Q1. (a) Since λ is an eigenvalue of A $\det(A - \lambda I) = 0$ for matrix $I - A$

$$\begin{aligned}\det(I - A - (1 \cdot \lambda)I) &= \det(I - A - I + \lambda I)\\ &= \det(-A + \lambda I)\\ &= -\det(A - \lambda I)\\ &= -0\\ &= 0\end{aligned}$$

Hence $1 - \lambda$ is an eigenvalue of matrix $I - A$.

(b) Since μ is an eigenvalue of matrix $I - A$

$$\begin{aligned}&\det(I - A - \mu I) = 0.\\ &\det(-A + (I - \mu I)) = 0.\\ &\det(A - (1 - \mu)I) = -0 = 0.\end{aligned}$$

If λ is an eigenvalue of A then $\det(A - \lambda I) = 0$ so there exist λ such that

$$\begin{aligned}-\lambda &= -(1 - \mu)\\ \mu - 1 &= -\lambda\\ \mu &= 1 - \lambda\end{aligned}$$

(c) Proof by Contradiction
Assume $I - A$ is not invertible then $I - A$ must have an eigenvalue where

$$\mu = 0$$

According to Q1.b, there exist an eigenvalue λ tor A where

$$\begin{aligned}&\mu = 1 - \lambda\\ &0 = 1 - \lambda\\ &\lambda = 1\end{aligned}$$

Since $|\lambda| < 1$ for all λ
$I - A$ is invertible, hence the contradiction and the statement is true.

Q2. (a) When $n = 0$.

$$a_n = a_0 = 7$$
$$a_{n+1} = a_1 = 26.$$
$$A^n \cdot \vec{v} = A^0 \cdot \vec{v} = I_2 \cdot \vec{v} = \vec{v} = \begin{bmatrix} 26 \\ 7 \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$$

Assume that when $n = k \quad k \in \mathbb{R} \wedge k \geqslant 0$.

$$A^k\vec{v} = \begin{bmatrix} a_{k+1} \\ a_k \end{bmatrix}$$

when $n = k + 1$

$$\begin{aligned} A^{k+1}\vec{v} &= A \cdot A^k\vec{v} \\ &= \begin{bmatrix} 7 & -10 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{k+1} \\ a_k \end{bmatrix} \\ &= \begin{bmatrix} 7a_{k+1} - 10a_k \\ a_{k+1} \end{bmatrix} \end{aligned}$$

As provided, $a_n = 7a_{n-1} - 10a_{n-2}$

$$7a_{k+1} - 10a_k = a_{k+1+1} = a_{k+2}.$$

$$7a_{k+1} - 10a_k = a_{k+1+1} = a_{k+2}.$$

$$\text{so } A^{k+1}\vec{v} = \begin{bmatrix} a_{k+2} \\ a_{k+1} \end{bmatrix}$$

so statement is also true for $n = k + 1$
Hence by the principle of mathematical induction, the statement is true.

(b) For $A = \begin{bmatrix} 1 & -10 \\ 1 & 0 \end{bmatrix}$
Consider $\det(A - \lambda I) = 0$

$$\begin{aligned} \det\left(\begin{bmatrix} 7-\lambda & -10 \\ 1 & -\lambda \end{bmatrix}\right) &= 0 \\ (7-\lambda)(-\lambda) - 1 \times (-10) &= 0 \\ \lambda^2 - 7\lambda + 10 &= 0 \\ (\lambda - 2)(\lambda - 5) &= 0 \\ \lambda_1 = 2; \lambda_2 &= 5 \end{aligned}$$

For $\lambda_1 = 2$.

Consider $(A - \lambda I)\vec{v} = \overrightarrow{0}$

$$\begin{bmatrix} 5 & -10 \\ 1 & -2 \end{bmatrix} \vec{v} = \overrightarrow{0}$$

$$r_1 \to r_1 - 4r_2$$

$$r_2 \to r_2 - \frac{1}{5}r_1$$

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \vec{v} = \overrightarrow{0}$$

$$x_1 - 2x_2 = 0$$

$$x_2 = x_2$$

$$\Rightarrow \vec{v} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

so $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector of A for $\lambda_1 = 2$. For $\lambda_2 = 5$.
Consider $(A - \lambda I)\vec{v} = \overrightarrow{0}$

$$\begin{bmatrix} 2 & -10 \\ 1 & -5 \end{bmatrix} \vec{v} = \overrightarrow{0}.$$

$$\begin{bmatrix} 1 & -5 \\ 0 & 0 \end{bmatrix} \vec{v} = \overrightarrow{0}$$

$$x_1 - 5x_2 = 0$$

$$x_2 = x_2$$

$$\vec{v} = x_2 \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

so $\vec{v}_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ is an eigenvector of A for $\lambda_2 = 5$.

$$\begin{aligned}
&\text{so } p = \begin{bmatrix} 2 & 5 \\ 1 & 1 \end{bmatrix} \\
&p^{-1} = \frac{1}{\det p} \cdot \begin{bmatrix} 1 & -5 \\ -1 & 2 \end{bmatrix} \\
&= \frac{1}{2-5} \cdot \begin{bmatrix} 1 & -5 \\ -1 & 2 \end{bmatrix} \\
&= -\frac{1}{3} \cdot \begin{bmatrix} 1 & -5 \\ 1 & 2 \end{bmatrix}. \\
&\text{us } A = PDP^{-1} \\
&P^{-1}AP = P^{-1}PDP^{-1}P \\
&D = P^{-1}AP \\
&= -\frac{1}{3} \cdot \begin{bmatrix} 1 & -5 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 7 & -10 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 5 \\ 1 & 1 \end{bmatrix} \\
&= -\frac{1}{3} \begin{bmatrix} 2 & -10 \\ -5 & 10 \end{bmatrix} \cdot \begin{bmatrix} 2 & 5 \\ 1 & 1 \end{bmatrix} \\
&= -\frac{1}{3} \begin{bmatrix} -6 & 0 \\ 0 & -15 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}
\end{aligned}$$

(c) According to Q2.c

$$A^n\vec{v} = \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$$

$$\text{so } a_n = \begin{bmatrix} 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a_{w_n} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \cdot A^n \cdot \begin{bmatrix} 26 \\ 7 \end{bmatrix}$$

according to (b)

$$a_n = \begin{bmatrix} 0 & 1 \end{bmatrix} \left(pOP^{-1}\right)^n \begin{bmatrix} 26 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \underbrace{\left(PDP^{-1}\right)\left(BDP^{-1}\right)\cdots\left(PDP^{-1}\right)}_{PDP^{-1}\times n} \begin{bmatrix} 26 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} P \cdot D^n \cdot P^{-1} \cdot \begin{bmatrix} 26 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 5 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}^n \cdot \left(-\frac{1}{3}\right) \begin{bmatrix} 1 & -5 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 26 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2^n & 0 \\ 0 & 5^n \end{bmatrix} \cdot \left(-\frac{1}{3}\right) \cdot \begin{bmatrix} -9 \\ -12 \end{bmatrix}$$

$$= \begin{bmatrix} 2^n & 5^n \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$a_n = 3 \cdot 2^n + 4 \cdot 5^n$$

Hence the statement is true.

Q3. (a) As T is a linear transformation. If $T(\vec{u}) \neq \vec{o}$ then $\vec{u} \neq \overrightarrow{0}$. Therefore, for all $\vec{w} \in w : \vec{w} \neq \overrightarrow{0}$. $u \neq \overrightarrow{0}$, which means $\overrightarrow{0} \notin S_{\vec{w}}$, so the statement is false.

(b) Since T is a linear transformation.

$$T(\vec{O}) = \overrightarrow{0}$$

Since u and w are subspaces of $\mathbb{F}^n$ $\overrightarrow{0} \in U$ and $\overrightarrow{0} \in W$

$$\text{so} \quad \overrightarrow{0} \in S_w$$

For $\vec{x}, \vec{y} \in Sw$.

$$\begin{aligned} \text{so } \vec{x} &\in U; T(\vec{x}) \in W \\ \vec{y} &\in U; T(\vec{y}) \in W \end{aligned}$$

Consider $c\vec{x} + \vec{y} : c \in \mathbb{R}$
Since u is a subspace of $\mathbb{F}^n$, $c\vec{x} + \vec{y} \in u$
Since W is a subspace of $\mathbb{F}^n$
$C\,(\vec{x}') + T(\vec{y}) \in W$
Since T is a linear transformation.

$$cT(\vec{x}) + T(\vec{y}) = T((\vec{x} + \vec{y})$$

Hence $T(c\vec{x} + \vec{y}) \in W$, $c\vec{x} + \vec{y} \in S_w$
Therefore, the statement is true.

Q4. (a) Since $\left\{\overrightarrow{b_1}, \cdots, \overrightarrow{b_k}\right\}$ is linearly independent then $c_1\vec{b}_1 + \cdots + c_k\vec{b}_k = \overrightarrow{0}$ only has trivial solution. Since $\vec{x}_i$ is a solution to $A\vec{x} = \vec{b}_i$ for $i = 1 \ldots, k$ $A\overrightarrow{x_i} = \overrightarrow{b_1}$
$c_1 A\vec{x}_1 + \cdots + c_k A\vec{x}_k = \overrightarrow{0}$ only has trivial solution.
$A\left(c_1\vec{x}_1 + \cdots + c_k\vec{x}_k\right) = \overrightarrow{0}$ only has trivial solution.
Since $\left\{\overrightarrow{b_1} : \cdots, \overrightarrow{b_{11}}\right\}$ is linearly independent, so there must exist $\overrightarrow{b_i} \neq 0$.
Also we have $A\vec{x}_i = \vec{b}$, so $\operatorname{rank}(A) \neq 0$.
Modifying $A\left(c_1\overrightarrow{x_1} + \cdots + c_k\vec{x}_k\right) = \overrightarrow{0}$ we can have $c_1\vec{x} + \cdots + c_k\overrightarrow{x_k} = \overrightarrow{0}$ has only trivial solution
hence $\{\vec{x}_1, \cdots, \vec{x}_k\}$ is linearly independent.

(b) consider $c_1\vec{b}_1 + \cdots + c_k\vec{b}_k = \overrightarrow{0}$.

$$
\begin{aligned}
c_1 A\overrightarrow{x_1} + \cdots + c_k A\overrightarrow{x_k} &= \overrightarrow{0}. \\
A\left(c_1\overrightarrow{x_1} + \cdots + c_k\overrightarrow{x_k}\right) &= \overrightarrow{0} \\
c_1\overrightarrow{x_1} + \cdots + c_k\overrightarrow{x_k} &\in \operatorname{Null}(A)
\end{aligned}
$$

Since $\operatorname{rank}(A) = n$, Null $(A) = \{\overrightarrow{0}\}$
so $A\left(c_1\vec{x}_1 + \cdots + c_k\overrightarrow{x_k}\right) = \overrightarrow{0}$, if and only if $c_1\overrightarrow{x_1} + \cdots + c_k\overrightarrow{x_k} = \overrightarrow{0}$. as $\{\vec{x}_1, \cdots, \vec{x}_k\}$ is linearly independent.
So $A\left(c_1\overrightarrow{x_1} + \cdots + c_k\vec{x}_k\right) = 0$ has only trivial solution, therefore, $\left\{\overrightarrow{b_1}, \cdots, \overrightarrow{b_k}\right\}$ is linearly independent.