Instructions

- Please upload your solutions into the appropriate slot on Crowdmark.
- The **coverage** for this assignment is up to section 5.4 (inclusive). Your solutions should not use material from any later sections. You are allowed to use material from earlier sections. You are also allowed to use any results that appear in Practice Problem lists 1–6 (but please make sure to clearly cite them).
- You can earn a 0.25 course grade bonus for typesetting your solutions in LaTeX (or equivalent typesetting software). Please see the course outline for more details.

Problems

Q1. Let $A = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}$, where $a \in \mathbb{F}$. Prove that

$$A^n = \begin{bmatrix} a^n & na^{n-1} & \frac{n(n-1)}{2}a^{n-2} \\ 0 & a^n & na^{n-1} \\ 0 & 0 & a^n \end{bmatrix} \quad \text{for all integers } n \geq 1.$$

Q2. We say that a function $T\colon \mathbb{F}^n \to \mathbb{F}^m$ is **invertible** if there exists a function $L\colon \mathbb{F}^m \to \mathbb{F}^n$ such that

$$L(T(\vec{v})) = \vec{v} \quad \text{for all } \vec{v} \in \mathbb{F}^n$$

and

$$T(L(\vec{u})) = \vec{u} \quad \text{for all } \vec{u} \in \mathbb{F}^m.$$

Such a function L, if it exists, is said to be an **inverse** of T.

(a) Let $T\colon \mathbb{F}^n \to \mathbb{F}^m$ be an invertible function. Prove that if $L_1\colon \mathbb{F}^m \to \mathbb{F}^n$ and $L_2\colon \mathbb{F}^m \to \mathbb{F}^n$ are inverses of T, then

$$L_1(\vec{u}) = L_2(\vec{u}) \quad \text{for all } \vec{u} \in \mathbb{F}^m.$$

[Thus, L_1 and L_2 are the same function. So an inverse of T, if it exists, is unique, and we may call it **the** inverse of T.]

(b) Prove that if $T\colon \mathbb{F}^n \to \mathbb{F}^m$ is an invertible *linear transformation*, and if $L\colon \mathbb{F}^m \to \mathbb{F}^n$ is the inverse of T, then L is a linear transformation.

(c) Let $T_A\colon \mathbb{F}^n \to \mathbb{F}^n$ be the linear transformation determined by a matrix $A \in M_{n\times n}(\mathbb{F})$. Prove that

$$T_A \text{ is invertible if and only if } A \text{ is invertible.}$$

In the case where T_A is invertible, determine the inverse of T_A.

Q3. Let $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{F}^n$ be such that $\mathbb{F}^n = \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_k\}$. Let $T\colon \mathbb{F}^n \to \mathbb{F}^m$ and $S\colon \mathbb{F}^n \to \mathbb{F}^m$ be linear transformations that satisfy

$$T(\vec{v}_i) = S(\vec{v}_i) \quad \text{for all } i = 1, \ldots, k.$$

Prove that

$$T(\vec{v}) = S(\vec{v}) \quad \text{for all } \vec{v} \in \mathbb{F}^n.$$

Q4. Prove/disprove:

(a) The function $T\colon \mathbb{R}^n \to \mathbb{R}$ defined by $T(\vec{v}) = \|\vec{v}\|$ is a linear transformation.

(b) Let $\vec{x} \in \mathbb{R}^n$ be a fixed vector. The function $T\colon \mathbb{R}^n \to \mathbb{R}$ defined by $T(\vec{v}) = \vec{v} \cdot \vec{x}$ is a linear transformation.

(c) Let $A \in M_{n\times n}(\mathbb{F})$ be an *invertible* matrix, and let $T_A\colon \mathbb{F}^n \to \mathbb{F}^n$ be the linear transformation determined by A. For all $\vec{y} \in \mathbb{F}^n$, there exists a *unique* $\vec{x} \in \mathbb{F}^n$ such that $T_A(\vec{x}) = \vec{y}$.

(d) Let $A, B \in M_{n\times n}(\mathbb{F})$. Suppose that $AB = A^T$ and that A is invertible. Then B must be invertible.