Attendance	10%	
Assignment	20%	(ใช้วิธีที่สอนในคาบเท่านั้น)
Mid-term	30%	
Finals	40%	

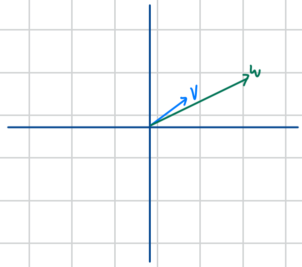
Reference : Introduction to linear Algebra 5^{th} edition

Week 1

Linear Combination

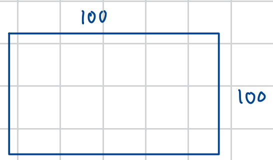
เวลา + − vector เช่น $v = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ $w = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

x, y



เราเอาเส้น 2 ตัวมาบวกรวมกัน

$v + w = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$



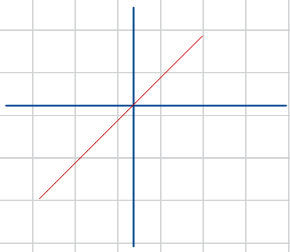
the dimension of this box in 2d is 10,000

* why is it 10,000 dimension

if we have 1 vector as cu, the most we can get is a line

if 2 vector as $cu + dv$, we get a plane

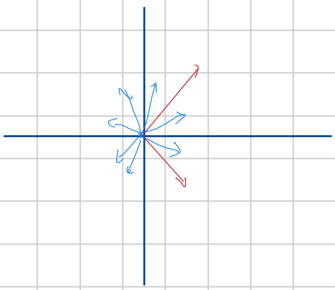
if 3 as $cu + dv + ew$, we get 3 dimentional space



$cu \Rightarrow$ c could be −2

c^2 will be 4

thus creating the line



The combination of $cu + dv$ could go into any direction we want thus creating the plane

* if cu and dv is the same direction plane will not be form.

Unit vector

if $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ the unit vector of $v = \dfrac{v}{\|v\|^2}$

$= \dfrac{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}{\|v\|^2} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$

$\|u\| = (1/\sqrt{5})(1/\sqrt{5}) + (2/\sqrt{5})(2/\sqrt{5})$

$= \frac{1}{5} + \frac{4}{5} = 1$

dot product of unit vector (always = $\cos\theta$)

$i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $u = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

$u = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$ $U = \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix}$

$u \cdot U = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$= \cos(\beta - \alpha) = \cos\theta$

---

if we buy and sell product with

p: price and q: quantity

so if $p = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$ $q = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ $p \cdot q = 10(1) + 20(1) + 30(2)$

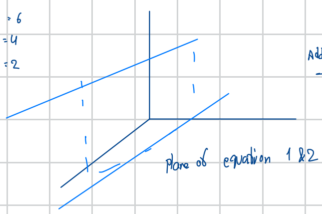
$= 10 + 20 + 60 = 90$ baht

Row picture with 3 viables

$x + 2y + 3z = 6$

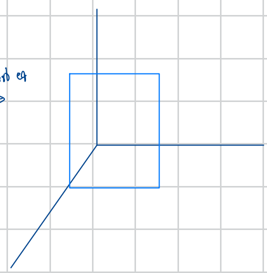
$2x + 5y + 2z = 4$

$6x - 3y + z = 2$



Add last eq $\rightarrow$

now we can find solution



Column with 3

$x\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + y\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + z\begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ -12 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & -2 \\ 3 & -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ -12 \end{bmatrix}$

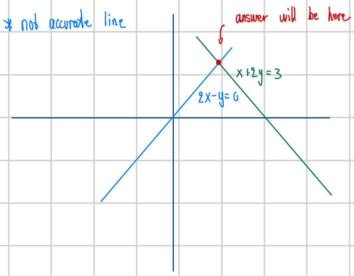
Row picture

$2x - y = 0$

$x + 2y = 3$ $\rightarrow \begin{bmatrix} [2 & -1] \\ [1 & 2] \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

* not accurate line

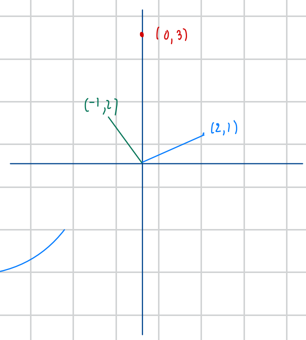
answer will be here

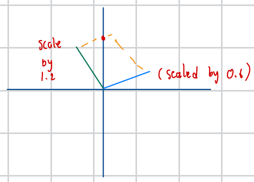


Column picture

$2x - y = 0$

$x + 2y = 3$ $\rightarrow x\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$





Guassion - Jordan Elimination

$$
\begin{aligned}
-x + 2y + 2z &= -24 \\
x + y + z &= 48 \\
x - 6y + 4z &= 12
\end{aligned}
=
\begin{bmatrix} -1 & 2 & 2 & -24 \\ 1 & 1 & 1 & 48 \\ 1 & -6 & 4 & 12 \end{bmatrix}
\xrightarrow{R_1 + R_2}
\begin{bmatrix} 0 & 3 & 3 & 24 \\ 1 & 1 & 1 & 48 \\ 1 & -6 & 4 & 12 \end{bmatrix}
\xrightarrow{R_1 \leftrightarrow R_2}
\begin{bmatrix} 1 & 1 & 1 & 48 \\ 0 & 3 & 3 & 24 \\ 1 & -6 & 4 & 12 \end{bmatrix}
$$

$$
\xrightarrow{R_1 - \frac{1}{3}R_2} \begin{bmatrix} 1 & 0 & 0 & 40 \\ 0 & 3 & 3 & 24 \\ 1 & -6 & 4 & 12 \end{bmatrix}
\xrightarrow{\frac{1}{3}R_2}
\begin{bmatrix} 1 & 0 & 0 & 40 \\ 0 & 1 & 1 & 8 \\ 1 & -6 & 4 & 12 \end{bmatrix}
$$

$$
\xrightarrow{R_3 - R_1}
\begin{bmatrix} 1 & 0 & 0 & 40 \\ 0 & 1 & 1 & 8 \\ 0 & -6 & 4 & -28 \end{bmatrix}
\xrightarrow{R_3 + 6R_2}
\begin{bmatrix} 1 & 0 & 0 & 40 \\ 0 & 1 & 1 & 8 \\ 0 & 0 & 10 & 20 \end{bmatrix}
$$

$$
\xrightarrow{R_3/10}
\begin{bmatrix} 1 & 0 & 0 & 40 \\ 0 & 1 & 1 & 8 \\ 0 & 0 & 1 & 2 \end{bmatrix}
\xrightarrow{R_2 - R_3}
\begin{bmatrix} 1 & 0 & 0 & 40 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}
$$

$$x_1 = 40$$

$$x_2 + x_3 = 8$$

$R_1 - 2R_2$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} -1 & 2 & 2 & -24 \\ 1 & 1 & 1 & 48 \\ 1 & -6 & 4 & 12 \end{array}\right] \longrightarrow$$

$\frac{1}{3}R_1$

$$\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} -3 & 0 & 0 & -120 \\ 1 & 1 & 1 & 48 \\ 1 & -6 & 4 & 12 \end{array}\right]$$

$R_2 - R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 40 \\ 1 & 1 & 1 & 48 \\ 1 & -6 & 4 & 12 \end{array}\right] \longrightarrow$$

$R_3 - R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 40 \\ 0 & 1 & 1 & 8 \\ 1 & -6 & 4 & 12 \end{array}\right]$$

$R_3 + 6R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 6 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 40 \\ 0 & 1 & 1 & 8 \\ 0 & -6 & 4 & -28 \end{array}\right] \longrightarrow$$

$\frac{R_3}{10}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{10} \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 40 \\ 0 & 1 & 1 & 8 \\ 0 & 0 & 10 & 20 \end{array}\right]$$

$R_2 - R_3$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 40 \\ 0 & 1 & 1 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right] \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 40 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Week 2

also called Augmented matrix

Elimination method
- The first step is to check if the matrix is in ladder form, if not, try to swap row to make it possible to be in ladder
- Then do row operation to get row echelon form
- then back substitution

upper triangle form

Matrix wants to be in ladder form so it's easier to do back substitution, in the case of coding, you don't have to write lots of if/else to complete the matrix

Computation Complexity - How hard it is for computer to compute (Big O) → in 4x4 matrix $\begin{bmatrix} 1 & 1 & 1 & 1 \\ \times & 1 & 1 & 1 \\ \times & \times & 1 & 1 \\ \times & \times & \times & 1 \end{bmatrix}$ 6 row operation is required so complexity is $(n-1)!$ in this case $(6-1)!$

then +n (number of back substitution) and + number of row swap

Ex. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ this is very hard to compute because row swap is needed in every step

pivot - number of viable (if 3x3 matrix have 3 viable then you can solve for all viable)

Invert matrix → $\begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix}$ can be done by $\left[\begin{array}{cc|cc} 4 & 5 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{array}\right]$ this will be the inverted matrix

make this into reduced row echelon form

$ABB^{-1}A^{-1} = I$

$B^{-1}A^{-1}AB = I$

Invert of transpose

$I = I^T$

$= (A^{-1}A)^T$

$= A^T (A^{-1})^T$

Non-Invertible

Invert matrix need to be multiply by $\frac{1}{|A|}$ (det of matrix) → so if the det is 0 then matrix is not invertible or matrix is singular

Find x to make $Ax = 0$

$$\overset{A}{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}} \overset{x}{\begin{bmatrix} -2 \\ 1 \end{bmatrix}} = 0$$

$$\begin{bmatrix} 3 & 8 & 1 \\ -4 & 1 & 1 \\ -4 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

↓

$$\begin{bmatrix} 1 & \frac{8}{3} & \frac{1}{3} \\ 0 & \frac{35}{3} & \frac{7}{3} \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{array}{l} x + \frac{8}{3}y + \frac{1}{3}z = 0 \\ \frac{35}{3}y + \frac{7}{3}z = 0 \end{array}$$

$35y + 7z = 0 \rightarrow x + \frac{8}{3}y - \frac{5}{3}y = 0 \rightarrow x = -y$

$5y = -z$; if z is 1 → $y = -\frac{1}{5}$, $x = \frac{1}{5}$

* z is free viable since whatever z is, the $Ax = 0$ will still be true

$Ax = b$

if matrix can find x in $Ax = 0$ then it's non invertible

$$A = \begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix} \rightarrow \text{upper} \begin{bmatrix} u & u & u \\ 0 & u & u \\ 0 & 0 & u \end{bmatrix} \quad \text{so } A = LU$$

lower $\begin{bmatrix} L & 0 & 0 \\ L & L & 0 \\ L & L & L \end{bmatrix}$

if $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$

and $E_{32}E_{31}E_{21}A$ = upper → $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

if I want to find the lower

↓

$E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}A = L$ ← Lower

if $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $(R_2 - 3R_1)$

$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $(R_2 + 3R_1)$

} undo the change

Permutation matrix (row switching)

PA = LU

Solve the linear equation using LU

$$\begin{aligned} 2x - y - 2z &= 1 \\ -4x + 6y + 3z &= 2 \\ -4x - 2y + 8z &= 1 \end{aligned} \rightarrow \begin{bmatrix} 2 & -1 & -2 & 1 \\ -4 & 6 & 3 & 2 \\ -4 & -2 & 8 & 1 \end{bmatrix} \rightarrow \overset{\frac{1}{2}R_1}{\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 2 & -1 & -2 & 1 \\ -4 & 6 & 3 & 2 \\ -4 & -2 & 8 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} & -1 & \frac{1}{2} \\ -4 & 6 & 3 & 2 \\ -4 & -2 & 8 & 1 \end{bmatrix}$$

$$\overset{R_2 + 4R_1}{\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & -\frac{1}{2} & -1 & \frac{1}{2} \\ -4 & 6 & 3 & 2 \\ -4 & -2 & 8 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 4 & -1 & 4 \\ -4 & -2 & 8 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 4 & -1 & 4 \\ -4 & -2 & 8 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{4} & 1 \\ -4 & -2 & 8 & 1 \end{bmatrix}$$

$$\overset{R_3 + 4R_1}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & -\frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{4} & 1 \\ -4 & -2 & 8 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{4} & 1 \\ 0 & -4 & 6 & 3 \end{bmatrix} \rightarrow \overset{R_3 + 4R_2}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}} \begin{bmatrix} 1 & -\frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{4} & 1 \\ 0 & -4 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 1 & \frac{1}{4} & 1 \\ 0 & 0 & 5 & 7 \end{bmatrix}$$

$$\overset{\frac{R_3}{5}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}} \begin{bmatrix} 1 & -\frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{4} & 1 \\ 0 & 0 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{4} & 1 \\ 0 & 0 & 1 & \frac{7}{5} \end{bmatrix}$$

=

Ajarn Ver

$$\begin{bmatrix} 2 & -1 & 2 \\ -4 & 6 & 3 \\ -4 & 2 & 8 \end{bmatrix} \rightarrow \frac{R_1}{2} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & -1 \\ -4 & 6 & 3 \\ -4 & 2 & 8 \end{bmatrix}$$

(inverted)

$R_2 + 4R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 1 \\ 0 & 4 & -1 \\ -4 & -2 & 8 \end{bmatrix}$$

$R_3 + 4R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 \\ -4 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 1 \\ 0 & 4 & -1 \\ 0 & -4 & 4 \end{bmatrix}$$

$\frac{R_2}{4}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 \\ -4 & 4 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & -1 \\ 0 & 1 & -\frac{1}{4} \\ 0 & -4 & 4 \end{bmatrix}$$

$R_3 + 4R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 \\ -4 & 4 & 0 \\ -4 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & -1 \\ 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 3 \end{bmatrix}$$

$$A = LU \rightarrow \begin{bmatrix} 2 & -1 & 2 \\ -4 & 6 & 3 \\ -4 & 2 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ -4 & 4 & 0 \\ -4 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & -1 \\ 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 3 \end{bmatrix}$$

$$E_{32} E_2 E_{31} E_{21} E_1 A = U$$

$$A = E_1^{-1} E_{21}^{-1} E_{31}^{-1} E_2^{-1} E_{32}^{-1}$$

Forward substitution

$Ax = b$

$A = LU$

$LUx = b$

if $w = Ux$

$LW = b$

$$\rightarrow \begin{bmatrix} 2 & 0 & 0 \\ -4 & 4 & 0 \\ -4 & -4 & 3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$2w_1 = 1$

$-4w_1 + 4w_2 = 2$

$-4w_1 - 4w_2 + 3w_3 = 1$

$w_1 = \frac{1}{2}$

$-2 + 4w_2 = 2$

$4w_2 = 4$

$w_2 = 1$

$\rightarrow -2 - 4 + 3w_3 = 1$

$3w_3 = 7$

$w_3 = \frac{7}{3}$

$Ux = w$

$$\begin{bmatrix} 1 & -\frac{1}{2} & 1 \\ 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ \frac{7}{3} \end{bmatrix}$$

$x - \frac{1}{2}y + z = \frac{1}{2}$

$y - \frac{1}{4}z = 1 \rightarrow y - \frac{1}{4}\left(\frac{7}{3}\right) = 1$

$z = \frac{7}{3}$

$y - \frac{7}{12} = 1$

$y = \frac{19}{12}$

$x - \frac{1}{2}\left(\frac{19}{12}\right) + \frac{7}{3} = \frac{1}{2}$

$x - \frac{19}{24} + \frac{7}{3} = \frac{1}{2} \rightarrow x + \frac{37}{24} = \frac{1}{2} \rightarrow x = -\frac{25}{24}$

Week 3

Gussian elimination

Matrix

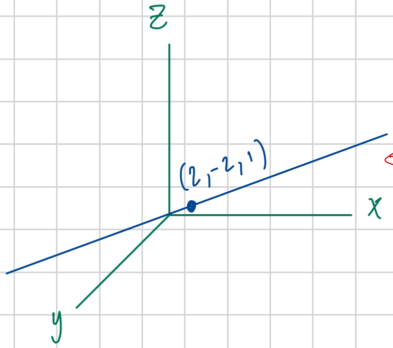
LU decomposition ← if B change then you just need to redo forward - backward substitution.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 5 & 3 \end{array}\right] \rightarrow \text{infinite answer}$$

$x + y + 2z = 2$
$y + 5z = 3$ →

if $z = 1$; $y = -2$ $x = 2$
$z = 0$; $y = 3$ $x = -1$
$z = -1$; $y = 8$ $x = -4$

So you can write it as $C = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$



← every point in this line satisfy $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 5 & 3 \end{array}\right]$

you can also say that z is a free viable (other value change base on z)

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 4 & 0 \\ 3 & 8 & 6 & 16 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 2 & 4 & 0 \\ 0 & 2 & 0 & 4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \end{array}\right]$$

Pivot

Free viable

Free viable

can't change any more so these 2 are free viable

$x_1 + 2x_3 = 0$
$x_2 + 2x_4 = 0$

if $x_3 = 1, x_4 = 0 \rightarrow x_1 = -2, x_2 = 0$
$x_3 = 0, x_4 = 1 \rightarrow x_1 = 0, x_2 = -2$

$$C_1 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \rightarrow \text{if } C_1 = 2,\ C_2 = 1 \rightarrow \begin{bmatrix} -4 \\ -2 \\ 2 \\ 1 \end{bmatrix}$$

Vector space

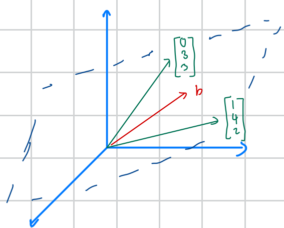
Subspace vector is a vector that go through (0,0)

So if we are in R^2 → every vector have to pass $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to be subspace vector

Column space

– Space that is a combination of column vector

$$A = \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix} \rightarrow x_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$$



b have to be in column space of A in order for the equation to have answer

$C(A^T)$ → row space

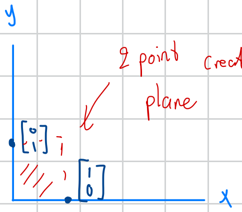
$C(A)$ → column space

$N(A)$ → null space

$N(A^T)$ → left Nullspace

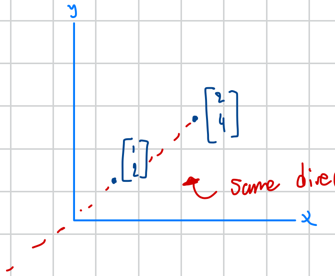
Column space

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$



$\rightarrow$ so $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ represent the whole plane of R^2

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$



$\rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ represent line in R^2

every point in line is b (answer to equation)

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \leftarrow$ there are free viable so this represent the whole R^2

there are infinite solution

Null space

$\leftarrow$ all solution that solve $Ax = 0$ including $x = 0$

$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ $\begin{matrix} x_1 + 2x_2 = 0 \\ 3x_1 + 6x_2 = 0 \end{matrix} \rightarrow \begin{matrix} x_1 + 2x_2 = 0 \\ 0 = 0 \end{matrix} \rightarrow \begin{matrix} x_1 = -2 \\ x_2 = 1 \end{matrix} \rightarrow c\begin{bmatrix} -2 \\ 1 \end{bmatrix} \leftarrow$ line in R^2 that is null space of A

The null space for $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ will always be $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ = have 2 pivot, no free viable and $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \rightarrow$ 2 pivot and 2 free viable

$c_1 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $c_2 \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

Full rank matrix

- Pivot in every row

$\hookrightarrow$ if there are free viable $\rightarrow$ not full rank

if matrix is full rank $\rightarrow$ 1 solution

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$ Find N(A) $\rightarrow$ in R^3

$\begin{matrix} x_1 + 3x_3 = 0 \\ x_2 - 2x_3 = 0 \end{matrix}$ if $x_3 = 1 \rightarrow x_1 = -3, 2 \rightarrow c\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \leftarrow$ line in R

x pivot

$Ax = b$

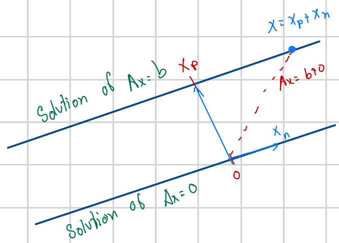
$Ax_p + Ax_n = b$ $\leftarrow$ x null space

$A(x_p + x_n) = b \leftarrow$ if there's no free variable, $x_n = 0$ so $Ax_p = b$

if $b = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & -1 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & -2 & 1 \end{array}\right] \rightarrow x_p = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \rightarrow$ from $A(x_p + x_n) = b$

$\hookrightarrow \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$



Complete solution = 1 solution + All null solution

Solution of System

$r = m = n$	$r = n$, $m > n$	$r = m$, $m < n$	$r < m$, $r < n$
$R = [I]$	$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$	$R = \begin{bmatrix} I & F \end{bmatrix}$	$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$
1 solution	0 or 1 solution	many solution	0 or many solution

- R = rank
- m = row
- n = column
- I = identity matrix
- F = free viable

Week 4

week 3 Hw

1. From matrix A:

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \\ 3 & 9 & 6 \end{bmatrix}$$

   a. If we have linear equation system Ax=b, give the sample of valid vector b and explain why.
   b. Find all possible solutions with vector b and plot a graph to visualize the solutions.

$$\begin{array}{c} x_1 \\ \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \\ 3 & 9 & 6 \end{bmatrix} \end{array} \rightarrow \begin{bmatrix} x_1 & 3x_1 & 2x_1 \end{bmatrix}$$

Same → So the matrix have 1 pivot

So b will be a line that pass through $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Vector b is combination of column vector

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} x_1 + \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} x_2 + \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} x_3 \rightarrow \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} x_1 + 3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} x_1 + 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} x_1 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

So All possible b is Column space of A

2. The column picture for the matrix is

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} v + \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} w = b$$

What are all the solutions (u, v, w) if b is the zero vector (0, 0, 0) ?

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{array}{cc} I & F \\ \left[\begin{matrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & \boxed{\begin{matrix} -1 \\ 2 \end{matrix}} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{matrix}\right] & \end{array}$$

Infinite solution

$x_1 - x_3 = 0$

$x_2 + 2x_3 = 0$

→ if $x_3 = 1$; $x_1 = 1$, $x_2 = -2$

$$= C\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

3. Compute the matrix I (pivots) and F (free variables) from

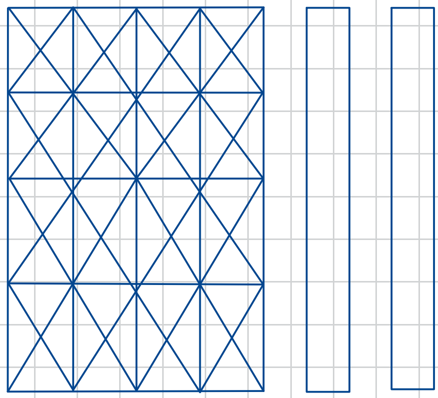
$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 2 & 1 \\ 3 & 6 & 6 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 3 & 2 & 1 \\ 3 & 6 & 6 \\ 1 & 0 & 1 \end{bmatrix} \rightarrow \overset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \rightarrow \begin{bmatrix} I \\ 0 \end{bmatrix} \qquad N(A) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

4. From 3., what is the column space and null space for the matrix A?

$C(A) = R^4$

$N(A) = R^3$



No. 5 = Already correct

แทน Free viable เป็น 0

$$Ax_p + Ax_n = b \rightarrow A(x_p + 0) = b$$

Linear Independence

the sequence of vector that can only combined 0 to get 0

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \rightarrow 0\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

only 0 can make this equation 0

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \rightarrow 3\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 1\begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} = 0$$

Since there are combination that can archive 0 other than 0 so this is linearly dependent

Orthogonal vector → Vector ตั้งฉาก

if $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rightarrow \|x\| = \sqrt{1^2+1^2+1^2} \rightarrow \|x\|^2 = 1^2+1^2+1^2$ same

$x^T = [1\ 1\ 1] \rightarrow x^Tx = [1\ 1\ 1]\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1^2+1^2+1^2$

Week 5

(Recap)

Possible x that make $Ax = 0$

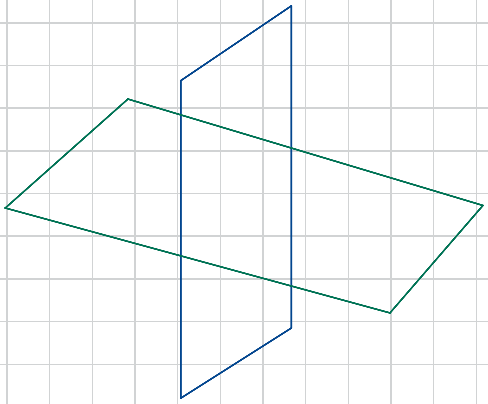
null space $\rightarrow$ $Ax = 0$

column space $\rightarrow$ $Ax = b$ $\leftarrow$ Possible b

} non-invertible, non square

Row operation $\leftarrow$ only use when A is invertible, no free viable, full rank, Square

to prove that something is orthogonal $\perp$ $\rightarrow$ $A \cdot (\vec{v}) = 0 \rightarrow A^T v = 0$



everything in plane Blue will be orthogonal to plane green

Ex. Find the orthogonal complement of $V = \text{span}\left(\begin{bmatrix}1\\-3\\2\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right)$

$$A = \begin{bmatrix}1 & 0\\-3 & 1\\2 & 1\end{bmatrix} \qquad A^T = \begin{bmatrix}1 & -3 & 2\\0 & 1 & 1\end{bmatrix}$$

First - Find null space

Free viable

$$\begin{bmatrix}1 & -3 & 2\\0 & 1 & 1\end{bmatrix} \xrightarrow[\text{row}]{\text{Reduce}} \begin{bmatrix}1 & 0 & 5\\0 & 1 & 1\end{bmatrix} \rightarrow \begin{matrix}x_1 + 5x_3 = 0\\x_2 + x_3 = 0\end{matrix}$$

if $x_3 = 1 \rightarrow x_1 = -5 \quad x_2 = -1 \rightarrow c\begin{bmatrix}-5\\-1\\1\end{bmatrix}, \; c \in \mathbb{R}$ $\leftarrow$ c can be any real number

Null space : orthogonal complement

since $A^T v = 0$

so if $A^T = \begin{bmatrix}1 & -3 & 2\\0 & 1 & 1\end{bmatrix} v = \begin{bmatrix}-5\\-1\\1\end{bmatrix}$

then $A^T v = 0$

null space = orthogonal

Short cut for free viable $\longleftarrow$ Prove of concept

$$\begin{bmatrix}1 & 0 & 5\\0 & 1 & 1\end{bmatrix} \qquad x = \begin{bmatrix}-5\\-1\\1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 3 & 5\\0 & 1 & 1 & 2\end{bmatrix} \rightarrow f_1\, f_2 = \begin{bmatrix}-3\\-1\\1\\0\end{bmatrix} \begin{bmatrix}-5\\-2\\0\\1\end{bmatrix}$$

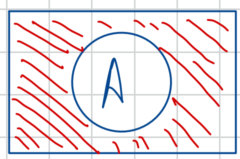
Identity

$$[I \;\; F]\, x_n = 0$$

$$x_n = \begin{bmatrix}-F\\I\end{bmatrix}$$

what is complement?

if A is matrix



complement is everything not in A

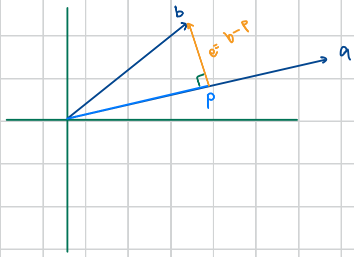
and if something is perpendicular it will not have any relationship between them

Projection (Line)

Projection is when we want to project some vector into a choosen plane to see it's component

Ex. If I want to project P_1 into z plane

$P_1 = \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$ if P_2 in xy- plane $P_2 = \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$



$p = \hat{x}a \rightarrow \hat{x}$: scalar value

we know that $e \cdot a = 0 \rightarrow e$ is orthogonal to a

then $e = b - p$ and $p = \hat{x}a$

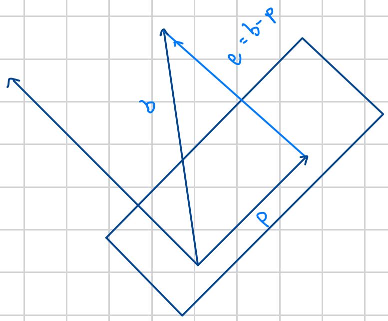
$e = b - \hat{x}a$

so $(b - \hat{x}a)\cdot a = 0 \rightarrow b\cdot a - \hat{x}a \cdot a = 0$

$\hat{x} = \dfrac{a\cdot b}{a \cdot a} \rightarrow \hat{x} = \dfrac{a^Tb}{a^Ta}$

Ex. if we project $b = \begin{bmatrix}1\\1\\1\end{bmatrix}$ into $a = \begin{bmatrix}1\\2\\2\end{bmatrix}$

Projection (Subspace)



$p = \hat{x}_1 a_1 + \hat{x}_2 a_2 + \ldots\ldots \hat{x}_n a_n$

$p = A\hat{x}$

$A^T(b - A\hat{x}) = 0$

$A^Tb - A^TA\hat{x} = 0$

$\hat{x} = (A^TA)^{-1}A^Tb$

$p = A\hat{x} = A(A^TA)^{-1}A^T\hat{x}$

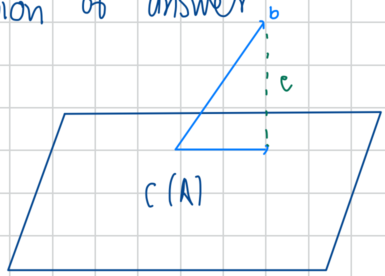
$P = Pb$

$P = A(A^TA)^{-1}A^T$

(Linear Regression)

with no solution $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}$ ← No solution $b \notin C(A)$

Although there's no exact solution but we can still find Approximation of answer



← we can project b into column space to minimize the e as much as possible to get the smallest error and e should be perpendicular to C(A)

when $Ax = b$ has no solution, $\hat{x}$ is the solution

$A^T(b - A\hat{x}) = 0 \rightarrow \hat{x} = (A^TA)^{-1}A^Ty$

$\|b - A\hat{x}\|^2 = \text{minimum}$

$b = A\hat{x}$

$$\underset{b}{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}} = \underset{A}{\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ x_3 & 1 \end{bmatrix}} \underset{\hat{x}}{\begin{bmatrix} m \\ c \end{bmatrix}}$$

* Gram-schmidt ไม่ออก

Orthonormal $= \dfrac{b}{|b|}$ ← orthogonal but with unit vector

Determinant (det) → Area of something

- Singular matrix det = 0
  (not-full rank

* Trace and det
ออกสอบ

Eigenvector and Eigenvalve

A as transformation matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $Ax = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$

not Eigenvector

Eigenvector is the vector x that multiply A and got the same direction as before

Ex. $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$ ← $\begin{bmatrix} 5 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a same direction but $\begin{bmatrix} 5 \\ 5 \end{bmatrix}$ got scaled up

so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is Eigenvector

Eigenvalve

$Ax = \lambda x \rightarrow \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Eigenvector

$Ax = \lambda x$

$Ax = \lambda I x$

$Ax - \lambda I x = 0$

$(A - \lambda I)x = 0$

(singular matrix) ← หา Null space โดยการทำ det = 0

$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

$(A - \lambda I)x = 0$

$\begin{bmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{bmatrix} x = 0$

$(1-\lambda)($

$\lambda = 5, -2$

$\begin{bmatrix} 1-5 & 4 \\ 3 & 2-5 \end{bmatrix}\begin{bmatrix} i \\ j \end{bmatrix} = 0$ $\begin{bmatrix} 1-(-2) & 4 \\ 3 & 2-(-2) \end{bmatrix}\begin{bmatrix} i \\ j \end{bmatrix} = 0$

$\begin{bmatrix} -4 & 4 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$ $\begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -4 \\ 3 \end{bmatrix} = 0$

null null

$\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -4 \\ 3 \end{bmatrix} = -2\begin{bmatrix} -4 \\ 3 \end{bmatrix}$

Eigenvalve

Eigenvector

$Ax = x\Lambda$

$Ax = [\vec{x}_1, \vec{x}_2]\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \lambda_1 x_1 + \lambda_2 x_2$

$A = X\Lambda X^{-1}$

$A^2 = (X\Lambda X^{-1})(X\Lambda X^{-1})$

$A^2 = X\Lambda^2 X^{-1}$

thus $A^k = X\Lambda^k X^{-1}$

$A^k = \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}$

Eigenvalve is always diagonal matrix

$Ax = X\Lambda$

Eigenvalve

$\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} x = \begin{bmatrix} 1 & -4 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$

Eigen vector

Week 6

Any vector can be decompose into Eigenvector

Symmetric matrices that have positive eigenvalue

Linear mapping (Linear transformation) ← very important — ออกสอบ

$$\phi\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1 + ax_2$$

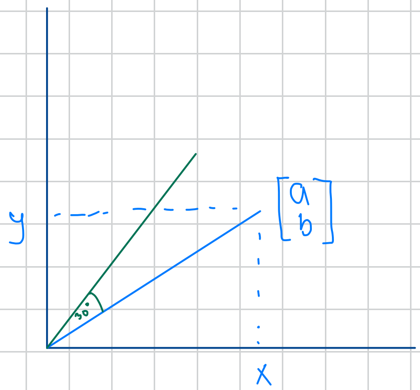
Function taking vector as input

$$\phi\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = (x_1 + y_1) + a(x_2 + y_2)$$

$$= x_1 + ax_2 + y_1 + ay_2$$

$$= \phi\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) + \phi\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) \rightarrow T(x+y) = Tx + Ty = \text{Linear transformation}$$

Rotation matrix



$$a' = a\cos\theta - b\sin\theta$$
$$b' = a\sin\theta + b\cos\theta$$

└ Proof of concept

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3D Rotation

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \quad R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

Order of rotation

$$R_z(\theta) R_y(\theta) R_x(\theta) \begin{bmatrix} a \\ b \end{bmatrix}$$

(ตัวติด matrix เริ่มต้นคือ action ที่ทำก่อน)

Start From right to left

In this case $R_x \rightarrow R_y \rightarrow R_z$

matrix of linear transformation — ทุกการคูณ matrix : Linear Transformation

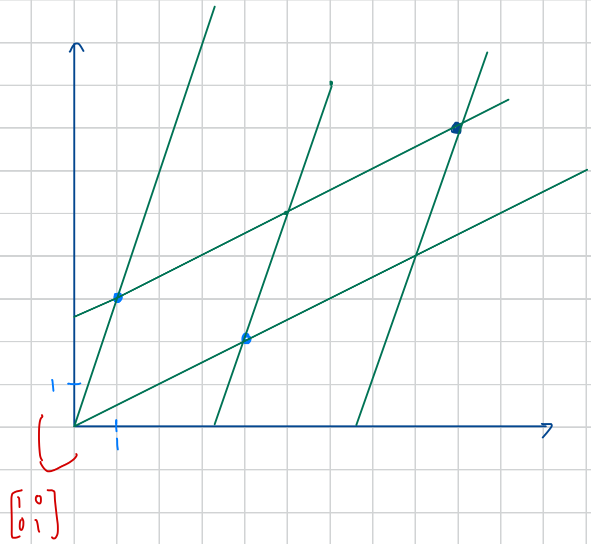
$$\overset{A}{\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}} \overset{V}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ → $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ → linear transform $T(V) = AV$

basis R^n basis R^m

$\begin{bmatrix} 9 \\ 7 \end{bmatrix}$ ← $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

เช่น



ใน Basis ของ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

แต่ใน Basis ใหม่ $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

ถ้าเป็น $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ในแกนใหม่ $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

สีเขียวคือแกนใหม่ $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

ถ้ามองในแกนใหม่ $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ จะเท่ากับ $\begin{bmatrix} 9 \\ 7 \end{bmatrix}$ ในแกนเก่า

Ex. Transform $v_1 = (1,0)$ $v_2 = (0,1)$ to $T(v_1) = (2,3,4)$, $T(v_2) = (5,5,5)$

$A = \begin{bmatrix} 2 & 5 \\ 3 & 5 \\ 4 & 5 \end{bmatrix}$, $T(v_1+v_2) = \begin{bmatrix} 2 & 5 \\ 3 & 5 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$

มาจาก $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Ex. Find transformation matrix $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ with changing standard basis to new basis

$W = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$WB = V \rightarrow B = W^{-1}V$

$A_{new} = B^{-1}AB$

V เป็น I เพราะโจทย์บอกว่าเปลี่ยนจาก standard basis

$B = \begin{bmatrix} -1 & 3 \\ 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 2 & 4 \end{bmatrix}$

Cholesky Decomposition

$$L_{ii} = \sqrt{a_{ii} - \Sigma_{k=1}^{i-1} L_{ik}^2} \rightarrow \text{Diagonal term}$$

$$L_{ij} = \frac{1}{L_{jj}} \left(A_{ij} - \Sigma_{k=1}^{j-1} L_{ik} L_{jk} \right) \quad \text{for } i > j$$

Ex. $A = \begin{bmatrix} 4 & 10 & 8 \\ 10 & 26 & 26 \\ 8 & 26 & 61 \end{bmatrix}$

$L_{11} \rightarrow$ diagonal $= \sqrt{4 - 0} = \sqrt{4} = 2$

$L_{21} \rightarrow$ non-diagonal $= L_{21} = \frac{1}{L_{11}} (A_{21} - 0) = \frac{1}{2}(10) = 5$

$L_{31} = \frac{1}{L_{11}} (A_{31} - 0) = \frac{1}{2}(8) = 4$

$L_{22} \rightarrow \sqrt{26 - 5^2} = 1$

$L_{12} \rightarrow \frac{1}{L_{22}} (A_{12} - 2(5)) \Rightarrow \frac{1}{1}(10 - 10) = 0$

$L_{32} \Rightarrow \frac{1}{}$

Midterm

Row operation

LU decomposition

Vector space

solving $Ax = b$ & Find complete solution

orthogonal and projection

Eigenvector & Eigenvalue

Linear Transformation

Make this singular system
- → det = 0
- → not full rank
- → not invertible
- → dependent
- → have free viable

$3x + 2y = 10$

$6x + 4y = 20$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \_ & \_ & \_ \end{bmatrix}$$

→ to make this dependent (1 free viable) the last row can be combination of 1st and 2nd

Ex. 5 7 9

# of free viable = $n - r$ → Rank

n ↓ Dimension of matrix (row)

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}^{-1}$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\frac{R_1}{2}} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$R_2 - R_1$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & 1 & -\frac{1}{2} & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \rightarrow$$

$\frac{2}{3} R_2$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$R_1 - 2R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{4}{6} & \frac{5}{6} & -\frac{4}{3} & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \rightarrow$$

$R_3 - R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{4}{6} & \frac{5}{6} & -\frac{4}{3} & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & -\frac{2}{3} & 1 \end{array}\right]$$

$\frac{3}{4} R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{4}{6} & \frac{5}{6} & -\frac{4}{3} & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \end{array}\right]$$

$R_2 - \frac{2}{3} R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{2}{3} & \frac{5}{6} & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{3} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \end{array}\right]$$

$R_1 + \frac{2}{3} R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{4}{3} & -\frac{5}{3} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{3} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \end{array}\right]$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 7 & 9 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 7 & 9 \end{bmatrix}$$

$R_3 - 2R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 7 \end{bmatrix}$$

$R_3 - 3R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\frac{R_3}{4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$PA$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 7 & 7 & 9 \end{bmatrix}$$

$$L \qquad U$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Describe subspace S of each vector space V and then subspace SS of S

if V is $\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix},$ Subspace of V is a cube in R^4

and subspace SS $c\begin{bmatrix}1\\1\\0\\0\end{bmatrix} c\begin{bmatrix}1\\1\\1\\0\end{bmatrix}$ of S is plane in R^4

$V_2 = \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}0\\2\\1\end{bmatrix}$ → subspace of V_2 = plane in R^3

and subspace of subspace of V_2 is a line in R^3

↓

$c\begin{bmatrix}1\\2\\0\end{bmatrix}$ or $c\begin{bmatrix}0\\2\\1\end{bmatrix}$

orthogonal complement of $(1,2,1)$

$\begin{bmatrix}1\\2\\1\end{bmatrix} \to A^TA \quad [1, 2, 1]\begin{bmatrix}1\\2\\1\end{bmatrix}$

$\begin{array}{cc} F_2 & F_1 \\ \downarrow & \downarrow \end{array}$

then find null(A^T) → $\begin{bmatrix}1 & 2 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$ → $x_1 + 2x_2 + x_3 = 0$

if $x_3 = 1$ $x_2 = 0$

so null of A^T = $c_1\begin{bmatrix}-1\\0\\1\end{bmatrix} + c_2\begin{bmatrix}-2\\1\\0\end{bmatrix}$

$x_1 = -1$

if $x_3 = 0$ $x_2 = 1$

$x_1 = -2$

span$\left(\begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}-2\\1\\0\end{bmatrix}\right)$

1. Construct a matrix with column space contains $\begin{bmatrix}1\\2\\-3\end{bmatrix}$ and $\begin{bmatrix}2\\-3\\5\end{bmatrix}$, nullspace contains $\begin{bmatrix}1\\1\\1\end{bmatrix}$.

$$\begin{bmatrix}1 & 2 & -3\\2 & -3 & 1\\-3 & 5 & -2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

Linear combination

So $x_3 = c_1x_1 + c_2x_2 \rightarrow 0 = c_1x_1 + c_2x_2 - x_3$

$0 = 1 + 2 + x_3 \rightarrow x_3 = -3$

$0 = 2 - 3 + x_3 \rightarrow x_3 = 1$

$0 = -3 + 5 + x_3 \rightarrow x_3 = -2$

$$\begin{bmatrix}1 & 2 & x\\2 & -3 & y\\-3 & 5 & z\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}$$

$1a + 2b + xc = 0$

$2a - 3b + yc = 0$

$-3a + 5b + zc = 0$

$\rightarrow$ แทน $\begin{bmatrix}a\\b\\c\end{bmatrix}$ เป็น null ที่ให้มาเช่น $\begin{bmatrix}1\\1\\1\end{bmatrix}$

C null