Instructions

- Please upload your solutions into the appropriate slot on Crowdmark.
- The **coverage** for this assignment is up to section 6.3 (inclusive). Your solutions should not use material from any later sections. You are also allowed to use any results that appear in Practice Problem lists 1–7 (but please make sure to clearly cite them).
- You can earn a 0.25 course grade bonus for typesetting your solutions in LaTeX (or equivalent typesetting software). Please see the course outline for more details.

Problems

Q1. Let $\vec{v}, \vec{w} \in \mathbb{R}^3$ be non-zero **orthogonal** vectors and let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(\vec{x}) = 2\,\mathrm{proj}_{\vec{v}}(\vec{x}) + 2\,\mathrm{proj}_{\vec{w}}(\vec{x}) - \vec{x}$$

and let $A = [T]_{\mathcal{E}}$ be its standard matrix.

(a) Prove that $A^2 = I_3$. [**Hint:** Do **not** attempt to "compute" A. Instead, think about how A^2 is related to T.]

(b) Using (a), determine whether T is one-to-one and/or onto. Justify your answers.

(c) Consider the case where $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Determine $T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right)$ explicitly as a vector in $\mathbb{R}^3$. Hence, determine A in this particular case.

(d) Returning now to to the general case, let $\mathcal{P} = \mathrm{Span}\{\vec{v}, \vec{w}\}$ be the plane in $\mathbb{R}^3$ spanned by $\vec{v}$ and $\vec{w}$. Referring to $\mathcal{P}$, give a one-sentence description of what T does geometrically. No justification is necessary. [**Hint:** Use part (c) for inspiration. A correct description will allow you to immediately see why $A^2 = I_3$ is true.]

Q2. Let A_n be the $2n \times 2n$ matrix whose (i, j)th entry is given by

$$a_{ij} = \begin{cases} 3 & \text{if } i < j \\ 0 & \text{if } i = j \\ -3 & \text{if } i > j. \end{cases}$$

(a) Write down A_1 and A_2.

(b) Determine $\det(A_n)$ for all $n \geq 1$. Justify your answer.

(c) Based on your answer to part (b), determine all values of $n \geq 1$ for which A_n is invertible. Justify your answer.

Q3. Consider the function $f\colon \mathbb{C} \to M_{2\times 2}(\mathbb{R})$ defined by

$$f(a+bi) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

(a) Prove that f is *linear over* $\mathbb{R}$ by proving that:

 (i) $f(z+w) = f(z) + f(w)$ for all $z, w \in \mathbb{C}$.

 (ii) $f(tz) = tf(z)$ for all $z \in \mathbb{C}$ and $t \in \mathbb{R}$. (Pay close attention to the fact that $t \in \mathbb{R}$!)

(b) Prove that f is *multiplicative* by proving that:

$$f(zw) = f(z)f(w) \quad \text{for all } z, w \in \mathbb{C}.$$

(c) Prove that f is one-to-one. [*Warning:* Since f is not a linear transformation as per our definition in Chapter 5, you shouldn't apply any of the one-to-one criteria given there. However, the definition of *one-to-one* given in Chapter 5 can be applied to any function. That is, f is *one-to-one* if whenever $f(z) = f(w)$ then $z = w$.]

[*Note:* This problem shows that we can "identify" $\mathbb{C}$ with the set of 2×2 real matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, in the sense that every complex number corresponds uniquely to such a matrix, and moreover the addition and multiplication of complex numbers matches the addition and multiplication of these corresponding matrices! This is an example of an *isomorphism*.]

Q4. Let $A \in M_{n\times n}(\mathbb{C})$.

(a) Prove that if $A^2 = -I_n$ and n is odd, then A must have at least one non-real entry.

(b) Show that there is a real 2×2 matrix A such that $A^2 = -I_2$. [**Hint:** Think about **Q3**.]

(c) **Bonus:** Show that if n is even then there exists an $A \in M_{n\times n}(\mathbb{R})$ such that $A^2 = -I_n$.