Q1. (a)

$$\begin{aligned} A^2 &= [T]_\varepsilon^2 \\ &= [T \circ T]_\varepsilon \vec{x} \end{aligned} \tag{1}$$

$$= 2\,\mathrm{proj}_{\vec{v}}\,(2\,\mathrm{proj}_{\vec{v}}\,\vec{x} + 2\,\mathrm{proj}_{\vec{w}}\,\vec{x} - \vec{x}) + 2\,\mathrm{proj}_{\vec{w}}\,(2\,\mathrm{proj}_{\vec{v}}\,\vec{x} + 2\,\mathrm{proj}_{\vec{w}}\,\vec{x} - \vec{x}) - \vec{x} \tag{2}$$

$$= 2\,\mathrm{proj}_{\vec{v}}\,(2\,\mathrm{proj}_{\vec{v}}\,\vec{x}) + 2\,\mathrm{proj}_{\vec{v}}\,(2\,\mathrm{proj}_{\vec{w}}\,\vec{x}) - 2\,\mathrm{proj}_{\vec{v}}\,\vec{x} + 2\,\mathrm{proj}_{\vec{w}}\,(2\,\mathrm{proj}_{\vec{v}}\,\vec{x}) + 2\,\mathrm{prog}_{\vec{w}}\,(2\,\mathrm{proj}_{\vec{w}}\,\vec{x}) - 2\,\mathrm{proj}_{\vec{w}}\,\vec{x} - \vec{x} \tag{3}$$

$$= 4\,\mathrm{proj}_{\vec{v}}\,\vec{x} + \overrightarrow{0} - 2\,\mathrm{proj}_{\vec{v}}\,\vec{x} + \overrightarrow{0} + 4\,\mathrm{proj}_{\vec{w}}\,\vec{x} - 2\,\mathrm{proj}_{\vec{w}}\,\vec{x} - \vec{x} \tag{4}$$

$$= 2\,(\mathrm{proj}_{\vec{v}}\,\vec{x} + \mathrm{proj}_{\vec{w}}\,\vec{x}) - \vec{x} \tag{5}$$

$$= 2\vec{x} - \vec{x} = \bar{x} \tag{6}$$

(b) From part a we know there exist $A^{-1} = A$ s.t.

$$A^{-1}A = [T \circ T]_\varepsilon = AA^{-1} = I_3$$

$\Rightarrow A$ is invertible Therefore, by invertibility criteria - second version, T_A is onto and one to one.

(c)

$$\begin{aligned} T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) &= 2\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} - \begin{bmatrix} a \\ b \\ c \end{bmatrix} \\ &= \begin{bmatrix} 2a \\ 2b \\ 0 \end{bmatrix} - \begin{bmatrix} a \\ b \\ c \end{bmatrix} \\ &= \begin{bmatrix} a \\ b \\ -c \end{bmatrix} \end{aligned} \tag{7}$$

(d) Reflects the vector across one of the x, y or z axis.

Q2. (a)

$$A_1 = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 3 & 3 & 3 \\ -3 & 0 & 3 & 3 \\ -3 & -3 & 0 & 3 \\ -3 & -3 & -3 & 0 \end{bmatrix} \tag{8}$$

(b) $b_1 \det(A_n) = 3^{2n}$ Let $A_n^{-1} = -\frac{1}{3}A_n, A_n^{-1}$ is the inverse of A_n meaning A_n is invertible $\Rightarrow A_n$ is row equivalent to I_{2n}
I_{2n} can undergo $2n$ number of Row scale to become the following

$$\begin{bmatrix} 3 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 3 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 3 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & 0 & \cdots & 3 \end{bmatrix}$$

It is apparent after some row addition, n row swaps and n row scale with -1 following the Gauss-Jordan elimination method.
By Theorem 6.2.1, Effect of ERO on Determinent, we can conclude $\det(A_n) = 1 \cdot 3^{2n} \cdot (-1)^{2n} = 3^{2n}$

(c) By Theorem 6.3.1, since $\det(A_n) \neq 0$,
$\Rightarrow A_n$ is always invertible.

Q3. (a)

$$\begin{aligned}
&\text{i. Let } z = a + b_i \quad w = c + di \\
&f(2 + w) = f(a + bi + c + di) = f((a + c) + (b + d)i) \\
&= \begin{bmatrix} a + c & -(b + d) \\ b + d & a + c \end{bmatrix} \\
&= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \\
&= f(z) + f(w) \\
&\text{ii. Let } 2 = a + bi \\
&f(tz) = f(ta + tbi) \\
&= \begin{bmatrix} ta & -tb \\ tb & ta \end{bmatrix} = t \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \\
&= tf(z)
\end{aligned} \tag{9}$$

(b)

$$\text{b. Let} \quad \begin{aligned}
z &= a + bi \quad w = c + di \\
f(zw) &= f(ca + bi)(c + di)) \\
&= f(ac + adi + bci - bd) \\
&= f((ac - bd) + (ad + bc)i) \\
&= \begin{bmatrix} a(-bd & -ad - cb \\ ad + cb & ac - bd \end{bmatrix} \\
&= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \\
&= f(z) \cdot f(w)
\end{aligned} \tag{10}$$

(c)

$$\begin{aligned}
&\text{C. Let } z = a + bi \quad w = c + di \quad \text{ and } f(z) = f(w) \\
&f\,(a + b_i) = f\,(c + d_i) \\
&\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \\
&\Rightarrow a = c, b = d \\
&\Rightarrow a + bi = c + di \\
&z = w
\end{aligned}$$

Hence, f is one to one

Q4. (a) Proof by contradiction
Assume all entries of A are real and $A^2 = -I_n$
$\det\left(A^2\right) = \det\left(-I_n\right)$

$$
\begin{aligned}
&= -1(n \text{ is odd}) \\
\det\left(A^2\right) &= \det(A)\det(A) = (\det(A))^2 = -1 \\
\det A &= i
\end{aligned}
$$

However, this contradict gar assumption that all entries are veal, otherwise we cant produce a determinant of i, hence the contradiction.
QED

(b)

$$
\begin{aligned}
\text{Let } A &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\
A^2 &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\
&= -I_2
\end{aligned} \tag{11}
$$

(c) Let $A \in M_{h\times n}(\mathbb{R})$ whose i, j th entries ave given by

$$
a_{i,j} = \begin{cases} -1 & \text{if } i < j \text{ and } j = n+1-i \\ 1 & \text{if } i > j \text{ and } i = n+1-j \\ 0 & \text{otherwise} \end{cases}
$$

$$
\begin{aligned}
A^2 = AA &= \begin{bmatrix} 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & -1 & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & -1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & \cdots & 0 & -1 \end{bmatrix} = -I_n
\end{aligned} \tag{12}
$$