

Relations

Puttachat Khunontong

Relations

- Relation and Their Properties
 - Relations on a Set
 - Properties of Relations
 - Reflexive
 - Irreflexive
 - Symmetric
 - Anti-symmetric
 - Transitive
 - Combining Relations
- N-ary Relations and Their Applications
 - N-ary Relations
 - Databased and Relations
 - Operations on n-ary Relations
 - SQL

Cartesian product (^{Larger set}review)

Let $A = \{a_1, a_2, \dots, a_k\}$ and $B = \{b_1, b_2, \dots, b_m\}$.

The Cartesian product $A \times B$ is defined by a set of pairs
 $\{(a_1, b_1), (a_1, b_2), \dots, (a_1, b_m), \dots, (a_k, b_m)\}$.

Cartesian product defines a product set, or a set of all ordered arrangements of elements in sets in the Cartesian product.

Binary relation

Definition: Let A and B be two sets. A **binary relation from A to B** is a subset of a Cartesian product $A \times B$.

- Let $R \subseteq A \times B$ means R is a set of ordered pairs of the form (a,b) where $a \in A$ and $b \in B$.
- We use the notation **a R b** to denote $(a,b) \in R$ and **a R̄ b** to denote $(a,b) \notin R$. If $a R b$, we say a is related to b by R.

Example: Let $A=\{a,b,c\}$ and $B=\{1,2,3\}$. $A \rightarrow B$

- Is $R=\{(a,1),(b,2),(c,2)\}$ a relation from A to B? Yes
- Is $Q=\{(1,a),(2,b)\}$ a relation from A to B? $\begin{array}{l} A \rightarrow B \\ A \rightarrow A \end{array}$ No
- Is $P=\{(a,a),(b,c),(b,a)\}$ a relation from A to A? Yes

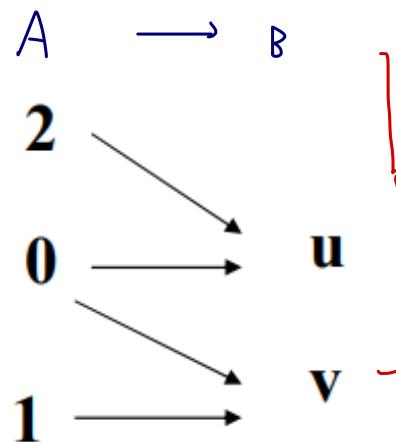
Representing binary relations

- We can graphically represent a binary relation R as follows:
 - if $a R b$ then draw an arrow from a to b .

$$a \rightarrow b$$

Example:

- Let $A = \{0, 1, 2\}$, $B = \{u, v\}$ and $R = \{ (0,u), (0,v), (1,v), (2,u) \}$
- Note: $R \subset A \times B$.
- **Graph:**



Representing binary relations

- We can represent a binary relation R by a **table** showing (marking) the ordered pairs of R .

Example:

- Let $A = \{0, 1, 2\}$, $B = \{u, v\}$ and $R = \{(0,u), (0,v), (1,v), (2,u)\}$

- **Table:**

B

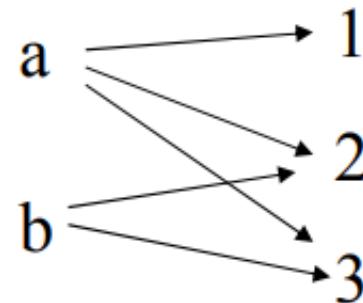
R		u	v
		x	x
A	0	x	x
	1		x
2	x		

or

R		u	v
		1	1
A	0	1	1
	1	0	1
2	1	0	

Relations and functions

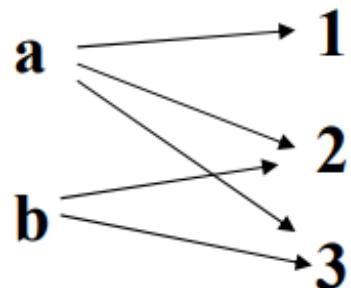
- Relations represent **one to many relationships** between elements in A and B.
- **Example:**



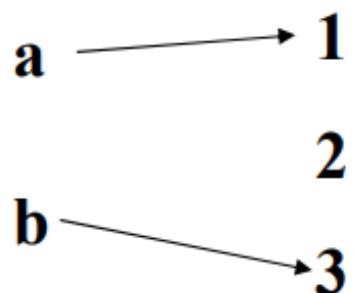
- What is the difference between a **relation** and a **function** from **A to B**?

Relations and functions

- Relations represent **one to many relationships** between elements in A and B.
- **Example:**



- What is the difference between a **relation and a function from A to B?** A function defined on sets A,B $A \rightarrow B$ assigns to each element in the domain set A exactly one element from B. So it is a **special relation.**



Relation on the set

Definition: A relation on the set A is a relation from A to itself.

Example 1:

- Let $A = \{1,2,3,4\}$ and $R_{\text{div}} = \{(a,b) | a \text{ divides } b\}$
- What does R_{div} consist of?
- $R_{\text{div}} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

- | R | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | x | x | x | x |
| 2 | | x | | x |
| 3 | | | x | |
| 4 | | | | x |

Relation on the set

Example:

- Let $A = \{1, 2, 3, 4\}$.
- Define $a R_{\neq} b$ if and only if $a \neq b$.

$$R_{\neq} = \{(1,2), (1,3), (1,4), (2,1), (2,3), (2,4), (3,1), (3,2), (3,4), (4,1), (4,2), (4,3)\}$$

- | R | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | | X | X | X |
| 2 | X | | X | X |
| 3 | X | X | | X |
| 4 | X | X | X | |

Binary relations

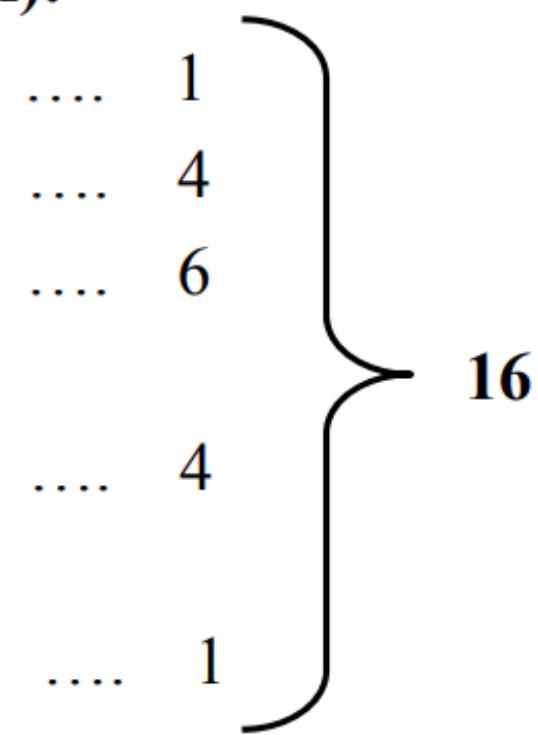
- **Theorem:** The number of binary relations on a set A, where $|A| = n$ is:

$$2^{n^2}$$

- **Proof:**
 - If $|A| = n$ then the cardinality of the Cartesian product $|A \times A| = n^2$.
 - R is a binary relation on A if $R \subseteq A \times A$ (that is, R is a subset of $A \times A$).
 - The number of subsets of a set with k elements : 2^k
 - The number of subsets of $A \times A$ is : $2^{|A \times A|} = 2^{n^2}$

Binary relations

- **Example:** Let $A = \{1,2\}$
- What is $A \times A = \{(1,1), (1,2), (2,1), (2,2)\}$
- **List of possible relations (subsets of $A \times A$):**

- \emptyset
 - $\{(1,1)\} \quad \{(1,2)\} \quad \{(2,1)\} \quad \{(2,2)\} \quad \dots \quad 1$
 - $\{(1,1), (1,2)\} \quad \{(1,1), (2,1)\} \quad \{(1,1), (2,2)\} \quad \dots \quad 4$
 - $\{(1,2), (2,1)\} \quad \{(1,2), (2,2)\} \quad \{(2,1), (2,2)\} \quad \dots \quad 6$
 - $\{(1,1), (1,2), (2,1)\} \quad \{(1,1), (1,2), (2,2)\} \quad \dots \quad 4$
 - $\{(1,1), (2,1), (2,2)\} \quad \{(1,2), (2,1), (2,2)\}$
 - $\{(1,1), (1,2), (2,1), (2,2)\} \quad \dots \quad 1$
- 

- Use formula: $2^4 = 16$

Exercises

1. List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$, where $(a, b) \in R$ if and only if
- a) $a = b$.
 - b) $a + b = 4$.
 - c) $a > b$.
 - d) $a | b$.
 - e) $\gcd(a, b) = 1$.
 - f) $\text{lcm}(a, b) = 2$.

Exercises

1. List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$, where $(a, b) \in R$ if and only if
- a) $a = b$. b) $a + b = 4$.

Exercises

1. List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$, where $(a, b) \in R$ if and only if
- c) $a > b.$ d) $a | b.$

Exercises

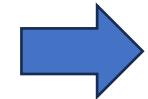
1. List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$, where $(a, b) \in R$ if and only if
- e) $\gcd(a, b) = 1.$ f) $\operatorname{lcm}(a, b) = 2.$

Exercises

1. List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$, where $(a, b) \in R$ if and only if
- e) $\gcd(a, b) = 1.$ f) $\operatorname{lcm}(a, b) = 2.$

Relations

- Relation and Their Properties
 - Relations on a Set
 - **Properties of Relations**
 - Reflexive
 - Irreflexive
 - Symmetric
 - Anti-symmetric
 - Transitive
 - Combining Relations
- N-ary Relations and Their Applications
 - N-ary Relations
 - Databased and Relations
 - Operations on n-ary Relations
 - SQL



Properties of relations

Definition (reflexive relation) : A relation R on a set A is called **reflexive** if $(a,a) \in R$ for every element $a \in A$.

Example 1:

- Assume relation $R_{\text{div}} = \{(a|b), \text{ if } a|b\}$ on $A = \{1,2,3,4\}$
- Is R_{div} reflexive?
- $R_{\text{div}} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$
- **Answer:** Yes. $(1,1), (2,2), (3,3)$, and $(4,4) \in A$.

	1	2	3	4
1	x	x	x	x
2		x		x
3			x	
4				x

Reflexive relation

Reflexive relation

- $R_{\text{div}} = \{(a, b), \text{ if } a | b\}$ on $A = \{1, 2, 3, 4\}$
- $R_{\text{div}} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

$$\begin{array}{cc} & \begin{matrix} 1 & 1 & 1 & 1 \end{matrix} \\ \text{MR}_{\text{div}} = & \begin{matrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \end{array}$$

- A relation R is reflexive if and only if MR has 1 in every position on its main diagonal.

Properties of relations

Definition (reflexive relation) : A relation R on a set A is called **reflexive** if $(a,a) \in R$ for every element $a \in A$.

Example 2:

- Relation R_{fun} on $A = \{1,2,3,4\}$ defined as:
 - $R_{\text{fun}} = \{(1,2),(2,2),(3,3)\}$.
- **Is R_{fun} reflexive?**
- **No.** It is not reflexive since $(1,1) \notin R_{\text{fun}}$.

Properties of relations

Reflexive

Definition (irreflexive relation): A relation R on a set A is called **irreflexive** if $(a,a) \notin R$ for every $a \in A$.

Example 1:

- Assume relation R_{\neq} on $A=\{1,2,3,4\}$, such that $a R_{\neq} b$ if and only if $a \neq b$.
- Is R_{\neq} irreflexive?
- $R_{\neq} = \dots$

Reflexive relation

Reflexive relation

- $R_{\text{div}} = \{(a, b), \text{ if } a | b\}$ on $A = \{1, 2, 3, 4\}$
- $R_{\text{div}} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

$$\begin{array}{c} & & 1 & 1 & 1 & 1 \\ \text{MR}_{\text{div}} & = & 0 & 1 & 0 & 1 \\ & & 0 & 0 & 1 & 0 \\ & & 0 & 0 & 0 & 1 \end{array}$$

- A relation **R** is reflexive if and only if MR has 1 in every position on its main diagonal.

Reflexive

Reflexive relation

If $A = \{a,b\}$ then R is **reflexive** if:

Roster Method

$$R = \{(a,a), (b,b)\}$$

Directed Graph

Every vertex has a self-loop



Incidence Matrix

Main diagonal all “1s“

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

EXAMPLE

Consider the following relations on $\{1, 2, 3, 4\}$:

Reflexive

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$

$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$

$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}, \quad /$$

$$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$$

$$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}, \quad /$$

$$R_6 = \{(3, 4)\}.$$

Which of these relations are reflexive?

Properties of relations

Definition (irreflexive relation): A relation R on a set A is called **irreflexive** if $(a,a) \notin R$ for every $a \in A$.

Example 1:

- Assume relation R_{\neq} on $A=\{1,2,3,4\}$, such that $a R_{\neq} b$ if and only if $a \neq b$.
- **Is R_{\neq} irreflexive?**
- $R_{\neq}=\{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}$
- **Answer:** Yes. Because $(1,1),(2,2),(3,3)$ and $(4,4) \notin R_{\neq}$

Irreflexive relation

Irreflexive relation

- R_{\neq} on $A=\{1,2,3,4\}$, such that $a R_{\neq} b$ if and only if $a \neq b$.
- $R_{\neq}=\{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}$

$$\text{MR} = \begin{matrix} & \begin{matrix} 0 & 1 & 1 & 1 \end{matrix} \\ \begin{matrix} 1 & 0 & 1 & 1 \end{matrix} & \begin{matrix} 1 & 1 & 0 & 1 \end{matrix} \\ \begin{matrix} 1 & 1 & 1 & 0 \end{matrix} & \end{matrix}$$

- A relation R is irreflexive if and only if MR has 0 in every position on its main diagonal.

Irreflexive

Irreflexive relation

If set $A = \{a,b\}$ then R is **irreflexive** if:

Roster Method

$$(a,a) \notin R \text{ and } (b,b) \notin R$$

Directed Graph

Every vertex does not have a self-loop



Incidence Matrix

Main diagonal all “0s”

$$\begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix}$$

Properties of relations

Definition (irreflexive relation): A relation R on a set A is called **irreflexive** if $(a,a) \notin R$ for every $a \in A$.

Example 2:

- R_{fun} on $A = \{1,2,3,4\}$ defined as:
 - $R_{\text{fun}} = \{(1,2),(2,2),(3,3)\}$.
- **Is R_{fun} irreflexive?**
- **Answer: No.** Because $(2,2)$ and $(3,3) \in R_{\text{fun}}$

Properties of relations

Definition (symmetric relation): A relation R on a set A is called **symmetric** if

$$\forall a, b \in A \quad (a,b) \in R \rightarrow (b,a) \in R. \quad \text{No sym}$$

Example 1:

- $R_{\text{div}} = \{(a|b), \text{ if } a|b\}$ on $A = \{1,2,3,4\}$
- **Is R_{div} symmetric?**
- $R_{\text{div}} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$
- **Answer: No.** It is not symmetric since $(1,2) \in R$ but $(2,1) \notin R$.

X	X	X	X
Ø	X	Ø	X
Ø	Ø	X	Ø
Ø	Ø	Ø	X

Symmetric relation

Symmetric

Symmetric relation:

- R_{\neq} on $A=\{1,2,3,4\}$, such that $a R_{\neq} b$ if and only if $a \neq b$.
- $R_{\neq}=\{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}$

$$MR = \begin{matrix} & \begin{matrix} 0 & 1 & 1 & 1 \end{matrix} \\ \begin{matrix} 1 & 0 & 1 & 1 \end{matrix} & \begin{matrix} 1 & 1 & 0 & 1 \end{matrix} \\ \begin{matrix} 1 & 1 & 1 & 0 \end{matrix} & \end{matrix}$$

- A relation R is symmetric if and only if $m_{ij} = m_{ji}$ for all i,j .

Symmetric relation

Symmetric

If set $A = \{a, b\}$ then R is **symmetric** if:

Roster Method

$$(a, b) \in R \text{ and } (b, a) \in R$$

Directed Graph

Every pair of vertices is connected by none or exactly two directed lines in opposite directions



Incidence Matrix

$$m_{ij} = m_{ji} \text{ off the main diagonal}$$

$$\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & 0 \end{bmatrix}$$

Properties of relations

Properties of relations on A:

- Reflexive ✓
- Irreflexive ✓
- Symmetric ✓
- **Anti-symmetric**
- **Transitive**

Anti-symmetric relation

Anti-symmetric

Definition (anti-symmetric relation): A relation on a set A is called **anti-symmetric** if

- $[(a,b) \in R \text{ and } (b,a) \in R] \rightarrow a = b \text{ where } a, b \in A.$

Example 3:

- Relation R_{fun} on $A = \{1,2,3,4\}$ defined as:
 - $R_{\text{fun}} = \{(1,2),(2,2),(3,3)\}.$
- Is R_{fun} anti-symmetric?
- Answer: Yes. It is anti-symmetric

Anti-symmetric relation

Antisymmetric relation

- relation $R_{\text{fun}} = \{(1,2),(2,2),(3,3)\}$

$$MR_{\text{fun}} =$$

0	1	0	0
0	1	0	0
0	0	1	0
0	0	0	0

- A relation is **antisymmetric** if and only if $m_{ij} = 1 \rightarrow m_{ji} = 0$ for $i \neq j$.

Anti-symmetric relation

Anti-symmetric

If set $A = \{a,b\}$ then R is **antisymmetric** if:

Roster Method

if $(a,b) \in R$ and $(b,a) \in R$ then $a = b$

Directed
Graph

Every pair of vertices is connected by none
or exactly one directed lines



Incidence Matrix

$m_{ij} \neq m_{ji}$ off the main diagonal

$$\begin{bmatrix} & & 0 \\ 1 & & 0 \\ & 0 & & 1 \\ & & 0 & \end{bmatrix}$$

Exercise

Which of the relations from Relations below are symmetric and which are antisymmetric?

Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$

$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$

$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$$

$$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$$

$$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$$

$$R_6 = \{(3, 4)\}.$$

Transitive relation

Definition (transitive relation): A relation R on a set A is called **transitive** if

- $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R \text{ for all } a, b, c \in A.$
- **Example 1:**
- $R_{\text{div}} = \{(a|b), \text{ if } a|b\} \text{ on } A = \{1,2,3,4\}$
- $R_{\text{div}} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$
- **Is R_{div} transitive?**
- **Answer:**

Transitive relation

Definition (transitive relation): A relation R on a set A is called **transitive** if

- $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R \text{ for all } a, b, c \in A.$

- **Example 1:**

- $R_{\text{div}} = \{(a|b), \text{ if } a|b\} \text{ on } A = \{1,2,3,4\}$
- $R_{\text{div}} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

- **Is R_{div} transitive?**

- **Answer:** Yes.

Transitive relation

Definition (transitive relation): A relation R on a set A is called **transitive** if

- $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R \text{ for all } a, b, c \in A.$
- **Example 2:** \neq
- R_{\neq} on $A=\{1,2,3,4\}$, such that $a R_{\neq} b$ if and only if $a \neq b$.
- $R_{\neq} = \{(1,2), (1,3), (1,4), (2,1), (2,3), (2,4), (3,1), (3,2), (3,4), (4,1), (4,2), (4,3)\}$
- Is R_{\neq} transitive ?
- Answer:

Transitive relation

Definition (transitive relation): A relation R on a set A is called **transitive** if

- $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R \text{ for all } a, b, c \in A.$
- **Example 2:**
- R_{\neq} on $A=\{1,2,3,4\}$, such that $a R_{\neq} b$ if and only if $a \neq b$.
- $R_{\neq}=\{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}$
- **Is R_{\neq} transitive?**
- **Answer: No.** It is not transitive since $(1,2) \in R$ and $(2,1) \in R$ but $(1,1)$ is not an element of R.

Transitive relations

Definition (transitive relation): A relation R on a set A is called **transitive** if

- $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R \text{ for all } a, b, c \in A.$
- **Example 3:**
- Relation R_{fun} on $A = \{1,2,3,4\}$ defined as:
 - $R_{\text{fun}} = \{(1,2),(2,2),(3,3)\}.$
- **Is R_{fun} transitive?**
- **Answer:**

Transitive relations

Definition (transitive relation): A relation R on a set A is called **transitive** if

- $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R \text{ for all } a, b, c \in A.$
- **Example 3:**
- Relation R_{fun} on $A = \{1,2,3,4\}$ defined as:
 - $R_{\text{fun}} = \{(1,2),(2,2),(3,3)\}.$
- **Is R_{fun} transitive?**
- **Answer:** Yes. It is transitive.

Exercise

Determine whether the relation R on the set of all integers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if

- a)** $x \neq y$.
- b)** $xy \geq 1$.
- c)** $x = y + 1$ or $x = y - 1$.

Exercise

Determine whether the relation R on the set of all integers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if

a) $x \neq y.$

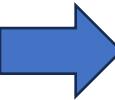
b) $xy \geq 1.$

Exercise

Determine whether the relation R on the set of all integers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if

- c) $x = y + 1$ or $x = y - 1$.

Relations

- Relation and Their Properties
 - Relations on a Set
 - Properties of Relations
 - Reflexive
 - Irreflexive
 - Symmetric
 - Anti-symmetric
 - Transitive
 - N-ary Relations and Their Applications
 - N-ary Relations
 - Databased and Relations
 - Operations on n-ary Relations
 - SQL
-  • **Combining Relations**

Cartesian product (review)

Let $A = \{a_1, a_2, \dots, a_k\}$ and $B = \{b_1, b_2, \dots, b_m\}$.

The Cartesian product $A \times B$ is defined by a set of pairs
 $\{(a_1, b_1), (a_1, b_2), \dots, (a_1, b_m), \dots, (a_k, b_m)\}$.

Cartesian product defines a product set, or a set of all ordered arrangements of elements in sets in the Cartesian product.

Combining relations

Definition: Let A and B be sets. A **binary relation from A to B** is a subset of a Cartesian product $A \times B$.

- Let $R \subseteq A \times B$ means R is a set of ordered pairs of the form (a,b) where $a \in A$ and $b \in B$.

Combining Relations

- **Relations are sets → combinations via set operations**
- Set operations of: **union, intersection, difference and symmetric difference.**

Combining relations

Example:

- Let $A = \{1,2,3\}$ and $B = \{u,v\}$ and
- $R1 = \{(1,u), (2,u), (2,v), (3,u)\}$
- $R2 = \{(1,v),(3,u),(3,v)\}$

What is:

- $R1 \cup R2 = ?$

Combining relations

Example:

- Let $A = \{1,2,3\}$ and $B = \{u,v\}$ and
- $R1 = \{(1,u), (2,u), (2,v), (3,u)\}$
- $R2 = \{(1,v),(3,u),(3,v)\}$

What is:

- $R1 \cup R2 = \{(1,u),(1,v),(2,u),(2,v),(3,u),(3,v)\}$
- $R1 \cap R2 = ?$

Combining relations

Example:

- Let $A = \{1,2,3\}$ and $B = \{u,v\}$ and
- $R1 = \{(1,u), (2,u), (2,v), (3,u)\}$
- $R2 = \{(1,v),(3,u),(3,v)\}$

What is:

- $R1 \cup R2 = \{(1,u),(1,v),(2,u),(2,v),(3,u),(3,v)\}$
- $R1 \cap R2 = \{(3,u)\}$
- $R1 - R2 = ?$

Combining relations

Example:

- Let $A = \{1,2,3\}$ and $B = \{u,v\}$ and
- $R_1 = \{(1,u), (2,u), (2,v), (3,u)\}$
- $R_2 = \{(1,v), (3,u), (3,v)\}$

What is:

- $R_1 \cup R_2 = \{(1,u), (1,v), (2,u), (2,v), (3,u), (3,v)\}$
- $R_1 \cap R_2 = \{(3,u)\}$
- $R_1 - R_2 = \{(1,u), (2,u), (2,v)\}$
- $R_2 - R_1 = ?$

Combining relations

Example:

- Let $A = \{1,2,3\}$ and $B = \{u,v\}$ and
- $R1 = \{(1,u), (2,u), (2,v), (3,u)\}$
- $R2 = \{(1,v),(3,u),(3,v)\}$

What is:

- $R1 \cup R2 = \{(1,u),(1,v),(2,u),(2,v),(3,u),(3,v)\}$
- $R1 \cap R2 = \{(3,u)\}$
- $R1 - R2 = \{(1,u),(2,u),(2,v)\}$
- $R2 - R1 = \{(1,v),(3,v)\}$

Combination of relations

Representation of operations on relations:

- **Question:** Can the relation be formed by taking the union or intersection or composition of two relations R1 and R2 be represented in terms of matrix operations?
- **Answer: Yes**

Combination of relations: implementation

Definition. The **join**, denoted by \vee , of two m-by-n matrices (a_{ij}) and (b_{ij}) of 0s and 1s is an m-by-n matrix (m_{ij}) where

- $m_{ij} = a_{ij} \vee b_{ij}$ for all i,j
- = pairwise or (disjunction)

- **Example:**
- Let $A = \{1,2,3\}$ and $B = \{u,v\}$ and
- $R1 = \{(1,u), (2,u), (2,v), (3,u)\}$
- $R2 = \{(1,v),(3,u),(3,v)\}$

- $MR1 = \begin{matrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{matrix}$ $MR2 = \begin{matrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{matrix}$ $M(R1 \vee R2) = \begin{matrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{matrix}$

Combination of relations: implementation

Definition. The **meet**, denoted by \wedge , of two m-by-n matrices (a_{ij}) and (b_{ij}) of 0s and 1s is an m-by-n matrix (m_{ij}) where

- $m_{ij} = a_{ij} \wedge b_{ij}$ for all i,j
- **= pairwise and (conjunction)**

- **Example:**
- Let $A = \{1,2,3\}$ and $B = \{u,v\}$ and
- $R1 = \{(1,u), (2,u), (2,v), (3,u)\}$
- $R2 = \{(1,v),(3,u),(3,v)\}$

• MR1	$\begin{matrix} 1 & 0 \end{matrix}$	MR2	$\begin{matrix} 0 & 1 \end{matrix}$	MR1 \wedge MR2	$\begin{matrix} 0 & 0 \end{matrix}$
	$\begin{matrix} 1 & 1 \end{matrix}$		$\begin{matrix} 0 & 0 \end{matrix}$		$\begin{matrix} 0 & 0 \end{matrix}$
	$\begin{matrix} 1 & 0 \end{matrix}$		$\begin{matrix} 1 & 1 \end{matrix}$		$\begin{matrix} 1 & 0 \end{matrix}$

Composite of relations

Definition: Let R be a relation from a set A to a set B and S a relation from B to a set C . The **composite of R and S** is the relation consisting of the ordered pairs (a,c) where $a \in A$ and $c \in C$, and for which there is a $b \in B$ such that $(a,b) \in R$ and $(b,c) \in S$. We denote the composite of R and S by $S \circ R$.

Examples:

- Let $A = \{1,2,3\}$, $B = \{0,1,2\}$ and $C = \{a,b\}$.
- $R = \{(1,0), (1,2), (3,1),(3,2)\}$
- $S = \{(0,b),(1,a),(2,b)\}$
- $S \circ R = ?$

Composite of relations

Definition: Let R be a relation from a set A to a set B and S a relation from B to a set C. The **composite of R and S** is the relation consisting of the ordered pairs (a,c) where $a \in A$ and $c \in C$, and for which there is a $b \in B$ such that $(a,b) \in R$ and $(b,c) \in S$. We denote the composite of R and S by **S o R**.

Examples:

- Let $A = \{1,2,3\}$, $B = \{0,1,2\}$ and $C = \{a,b\}$.
- $R = \{(1,0), (1,2), (3,1),(3,2)\}$
- $S = \{(0,b),(1,a),(2,b)\}$
- $S o R = \{(1,b),(3,a),(3,b)\}$

Implementation of composite

Definition. The **Boolean product**, denoted by \odot , of an m-by-n matrix (a_{ij}) and n-by-p matrix (b_{jk}) of 0s and 1s is an m-by-p matrix (m_{ik}) where

- $m_{ik} = \begin{cases} 1, & \text{if } a_{ij} = 1 \text{ and } b_{jk} = 1 \text{ for some } k=1,2,\dots,n \\ 0, & \text{otherwise} \end{cases}$

Examples:

- Let $A = \{1,2,3\}$, $B = \{0,1,2\}$ and $C = \{a,b\}$.
- $R = \{(1,0), (1,2), (3,1),(3,2)\}$
- $S = \{(0,b),(1,a),(2,b)\}$
- $S \circ R = \{(1,b),(3,a),(3,b)\}$

Implementation of composite

Examples:

- Let $A = \{1,2\}$, $B = \{1,2,3\}$ $C = \{a,b\}$
- $R = \{(1,2), (1,3), (2,1)\}$ is a relation from A to B
- $S = \{(1,a), (3,b), (3,a)\}$ is a relation from B to C .
- $S \circ R = \{(1,b), (1,a), (2,a)\}$

$$M_R = \begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{matrix} \quad M_S = \begin{matrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{matrix}$$
$$M_R \odot M_S = ?$$

Implementation of composite

Examples:

- Let $A = \{1,2\}$, $\{1,2,3\}$ $C = \{a,b\}$
- $R = \{(1,2),(1,3),(2,1)\}$ is a relation from A to B
- $S = \{(1,a),(3,b),(3,a)\}$ is a relation from B to C.
- $S \circ R = \{(1,b),(1,a),(2,a)\}$

$$M_R = \begin{matrix} & \begin{matrix} 0 & 1 & 1 \end{matrix} \\ \begin{matrix} 0 \\ 1 \end{matrix} & \begin{matrix} 1 & 0 & 0 \end{matrix} \end{matrix} \quad M_S = \begin{matrix} & \begin{matrix} 1 & 0 \end{matrix} \\ \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} & \begin{matrix} 0 & 0 & 1 \end{matrix} \end{matrix}$$
$$M_R \odot M_S = \begin{matrix} & \begin{matrix} x & x \end{matrix} \\ \begin{matrix} x \\ x \end{matrix} & \begin{matrix} x & x \end{matrix} \end{matrix}$$

Implementation of composite

Examples:

- Let $A = \{1,2\}, \{1,2,3\}$ $C = \{a,b\}$
- $R = \{(1,2), (1,3), (2,1)\}$ is a relation from A to B
- $S = \{(1,a), (3,b), (3,a)\}$ is a relation from B to C.
- $S \circ R = \{(1,b), (1,a), (2,a)\}$

$$M_R = \begin{matrix} & \boxed{0} & 1 & 1 \\ \begin{matrix} 1 & 0 & 0 \end{matrix} & & & \end{matrix} \quad M_S = \begin{matrix} & \boxed{1} & 0 \\ 0 & 0 & \\ 1 & 1 & \end{matrix}$$
$$M_R \odot M_S = \begin{matrix} & \textcolor{red}{1} & x \\ x & x & \end{matrix}$$

Implementation of composite

Examples:

- Let $A = \{1,2\}$, $\{1,2,3\}$ $C = \{a,b\}$
- $R = \{(1,2),(1,3),(2,1)\}$ is a relation from A to B
- $S = \{(1,a),(3,b),(3,a)\}$ is a relation from B to C.
- $S \circ R = \{(1,b),(1,a),(2,a)\}$

$$M_R = \begin{matrix} & \boxed{\begin{matrix} 0 & 1 & 1 \end{matrix}} \\ \begin{matrix} 0 & 1 & 0 \end{matrix} & \end{matrix} \quad M_S = \begin{matrix} & \begin{matrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{matrix} \\ \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{matrix} & \end{matrix}$$
$$M_R \odot M_S = \begin{matrix} & \begin{matrix} 1 & \textcolor{red}{1} \\ x & x \end{matrix} \\ \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} & \end{matrix}$$

Implementation of composite

Examples:

- Let $A = \{1,2\}, \{1,2,3\}$ $C = \{a,b\}$
- $R = \{(1,2), (1,3), (2,1)\}$ is a relation from A to B
- $S = \{(1,a), (3,b), (3,a)\}$ is a relation from B to C.
- $S \circ R = \{(1,b), (1,a), (2,a)\}$

$$M_R = \begin{matrix} & 0 & 1 & 1 \\ \boxed{1} & & 0 & 0 \end{matrix} \quad M_S = \begin{matrix} & 1 & 0 \\ 0 & & 0 \\ & 1 & 1 \end{matrix}$$
$$M_R \odot M_S = \begin{matrix} & 1 & 1 \\ \textcolor{red}{1} & & x \end{matrix}$$

Implementation of composite

Examples:

- Let $A = \{1,2\}$, $\{1,2,3\}$ $C = \{a,b\}$
- $R = \{(1,2),(1,3),(2,1)\}$ is a relation from A to B
- $S = \{(1,a),(3,b),(3,a)\}$ is a relation from B to C.
- $S \circ R = \{(1,b),(1,a),(2,a)\}$

$$M_R = \begin{matrix} & \begin{matrix} 0 & 1 & 1 \end{matrix} \\ \begin{matrix} 0 \\ 1 \end{matrix} & \boxed{\begin{matrix} 1 & 0 & 0 \end{matrix}} \end{matrix} \quad M_S = \begin{matrix} & \begin{matrix} 1 & 0 \end{matrix} \\ \begin{matrix} 1 \\ 0 \\ 1 \end{matrix} & \boxed{\begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix}} \end{matrix}$$

$$M_R \odot M_S = \begin{matrix} & \begin{matrix} 1 & 1 \\ 1 & \textcolor{red}{0} \end{matrix} \end{matrix}$$

$$M_{S \circ R} = ?$$

Implementation of composite

Examples:

- Let $A = \{1,2\}$, $\{1,2,3\}$ $C = \{a,b\}$
- $R = \{(1,2),(1,3),(2,1)\}$ is a relation from A to B
- $S = \{(1,a),(3,b),(3,a)\}$ is a relation from B to C.
- $S \circ R = \{(1,b),(1,a),(2,a)\}$

$$M_R = \begin{matrix} 0 & 1 & 1 \end{matrix} \quad M_S = \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix}$$

$$M_R \odot M_S = \begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}$$

$$M_{S \circ R} = \begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}$$

$$M_{S \circ R} = \begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}$$

Composite of relations

Definition: Let R be a relation on a set A . The **powers R^n** , $n = 1, 2, 3, \dots$ is defined inductively by

- $R^1 = R$ and $R^{n+1} = R^n \circ R$.

Examples

- $R = \{(1,2), (2,3), (2,4), (3,3)\}$ is a relation on $A = \{1,2,3,4\}$.
- $R^1 = ?$

Composite of relations

Definition: Let R be a relation on a set A . The **powers** R^n , $n = 1, 2, 3, \dots$ is defined inductively by

- $R^1 = R$ and $R^{n+1} = R^n \circ R$.

Examples

- $R = \{(1,2), (2,3), (2,4), (3,3)\}$ is a relation on $A = \{1,2,3,4\}$.
- $R^1 = R = \{(1,2), (2,3), (2,4), (3,3)\}$
- $R^2 = ?$

Composite of relations

Definition: Let R be a relation on a set A . The **powers R^n** , $n = 1, 2, 3, \dots$ is defined inductively by

- $R^1 = R$ and $R^{n+1} = R^n \circ R$.

Examples

- $R = \{(1,2), (2,3), (2,4), (3,3)\}$ is a relation on $A = \{1,2,3,4\}$.
- $R^1 = R = \{(1,2), (2,3), (2,4), (3,3)\}$
- $R^2 = \{(1,3), (1,4), (2,3), (3,3)\}$
- $R^3 = ?$

Composite of relations

Definition: Let R be a relation on a set A . The **powers R^n** , $n = 1, 2, 3, \dots$ is defined inductively by

- $R^1 = R$ and $R^{n+1} = R^n \circ R$.

Examples

- $R = \{(1,2), (2,3), (2,4), (3,3)\}$ is a relation on $A = \{1,2,3,4\}$.
- $R^1 = R = \{(1,2), (2,3), (2,4), (3,3)\}$
- $R^2 = \{(1,3), (1,4), (2,3), (3,3)\}$
- $R^3 = \{(1,3), (2,3), (3,3)\}$
- $R^4 = ?$

Composite of relations

Definition: Let R be a relation on a set A . The **powers R^n** , $n = 1, 2, 3, \dots$ is defined inductively by

$$\bullet \quad R^1 = R \quad \text{and} \quad R^{n+1} = R^n \circ R.$$

Examples

- $R = \{(1,2), (2,3), (2,4), (3,3)\}$ is a relation on $A = \{1,2,3,4\}$.
- $R^1 = R = \{(1,2), (2,3), (2,4), (3,3)\}$
- $R^2 = \{(1,3), (1,4), (2,3), (3,3)\}$
- $R^3 = \{(1,3), (2,3), (3,3)\}$
- $R^4 = \{(1,3), (2,3), (3,3)\}$
- $R^k = ?$, $k > 3$.

Composite of relations

Definition: Let R be a relation on a set A . The **powers R^n** , $n = 1, 2, 3, \dots$ is defined inductively by

- $R^1 = R$ and $R^{n+1} = R^n \circ R$.

Examples

- $R = \{(1,2), (2,3), (2,4), (3,3)\}$ is a relation on $A = \{1,2,3,4\}$.
- $R^1 = R = \{(1,2), (2,3), (2,4), (3,3)\}$
- $R^2 = \{(1,3), (1,4), (2,3), (3,3)\}$
- $R^3 = \{(1,3), (2,3), (3,3)\}$
- $R^4 = \{(1,3), (2,3), (3,3)\}$
- $R^k = R^3, k > 3.$

Transitive relation

Definition (transitive relation): A relation R on a set A is called **transitive** if

- $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R \text{ for all } a, b, c \in A.$
- **Example 1:**
- $R_{\text{div}} = \{(a|b), \text{ if } a|b\} \text{ on } A = \{1,2,3,4\}$
- $R_{\text{div}} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$
- **Is R_{div} transitive?**
- **Answer:** ?

Transitive relation

Definition (transitive relation): A relation R on a set A is called **transitive** if

- $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R \text{ for all } a, b, c \in A.$
- **Example 1:**
- $R_{\text{div}} = \{(a|b), \text{ if } a|b\} \text{ on } A = \{1,2,3,4\}$
- $R_{\text{div}} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$
- **Is R_{div} transitive?**
- **Answer:** Yes.

Connection to R^n

Theorem: The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \dots$.

Proof: biconditional (if and only if)

(\leftarrow) Suppose $R^n \subseteq R$, for $n = 1, 2, 3, \dots$.

- Let $(a, b) \in R$ and $(b, c) \in R$
- by the definition of $R \circ R$, $(a, c) \in R \circ R = R^2 \subseteq R$
- Therefore R is transitive.

Connection to R^n

Theorem: The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \dots$.

Proof: biconditional (if and only if)

(\rightarrow) Suppose R is transitive. Show $R^n \subseteq R$, for $n = 1, 2, 3, \dots$.

- Let $P(n) : R^n \subseteq R$. Mathematical induction.
- **Basis Step:**

Connection to R^n

Theorem: The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \dots$.

Proof: biconditional (if and only if)

(\rightarrow) Suppose R is transitive. Show $R^n \subseteq R$, for $n = 1, 2, 3, \dots$.

- Let $P(n) : R^n \subseteq R$. Mathematical induction.
- **Basis Step:** $P(1)$ says $R^1 = R$ so, $R^1 \subseteq R$ is true.

Connection to \mathbf{R}^n

Theorem: The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \dots$.

Proof: biconditional (if and only if)

(\rightarrow) Suppose R is transitive. Show $R^n \subseteq R$, for $n = 1, 2, 3, \dots$.

- Let $P(n) : R^n \subseteq R$. Mathematical induction.
- **Basis Step:** $P(1)$ says $R^1 = R$ so, $R^1 \subseteq R$ is true.
- **Inductive Step:** show $P(n) \rightarrow P(n+1)$
- Want to show if $R^n \subseteq R$ then $R^{n+1} \subseteq R$.

Connection to R^n

Theorem: The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \dots$.

Proof: biconditional (if and only if)

(\rightarrow) Suppose R is transitive. Show $R^n \subseteq R$, for $n = 1, 2, 3, \dots$.

- Let $P(n) : R^n \subseteq R$. Mathematical induction.
- **Basis Step:** $P(1)$ says $R^1 = R$ so, $R^1 \subseteq R$ is true.
- **Inductive Step:** show $P(n) \rightarrow P(n+1)$
- Want to show if $R^n \subseteq R$ then $R^{n+1} \subseteq R$.
- Let $(a, b) \in R^{n+1}$ then by the definition of $R^{n+1} = R^n \circ R$ there is an element $x \in A$ so that $(a, x) \in R$ and $(x, b) \in R^n \subseteq R$ (inductive hypothesis). In addition to $(a, x) \in R$ and $(x, b) \in R$, R is transitive; so $(a, b) \in R$.
- Therefore, $R^{n+1} \subseteq R$.

Number of reflexive relations

Theorem: The number of reflexive relations on a set A, where $|A| = n$ is: $2^{n(n-1)}$.

Proof:

- A reflexive relation R on A **must contain** all pairs (a,a) where $a \in A$.
- All other pairs in R are of the form (a,b) , $a \neq b$, such that $a, b \in A$.
- How many of these pairs are there? Answer: $n(n-1)$.
- How many subsets on $n(n-1)$ elements are there?
- **Answer:** $2^{n(n-1)}$.

