Relations

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Relations

- Relation and Their Properties
  - Relations on a Set
  - Properties of Relations
    - Reflexive
    - Irreflexive
    - Symmetric
    - Anti-symmetric
    - Transitive
  - Combining Relations
- N-ary Relations and Their Applications
  - N-ary Relations
  - Databased and Relations
  - Operations on n-ary Relations
  - SQL

Cartesian product (review)

Let $A=\{a_1, a_2, ..a_k\}$ and $B=\{b_1,b_2,..b_m\}$.

The Cartesian product A x B is defined by a set of pairs
$\{(a_1\ b_1), (a_1, b_2), \ldots (a_1, b_m), \ldots, (a_k,b_m)\}$.

Cartesian product defines a product set, or a set of all ordered arrangements of elements in sets in the Cartesian product.

Binary relation

Definition: Let A and B be two sets. A **binary relation from A to B** is a subset of a Cartesian product A x B.

- Let $R \subseteq A \times B$ means R is a set of ordered pairs of the form (a,b) where $a \in A$ and $b \in B$.
- We use the notation **a R b to denote (a,b) $\in$ R** and **a $\not R$ b to denote (a,b) $\notin$ R**. If **a R b**, we say a is related to b by R.

Example: Let A={a,b,c} and B={1,2,3}.

- Is R={(a,1),(b,2),(c,2)} a relation from A to B?
- Is Q={(1,a),(2,b)} a relation from A to B?
- Is P={(a,a),(b,c),(b,a)} a relation from A to A?

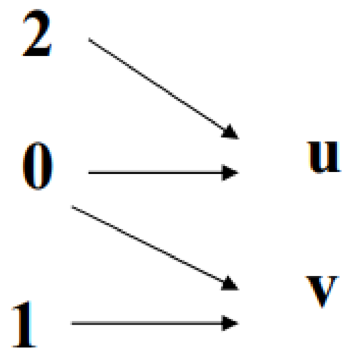
Representing binary relations

- We can graphically represent a binary relation R as follows:
  - if **a R b** then draw an arrow from a to b.

$$\mathbf{a \rightarrow b}$$

Example:

- Let $A = \{0, 1, 2\}$, $B = \{u,v\}$ and $R = \{ (0,u), (0,v), (1,v), (2,u) \}$
- Note: $R \subset A$ x B.
- **Graph:**



Representing binary relations

- We can represent a binary relation R by a **table** showing (marking) the ordered pairs of R.

Example:

- Let A = {0, 1, 2}, B = {u,v} and R = { (0,u), (0,v), (1,v), (2,u) }

- **Table:**

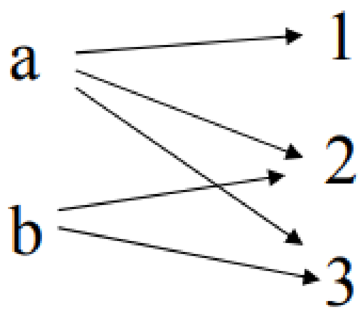
R	u	v
0	x	x
1		x
2	x	

or

R	u	v
0	1	1
1	0	1
2	1	0

Relations and functions

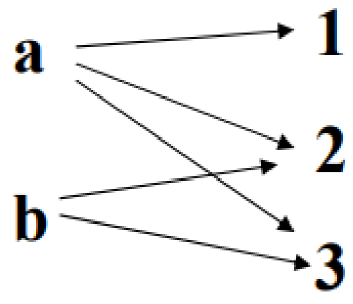
- Relations represent **one to many relationships** between elements in A and B.
- **Example:**



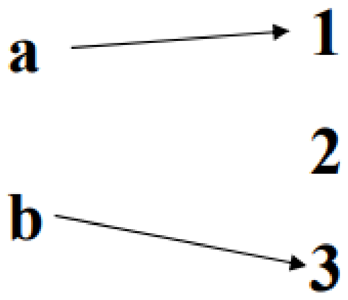
- What is the difference between a **relation and a function from A to B**?

Relations and functions

- Relations represent **one to many relationships** between elements in A and B.
- **Example:**



- What is the difference between a **relation and a function from A to B**? A function defined on sets A,B A → B assigns to each element in the domain set A exactly one element from B. So it is **a special relation.**



Relation on the set

<u>Definition:</u> **A relation on the set A** is a relation from A to itself.

Example 1:

- Let A = {1,2,3,4} and R_{div} = {(a,b)| a divides b}
- What does R_{div} consist of?
- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}
-

R	1	2	3	4
1	x	x	x	x
2		x		x
3			x	
4				x

Relation on the set

Example:

- Let A = {1,2,3,4}.
- Define a $R_{\neq}$ b if and only if a $\neq$ b.

$R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}

-

R	1	2	3	4
1		x	x	x
2	x		x	x
3	x	x		x
4	x	x	x	

Binary relations

- **Theorem:** The number of binary relations on a set A, where $|A| = n$ is:

$$2^{n^2}$$

- **Proof:**
- If $|A| = n$ then the cardinality of the Cartesian product $|A \times A| = n^2$.
- R is a binary relation on A if $R \subseteq A \times A$ (that is, R is a subset of A x A).
- The number of subsets of a set with k elements : 2^k
- The number of subsets of A x A is : $2^{|AxA|} = 2^{n^2}$

Binary relations

- **Example**: Let A = {1,2}
- What is A x A = **{(1,1),(1,2),(2,1),(2,2)}**
- **List of possible relations (subsets of A x A):**
- $\varnothing$ …. 1
- {(1,1)} {(1,2)} {(2,1)} {(2,2)} …. 4
- {(1,1), (1,2)} {(1,1),(2,1)} {(1,1),(2,2)} …. 6
  {(1,2),(2,1)} {(1,2),(2,2)} {(2,1),(2,2)}
- {(1,1),(1,2),(2,1)} {(1,1),(1,2),(2,2)} …. 4
  {(1,1),(2,1),(2,2)} {(1,2),(2,1),(2,2)}
- {(1,1),(1,2),(2,1),(2,2)} …. 1

16

- Use formula: $\mathbf{2^4 = 16}$

Exercises

1. List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$, where $(a, b) \in R$ if and only if

a) $a = b$.

b) $a + b = 4$.

c) $a > b$.

d) $a \mid b$.

e) $\gcd(a, b) = 1$.

f) $\operatorname{lcm}(a, b) = 2$.

Exercises

1. List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$, where $(a, b) \in R$ if and only if

a) $a = b$.

0,0 1,1 2,2 3,3

b) $a + b = 4$.

1,3 2,2 3,1 4,0

Exercises

1. List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$, where $(a, b) \in R$ if and only if

c) $a > b$.

d) $a \mid b$.

1,0 1,1 1,2 1,3
2,0 2,2 3,0 3,3
4,0

1,0 2,1 2,0 3,0 3,1 3,2 4,0 4,1 4,2 4,3

Exercises

1. List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$, where $(a, b) \in R$ if and only if

e) $\gcd(a, b) = 1$. **f)** $\operatorname{lcm}(a, b) = 2$.

1,2 2,1 2,2

(0,1) (1,0), 1,1
1,2 1,3 2,1 2,3
3,1 3,2 4,1 4,3

Exercises

1. List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$, where $(a, b) \in R$ if and only if

e) $\gcd(a, b) = 1$. **f)** $\text{lcm}(a, b) = 2$.

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Properties of relations

Definition (reflexive relation) : A relation R on a set A is called **reflexive** if $(a,a) \in R$ for every element $a \in A$.

Example 1:

- Assume relation R_{div} =$\{(a\ b)$, if $a \mid b\}$ on $A = \{1,2,3,4\}$
- **Is R_{div} reflexive?**
- $R_{div} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$
- **Answer:** Yes. (1,1), (2,2), (3,3), and $(4,4) \in A$.

Reflexive relation

Reflexive relation

- R_{div} ={(a b), if a |b} on A = {1,2,3,4}
- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}

$$MR_{div} = \begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

- **A relation R is reflexive** if and only if MR has 1 in every position on its main diagonal.

Properties of relations

Definition (reflexive relation) : A relation R on a set A is called **reflexive** if $(a,a) \in R$ for every element $a \in A$.

Example 2:

- Relation R_{fun} on $A = \{1,2,3,4\}$ defined as:
  - $R_{fun} = \{(1,2),(2,2),(3,3)\}$.
- **Is R_{fun} reflexive?**
- **No.** It is not reflexive since $(1,1) \notin R_{fun}$.

Properties of relations

Reflexive

Definition (irreflexive relation): A relation R on a set A is called **irreflexive** if **(a,a) $\notin$ R for every a $\in$ A.**

Example 1:

- Assume relation $R_{\neq}$ on A={1,2,3,4}, such that **a $R_{\neq}$ b** if and only if a $\neq$ b.
- **Is $R_{\neq}$ irreflexive?**
- $R_{\neq}$ = …

Reflexive relation

Reflexive relation

- R_{div} ={(a b), if a |b} on A = {1,2,3,4}
- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}

$$MR_{div} = \begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

- **A relation R is reflexive** if and only if MR has 1 in every position on its main diagonal.

Reflexive relation

Reflexive

If A = {a,b} then R is **reflexive** if:

Roster Method	R = {(a,a),(b,b)}
Directed Graph	Every vertex has a <u>self-loop</u> 
Incidence Matrix	Main diagonal all “1s“ $\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$

Reflexive

EXAMPLE Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$
$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$
$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$$
$$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$$
$$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$$
$$R_6 = \{(3, 4)\}.$$

Which of these relations are reflexive?

Properties of relations

Definition (irreflexive relation): A relation R on a set A is called **irreflexive** if **(a,a) $\notin$ R for every a $\in$ A.**

Example 1:

- Assume relation $R_{\neq}$ on A={1,2,3,4}, such that **a $R_{\neq}$ b** if and only if a $\neq$ b.
- **Is $R_{\neq}$ irreflexive?**
- **$R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}**
- **Answer:** Yes. Because (1,1),(2,2),(3,3) and (4,4) $\notin$ $R_{\neq}$

Irreflexive relation

Irreflexive relation

- $R_{\neq}$ on A={1,2,3,4}, such that **a $R_{\neq}$ b** if and only if a $\neq$ b.
- **$R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}**

$$MR = \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}$$

- **A relation R is irreflexive** if and only if MR has 0 in every position on its main diagonal.

Irreflexive relation

If set A = {a,b} then R is **irreflexive** if:

Roster Method	$(a,a) \notin R$ *and* $(b,b) \notin R$	
Directed Graph	Every vertex does not have a self-loop	a $\bullet$ $\bullet$ b
Incidence Matrix	Main diagonal all "0s"	$\begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix}$

Properties of relations

Definition (irreflexive relation): A relation R on a set A is called **irreflexive** if **(a,a) $\notin$ R for every a $\in$ A.**

Example 2:

- R_{fun} on A = {1,2,3,4} defined as:
  - $R_{fun} = \{(1,2),(2,2),(3,3)\}$.
- **Is R_{fun} irreflexive?**

- **Answer: No**. Because (2,2) and (3,3) $\in$ R_{fun}

Symmetric

Properties of relations

Definition (symmetric relation): A relation R on a set A is called **symmetric** if

$$\forall\, a, b \in A \quad (a,b) \in R \rightarrow (b,a) \in R.$$

Example 1:

- R_{div} ={(a b), if a |b} on A = {1,2,3,4}
- **Is R_{div} symmetric?**
- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}
- **Answer: No.** It is not symmetric since $(1,2) \in R$ but $(2,1) \notin R$.

Symmetric

Symmetric relation

Symmetric relation:

- $R_{\neq}$ on A={1,2,3,4}, such that **a $R_{\neq}$ b** if and only if $a \neq b$.
- **$R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}**

$$
MR = \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}
$$

- **A relation R is symmetric** if and only if $m_{ij} = m_{ji}$ for all i,j.

Symmetric relation

If set A = {a,b} then R is **symmetric** if:

Roster Method	$(a,b) \in R$ *and* $(b,a) \in R$
Directed Graph	Every pair of vertices is connected by <u>none</u> or exactly <u>two directed lines</u> in opposite directions 
Incidence Matrix	$m_{ij} = m_{ji}$ off the main diagonal $\begin{bmatrix} \ddots & 1 & \\ 1 & \ddots & 0 \\ & 0 & \ddots \end{bmatrix}$

Properties of relations

Properties of relations on A:

- Reflexive ✓
- Irreflexive ✓
- Symmetric ✓
- **Anti-symmetric**
- **Transitive**

Anti-symmetric relation

Definition (anti-symmetric relation): A relation on a set A is called **anti-symmetric** if

- $[(a,b) \in R \text{ and } (b,a) \in R] \rightarrow a = b$ where $a, b \in A$.

Example 3:

- Relation R_{fun} on $A = \{1,2,3,4\}$ defined as:
  - $R_{fun} = \{(1,2),(2,2),(3,3)\}$.
- **Is R_{fun} anti-symmetric?**
- **Answer: Yes.** It is anti-symmetric

Anti-symmetric relation

Antisymmetric relation

- relation $R_{fun} = \{(1,2),(2,2),(3,3)\}$

$$MR_{fun} = \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

- A relation is **antisymmetric** if and only if $m_{ij} = 1 \rightarrow m_{ji} = 0$ for $i \neq j$.

Anti-symmetric relation

If set A = {a,b} then R is **antisymmetric** if:

Roster Method	$if\ (a,b) \in R\ and\ (b,a) \in R\ then\ a = b$
Directed Graph	Every pair of vertices is connected by <u>none</u> or exactly <u>one directed lines</u> 
Incidence Matrix	$m_{ij} \neq m_{ji}$ off the main diagonal $\begin{bmatrix} & 0 & & \\ 1 & & 0 & \\ & 0 & & 1 \\ & & 0 & \end{bmatrix}$

Exercise

Which of the relations from Relations below are symmetric and which are antisymmetric?

Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$
$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$
$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$$
$$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$$
$$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$$
$$R_6 = \{(3, 4)\}.$$

R2 and R3 are symmetric because (b,a) belong to the relation whenever (a,b) does

R4 R5 and R6 are all antisymmetric

Transitive relation

Definition (transitive relation): A relation R on a set A is called **transitive** if

- $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R$ for all $a, b, c \in A$.

- **Example 1:**
- R_{div} ={(a b), if a |b} on A = {1,2,3,4}
- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}
- **Is R_{div} transitive?**
- **Answer:**

Transitive relation

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- $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R$ for all $a, b, c \in A$.

- **Example 1:**
- $R_{div} = \{(a\ b), \text{ if } a \mid b\}$ on $A = \{1,2,3,4\}$
- $R_{div} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$
- **Is R_{div} transitive?**
- **Answer: Yes.**

Transitive relation

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- $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R$ for all $a, b, c \in A$.

- **Example 2**:
- $R_{\neq}$ on $A=\{1,2,3,4\}$, such that **a $R_{\neq}$ b** if and only if $a \neq b$.
- $R_{\neq}=\{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}$
- **Is $R_{\neq}$ transitive ?**
- **Answer:**

Transitive relation

Definition (transitive relation): A relation R on a set A is called **transitive** if

- $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R$ for all $a, b, c \in A$.

- **Example 2**:
- $R_{\neq}$ on $A=\{1,2,3,4\}$, such that **a $R_{\neq}$ b** if and only if $a \neq b$.
- $\mathbf{R_{\neq}=\{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}}$
- **Is $R_{\neq}$ transitive?**
- **Answer: No.** It is not transitive since $(1,2) \in R$ and $(2,1) \in R$ but $(1,1)$ is not an element of R.

Transitive

Transitive relations

Definition (transitive relation): A relation R on a set A is called **transitive** if

- $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R$ for all $a, b, c \in A$.

- **Example** 3:
- Relation R_{fun} on $A = \{1,2,3,4\}$ defined as:
  - $R_{fun} = \{(1,2),(2,2),(3,3)\}$.
- **Is R_{fun} transitive?**
- **Answer:** Yes

Transitive relations

Definition (transitive relation): A relation R on a set A is called **transitive** if

- $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R$ for all $a, b, c \in A$.

- **Example** 3:
- Relation R_{fun} on $A = \{1,2,3,4\}$ defined as:
  - $R_{fun} = \{(1,2),(2,2),(3,3)\}$.
- **Is R_{fun} transitive?**
- **Answer: Yes.** It is transitive.

Exercise

Determine whether the relation R on the set of all integers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if

a) $x \neq y$.

b) $xy \geq 1$.

c) $x = y + 1$ or $x = y - 1$.

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Determine whether the relation R on the set of all integers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if

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Cartesian product (review)

Let $A=\{a_1, a_2, ..a_k\}$ and $B=\{b_1,b_2,..b_m\}$.

The Cartesian product A x B is defined by a set of pairs
$\{(a_1\ b_1), (a_1, b_2), \ldots (a_1, b_m), \ldots, (a_k,b_m)\}$.

Cartesian product defines a product set, or a set of all ordered arrangements of elements in sets in the Cartesian product.

Combining relations

Definition: Let A and B be sets. A **binary relation from A to B** is a subset of a Cartesian product A x B.

- Let $R \subseteq A \times B$ means R is a set of ordered pairs of the form (a,b) where $a \in A$ and $b \in B$.

Combining Relations

- **Relations are sets → combinations via set operations**
- Set operations of: **union, intersection, difference and symmetric difference.**

Combining relations

Example:

- Let $A = \{1,2,3\}$ and $B = \{u,v\}$ and
- $R1 = \{(1,u), (2,u), (2,v), (3,u)\}$
- $R2 = \{(1,v),(3,u),(3,v)\}$

What is:

- $R1 \cup R2 = ?$

Combining relations

Example:

- Let $A = \{1,2,3\}$ and $B = \{u,v\}$ and
- $R1 = \{(1,u), (2,u), (2,v), (3,u)\}$
- $R2 = \{(1,v),(3,u),(3,v)\}$

What is:

- $R1 \cup R2 = \{(1,u),(1,v),(2,u),(2,v),(3,u),(3,v)\}$
- $R1 \cap R2 = ?$

Combining relations

Example:

- Let A = {1,2,3} and B = {u,v} and
- R1 = {(1,u), (2,u), (2,v), (3,u)}
- R2 = {(1,v),(3,u),(3,v)}

What is:

- $R1 \cup R2 = \{(1,u),(1,v),(2,u),(2,v),(3,u),(3,v)\}$
- $R1 \cap R2 = \{(3,u)\}$
- R1 - R2 = ?

Combining relations

Example:

- Let $A = \{1,2,3\}$ and $B = \{u,v\}$ and
- $R1 = \{(1,u), (2,u), (2,v), (3,u)\}$
- $R2 = \{(1,v),(3,u),(3,v)\}$

What is:

- $R1 \cup R2 = \{(1,u),(1,v),(2,u),(2,v),(3,u),(3,v)\}$
- $R1 \cap R2 = \{(3,u)\}$
- $R1 - R2 = \{(1,u),(2,u),(2,v)\}$
- $R2 - R1 = ?$

Combining relations

Example:

- Let $A = \{1,2,3\}$ and $B = \{u,v\}$ and
- $R1 = \{(1,u), (2,u), (2,v), (3,u)\}$
- $R2 = \{(1,v),(3,u),(3,v)\}$

What is:

- $R1 \cup R2 = \{(1,u),(1,v),(2,u),(2,v),(3,u),(3,v)\}$
- $R1 \cap R2 = \{(3,u)\}$
- $R1 - R2 = \{(1,u),(2,u),(2,v)\}$
- $R2 - R1 = \{(1,v),(3,v)\}$

Combination of relations

Representation of operations on relations:

- **Question:** Can the relation be formed by taking the union or intersection or composition of two relations R1 and R2 be represented in terms of matrix operations?

- **Answer: Yes**

Combination of relations: implementation

Definition. The **join,** denoted by $\vee$, of two m-by-n matrices (a_{ij}) and (b_{ij}) of 0s and 1s is an m-by-n matrix (m_{ij}) where

- $m_{ij} = a_{ij} \vee b_{ij}$ for all i,j

= pairwise or (disjunction)

- **Example**:
- Let A = {1,2,3} and B = {u,v} and
- R1 = {(1,u), (2,u), (2,v), (3,u)}
- R2 = {(1,v),(3,u),(3,v)}

- **MR1 =** $\begin{matrix} \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{matrix}$ **MR2 =** $\begin{matrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} \end{matrix}$ **M(R1** $\vee$ **R2)=** $\begin{matrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{matrix}$

Combination of relations: implementation

Definition. The **meet,** denoted by $\wedge$, of two m-by-n matrices (a_{ij}) and (b_{ij}) of 0s and 1s is an m-by-n matrix (m_{ij}) where

- $m_{ij} = a_{ij} \wedge b_{ij}$ for all i,j

= **pairwise and (conjunction)**

- **Example**:
- Let A = {1,2,3} and B = {u,v} and
- R1 = {(1,u), (2,u), (2,v), (3,u)}
- R2 = {(1,v),(3,u),(3,v)}

- $\mathbf{MR1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \mathbf{MR2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \quad \mathbf{MR1} \wedge \mathbf{MR2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$

Composite of relations

Definition: Let R be a relation from a set A to a set B and S a relation from B to a set C. The **composite of R and S** is the relation consisting of the ordered pairs (a,c) where $a \in A$ and $c \in C$, and for which there is a $b \in B$ such that $(a,b) \in R$ and $(b,c) \in S$. We denote the composite of R and S by S o R.

Examples:

- Let A = {1,2,3}, B = {0,1,2} and C = {a,b}.
- R = {(1,0), (1,2), (3,1),(3,2)}
- S = {(0,b),(1,a),(2,b)}

- S o R = ?

Composite of relations

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- S = {(0,b),(1,a),(2,b)}

- S o R = {(1,b),(3,a),(3,b)}

Implementation of composite

Definition. The **Boolean product**, **denoted by ⊙,** of an m-by-n matrix (a_{ij}) and n-by-p matrix (b_{jk}) of 0s and 1s is an m-by-p matrix (m_{ik}) where

- m_{ik} = 1, if $a_{ij} = 1$ and $b_{jk} = 1$ for some k=1,2,...,n
  0, otherwise

Examples:

- Let A = {1,2,3}, B = {0,1,2} and C = {a,b}.
- R = {(1,0), (1,2), (3,1),(3,2)}
- S = {(0,b),(1,a),(2,b)}

- S o R = {(1,b),(3,a),(3,b)}

Implementation of composite

Examples:

- Let A = {1,2}, B= {1,2,3} C = {a,b}
- R = {(1,2),(1,3),(2,1)} is a relation from A to B
- S = {(1,a),(3,b),(3,a)} is a relation from B to C.
- $S \circ R$ = {(1,b),(1,a),(2,a)}

$$M_R = \begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{matrix} \qquad M_S = \begin{matrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{matrix}$$

$$M_R \odot M_S = ?$$

Implementation of composite

Examples:

- Let A = {1,2}, {1,2,3} C = {a,b}
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$$M_R \odot M_S = \begin{matrix} x & x \\ x & x \end{matrix}$$

Implementation of composite

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- Let A = {1,2}, {1,2,3} C = {a,b}
- R = {(1,2),(1,3),(2,1)} is a relation from A to B
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Implementation of composite

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- Let A = {1,2}, {1,2,3} C = {a,b}
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$$M_{S \circ R} = \quad ?$$

Implementation of composite

Examples:

- Let A = {1,2}, {1,2,3} C = {a,b}
- R = {(1,2),(1,3),(2,1)} is a relation from A to B
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$$M_R = \begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{matrix} \qquad M_S = \begin{matrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{matrix}$$

$$M_R \odot M_S = \begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}$$

$$M_{S \circ R} = \begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}$$

Composite of relations

Definition: Let R be a relation on a set A. The **powers R^n**, n = 1,2,3,... is defined inductively by

- $\mathbf{R^1 = R}$ and $\mathbf{R^{n+1} = R^n \circ R}$.

Examples

- $R = \{(1,2),(2,3),(2,4),(3,3)\}$ is a relation on $A = \{1,2,3,4\}$.
- $R^1 = ?$

Composite of relations

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Examples

- R = {(1,2),(2,3),(2,4), (3,3)} is a relation on A = {1,2,3,4}.
- $R^1 = R$ = {(1,2),(2,3),(2,4), (3,3)}
- $R^2 = ?$

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- $R^1 = R = \{(1,2),(2,3),(2,4),(3,3)\}$
- $R^2 = \{(1,3),(1,4),(2,3),(3,3)\}$
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Examples

- $R = \{(1,2),(2,3),(2,4), (3,3)\}$ is a relation on $A = \{1,2,3,4\}$.
- $R^1 = R = \{(1,2),(2,3),(2,4), (3,3)\}$
- $R^2 = \{(1,3), (1,4), (2,3), (3,3)\}$
- $R^3 = \{(1,3), (2,3), (3,3)\}$
- $R^4 = ?$

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- $\mathbf{R^1 = R}$ and $\mathbf{R^{n+1} = R^n \circ R}$.

Examples

- $R = \{(1,2),(2,3),(2,4),(3,3)\}$ is a relation on $A = \{1,2,3,4\}$.
- $R^1 = R = \{(1,2),(2,3),(2,4),(3,3)\}$
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- $R^3 = \{(1,3),(2,3),(3,3)\}$
- $R^4 = \{(1,3),(2,3),(3,3)\}$
- $R^k = ?$, $k > 3$.

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- $R^3 = \{(1,3),(2,3),(3,3)\}$
- $R^4 = \{(1,3),(2,3),(3,3)\}$
- $R^k = R^3$, $k > 3$.

Transitive relation

Definition (transitive relation): A relation R on a set A is called **transitive** if

- $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R$ for all $a, b, c \in A$.

- **Example 1:**
- $R_{div} = \{(a\ b), \text{ if } a \mid b\}$ on $A = \{1,2,3,4\}$
- $R_{div} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$
- **Is R_{div} transitive?**
- **Answer: ?**

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- $R_{div} = \{(a\ b), \text{ if } a \mid b\}$ on $A = \{1,2,3,4\}$
- $R_{div} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$
- **Is R_{div} transitive?**
- **Answer: Yes.**

Connection to R^n

Theorem: The relation R on a set A is transitive <u>if and only if</u> $R^n \subseteq R$ for n = 1,2,3,... .

Proof: biconditional (if and only if)

(←) Suppose $R^n \subseteq R$, for n =1,2,3,... .

- Let $(a,b) \in R$ and $(b,c) \in R$
- by the definition of $R \circ R$, $(a,c) \in R \circ R = R^2 \subseteq R$
- Therefore R is transitive.

Connection to R^n

Theorem: The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1,2,3,\ldots$.

Proof: biconditional (if and only if)

($\rightarrow$) Suppose R is transitive. Show $R^n \subseteq R$, for $n = 1,2,3,\ldots$.

- Let $P(n) : R^n \subseteq R$. Mathematical induction.
- **Basis Step:**

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- Let $P(n) : R^n \subseteq R$. Mathematical induction.
- **Basis Step:** P(1) says $R^1 = R$ so, $R^1 \subseteq R$ is true.

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- Want to show if $R^n \subseteq R$ then $R^{n+1} \subseteq R$.

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- Let $P(n) : R^n \subseteq R$. Mathematical induction.
- **Basis Step:** $P(1)$ says $R^1 = R$ so, $R^1 \subseteq R$ is true.
- **Inductive Step:** show $P(n) \rightarrow P(n+1)$
- Want to show if $R^n \subseteq R$ then $R^{n+1} \subseteq R$.
- Let $(a,b) \in R^{n+1}$ then by the definition of $R^{n+1} = R^n \circ R$ there is an element $x \in A$ so that $(a,x) \in R$ and $(x,b) \in R^n \subseteq R$ (inductive hypothesis). In addition to $(a,x) \in R$ and $(x,b) \in R$, R is transitive; so $(a,b) \in R$.
- Therefore, $R^{n+1} \subseteq R$.

Number of reflexive relations

Theorem: The number of reflexive relations on a set A, where $|A| = n$ is: $2^{n(n-1)}$.

Proof:

- A reflexive relation R on A **must contain** all pairs (a,a) where $a \in A$.
- All other pairs in R are of the form (a,b), $a \neq b$, such that $a, b \in A$.
- How many of these pairs are there? Answer: n(n-1).
- How many subsets on n(n-1) elements are there?
- **Answer:** $2^{n(n-1)}$.