Q1. Assume $\vec{u}$ and $\vec{v}$ ave non-zero vectors which ave not parallel. To prove parallelogram defined by $\vec{u}$ and $\vec{w} \Rightarrow$ orthogonal diagonals in the parallelogram. since $\vec{u}$ and $\vec{v}$ form diagonals of a rhombus d_1, d_2 can be witten as follow

$$\begin{aligned}
d_1 &= \vec{u} + \vec{w} \\
d_2 &= \vec{u} - \vec{w} \\
d_1 \cdot d_2 &= (\vec{u} + \vec{w}) \cdot (\vec{u} - \vec{w}) \\
&= \vec{u} \cdot \vec{u} - \vec{u}\vec{w} + \vec{u} \cdot \vec{w} - \vec{w} \cdot \vec{w} \\
&= \vec{u} \cdot \vec{u} - \vec{w} \cdot \vec{w} \\
&= \|\vec{u}\|^2 - \|\vec{w}\|^2 \quad \text{ since } \|\vec{w}\| = \|\vec{u}\|, \text{ so } \|\vec{u}\|^2 = \|\vec{w}\|^2 \\
&= 0
\end{aligned}$$

To prove $\Rightarrow$, if diagonals are orthogonal

$$\begin{aligned}
d_1 \cdot d_2 &= 0 \\
(\vec{u} + \vec{w})(\vec{u} - \vec{w}) &= 0 \\
\vec{u} \cdot \vec{u} - \vec{u} \cdot \vec{w} + \vec{w}\vec{w} - \vec{w} \cdot \vec{w} &= 0 \\
\vec{u} \cdot \vec{u} &= \vec{w} \cdot \vec{w} \\
||\vec{u}\|^2 &= \|\vec{w}\|^2
\end{aligned}$$

Hence $\vec{u}, \vec{w}$ have the same magnitude and ave non-parallel, so the formed parallelogram is a vombus.

Q2. a. Proof by Contradiction Assume $\begin{bmatrix}1\\2\\2\\3\end{bmatrix} \in \text{Span}\left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\2\\2\\0\end{bmatrix}\right\}$ in $\mathbb{R}^4$

$$\Rightarrow \begin{bmatrix}1\\2\\2\\3\end{bmatrix} = c_1\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + c_2\begin{bmatrix}0\\2\\2\\0\end{bmatrix}$$

$$\begin{aligned}1 &= c_1\\ 2 &= c_1 + 2c_2\\ 2 &= c_1 + 2c_2\\ 3 &= c_1\\ 1 &= c_1 = 3\end{aligned}$$

$\Rightarrow 1 = 3$ which is clearly false, hence the contradiction. QED

b. To prove span $\left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\2\\2\\0\end{bmatrix}\right\} \in \text{span}\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}\right\}$ Consider

$$c_1\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + c_2\begin{bmatrix}0\\2\\2\\0\end{bmatrix} = c_3\begin{bmatrix}1\\0\\0\\1\end{bmatrix} + c_4\begin{bmatrix}2\\0\\0\\2\end{bmatrix} + c_5\begin{bmatrix}0\\1\\1\\0\end{bmatrix}$$

$$\begin{aligned}\Rightarrow c_1 &= c_3 + 2c_4\\ c_1 + 2c_2 &= c_5\\ c_1 + 2c_2 &= c_5\\ c_1 &= c_3 + 2c_4\\ \text{Let } c_3 = 0, c_4 &= \frac{c_1}{2}, c_5 = c_1 + 2c_2\end{aligned}$$

$\Rightarrow$ span $\left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\2\\2\\0\end{bmatrix}\right\} = c_1\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + c_2\begin{bmatrix}0\\2\\2\\0\end{bmatrix} \in 0\begin{bmatrix}1\\0\\0\\1\end{bmatrix} + \frac{c_1}{2}\begin{bmatrix}2\\0\\0\\2\end{bmatrix} + (c_1 + 2c_2)\begin{bmatrix}0\\1\\1\\0\end{bmatrix} =$ span $\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\0\\2\end{bmatrix} \Bigg[\begin{bmatrix}0\\1\\1\\0\end{bmatrix}\right\}$ As desired

To prove $\text{span}\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\9\\2\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}\right\} \in \text{span}\left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\2\\2\\0\end{bmatrix}\right\}$

$\Rightarrow e_1 \begin{bmatrix}1\\0\\0\\1\end{bmatrix} + e_2 \begin{bmatrix}2\\0\\0\\2\end{bmatrix} + e_3 \begin{bmatrix}0\\1\\1\\0\end{bmatrix} = e_4 \begin{bmatrix}1\\1\\1\\1\end{bmatrix} + e_5 \begin{bmatrix}0\\2\\2\\0\end{bmatrix}$

$\Rightarrow \begin{array}{l} e_1 + 2e_2 = e_4 \\ e_3 = e_4 + 2e_5 \\ e_3 = e_4 + 2e_5 \\ e_1 + 2e_2 = e_4 \end{array}$

$e_3 = e_1 + 2e_2 + 2e_5$
$e_3 - 2e_2 - e_1 = 2e_5$
$e_5 = \frac{e_3 - 2e_2 - e_1}{2}$

Let $e_4 = e_1 + 2e_2, e_5 = \frac{e_3 - 2e_2 - e_1}{2}$

$\text{span}\left\{\begin{bmatrix}1\\0\\9\\1\end{bmatrix}, \begin{bmatrix}2\\9\\9\\2\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}\right\} = e_1 \begin{bmatrix}1\\0\\0\\1\end{bmatrix} + e_2 \begin{bmatrix}2\\0\\0\\2\end{bmatrix} + e_3 \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \in (e_1 + 2e_2) \begin{bmatrix}1\\1\\1\\1\end{bmatrix} +$
$\left(\frac{e_3 - 2e_2 - e_1}{2}\right) \begin{bmatrix}0\\2\\2\\0\end{bmatrix} = \text{span}\left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\2\\2\\0\end{bmatrix}\right\}$

since $\text{span}\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\0\\2\end{bmatrix} \begin{bmatrix}0\\1\\1\\0\end{bmatrix}\right\} \in \text{span}\left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\2\\2\\0\end{bmatrix}\right\}$ and $\text{span}\left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\2\\2\\0\end{bmatrix}\right\} \in$
$\text{span}\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}\right\}, \text{span}\left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \begin{bmatrix}0\\2\\2\\0\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}\right\}$

QED

Q3. a, Proof by Contradiction
Suppose $\vec{x} \in \operatorname{Span}\{\vec{v}_1, \cdots, \vec{v}_k\}$ $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + a_k\vec{v}_k$ for $a_1, \cdots, a_k \in \mathbb{F}$
since $\vec{x}$ is orthogonal to $\vec{v}_i$ for $1 \leqslant i \leqslant k$

$\Rightarrow \vec{x}\cdot\vec{v}_1 + \vec{x}\cdot\vec{v}_2 + \cdots + \vec{x}\cdot\vec{v}_k = 0$
$\vec{x}\cdot a_1\vec{v}_1 + \vec{x}\cdot a_2\vec{v}_2 + \cdots + \vec{x}\cdot a_k\cdot\vec{v}_k = 0$, since a_n are scalars, vectors remain orthogonal
$\vec{x}\cdot(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k) = 0$
$\vec{x}\cdot\vec{x} = 0$
$\vec{x} = \overrightarrow{0}$

$\Rightarrow$ Hence the contradiction that $\vec{x}$ is $\overrightarrow{0}$
QED
b. Proof by Contradiction
Assume $\operatorname{Span}\{\vec{v}\} = \mathbb{R}^3$ for some $\vec{v} \in \mathbb{R}$
Consider vectors

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow c_1 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow c_2 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$c_1 v_1 = 0, c_1 v_3 = 1 \Rightarrow v_1 = 0$$

However $c_2 \cdot v_1 = 1$, which contradicts $v_1 = 0$, hence by contradiction $\operatorname{span}\{\vec{v}\} \neq \mathbb{R}^3$
c. Proof by Contradiction Assume Span $\{\vec{v}_1, \vec{v}_2\} = \mathbb{R}^3$
Let $\vec{x} = \vec{v}_1 \times \vec{v}_2 \in \mathbb{R}^3 \Rightarrow \vec{x} \in \operatorname{Span}\{\vec{v}_1, \overrightarrow{v_2}\}$
Since $\vec{x}$ is cross product of $\vec{v}_1, \vec{v}_2, \vec{x}\cdot\vec{v}_1 = 0, \vec{x}\cdot\overrightarrow{v_2} = 0 \Rightarrow$ By part a, $\vec{x} = \overrightarrow{0}$.
By Weekly Practice $Q7$, since $\vec{v}_1 \times \vec{V}_2 = \overrightarrow{0}, \overrightarrow{v_1} // \vec{v}_2 \Rightarrow \vec{V}_1 = c\vec{V}_2$ $\operatorname{Span}\{\vec{v}_1, \vec{v}_2\} = \operatorname{Span}\{c\vec{v}_2, \vec{v}_2\} = \operatorname{Span}\{\vec{v}_2\}$
However, by part $b, \mathbb{R}^3 \neq \operatorname{Span}\{\vec{v}\}$, hence the contradiction.
QED

Q4. To prove $L \subseteq P \Rightarrow \vec{a} \in P \wedge d \in \operatorname{Span}\{\vec{V}, \vec{w}\}$

$$L = \{\vec{a} + t\vec{d} : t \in \mathbb{R}\}$$

Let $t = 0, L = \vec{a}$, since $L \subseteq P, \vec{a} \in P$ Let $L = p, \vec{a} + 0\vec{d} = \vec{b} + r\vec{v} + s_1\vec{w}$

$$\vec{a} = \vec{b} + r_1\vec{v} + s_1\vec{w}$$

for $t \neq 0$, substitute $\vec{a}$ with $\vec{b} + r, \vec{v} + s_1, \vec{w}$ $t\vec{d}\vec{b} + r_1\vec{v} + s_1\vec{w} = \cancel{\vec{b}} + r\vec{\nabla} + s\vec{w}$

$$\begin{aligned} t\vec{d} &= (r - r_1)\,\vec{V} + (s - s_1)\,\vec{w} \\ \vec{d} &= \frac{r - r_1}{t}\vec{v} + \frac{s - s_1}{t}\vec{w} \end{aligned}$$

since $r_1, r, s_1, s, t \in \mathbb{R}, \frac{r - v_1}{t_1}, \frac{s - s_1}{t} \in \mathbb{R} \Rightarrow d \in \operatorname{span}\{\vec{v}, \vec{w}\} \Rightarrow \vec{a} \in P \wedge \vec{d} \in \operatorname{span}\{\vec{v}, \vec{w}\}$
To prove $\vec{a} \in P \wedge \vec{d} \in \operatorname{span}\{\vec{v}, \vec{w}\} \Rightarrow L \subseteq p$

$$\begin{aligned} \vec{a} &= \vec{b} + r\vec{v} + s\vec{w}, \vec{d} = c\vec{v} + c_2\vec{w} \\ L &= \left\{\vec{b} + r\vec{v} + s\vec{w} + t\,(c_1\vec{v} + c_2\vec{w})\right\} \\ \text{Let} \quad &= \left\{\vec{b} + r\vec{v} + s\vec{w} + tc_1\vec{v} + tc_2\vec{w}\right\} \\ &= \left\{\vec{b} + (r + tc_1)\,\vec{v} + (s + tc_2)\,\vec{w}\right\} \end{aligned}$$

Since $r, t, c_1, c_2, s \in \mathbb{R}, (r + tc_1), (s + tc_2) \in \mathbb{R}$

$$\begin{aligned} &\Rightarrow L \subseteq P \\ &\Rightarrow L \subseteq P \text{ iff } \vec{a} \in P \wedge \vec{d} \in \operatorname{span}\{\vec{v}, \vec{w}\} \end{aligned}$$