Discrete Probability_02

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Discrete Probability

1. An Introduction to Discrete Probability

2. Probability Theory

3. Bayes’ Theorem

4. Expected Value and Variance

Bayes theorem

Definition: Let E and F be two events such that $P(F) > 0$. Then:

- $P(E| F) = P(F| E)P(E) / P(F)$

  $= P(F|E)P(E) / (P(F|E)P(E) + P(F|\sim E)P(\sim E))$

Proof:

$P(E| F) = P(E \cap F) / P(F)$

$= P(F| E) P(E) / P(F)$

$P(F) = P(F \cap E) + P(F \cap \sim E)$

$= P(F| E) P(E) + P(F| \sim E) P(\sim E)$

Hence:

$P(E| F) = P(F|E)P(E) / (P(F|E)P(E) + P(F|\sim E)P(\sim E))$

Idea: Simply switch the conditioning events.

Bayes theorem

Definition: Let E and F be two events such that P(F) > 0. Then:

- $P(E|F) = P(F|E)P(E) / P(F)$

  $= P(F|E)P(E) / (P(F|E)P(E) + P(F|{\sim}E)P({\sim}E))$

Example:

- Assume the probability of getting a flu is 0.2
- Assume the probability of getting a fever is 0.3
- Assume the probability of having a high fever given the flu: 0.9
- What is the probability of having a flu given the fever?

Bayes theorem

Definition: Let E and F be two events such that $P(F) > 0$. Then:

- $P(E|F) = P(F|E)P(E) / P(F)$

  $= P(F|E)P(E) / (P(F|E)P(E) + P(F|\sim E)P(\sim E))$

Example:

- Assume the probability of getting a flu is 0.2
- Assume the probability of getting a fever is 0.3
- Assume the probability of having a fever given the flu: 0.9
- What is the probability of having a flu given the fever?
- P(flu | fever) = P(fever | flu) P(flu) / P(fever) =

  = 0.9 x 0.2/0.3 = 0.18/0.3 = 0.6

Bayes theorem

Definition: Let E and F be two events such that $P(F) > 0$. Then:

- $P(E|F) = P(F|E)P(E) / P(F)$

  $= P(F|E)P(E) / (P(F|E)P(E) + P(F|\sim E)P(\sim E))$

Example (same as above but different probabilities are given):

- Assume the probability of getting a flu is 0.2
- Assume the probability of having a fever given the flu: 0.9
- Assume the probability of having a fever given no flue: 0.15
- What is the probability of having a flu given the fever?

Bayes theorem

Definition: Let E and F be two events such that P(F) > 0. Then:

- P(E| F) = P(F| E)P(E) / P(F)

  = P(F|E)P(E) / (P(F|E)P(E) + P(F|~E)P(~E))

Example:

- Assume the probability of getting a flu is 0.2
- Assume the probability of having a fever given the flu: 0.9
- Assume the probability of having a fever given no flue: 0.15
- What is the probability of having a flu given the fever?
- P(flu | fever) = P(fever | flu) P(flu) / P(fever)
- P(fever) = P(fever | flu) P(flu) + P(fever | ~flu) P(~flu)

  = 0.9*0.2 + 0.15*0.8 = 0.3

P(flu | fever) **= 0.9 x 0.2/0.3 = 0.18/0.3 = 0.6**

Bayes' theorem

This is related to conditional probability. We can make a realistic estimate when some extra information is available.

Problem 1.
There are two boxes.
Bob first chooses one of the two boxes at random.
He then selects one of the balls in this box at random.

If Bob **has selected a red ball**, what is the probability that ***he selected a ball from the first box?***

(See page 469 of your textbook)


Box 1


Box 2

Bayes' theorem

Let E= Bob chose a red ball. So E' = Bob chose a green ball
F= Bob chose from Box 1. So F' = Bob chose from Box 2
We have to compute p(F|E)

p(E|F) = 7/9, p(E|F') = 3/7

We have to find $p(F \mid E) = \dfrac{p(F \cap E)}{p(E)}$

p(F) = p(F') = 1/2

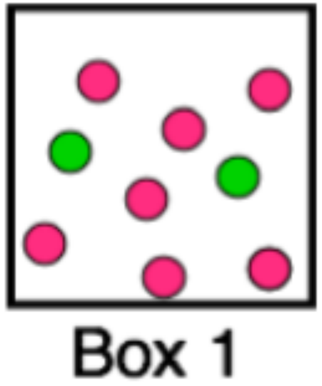
$$p(E \cap F) = p(E \mid F).p(F) = (7/9).(1/2) = 7/18$$

$$p(E \cap F') = p(E \mid F').p(F') = (3/7).(1/2) = 3/14$$

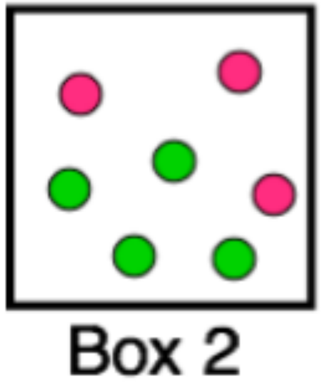
Bayes' theorem

$$p(E) = p(E \cap F) + p(E \cap F^{'}) == 7/18 + 3/14 = 38/63$$

$$p(F \mid E) = \frac{p(F \cap E)}{p(E)} = \frac{7/18}{38/63} = \frac{49}{76}$$



This is the probability that
Bob chose the ball from Box 1



Bayes' theorem

Let E and F be events from a sample space S such that p(E) ≠ 0 and p(F) ≠ 0. Then

given

$$p(F \mid E) = \frac{p(E \mid F).p(F)}{p(E \mid F)p(F) + p(E \mid \bar{F}).p(\bar{F})}$$

Compute this

Bayes' theorem

Problem 2

1. Suppose that one person in 100,000 has a particular rare disease for which there is a fairly accurate diagnostic test.

2. This test is correct 99.0% of the time when given to a person selected at random who has the disease;

3. The test is correct 99.5% of the time when given to a person selected at random who does not have the disease.

Find the probability that a person who tests positive for the disease really has the disease. (See page 471 of your textbook)

Bayes' theorem

- ✓ 1 in 100,000 has the rare disease (1)
- ✓ This test is 99.0% correct if actually infected; (2)
- ✓ The test is 99.5% correct if not infected (3)

Let F = event that a randomly chosen person has the disease
and E = event that a randomly chosen person tests positive

So, $p(F)= 0.00001$, $p(F') = 0.99999$ {from (1)}
Also, $p(E|F) = 0.99$, and $p(E'|F) = 1- 0.99 = 0.01$ {from (2)}
Also $p(E'|F') = 0.995$, and $p(E|F') = 1- 0.995 = 0.005$ {from (3)}

Bayes' theorem

$$p(F \mid E) = \frac{p(E \mid F).p(F)}{p(E \mid F)p(F) + p(E \mid \bar{F}).p(\bar{F})}$$

$$= \frac{0.99 \times 0.00001}{0.99 \times 0.00001 + 0.005 \times 0.99999} \simeq 0.002$$

So, the probability that a person "who tests positive for the disease" really has the disease is only 0.2%

Exercises

1. Suppose that E and F are events in a sample space and $p(E) = 1/3$, $p(F) = 1/2$, and $p(E \mid F) = 2/5$. Find $p(F \mid E)$

2. Suppose that Frida selects a ball by first picking one of two boxes at random and then selecting a ball from this box at random. The first box contains two white balls and three blue balls, and the second box contains four white balls and one blue ball. What is the probability that Frida picked a ball from the first box if she has selected a blue ball?

3. Suppose that 8% of all bicycle racers use steroids, that a bicyclist who uses steroids tests positive for steroids 96% of the time, and that a bicyclist who does not use steroids tests positive for steroids 9% of the time. What is the probability that a randomly selected bicyclist who tests positive for steroids actually uses steroids?

Exercises

1. Suppose that E and F are events in a sample space and $p(E) = 1/3$, $p(F) = 1/2$, and $p(E \mid F) = 2/5$. Find $p(F \mid E)$

Exercises

2. Suppose that Frida selects a ball by first picking one of two boxes at random and then selecting a ball from this box at random. The first box contains two white balls and three blue balls, and the second box contains four white balls and one blue ball. What is the probability that Frida picked a ball from the first box if she has selected a blue ball?

Exercises

3. Suppose that 8% of all bicycle racers use steroids, that a bicyclist who uses steroids tests positive for steroids 96% of the time, and that a bicyclist who does not use steroids tests positive for steroids 9% of the time. What is the probability that a randomly selected bicyclist who tests positive for steroids actually uses steroids?

Random variables

- **Definition: A random variable** is a function from the **sample space of an experiment** to the set of real numbers f: S → R. A random variable assigns a number to each possible outcome.

- **The distribution of a random variable X on the sample space S** is a set of pairs (r p(X=r)) for all r in S where r is the number and p(X=r) is the probability that X takes a value r.

Random variables

Example:

Let S be the outcomes of a two-dice roll

Let random variable X denotes the sum of outcomes

(1,1) → 2

(1,2) and (2,1) → 3

(1,3), (3,1) and (2,2) → 4

…

Distribution of X:

- 2 → 1/36,
- 3 → 2/36,
- 4 → 3/36 …
- 12 → 1/36

Expected value and variance

Definition: The expected value of the random variable X(s) on the sample space is equal to:

$$E(X) = \sum_{s \in S} p(s) X(s)$$

Example: roll of a dice

- Outcomes: 1 2 3 4 5 6
- Expected value:

E(X) = ?

Expected value and variance

Definition: The expected value of the random variable X(s) on the sample space is equal to:

$$E(X) = \sum_{s \in S} p(s) X(s)$$

Example: roll of a dice

- Outcomes: 1 2 3 4 5 6
- Expected value:

  E(X) = 1*1/6 + 2*1/6+3*1/6 + 4*1/6 + 5*1/6 +6* 1/6 = 7/2

Expected value

Example:

Flip a fair coin 3 times. The outcome of the trial X is the number of heads. What is the expected value of the trial?

Answer:

Possible outcomes:

= {HHH HHT HTH THH HTT THT TTH TTT}

3	**2**	**2**	**2**	**1**	**1**	**1**	**0**

E(X) = ?

Expected value

Example:

Flip a fair coin 3 times. The outcome of the trial X is the number of heads. What is the expected value of the trial?

Answer:

Possible outcomes:

= {HHH	**HHT**	**HTH**	**THH**	**HTT**	**THT**	**TTH**	**TTT}**
3	**2**	**2**	**2**	**1**	**1**	**1**	**0**

E(X) = 1/8 (3 + 3*2 +3*1 +0) = 12/8= 3/2

Expected value

- **Theorem:** If Xi i=1,2,3, n with n being a positive integer, are random variables on S, and a and b are real numbers then:
  - E(X1+X2+ ... Xn) = E(X1)+E(X2) + ... E(Xn)
  - E(aX+b) = aE(X) +b

Expected value

Example:

- Roll a pair of dices. What is the expected value of the sum of outcomes?
- **Approach 1:**
- Outcomes: (1,1) (1,2) (1,3) .... (6,1)... (6,6)

  2 3 4 7 12

Expected value: 1/36 (2*1 + ...) = 7

- Approach 2 (theorem):
- E(X1+X2) = E(X1) +E(X2)
- E(X1) =7/2 E(X2) = 7/2
- E(X1+X2) = 7

Expected value

Definition: The expected value of the random variable X(s) on the sample space is equal to:

$$E(X)=\sum_{s\in S} p(s)X(s)$$

Example: roll of a dice

- Outcomes: 1 2 3 4 5 6
- Expected value:

  E(X) = 1*1/6 + 2*1/6+3*1/6 + 4*1/6 + 5*1/6 +6* 1/6 = 7/2

Expected value

Investment problem:

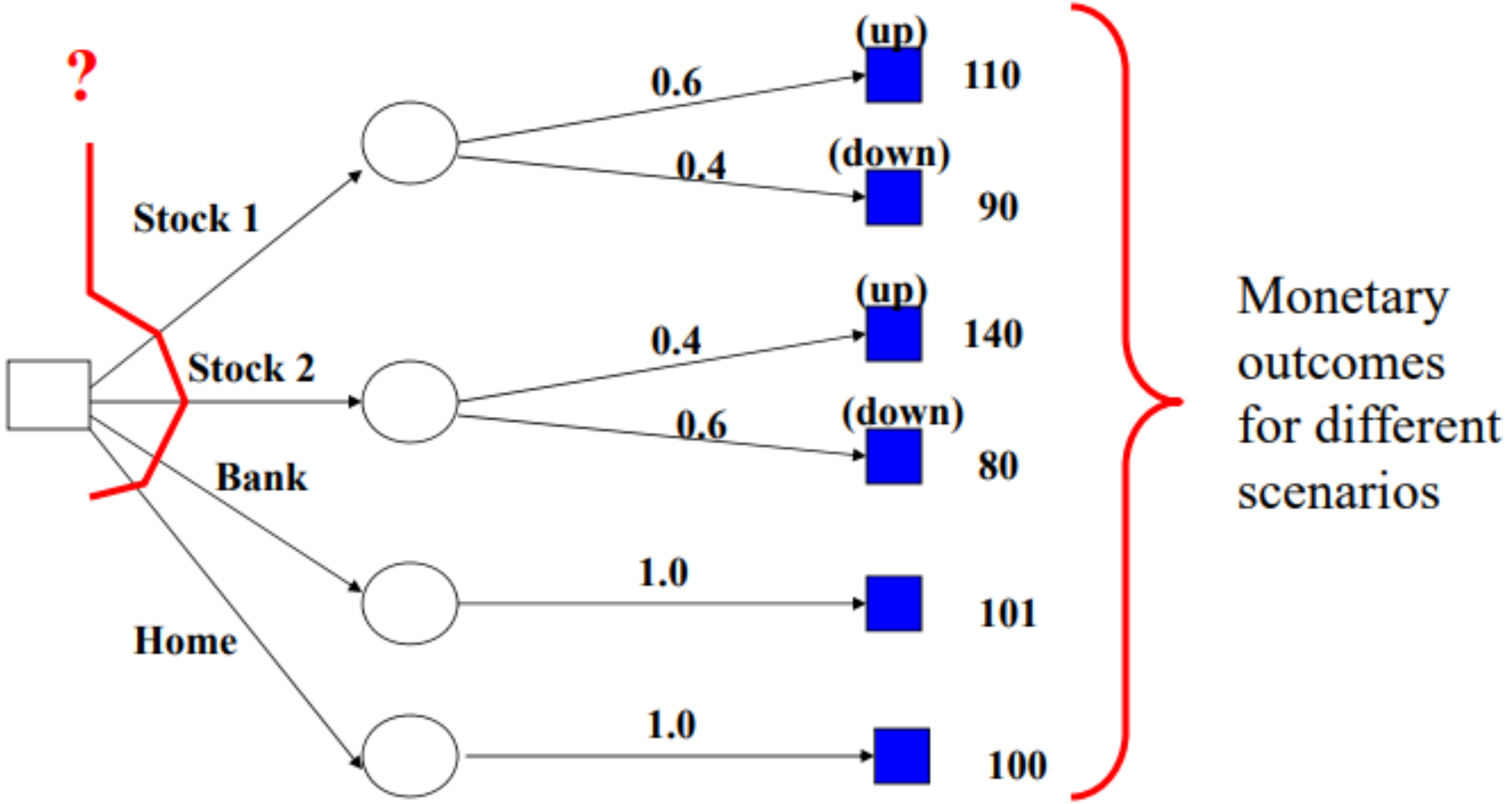
- You have 100 dollars and can invest into a stock. The returns are volatile and you may get either \$120 with probability of 0.4, or \$90 with probability 0.6.
- **What is the expected value of your investment?**
- $E(X) = 0.4*120+0.6*90=48+54=102$
- **Is it OK to invest?**

EXAMPLE 2

A fair coin is flipped three times. Let S be the sample space of the eight possible outcomes, and let X be the random variable that assigns to an outcome the number of heads in this outcome. What is the expected value of X?

Decision making example

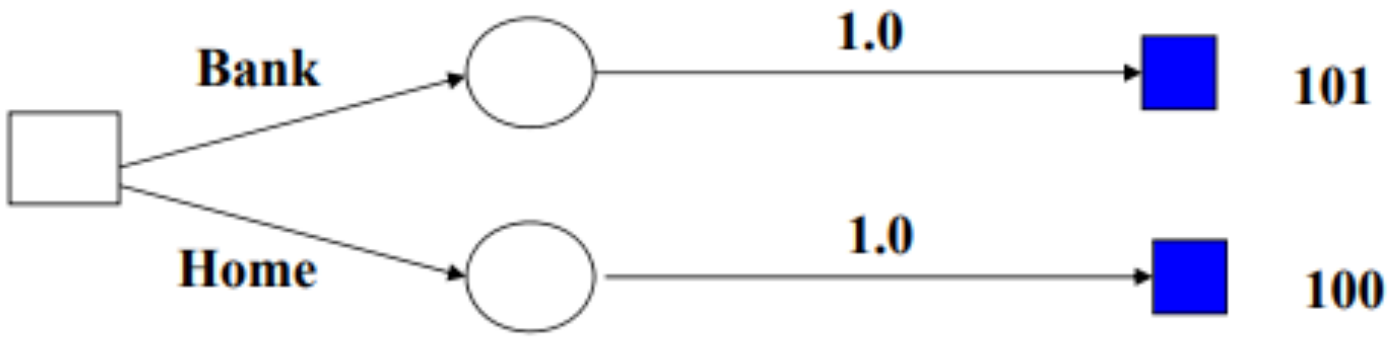
We need to make a choice whether to invest in Stock 1 or 2, put money into bank or keep them at home.



Decision making example.

Assume the simplified problem with the Bank and Home choices only.

The result is guaranteed – the outcome is deterministic

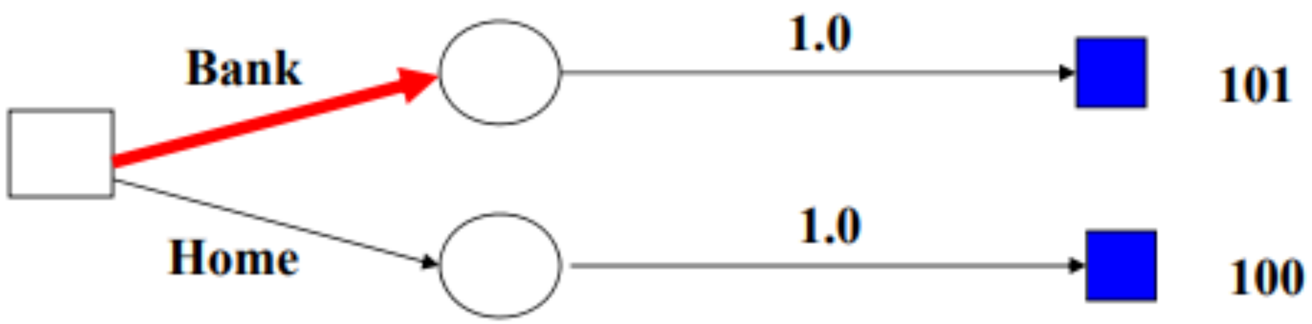


What is the rational choice assuming our goal is to make money?

Decision making. Deterministic outcome.

Assume the simplified problem with the Bank and Home choices only.

These choices are deterministic.

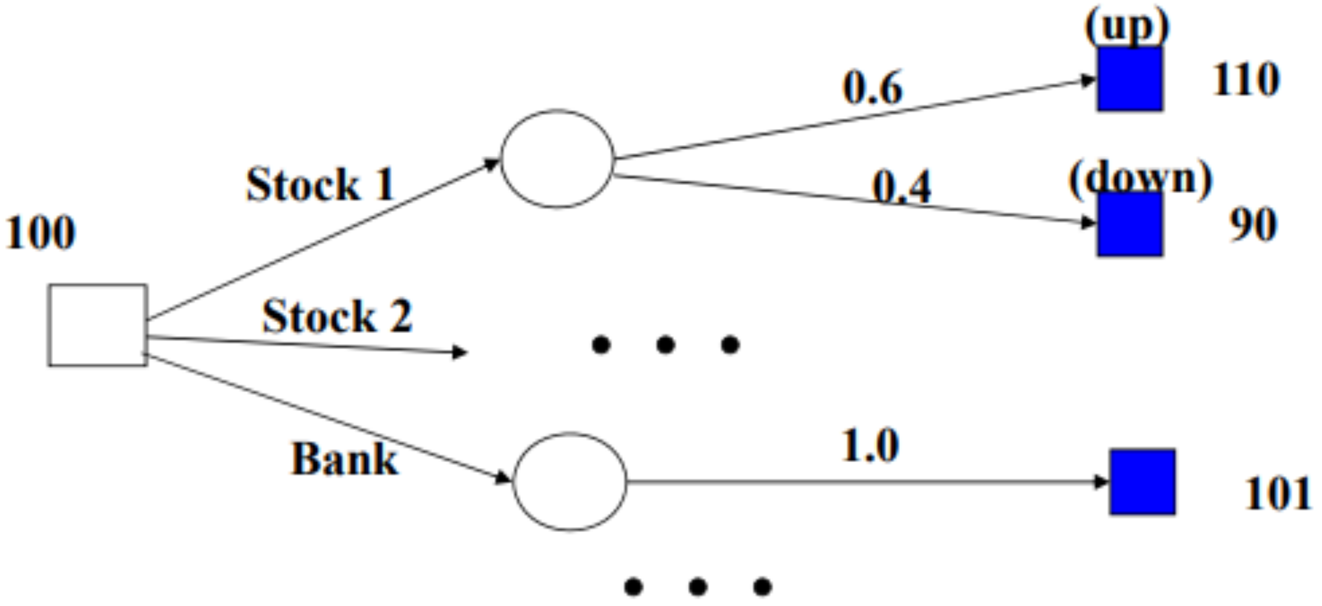


Our goal is to make money. What is the rational choice?

Answer: Put money into the bank. The choice is always strictly better in terms of the outcome

Decision making

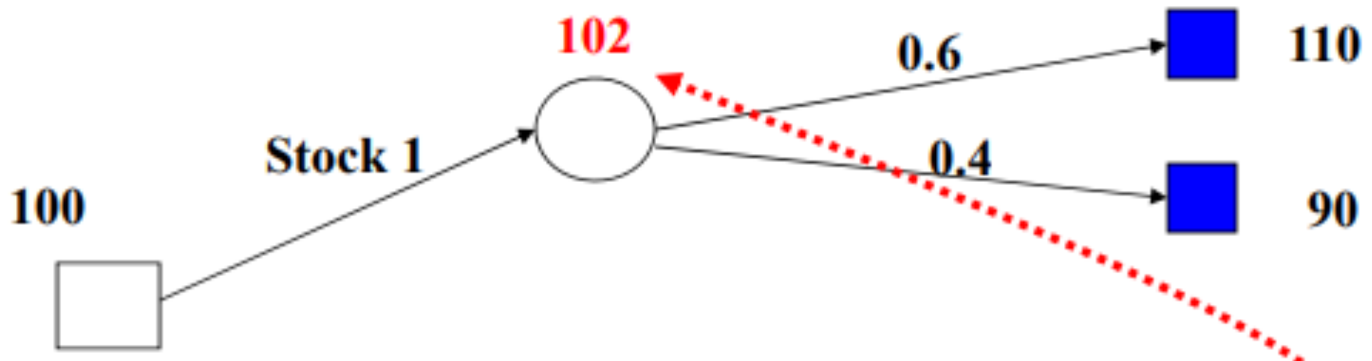
- How to quantify the goodness of the stochastic outcome?
  We want to compare it to deterministic and other stochastic outcomes.



Idea: Use the expected value of the outcome

Expected value

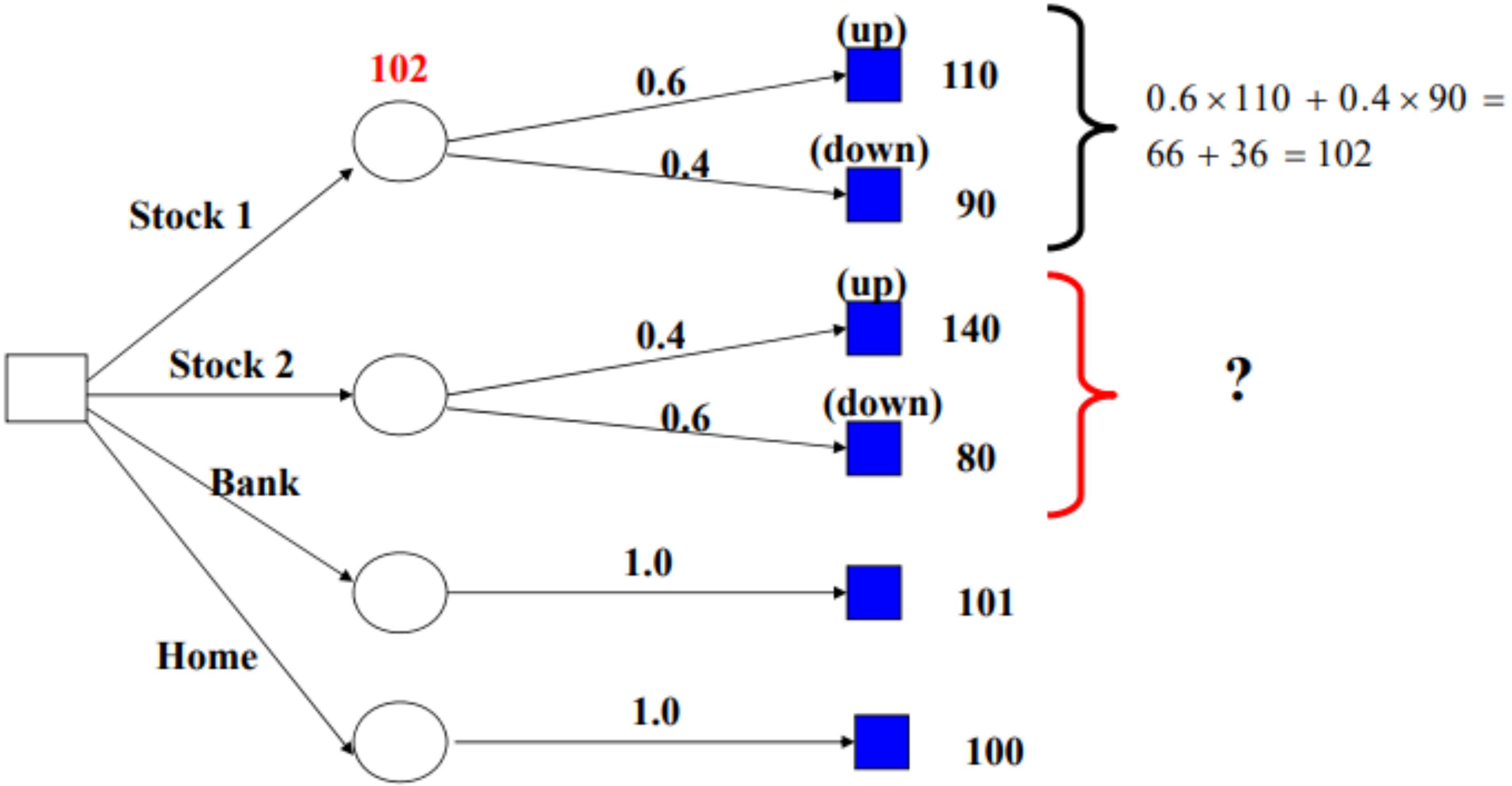
- **Expected value** summarizes all stochastic outcomes into a single quantity
  - Expected value for the outcome of the Stock 1 option is:



$$0.6 \times 110 + 0.4 \times 90 = 66 + 36 = \boxed{102}$$

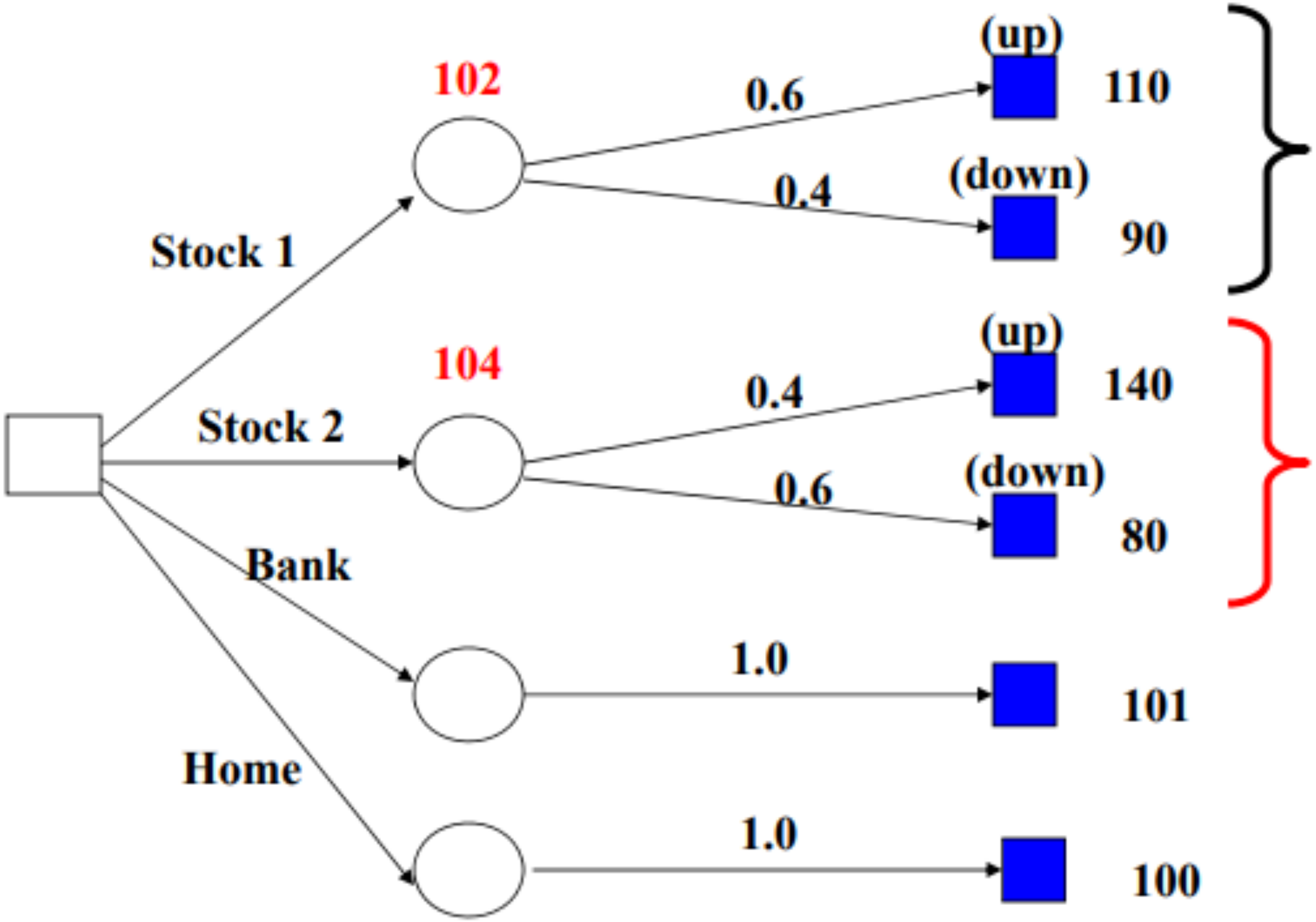
Expected values

Investing $100 for 6 months



Expected values

Investing $100 for 6 months

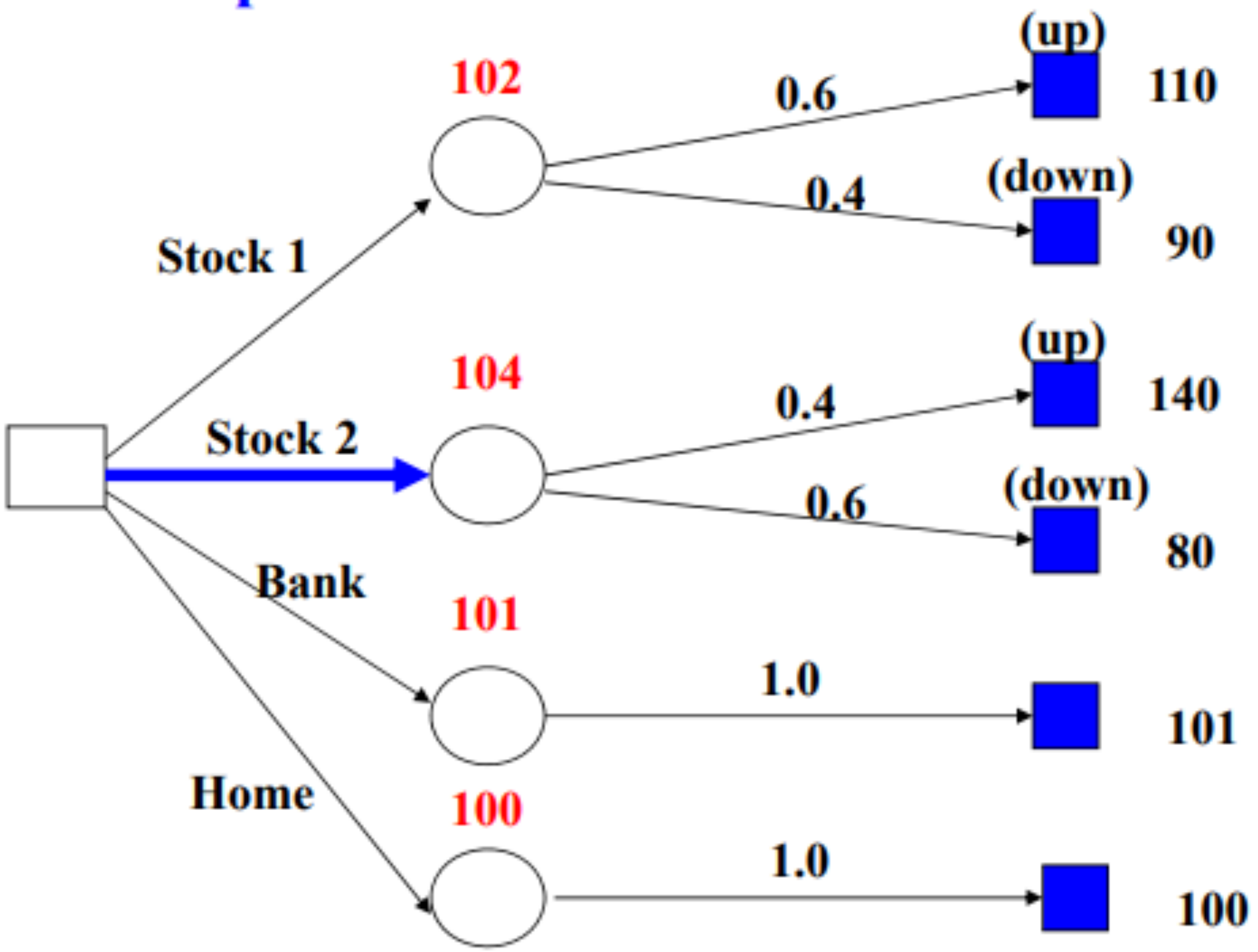


$$0.6 \times 110 + 0.4 \times 90 =$$
$$66 + 36 = 102$$

$$0.4 \times 140 + 0.6 \times 80 =$$
$$56 + 48 = 104$$

Selection based on expected values

The optimal action is the option that maximizes the expected outcome:



Variance of a Discrete Random Variable

The expected value of a random variable tells us its average value, but nothing about how widely its values are distributed.

For example, if X and Y are the random variables on the set $S = \{1, 2, 3, 4, 5, 6\}$,
with $X(s) = 0$ for all $s \in S$ and
$Y(s) = -1$ if $s \in \{1, 2, 3\}$ and
$Y(s) = 1$ if $s \in \{4, 5, 6\}$,
then the expected values of X and Y are both zero.

However, the random variable X never varies from 0, while the random variable Y always differs from 0 by 1.

The variance of a random variable helps us characterize how widely a random variable is distributed.

Variance of a Discrete Random Variable

The variance of a discrete random variable is given by:

$$\sigma^2 = \mathrm{Var}(X) = \sum (x_i - \mu)^2 f(x_i)$$

The formula means that we take each value of x, subtract the expected value, square that value and multiply that value by its probability. Then sum all of those values.

There is an easier form of this formula we can use.

$$\sigma^2 = \mathrm{Var}(X) = \sum x_i^2 f(x_i) - E(X)^2 = \sum x_i^2 f(x_i) - \mu^2$$

The formula means that first, we sum the square of each value times its probability then subtract the square of the mean. We will use this form of the formula in all of our examples.

Standard Deviation of a Discrete Random Variable

The standard deviation of a random variable, X, is the square root of the variance.

$$\sigma = \mathrm{SD}(X) = \sqrt{\mathrm{Var}}(X) = \sqrt{\sigma^2}$$

Standard Deviation of a Discrete Random Variable

Example : Standard Deviation

Consider the first example where we had the values 0, 1, 2, 3, 4. The PMF in tabular form was:

x	0	1	2	3	4
$f(x)$	1/5	1/5	1/5	1/5	1/5

Find the variance and the standard deviation of X.

Answer

$$\text{Var}(X) = \left[0^2\left(\frac{1}{5}\right) + 1^2\left(\frac{1}{5}\right) + 2^2\left(\frac{1}{5}\right) + 3^2\left(\frac{1}{5}\right) + 4^2\left(\frac{1}{5}\right)\right] - 2^2 = 6 - 4 = 2$$

$$\text{SD}(X) = \sqrt{2} \approx 1.4142$$

Standard Deviation of a Discrete Random Variable

Example : Prior Convictions

Number of prior convictions

Let X = number of prior convictions for prisoners at a state prison at which there are 500 prisoners. ($x = 0, 1, 2, 3, 4$)

$X = x$	0	1	2	3	4
$Number\ of\ Prisoners$	80	265	100	40	15
$f(x) = P(X = x)$	80/500	265/500	100/500	40/500	15/500
$f(x) = P(X = x)$	0.16	0.53	0.2	0.08	0.03

Standard Deviation of a Discrete Random Variable

Example : Prior Convictions

What is the expected value for number of prior convictions?

$X = x$	0	1	2	3	4
$Number\ of\ Prisoners$	80	265	100	40	15
$f(x) = P(X = x)$	80/500	265/500	100/500	40/500	15/500
$f(x) = P(X = x)$	0.16	0.53	0.2	0.08	0.03

Answer

For this we need a weighted average since not all the outcomes have equal chance of happening (i.e. they are not equally weighted). So, we need to find our expected value of X, or mean of X, or $E(X) = \Sigma f(x_i)(x_i)$. When we write this out it follows:

$$= (0.16)(0) + (0.53)(1) + (0.2)(2) + (0.08)(3) + (0.03)(4) = 1.29$$

Standard Deviation of a Discrete Random Variable

Example : Prior Convictions

Calculate the variance and the standard deviation for the Prior Convictions example:

$X = x$	0	1	2	3	4
$Number\ of\ Prisoners$	80	265	100	40	15
$f(x) = P(X = x)$	80/500	265/500	100/500	40/500	15/500
$f(x) = P(X = x)$	0.16	0.53	0.2	0.08	0.03

Answer

Using the data in our example we find that...

$$\begin{aligned} \mathrm{Var}(X) &= [0^2(0.16) + 1^2(0.53) + 2^2(0.2) + 3^2(0.08) + 4^2(0.03)]–(1.29)^2 \\ &= 2.53–1.66 \\ &= 0.87 \\ \mathrm{SD}(X) &= \sqrt{(0.87)} \\ &= 0.93 \end{aligned}$$

Assignment 4 : Discrete Probability

1. Find the probability of winning a lottery by selecting the correct six integers, where the order in which these integers are selected does not matter, from the positive integers not exceeding
a) 50. b) 52. c) 56. d) 60.

2. A pair of dice is loaded. The probability that a 4 appears on the first die is 2/7, and the probability that a 3 appears on the second die is 2/7. Other outcomes for each die appear with probability 1/7. What is the probability of 7 appearing as the sum of the numbers when the two dice are rolled?

3. What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up heads?

4. Suppose that 8% of all bicycle racers use steroids, that a bicyclist who uses steroids tests positive for steroids 96% of the time, and that a bicyclist who does not use steroids tests positive for steroids 9% of the time. What is the probability that a randomly selected bicyclist who tests positive for steroids actually uses steroids?

5. What is the expected sum of the numbers that appear on two dice, each biased so that a 3 comes up twice as often as each other number?