Counting

Puttachat Khuntontong

How many combination types we can have?

Counting

- Assume we have a set of objects with **certain properties**
- Counting is used to determine **the number of these objects**

Example

Number of available phone numbers with 10 digits in the local calling area

Number of possible football match starters given the number of team members and their positions

Counting

- Basic Counting
  - Product rule
  - Sum rule
  - More complex counting problems
  - Inclusion-Exclusion principle
  - Tree Diagrams
- Pigeonhole Principle

Basic counting rules

- **Basic Counting**
  - Product rule
  - Sum rule
  - More complex counting problems
  - Inclusion-Exclusion principle
  - Tree Diagrams
- Pigeonhole Principle

- **Counting problems may be very hard, not obvious**

- **Solution:**

– simplify the solution by decomposing the problem

Basic counting rules

- **Basic Counting**
  - Product rule
  - Sum rule
  - More complex counting problems
  - Inclusion-Exclusion principle
  - Tree Diagrams
- Pigeonhole Principle

- **Two basic decomposition rules:**
  - **Product rule**
    - A count decomposes into a sequence of dependent counts ("each element in the first count is associated with all elements of the second count")
  - **Sum rule**
    - A count decomposes into a set of independent counts ("elements of counts are alternatives")

Product rule

- **Basic Counting**
    - Product rule
    - Sum rule
    - More complex counting problems
    - Inclusion-Exclusion principle
    - Tree Diagrams
- Pigeonhole Principle

A count can be broken down into a sequence of dependent counts

- **"each element in the first count is associated with all elements of the second count"**

Example:

- Assume an auditorium with a seat labeled by a letter and numbers in between 1 to 50 (e.g. A23). We want the total number of seats in the auditorium.
- 26 letters and 50 numbers
- How to count?

- **One solution: write down all seats (objects) and count them**

A-1 A-2 A-3 …A-50 B-1… Z-49 Z-50

1 2 3 50 51 … (n-1) n ← eventually we get it

Product rule

A count can be broken down into a sequence of dependent counts

- **"each element in the first count is associated with all elements of the second count"**

Example:

- Assume an auditorium with a seat labeled by a letter and numbers in between 1 to 50 (e.g. A23). We want the total number of seats in the auditorium.
- 26 letters and 50 numbers
- How to count?
- **A better solution?**
- For each letter there are 50 numbers
- So the number of seats is 26*50 = 1300
- **Product rule:** number of letters * number of integers in [1,50]

Product rule

A count can be broken down into a sequence of dependent counts

- **“each element in the first count is associated with all elements of the second count”**

- **Product rule:** If a count of elements can be broken down into a sequence of dependent counts where the first count yields $n1$ elements, the second $n2$ elements, and kth count nk elements, by the product rule the total number of elements is:
    - $n = n1 * n2 * \ldots * nk$

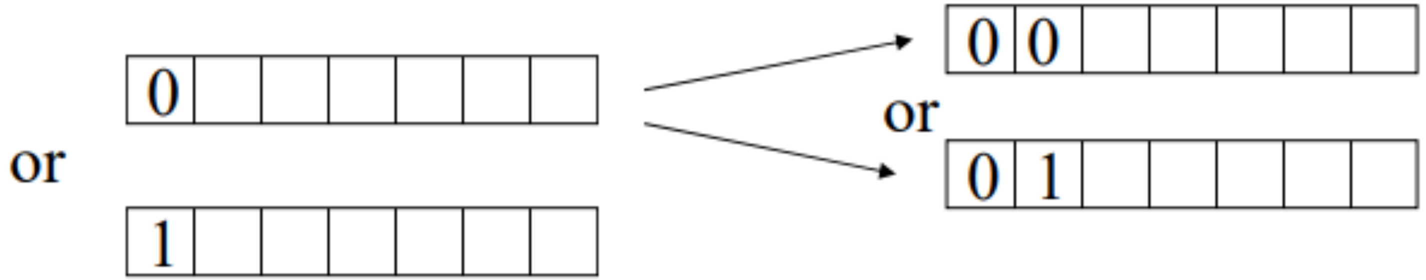
Product rule

Example:

- How many different bit strings of length 7 are there?
  - E.g. 1011010
- Is it possible to decompose the count problem and if yes how?

Yes.

- Count the number of possible assignments to bit 1
- For the first bit assignment (say 0) count assignments to bit 2



Total assignments to first 2 bits: 2*2=4

Product rule

Example:

- How many different bit strings of length 7 are there?
    - E.g. 1011010
- Is it possible to decompose the count problem and if yes how?

- **Yes.**
    - Count the number of possible assignments to bit 1
    - For the specific first bit count possible assignments to bit 2
    - For the specific first two bits count assignments to bit 3
    - Number of assignments to the first 3 bits: 2*2*2=8

Product rule

Example:

- How many different bit strings of length 7 are there?
  - E.g. 1011010
- Is it possible to decompose the count problem and if yes how?

- **Yes.**
  - Count the number of possible assignments to bit 1
  - For the specific first bit count possible assignments to bit 2
  - For the specific first two bits count assignments to bit 3
  - Gives a sequence of n dependent counts and by the product rule we have:

$$\mathbf{n = 2 * 2 * 2 * 2 * 2 * 2 * 2 = 2^7}$$

Product rule

Example:

The number of subsets of a set S with k elements.

- How to count them?
- **Hint:** think in terms of bitstring representation of a set?
- Assume each element in S is assigned a bit position.
- If A is a subset it can be encoded as a bitstring: if an element is in A then use 1 else put 0
- How many different bitstrings are there?
  - $n = \underbrace{2 * 2 * \ldots 2}_{\text{k bits}} = 2^k$

Product rule

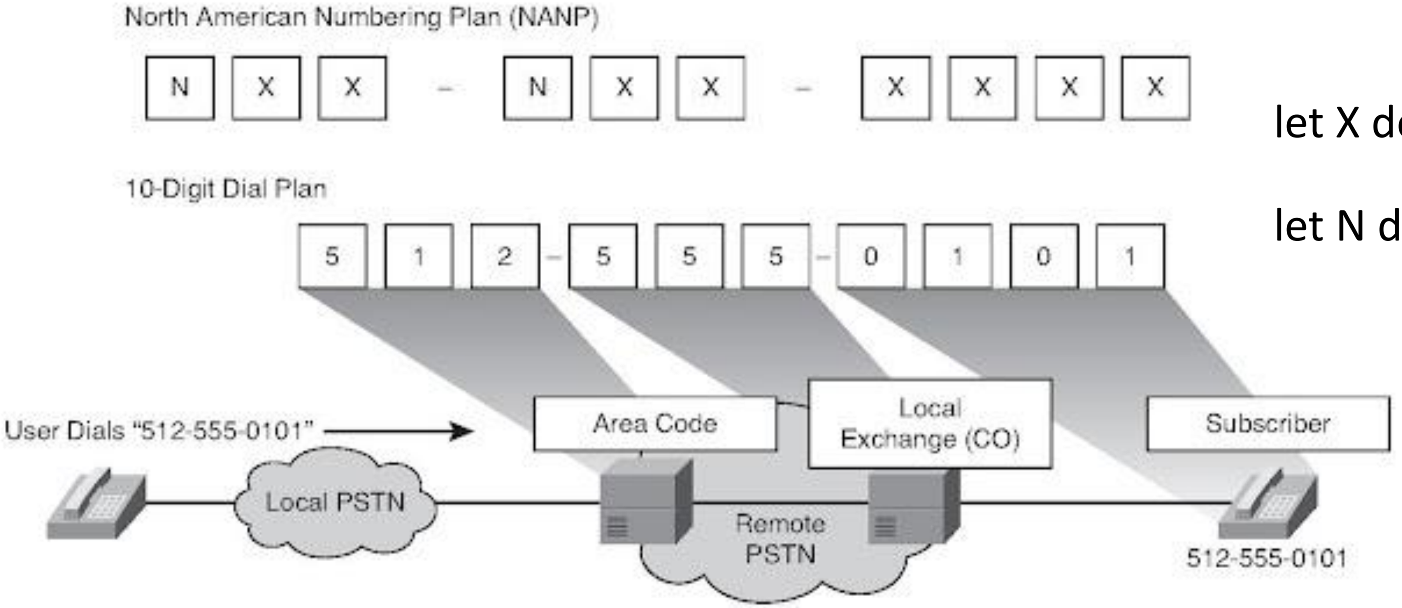
How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits (and no sequences of letters are prohibited, even if they are obscene)?

$$\underbrace{\_\_\ \ \_\_\ \ \_\_}_{\substack{\text{26 choices}\\ \text{for each}\\ \text{letter}}}\ \ \underbrace{\_\_\ \ \_\_\ \ \_\_}_{\substack{\text{10 choices}\\ \text{for each}\\ \text{digit}}}$$

1986 - 2000

Product rule

Example



let X denote a digit that can take any of the values 0 through 9

let N denote a digit that can take any of the values 2 through 9

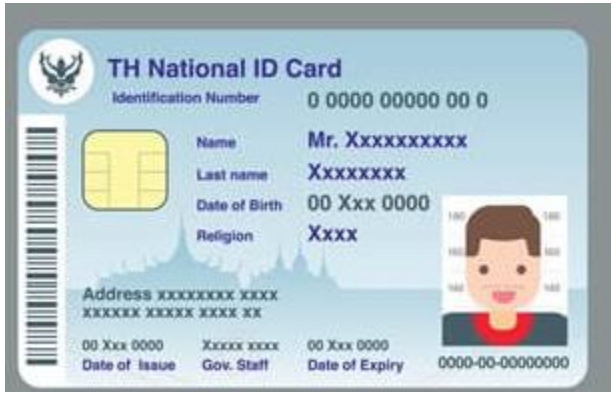
How many different North American telephone numbers are possible?

$8 \cdot 10 \cdot 10 = 800$ area codes with format NXX

$10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$ station codes with format XXXX.

different numbers available in North America. Under the new plan, there are

$$800 \cdot 800 \cdot 10{,}000 = 6{,}400{,}000{,}000$$


TH National ID Card
Identification Number
0 0000 00000 00 0
Name
Mr. Xxxxxxxxxx
Last name
Xxxxxxxx
Date of Birth
00 Xxx 0000
Religion
Xxxx
Address xxxxxxxx xxxx
xxxxxx xxxxx xxxx xx
00 Xxx 0000
Date of Issue
Xxxxx xxxx
Gov. Staff
00 Xxx 0000
Date of Expiry
0000-00-00000000


ABCD-00-00
ประเทศไทย
THAILAND
ABC - 0000000-12


#LAWGET
ตัวเลขต้องระวัง
ข้อมูลสำคัญในบัตรประชาชน
บัตรประจำตัวประชาชน Thai Nationnal ID Card
เลขประจำตัวประชาชน
Identification Number
1 23 45 67891 12 3
หลักที่ 1
ประเภทบุคคล
มี 9 ประเภท
หลักที่ 2-3
รหัสจังหวัดที่เกิด
หลักที่ 4-5
รหัสอำเภอหรือ
เทศบาลที่เกิด
หลักที่ 6-10
เล่มที่ของสูติบัตร
(ตามประเภทบุคคล)
หลักที่ 11-12
ใบที่ของสูติบัตรแต่ละเล่ม
(ตามประเภทบุคคล)
หลักที่ 13
ตัวเลขตรวจสอบ
ความถูกต้องของ 12 หลัก
(ใช้สูตรคำนวณเฉพาะ)
ระมัดระวังการเปิดเผยเลขบัตรประชาชนและข้อมูลส่วนตัวของคุณ
*ตัวอย่าง วิธีการที่มิจฉาชีพนำบัตรประชาชนไปใช้ทำผิดกฎหมาย
หลอกขอเลขบัตรประชาชน + รู้เบอร์โทร + รู้บัญชีธนาคาร
จากนั้น...โทรหลอก Call Center Cyber Banking ขอให้ปลดล็อค
และ Reset รหัสผ่านใหม่ โอนเงินออกจากบัญชีได้
สำนักงานกิจการยุติธรรม @weareoja กรมการปกครอง กระทรวงมหาดไทย

Sum rule

- **Basic Counting**
  - Product rule
  - Sum rule
  - More complex counting problems
  - Inclusion-Exclusion principle
  - Tree Diagrams
- Pigeonhole Principle

A count decomposes into a set of independent counts

- "elements of counts are alternatives", they do not depend on each other

Example:

- You need to travel in between city A and B. You can either fly, take a train, or a bus. There are 12 different flights in between A and B, 5 different trains and 10 buses. How many options do you have to get from A to B?
- We can take only one type of transportation and for each only one option. The number of options:
  - $n = 12+5+10$

Sum rule:

- n = number of flights + number of trains + number of buses

Sum rule

A count decomposes into a set of independent counts

- "elements of counts are alternatives"

- **Sum rule:** If a count of elements can be broken down into a set of independent counts where the first count yields $n1$ elements, the second $n2$ elements, and kth count nk elements, by the sum rule the total number of elements is:
  - $n = n1+n2+ \ldots + nk$

Sum rule

Example

A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

Beyond basic counting rules

- **Basic Counting**
  - Product rule
  - Sum rule
  - More complex counting problems
  - Inclusion-Exclusion principle
  - Tree Diagrams
- Pigeonhole Principle

- **More complex counting problems** typically require a combination of the sum and product rules.

Example: A login password:

- The minimum password length is 6 and the maximum is 8. The password can consist of either an uppercase letter or a digit. There must be at least one digit in the password.
- How many different passwords are there?

Beyond basic counting rules

Example: A password for the login name.

- The minimum password length is 6 and the maximum is 8. The password can consist of either an uppercase letter or a digit. There must be at least one digit in the password.
- How to compute the number of possible passwords?

Step 1:

- The password we select has either 6,7 or 8 characters.
- So the total number of valid passwords is by the sum rule:
  - P = P6+P7+P8

    The number of passwords of length 6,7 and 8 respectively

Beyond basic counting rules

Step 1:

- The password we select has either 6,7 or 8 characters.
- So the total number of valid passwords is by the sum rule:
  - P = P6+P7+P8

  The number of passwords of length 6,7 and 8 respectively

Step 2

- Assume passwords with 6 characters (upper-case letters):
- How many are there?
- If we let each character to be at any position we have:
  - P6-nodigits = 26^6 different passwords of length 6

Beyond basic counting rules

Step 1:

- The password we select has either 6,7 or 8 characters.
- So the total number of valid passwords is by the sum rule:
  - P = P6+P7+P8

  The number of passwords of length 6,7 and 8 respectively

Step 2

- Assume passwords with 6 characters

  (either digits + upper case letters):
- How many are there?
- If we let each character to be at any position we have:
  - P6-all = $(26+10)^6=(36)^6$ different passwords of length 6

Beyond basic counting rules

Step 2

But we must have a password with at least one digit. How to account for it?

A trick. Split the count of all passwords of length 6 into to two mutually exclusive groups:

- **P6-all = P6-digits + P6-nodigits**

1. P6-digits – count when the password has one or more digits
2. P6-nodigits – count when the password has no digits

- We know how to easily compute P6-all and P6-nodigits
  - **P6-all = 36^6 and P6-nodigits = 26^6**
  - Then **P6-digits = P6-all – P6-nodigits**

Beyond basic counting rules

Step 1:

the total number of valid passwords is by the sum rule:

- **P = P6+P7+P8**
- The number of passwords of length 6,7 and 8 respectively

Step 2

The number of valid passwords of length 6:

P6= P6-digits = P6-all – P6-nodigits

$$= 36^6 - 26^6$$

Analogically:

P7= P7-digits = P7-all – P7-nodigits

$$= 36^7 - 26^7$$

P8= P8-digits = P8-all – P8-nodigits

$$= 36^8 - 26^8$$

Inclusion-Exclusion principle

- **Basic Counting**
  - Product rule
  - Sum rule
  - More complex counting problems
  - Inclusion-Exclusion principle
  - Tree Diagrams
- Pigeonhole Principle

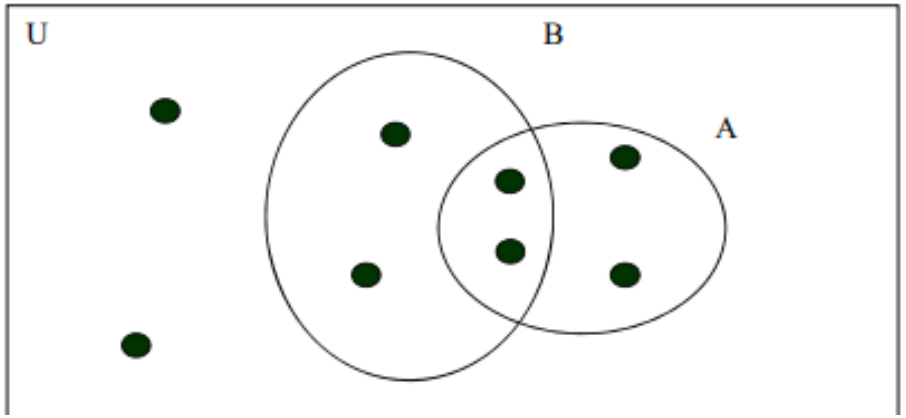
Used in counts where the decomposition yields two dependent count tasks with overlapping elements

- If we used the sum rule some elements would be counted twice

Inclusion-exclusion principle: uses a sum rule and then corrects for the overlapping elements.

We used the principle for the cardinality of the set union.

- $|A \cup B| = |A| + |B| - |A \cap B|$

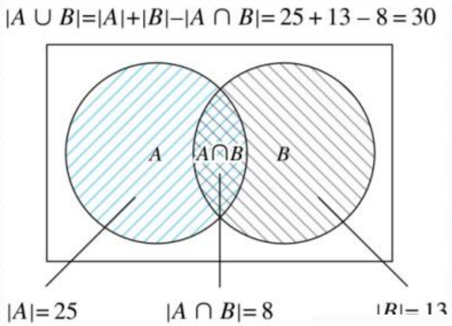


Inclusion-Exclusion principle

- **Basic Counting**
  - Product rule
  - Sum rule
  - More complex counting problems
  - Inclusion-Exclusion principle
  - Tree Diagrams
- Pigeonhole Principle

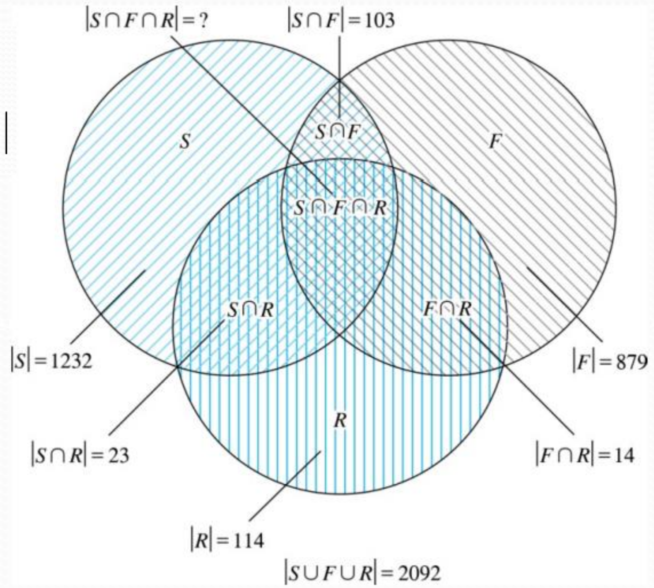
Two Finite Sets

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$= 25 + 13 - 8 = 30$$



Three Finite Sets

$$|A \cup B \cup C| =$$
$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



Inclusion-Exclusion principle

Two Finite Sets

Example: How many bitstrings of length 8 start either with a bit 1 or end with 00?

$$1\ \underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_}_{2^7 = 128 \text{ ways}}$$

$$\underbrace{\_\ \_\ \_\ \_\ \_\ \_}_{2^6 = 64 \text{ ways}}\ 0\ 0$$

$$1\ \underbrace{\_\ \_\ \_\ \_\ \_}_{2^5 = 32 \text{ ways}}\ 0\ 0$$

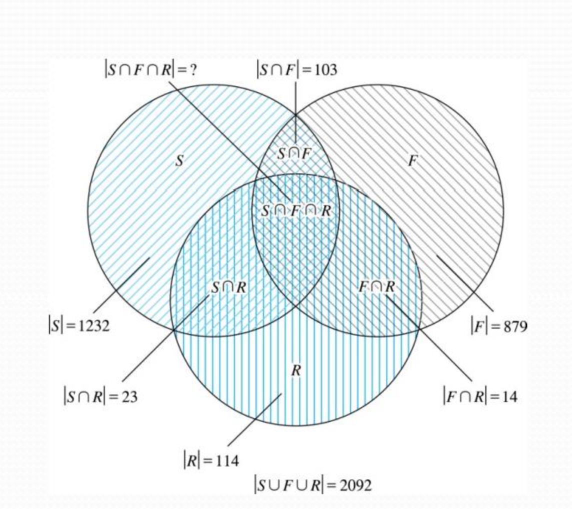
- It is easy to count **strings that start with 1**:
- How many are there?
- It is easy to count the **strings that end with 00**.
- How many are there?
- Is it OK to add the two numbers to get the answer?
- **No. Overcount**. There are some strings that can both start with 1 and end with 00. These strings are counted in twice.
- How to deal with it? How to correct for overlap?
- How many of strings were counted twice?
- Thus we can correct for the overlap simply by using:

Inclusion-Exclusion principle

Three Finite Sets : Example

A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken a course in at least one of Spanish French and Russian, how many students have taken a course in all 3 languages.

$$|A \cup B \cup C| =$$
$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



Inclusion-Exclusion principle

Three Finite Sets : Example (cont.)

Solution: Let S be the set of students who have taken a course in Spanish, F the set of students who have taken a course in French, and R the set of students who have taken a course in Russian.

Then, we have $|S| = 1232$, $|F| = 879$, $|R| = 114$, $|S \cap F| = 103$, $|S \cap R| = 23$, $|F \cap R| = 14$, and $|S \cup F \cup R| = 2092$.

Using the equation

$$|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |S \cap R| - |F \cap R| + |S \cap F \cap R|$$

$$2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |S \cap F \cap R|$$

Solving for $|S \cap F \cap R|$ yields 7.

Tree Diagarams

- **Basic Counting**
  - Product rule
  - Sum rule
  - More complex counting problems
  - Inclusion-Exclusion principle
  - Tree Diagrams
- Pigeonhole Principle

Tree: is a structure that consists of a root, branches and leaves.

- Can be useful to represent a counting problem and record the choices we made for alternatives. The count appears on the leaf nodes.

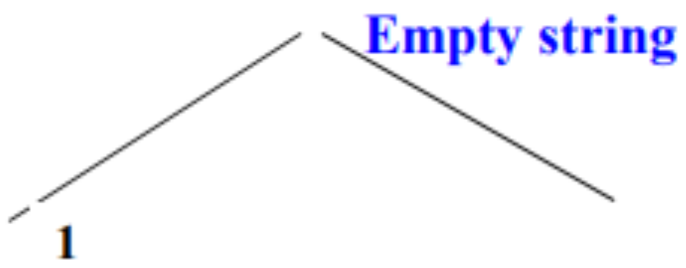
Example:

What is the number of bit strings of length 4 that do not have two consecutive ones.

Tree Diagarams

Example:

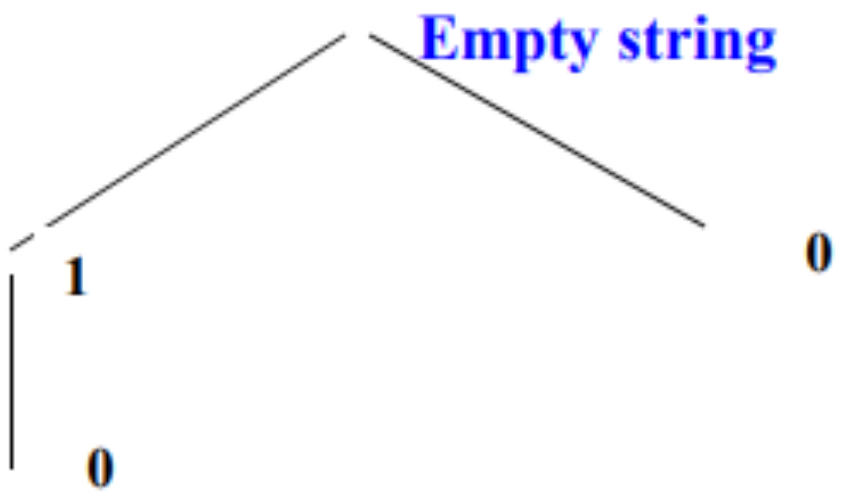
What is the number of bit strings of length 4 that do not have two consecutive ones?



Tree Diagarams

Example:

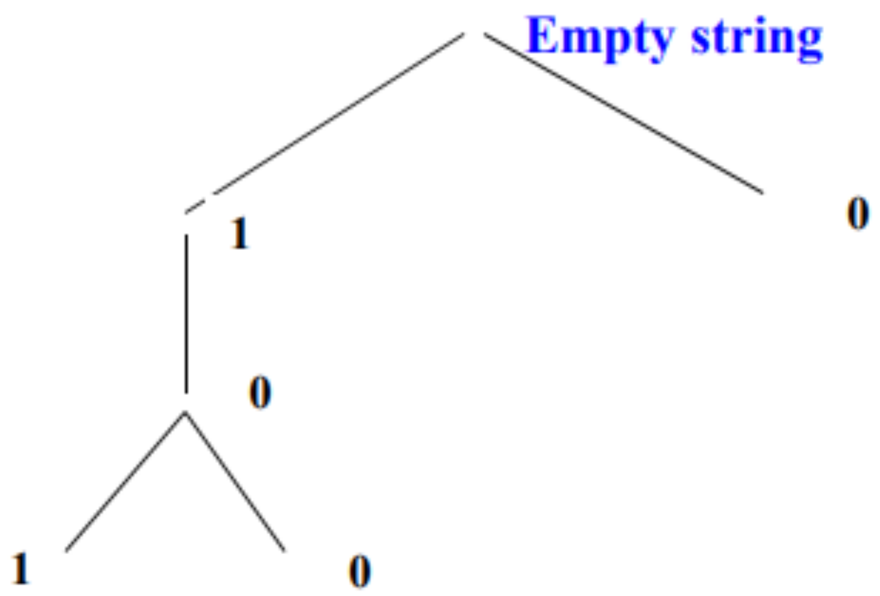
What is the number of bit strings of length 4 that do not have two consecutive ones?



Tree Diagarams

Example:

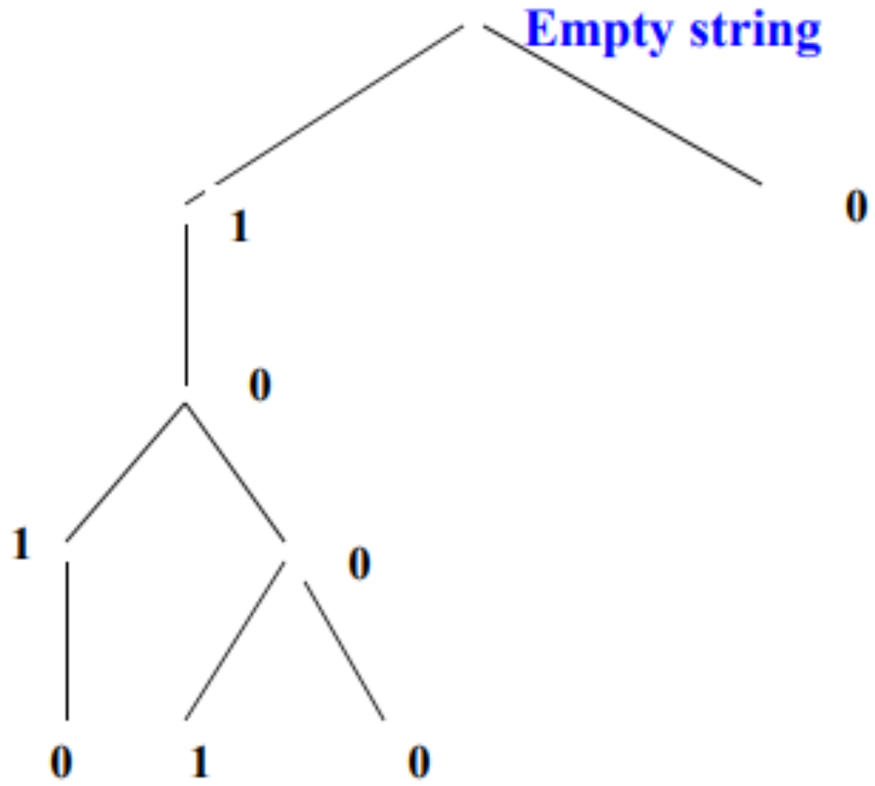
What is the number of bit strings of length 4 that do not have two consecutive ones?



Tree Diagarams

Example:

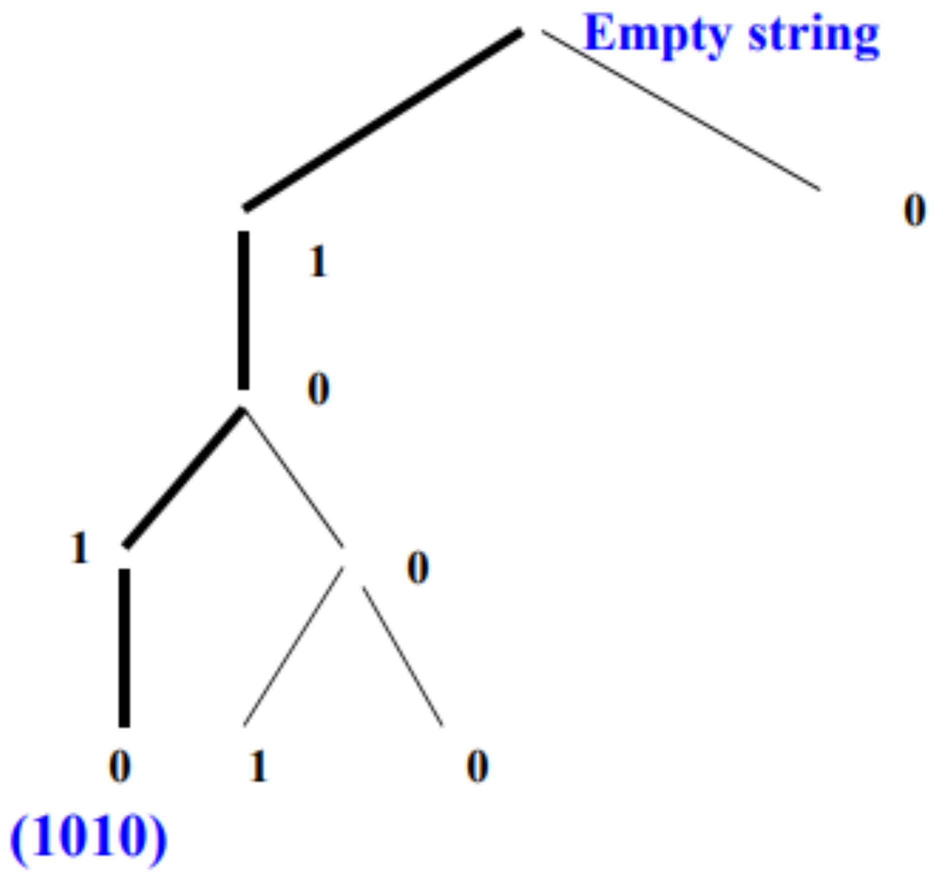
What is the number of bit strings of length 4 that do not have two consecutive ones?



Tree Diagarams

Example:

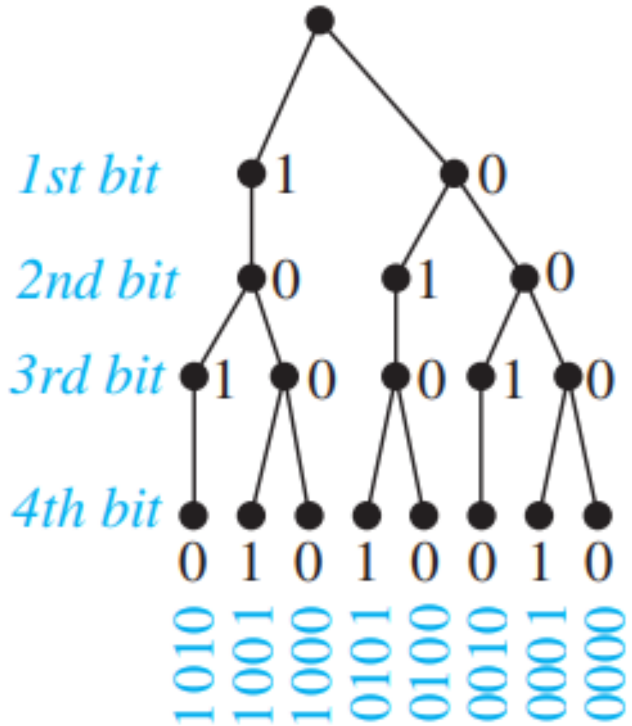
What is the number of bit strings of length 4 that do not have two consecutive ones?

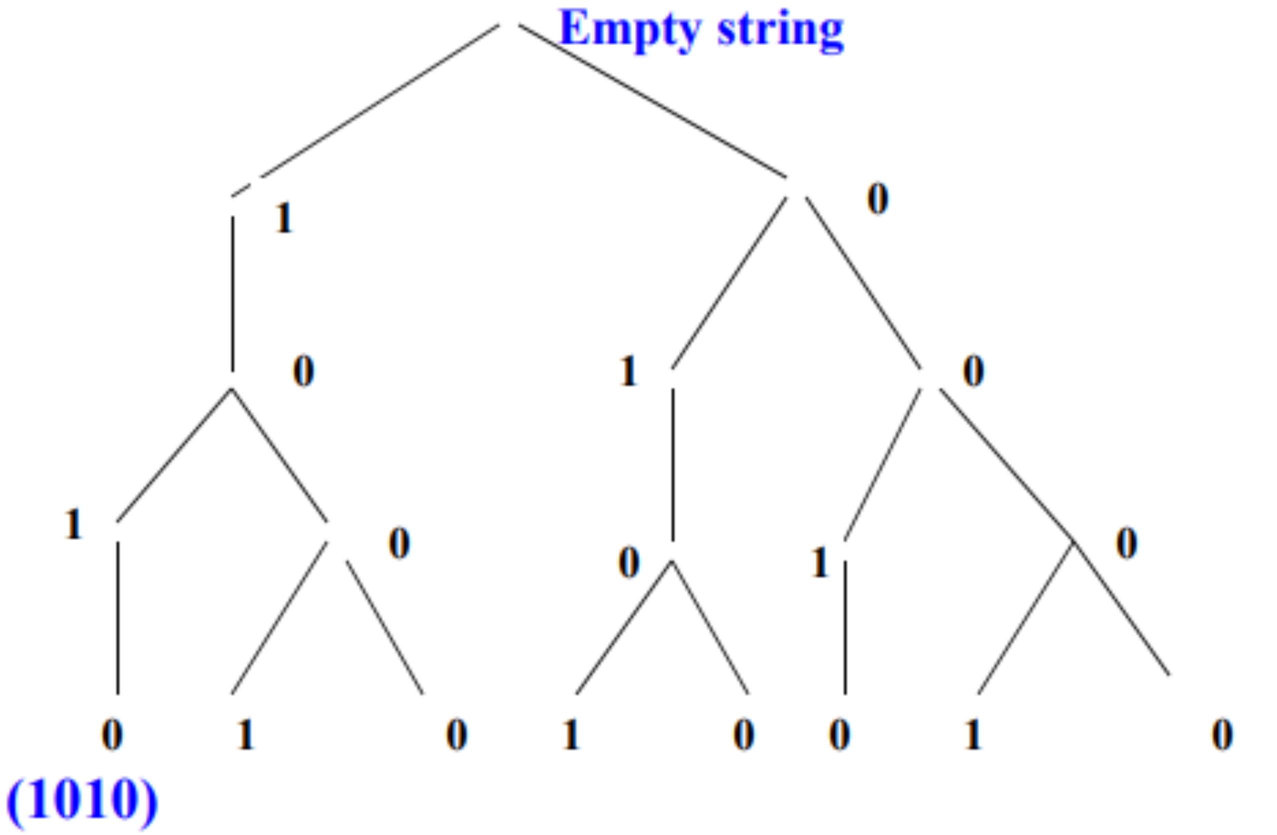


Tree Diagarams

Example:

What is the number of bit strings of length 4 that do not have two consecutive ones?

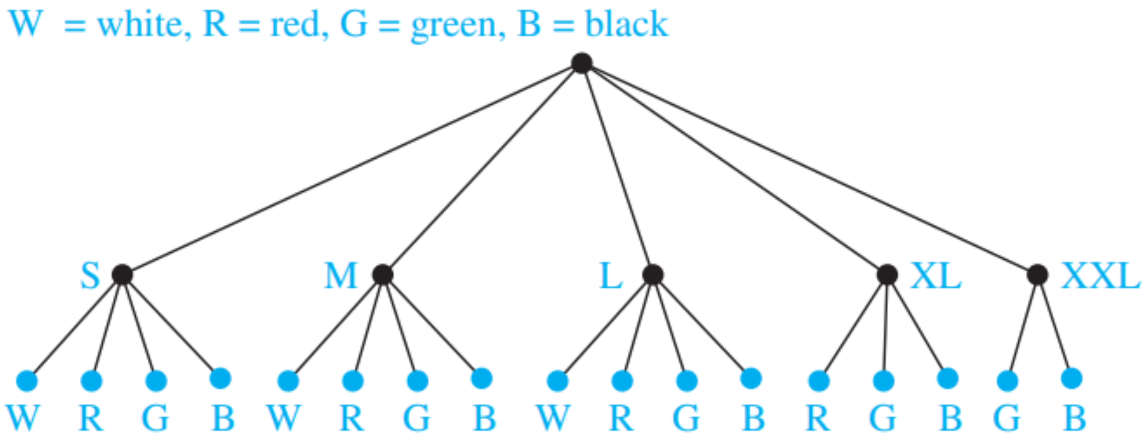




Tree Diagarams

Example

Suppose that "I Love New Jersey" T-shirts come in five different sizes: S, M, L, XL, and XXL. Further suppose that each size comes in four colors, white, red, green, and black, except for XL, which comes only in red, green, and black, and XXL, which comes only in green and black. How many different shirts does a souvenir shop have to stock to have at least one of each available size and color of the T-shirt?

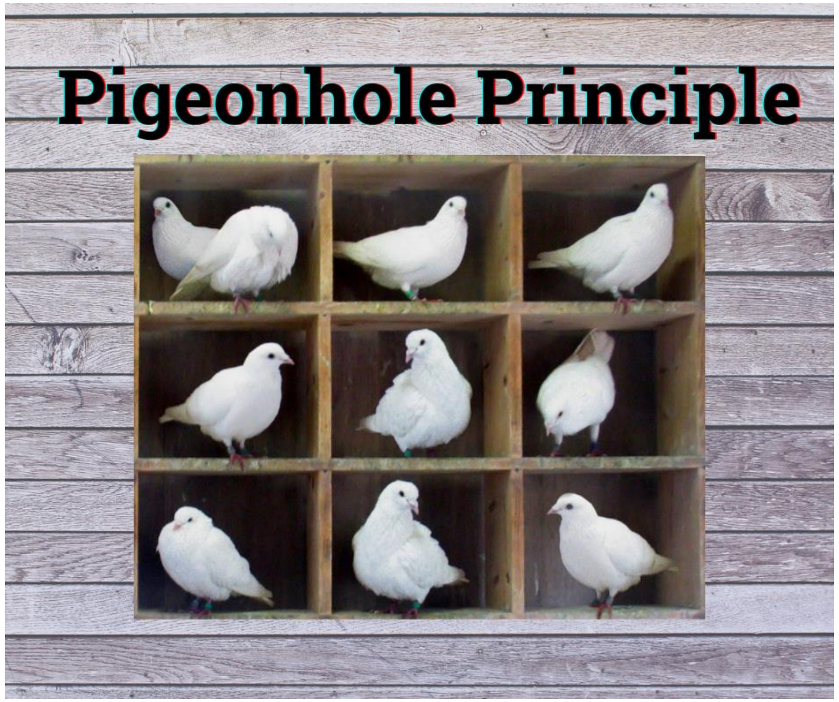


Solution: The tree diagram in Figure 4 displays all possible size and color pairs. It follows that the souvenir shop owner needs to stock 17 different T-shirts. ◀

Pigeonhole Principle

- **Basic Counting**
  - Product rule
  - Sum rule
  - More complex counting problems
  - Inclusion-Exclusion principle
  - Tree Diagrams
- Pigeonhole Principle

- Assume you have a set of objects and a set of bins used to store objects.
- **The pigeonhole principle** states that if there are more objects than bins then there is at least one bin with more than one object.

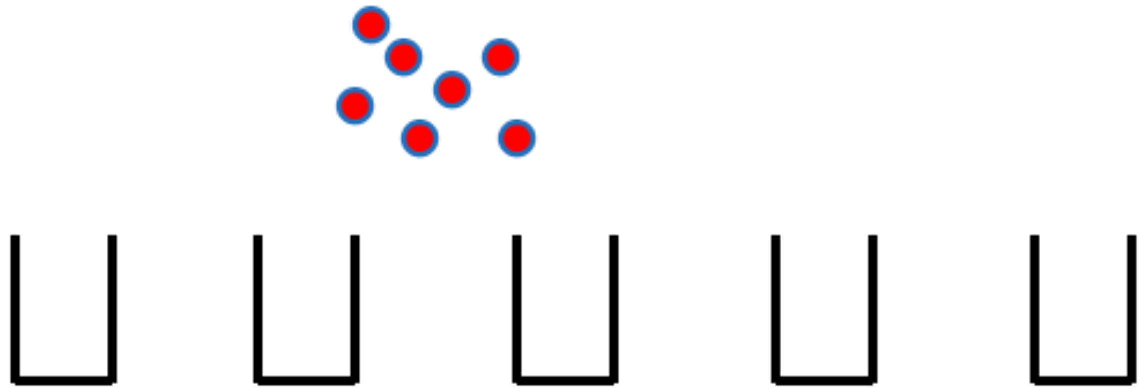


Pigeonhole Principle

- **Basic Counting**
  - Product rule
  - Sum rule
  - More complex counting problems
  - Inclusion-Exclusion principle
  - Tree Diagrams
- Pigeonhole Principle

- Assume you have a set of objects and a set of bins used to store objects.
- **The pigeonhole principle** states that if there are more objects than bins then there is at least one bin with more than one object.

- **Example:** 7 balls and 5 bins to store them



Pigeonhole Principle

- Assume you have a set of objects and a set of bins used to store objects.
- **The pigeonhole principle** states that if there are more objects than bins then there is at least one bin with more than one object.

- **Example:** 7 balls and 5 bins to store them
- At least one bin with more than 1 ball exists.

Pigeonhole Principle

- Assume you have a set of objects and a set of bins used to store objects. The pigeonhole principle states that if there are more objects than bins then there is at least one bin with more than one object.

- **Theorem. If there are k+1 objects and k bins. Then there is at least one bin with two or more objects.**

k bins

Pigeonhole Principle

- Assume you have a set of objects and a set of bins used to store objects. The pigeonhole principle states that if there are more objects than bins then there is at least one bin with more than one object.

- **Theorem. If there are k+1 objects and k bins. Then there is at least one bin with two or more objects.**

- **Proof. (by contradiction)**
- Assume that we have k + 1 objects and every bin has at most one element. Then the total number of elements is k which is a contradiction.
- End of proof

Pigeonhole Principle

Example:

- Assume 367 people. Are there any two people who has the same birthday?

- How many days are in the year? 365.
- Then there must be at least two people with the same birthday.

Generalized pigeonhole principle

- We can often say more about the number of objects.
- Say we have 5 bins and 12 objects. What is it we can say about the bins and number of elements they hold?

- There must be **a bin with at least 3 elements**.
- Why?
- Assume there is no bin with more than 3 elements. Then the max number of elements we can have in 5 bins is 10. We need to place 13 so at least one bin should have at least 3 elements.

Generalized pigeonhole principle

Theorem. If N objects are placed into k bins then there is at least one bin containing at least $\lceil N/k \rceil$ objects.

Example. Assume 100 people. Can you tell something about the number of people born in the same month.

- Yes. There exists a month in which at least $\lceil 100/12 \rceil = \lceil 8.3 \rceil = 9$ people were born.

Pigeonhole Principle

What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

Pigeonhole Principle

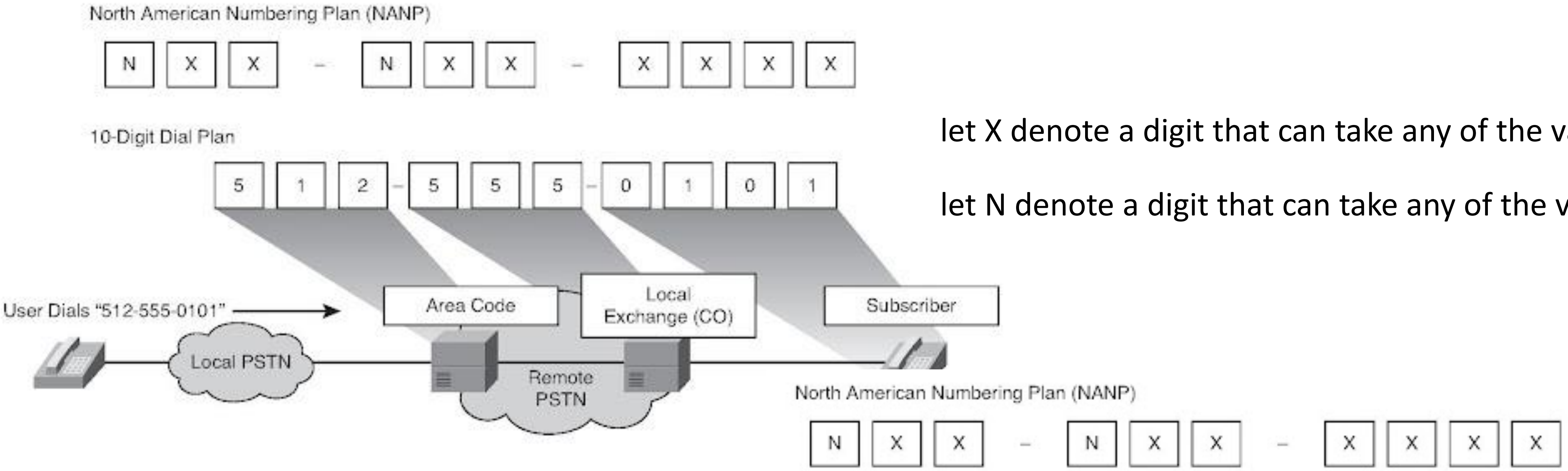
What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

Pigeonhole Principle

What is the least number of area codes needed to guarantee that the 25 million phones in a state can be assigned distinct 10-digit telephone numbers? (Assume that telephone numbers are of the form NXX-NXX-$XXXX$, where the first three digits form the area code, N represents a digit from 2 to 9 inclusive, and X represents any digit.)

Example

Product rule



let X denote a digit that can take any of the values 0 through 9

let N denote a digit that can take any of the values 2 through 9

How many different North American telephone numbers are possible?

$8 \cdot 10 \cdot 10 = 800$ area codes with format NXX

$10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$ station codes with format XXXX.

Different number of area code:
$800 \cdot 10{,}000 = 8{,}000{,}000$

Pigeonhole Principle

What is the least number of area codes needed to guarantee that the 25 million phones in a state can be assigned distinct 10-digit telephone numbers? (Assume that telephone numbers are of the form $NXX\text{-}NXX\text{-}XXXX$, where the first three digits form the area code, N represents a digit from 2 to 9 inclusive, and X represents any digit.)

Generalized pigeonhole principle

Example:

- How many students, each of whom comes from one of the 50 states, must be enrolled in a university to guaranteed that there are at least 100 who come from the same state?

Answer:

- Let there by 50 boxes, one per state.
- We want to find the minimal N so that $\lceil N/50 \rceil = 100$.
- Letting N=5000 is too much, since the remainder is 0.
- We want a remainder of 1 so that let N=50*99+1=4951.