Instructions

- Please upload your solutions into the appropriate slot on Crowdmark.
- The **coverage** for this assignment is up to 8.3 (inclusive). Your solutions should not use material from any later sections. You are also allowed to use any results that appear in Practice Problem lists 1–8 (but please make sure to clearly cite them).
- You can earn a 0.25 course grade bonus for typesetting your solutions in LaTeX (or equivalent typesetting software). Please see the course outline for more details.

Problems

Q1. Let $A \in M_{n\times n}(\mathbb{F})$.

(a) Prove that if λ is an eigenvalue of A, then $1-\lambda$ is an eigenvalue of $I-A$.

(b) Prove, conversely, that if μ is an eigenvalue of $I-A$, then there exists an eigenvalue λ of A such that $\mu = 1-\lambda$.

(c) Suppose that all the eigenvalues λ of A satisfy $|\lambda| < 1$. Prove that $I-A$ must be invertible.

Q2. A Fall 2022 MATH 135 assignment introduced the recursive sequence $(a_0, a_1, \ldots)$ defined by

$$a_0 = 7, \quad a_1 = 26 \quad \text{and} \quad a_n = 7a_{n-1} - 10a_{n-2} \text{ for } n \geq 2.$$

Let $A = \begin{bmatrix} 7 & -10 \\ 1 & 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 26 \\ 7 \end{bmatrix}$.

(a) Using the convention that $A^0 = I_2$, prove that $A^n\,\vec{v} = \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$ for all integers $n \geq 0$.

(b) Determine an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$.

(c) Using parts (a) and (b), prove that an explicit formula for the nth term of this recursive sequence is $a_n = 3 \cdot 2^n + 4 \cdot 5^n$.
[**Note:** This explicit formula was given on the MATH 135 assignment. In this problem, you must use parts (a) and (b) in order for your proof to earn any marks.]

Q3. Let U and W be subspaces of $\mathbb{F}^n$ and let $T\colon \mathbb{F}^n \to \mathbb{F}^n$ be a linear transformation. Prove/disprove:

(a) For all $\vec{w} \in W$, the set $S_{\vec{w}} = \{\vec{u} \in U : T(\vec{u}) = \vec{w}\}$ is a subspace of $\mathbb{F}^n$.

(b) The set $S_W = \{\vec{u} \in U : T(\vec{u}) \in W\}$ is a subspace of $\mathbb{F}^n$.

Q4. Let $A \in M_{m\times n}(\mathbb{F})$. Let $\{\vec{b}_1, \ldots, \vec{b}_k\} \subseteq \mathbb{F}^m$. Let $\vec{x}_i$ be a solution to $A\vec{x} = \vec{b}_i$ for $i = 1, \ldots, k$.

(a) Prove that if $\{\vec{b}_1, \ldots, \vec{b}_k\}$ is linearly independent, then $\{\vec{x}_1, \ldots, \vec{x}_k\}$ is linearly independent.

(b) Assume that $\operatorname{rank}(A) = n$. Prove that if $\{\vec{x}_1, \ldots, \vec{x}_k\}$ is linearly independent, then $\{\vec{b}_1, \ldots, \vec{b}_k\}$ is linearly independent.