-55
- You then pick one number from 1-42 (the powerball)
- By the product rule, you need to do both
- While there are many "sub" prizes, the probability for the jackpot is about 1 in 146 million

# Conditional probability

**Definition:** Let E and F be two events such that P(F) > 0. The **conditional probability** of E given F

- $P(E|F) = P(E \cap F) / P(F)$

**Example:**

- What is the probability that a family has two boys given that they have at least one boy. Assume the probability of having a girl or a boy is equal.
- **Possibilities. BB BG GB GG**
- **Probability of having two boys P(BB) = ¼**
- **Probability of having one boy P(one boy) = ¾**
- **P(BB|given a boy) = ¼ / ¾ = 1/3**

# Conditional probability

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- **Possibilities. BB BG GB GG**
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- **Probability of having a boy P(one boy) = ¾**
- **P(BB|given a boy) = ¼ / ¾ = 1/3**

**BB BG GB GG**

# Conditional probability

**Corrolary:** Let E and F be two events such that $P(F) > 0$. Then:

- $P(E \cap F) = P(E|F) * P(F)$

**Proof:**

- From the definition of the conditional probability:

  $P(E|F) = P(E \cap F) / P(F)$

  $\rightarrow$

  $P(E \cap F) = P(E|F) P(F)$

- **This result is also referred to as the product rule.**

# Conditional probability

**Corrolary:** Let E and F be two events such that P(F) > 0. Then:

- $P(E \cap F) = P(E|F)\, P(F)$

**Example:**

- Assume the probability of getting a flu is 0.2
- Assume the probability of having a high fever given the flu: 0.9

What is the probability of getting a flu with fever?

$P(\text{flu} \cap \text{fever}) =$

# Conditional probability

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- Assume the probability of getting a flu is 0.2
- Assume the probability of having a high fever given the flu: 0.9

What is the probability of getting a flu with fever?

$P(\text{flu} \cap \text{fever}) =$

- When is this useful?

Sometimes conditional probabilities are easier to estimate.

# Example of Conditional Probability

What is the probability that a family with two children has two boys, given that *they have at least one boy*?

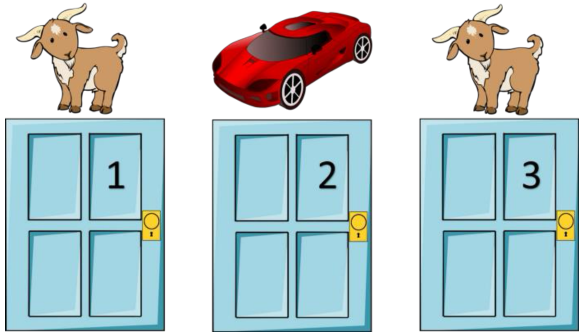
$F = \{BB, BG, GB\}$
$E = \{BB\}$

If the four events $\{BB, BG, GB, GG\}$ are equally likely, then

$$p(F) = 3/4, \text{ and } p(E \cap F) = 1/4$$

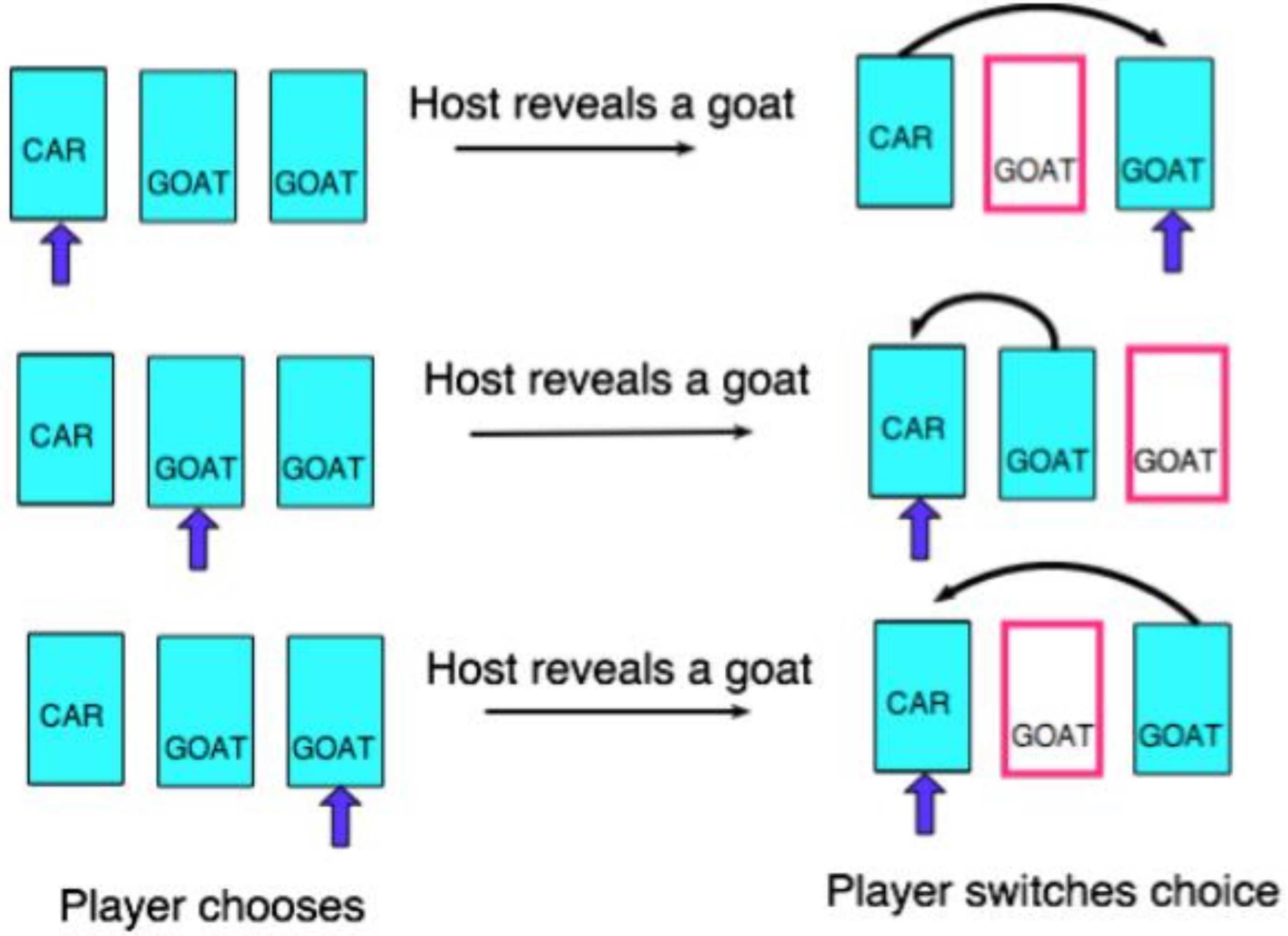
So the answer is ¼ divided by ¾ = 1/3

# Monty Hall 3-door Puzzle




- The Monty Hall problem paradox
  - Consider a game show where a prize (a car) is behind one of three doors
  - The other two doors do not have prizes (goats instead)
  - After picking one of the doors, the host (Monty Hall) opens a different door to show you that the door he opened is not the prize
  - Do you change your decision?
- Your initial probability to win (i.e. pick the right door) is 1/3
- What is your chance of winning if you change your choice after Monty opens a wrong door?
- After Monty opens a wrong door, if you change your choice, your chance of ***winning*** is 2/3
  - Thus, your chance of winning ***doubles*** if you change
  - Huh?

# What is behind the doors?



# Warm up

**Problem 1**. A sequence of 10 bits is randomly generated. What is the probability that at least one of these bits is 0?

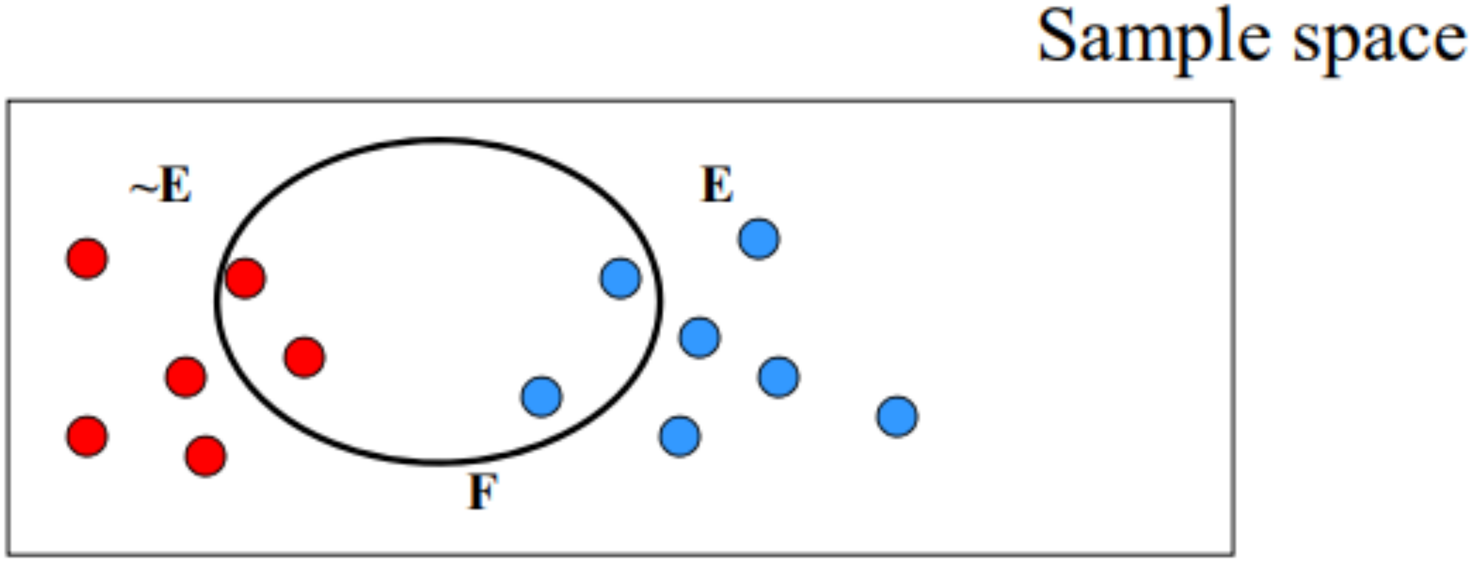
# Warm up

**Problem 2**. Find the probability of selecting none of the correct six integers in a lottery, (where the order in which these integers are selected does not matter) from the positive integers 1-40?

# Complements

Let E and F are two events. Then:

- $P(F) = P(F \cap E) + P(F \cap \sim E)$



# Bayes theorem

**Definition:** Let E and F be two events such that $P(F) > 0$. Then:

- $P(E|F) = P(F|E)P(E) / P(F)$

  $= P(F|E)P(E) / (P(F|E)P(E) + P(F|\sim E)P(\sim E))$

**Proof:**

$P(E|F) = P(E \cap F) / P(F)$

$= P(F|E)\, P(E) / P(F)$

$P(F) = P(F \cap E) + P(F \cap \sim E)$

$= P(F|E)\, P(E) + P(F|\sim E)\, P(\sim E)$

Hence:

$P(E|F) = P(F|E)P(E) / (P(F|E)P(E) + P(F|\sim E)P(\sim E))$

**Idea:** Simply switch the conditioning events.

# Bayes theorem

**Definition:** Let E and F be two events such that $P(F) > 0$. Then:

- $P(E|F) = P(F|E)P(E) / P(F)$

  $= P(F|E)P(E) / (P(F|E)P(E) + P(F|\sim E)P(\sim E))$

**Example:**

- Assume the probability of getting a flu is 0.2
- Assume the probability of getting a fever is 0.3
- Assume the probability of having a high fever given the flu: 0.9
- What is the probability of having a flu given the fever?

# Bayes theorem

**Definition:** Let E and F be two events such that $P(F) > 0$. Then:

- $P(E|F) = P(F|E)P(E) / P(F)$

  $= P(F|E)P(E) / (P(F|E)P(E) + P(F|\sim E)P(\sim E))$

**Example (same as above but different probabilities are given):**

- Assume the probability of getting a flu is 0.2
- Assume the probability of having a fever given the flu: 0.9
- Assume the probability of having a fever given no flue: 0.15
- What is the probability of having a flu given the fever?

- $P(\text{flu} \mid \text{fever}) = P(\text{fever} \mid \text{flu})\, P(\text{flu}) / P(\text{fever})$
- $P(\text{fever}) = P(\text{fever} \mid \text{flu})\, P(\text{flu}) + P(\text{fever} \mid \sim\text{flu})\, P(\sim\text{flu})$

# Bayes' theorem

This is related to conditional probability. We can make a realistic estimate when some extra information is available.

**Problem 1**.

There are two boxes.

Bob first chooses one of the two boxes at random.

He then selects one of the balls in this box at random.

If Bob **has selected a red ball**, what is the probability that ***he selected a ball from the first box?***

(See page 469 of your textbook)

Box 1

Box 2

# Bayes' theorem

Let E= Bob chose a red ball. So E' = Bob chose a green ball
F= Bob chose from Box 1. So F' = Bob chose from Box 2
We have to compute p(F|E)

Box 1

p(E|F) = 7/9, p(E|F') = 3/7

We have to find $p(F \mid E) = \dfrac{p(F \cap E)}{p(E)}$

p(F) = p(F') = 1/2

Box 2

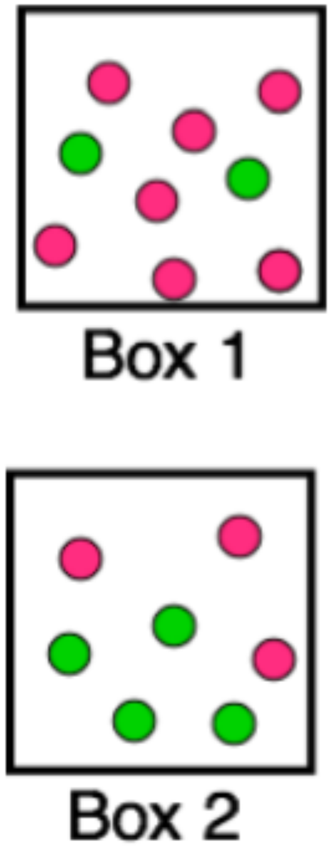
$$p(E \cap F) = p(E \mid F).p(F) = (7/9).(1/2) = 7/18$$

$$p(E \cap F') = p(E \mid F').p(F') = (3/7).(1/2) = 3/14$$

# Bayes' theorem

$$p(E) = p(E \cap F) + p(E \cap F') = 7/18 + 3/14 = 38/63$$

$$p(F \mid E) = \frac{p(F \cap E)}{p(E)} = \frac{7/18}{38/63} = \frac{49}{76}$$




This is the probability that
Bob chose the ball from Box 1

# Bayes' theorem

Let E and F be events from a sample space S such that p(E) ≠ 0 and p(F) ≠ 0. Then

given

$$p(F \mid E) = \frac{p(E \mid F).p(F)}{p(E \mid F)p(F) + p(E \mid \bar{F}).p(\bar{F})}$$

Compute this

# Bayes' theorem

**Problem 2**

1. Suppose that one person in 100,000 has a particular rare disease for which there is a fairly accurate diagnostic test.

2. This test is correct 99.0% of the time when given to a person selected at random who has the disease;

3. The test is correct 99.5% of the time when given to a person selected at random who does not have the disease.

Find the probability that a person who tests positive for the disease really has the disease. (See page 471 of your textbook)

# Independence

**Definition:** The events E and F are said to be **independent** if:

- $P(E \cap F) = P(E)P(F)$

**Example.** Assume that E denotes the family has three children of both sexes and F the fact that the family has at most one boy. Are E and F independent?

- All combos = {BBB, BBG, BGB, GBB,BGG,GBG,GGB,GGG} the number of elements = 8
- Both sexes = {BBG BGB GBB BGG GBG GGB} # = 6
- At most one boy = {GGG GGB GBG BGG} # = 4
- $E \cap F$ = {GGB GBG BGG} # = 3
- $P(E \cap F) = 3/8$ and $P(E)*P(F)= 4/8\ 6/8 = 3/8$
- **The two probabilities are equal → E and F are independent**

# Independence

**Example:**

- Assume the probability of getting a flu is 0.2
- Assume the probability of getting a fever is 0.3
- Assume the probability of having a fever given the flu: 0.9
- Are flu and fever independent ?
- P(**flu ∩ fever) = P( fever | flu) * P(flu) = 0.2 * 0.9 = 0.18**
- **P(flu) * P(fever) = 0.2 * 0.3 = 0.06**
- **Independent or not?**

# Bernoulli trials

An experiment with only two outcomes (like 0, 1 or T, F) is called a Bernoulli trial . Many problems need to compute the probability of exactly k successes when an experiment consists of n independent Bernoulli trials.

# Bernoulli trials

**Example.** A coin is *biased* so that the probability of *heads* is 2/3. What is the probability that exactly four heads come up when the coin is flipped exactly seven times?

# Bernoulli trials

The number of ways 4-out-of-7 flips can be heads is C(7,4).

**H H H H T T T**

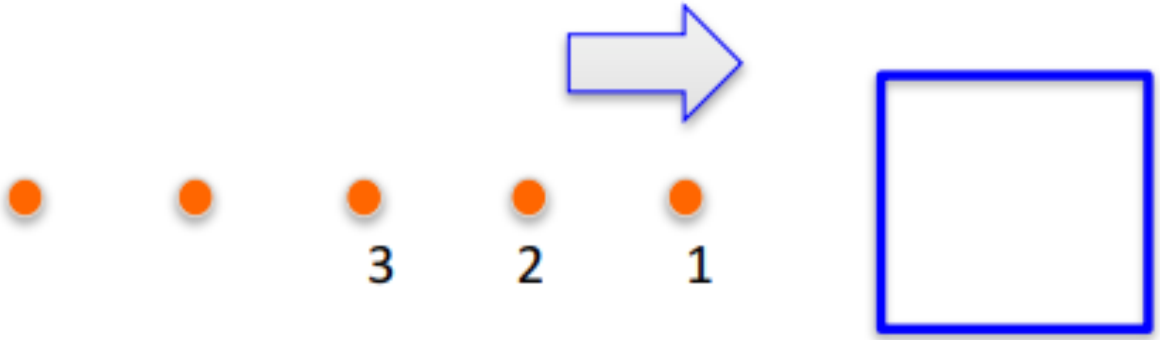
**T H H T H H T**

**T T T H H H H**

Each flip is an independent flips. For each such pattern, the probability of 4 heads (and 3 tails) = $(2/3)^4 . (1/3)^3$. So, in all, the probability of exactly 4 heads is $C(7,4). (2/3)^4 . (1/3)^3 = 560/2187$

# The Birthday Problem

**The problem**. What is the smallest number of people who should be in a room so that the probability that at least two of them have the same birthday is greater than ½?

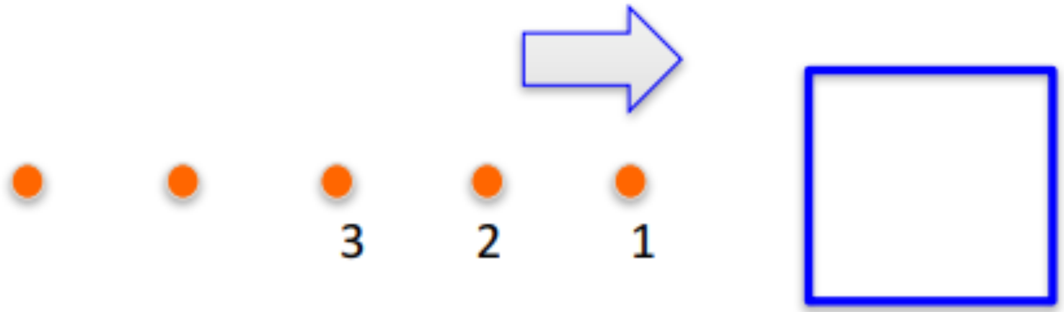



Consider people entering the room one after another. Assuming birthdays are randomly assigned dates, the probability that the second person has the same birthday as the first one is 1 - 365/366

Probability that third person has the same birthday as any one of the previous persons is 1 – 364/366 x 365/366

# The Birthday Problem

Continuing like this, probability that the $n^{th}$ person has the same birthday as one of the previous persons is $1 - 365/366 \times 364/366 \times \ldots \times (367-n)/366$




Solve the equation so that for the $n^{th}$ person, this probability exceeds ½. You will get $n = 23$

Also sometimes known as the **birthday paradox**.

# Bernoulli trial

**Assume:**

- **p = 0.6 is a probability of seeing head**
- **0.4 is the probability of seeing tail**

**Assume we see a sequence of independent coin flips:**

- **HHHTTHTHHT**
- **The probability of seeing this sequence:**

- **What is the probability of seeing a sequence of with 6 Heads and 4 tails?**

- **How many such sequences are there:** C(10,4)

# Random variables

- **Definition: A random variable** is a function from the **sample space of an experiment** to the set of real numbers f: S → R. A random variable assigns a number to each possible outcome.

- **The distribution of a random variable X on the sample space S** is a set of pairs (r p(X=r) ) for all r in S where r is the number and p(X=r) is the probability that X takes a value r.

# Bayes' theorem

- ✓ 1 in 100,000 has the rare disease (1)
- ✓ This test is 99.0% correct if actually infected; (2)
- ✓ The test is 99.5% correct if not infected (3)

Let F = event that a randomly chosen person has the disease
and E = event that a randomly chosen person tests positive

So, $p(F) = 0.00001$, $p(F') = 0.99999$ {from (1)}
Also, $p(E|F) = 0.99$ , and $p(E'|F) = 1 - 0.99 = 0.01$ {from (2)}
Also $p(E'|F') = 0.995$ , and $p(E|F') = 1 - 0.995 = 0.005$ {from (3)}

Now, plug into Bayes' theorem.

# Bayes' theorem

$$p(F \mid E) = \frac{p(E \mid F).p(F)}{p(E \mid F)p(F) + p(E \mid \bar{F}).p(\bar{F})}$$

$$= \frac{0.99 \times 0.00001}{0.99 \times 0.00001 + 0.005 \times 0.99999} \simeq 0.002$$

So, the probability that a person "who tests positive for the disease" really has the disease is only 0.2%

# Random variables

**Example:**

**Let S be the outcomes of a two-dice roll**

**Let random variable X denotes the sum of outcomes**

**(1,1)** → 2

(1,2) and (2,1) → 3

(1,3), (3,1) and (2,2) → 4

…

**Distribution of X:**

- 2 → 1/36,
- 3 → 2/36,
- 4 → 3/36 …
- 12 → 1/36

# Probabilities

- **Assume a repeated coin flip**
- P(head) =0.6 and the probability of a tail is 0.4. Each coin flip is independent of the previous.
- What is the probability of seeing:
  - HHHHH - 5 heads in a row
- **P(HHHHH) = $0.6^5$ =**
  - Assume the outcome is HHTTT
- **P(HHTTT)= $0.6*0.6*0.4^3 = 0.6^2*0.4^3$**
  - Assume the outcome is TTHHT
- **P(TTHHT)=$0.4^2*0.6^2*0.4 = 0.6^2*0.4^3$**
- What is the probability of seeing three tails and two heads?
- The number of two-head-three tail combinations = **C(5,2)**
- P(two-heads-three tails) = **C(5,2) $*0.6^2*0.4^3$**

# Probabilities

- **Assume a variant of a repeated coin flip problem**
- The space of possible outcomes is the count of occurrences of heads in 5 coin flips. For example:
- TTTTT yields outcome 0
- HTTTT or TTHTT yields 1
- HTHHT yields 3 …

- What is the probabilitv of an outcome 0?
- P(outcome=0) =
- P(outcome=1) =
- P(outcome =2) =
- P(outcome =3) =
- …

# Expected value and variance

**Definition: The expected value** of the random variable X(s) on the sample space is equal to:

$$E(X) = \sum_{s \in S} p(s) X(s)$$

**Example:** roll of a dice

- Outcomes: 1 2 3 4 5 6
- Expected value:

  **E(X) = ?**

# Expected value

**Example:**

Flip a fair coin 3 times. The outcome of the trial X is the number of heads. What is the expected value of the trial?

**Answer:**

**Possible outcomes:**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| **= {** | **HHH** | **HHT** | **HTH** | **THH** | **HTT** | **THT** | **TTH** | **TTT}** |
| | **3** | **2** | **2** | **2** | **1** | **1** | **1** | **0** |

**E(X) = ?**

# Expected value

- **Theorem:** If Xi i=1,2,3, n with n being a positive integer, are random variables on S, and a and b are real numbers then:
  - E(X1+X2+ ... Xn) = E(X1)+E(X2) + ... E(Xn)
  - E(aX+b) = aE(X) +b

# Expected value

**Example:**

- Roll a pair of dices. What is the expected value of the sum of outcomes?
- **Approach 1:**
- Outcomes: (1,1) (1,2) (1,3) .... (6,1)... (6,6)

  2 3 4 7 12

Expected value: 1/36 (2*1 + ....) = 7

- Approach 2 (theorem):
- $E(X1+X2) = E(X1) + E(X2)$
- $E(X1) = 7/2$  $E(X2) = 7/2$
- $E(X1+X2) = 7$