Q1. Proof by Induction

Base Case

$$n = 2$$

$$A^2 = AA = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix} \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}$$

$$= \begin{bmatrix} a^2 & 2a & 1 \\ 0 & a^2 & 2a \\ 0 & 0 & a^2 \end{bmatrix}$$

$$= \begin{bmatrix} a^2 & 2a^{2-1} & \frac{2(2-1)}{2}a^{2-2} \\ 0 & a^2 & 2a^{2-1} \\ 0 & 0 & a^2 \end{bmatrix}$$

Base case holds.

$$A^{k+1} = A^k A$$

$$= \begin{bmatrix} a^k & ka^{k-1} & \frac{k(k-1)}{2}a^{k-2} \\ 0 & a^k & ka^{k-1} \\ 0 & 0 & a^k \end{bmatrix} \cdot \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}$$

$$= \begin{bmatrix} a^{k+1} & a^k + ka^k & ka^{k-1} + \frac{k(k-1)}{2}a^{k-1} \\ 0 & a^{k+1} & a^k + ka^k \\ 0 & 0 & a^{k+1} \end{bmatrix}$$

$$= \begin{bmatrix} a^{k+1} & (k+1)a^k & \frac{(k+1)k}{2}a^{k-1} \\ 0 & a^{k+1} & (k+1)a^k \\ 0 & 0 & a^{k+1} \end{bmatrix}$$

Hence, by principle of mathematical induction, the statement is true.
QED

Q2. (a) Assume $L_1 : \mathbb{F}^m \to \mathbb{F}^n$ and $L_2 : \mathbb{F}^n \to \mathbb{F}^{\mathrm{m}}$ are inverses of T
$T\left(L_1(\vec{v})\right) = \vec{v}$, for all $\vec{v} \in \mathbb{F}^m$ (Definition of inverse)
$L_2\left(T\left(L_1(\vec{v})\right)\right) = L_2(\vec{v})$ (Apply L_2 to both sides of the equation)
$L_1\vec{v} = L_2\vec{v}$ (Definition of inverse)
QED

(b) Assume $T : \mathbb{F}^n \to \mathbb{F}^m$ is an invertible linear transformation and $L : \mathbb{F}^m \to \mathbb{F}^n$ is its inverse

$$L(T(\vec{x}+\vec{y}))$$
$$=L(T(\vec{x})+T(\vec{y})) \text{ (Definition of Linear Transformation)}$$
$$=\vec{x}+\vec{y}(\text{ Definition of Inverse)}$$
$$=L(T(\vec{x}))+L(T(\vec{y})) \text{ (Definition of Linear Transformation)}$$

Since we know T is linear transformation, it is also onto.

$$\Rightarrow \{T(\vec{x})+T(\vec{y})\} = \mathbb{F}^m$$
$$L(T(c\vec{x}))$$
$$= L(CT(\vec{x})) \text{ (Definition of Linear Transformation)}$$
$$= c\vec{x} \text{ (Definition of Inverse)}$$
$$= cL(T(\vec{x})) \text{ (Definition of Inverse)}$$

Similarly, since T is onto

$$\Rightarrow \{T(c\vec{x})\} = \mathbb{F}^m$$

since L satisfies the clefinition of a linear transformation, L is a linear transformation.
QED

(c) Assume $T_A : \mathbb{F}^n \to \mathbb{F}^m$ is a linear trans formation determined by $A \in M_{u\times n}(\mathbb{F})$
$\Rightarrow$
Assume T_A is invertible,

$$T_A = A\vec{x}, \vec{x} \in \mathbb{F}^n$$

Let L also be determined by a matrix B

$$L_B = B\vec{x}, \vec{x} \in \mathbb{F}^m$$
$$L_B\left(T_A(\vec{x})\right) = \vec{x} \text{ (Definition of Inverse)}$$
$$L_B(A\vec{x}) = \vec{x}$$
$$BA\vec{x} = \vec{x}$$
$$BA = I_n$$

$\Rightarrow A$ is invertible when L is determined by B, inverse of A
$\Leftarrow$
Assume A is invertable

$$AA^{-1} = I_n \text{ (Definition of Invert table)}$$
$$AA^{-1}\vec{x} = \vec{x}$$
$$T_A\left(A^{-1}\vec{x}\right) = \vec{x}$$

Also ,

$$A^{-1}A\vec{x} = \vec{x}$$
$$A^{-1}T_A(\vec{x}) = \vec{x}$$

Hence, we can create a matrix function $L : \mathbb{F}^n \to \mathbb{F}^m$ determined by matrix A^{-1} where when A^{-1} subbed into above equations

$$T_A\left(L_{A^{-1}}(\vec{x})\right) = \vec{x}$$
$$L_{A^{-1}}\left(T_A(\vec{x})\right) = \vec{x}$$

which satisfies the definition of invertability for T_A.
QED

Q3. Direct Proof

Let $\vec{u}$ be an aubitvary vector $\in \mathbb{F}^n$, since $\mathbb{F}^h = \operatorname{span}\{\vec{v}_1 \ldots v_k\}$

$$\vec{u} = a\vec{v}_1 + \cdots + k\vec{v}_k$$

$$\begin{aligned} T(\vec{u}) &= T\left(a\vec{v}_1 + \cdots + k\vec{v}_k\right) \\ &= aT\left(\vec{v}_1\right) + \cdots + kT\left(\vec{v}_k\right) \quad \text{(Definition of Linear Transformation)} \\ &= aS\left(\overrightarrow{v_1}\right) + \cdots + ks\left(\vec{v}_k\right) \text{ (Sub with } T\left(\vec{v}_j\right) = s\left(\vec{v}_i\right) \\ &= s\left(a\vec{v}_1\right) + \cdots + s\left(k\vec{v}_k\right) \quad \text{(Definition of Linear Transformation)} \\ &= s\left(a\vec{v}_1 + \cdots + k\vec{v}_k\right) \quad \text{(Definition of Linear Transformation)} \\ &= s(\vec{u}) \end{aligned}$$

since $\vec{u}$ is arbitrary,
$T(\vec{v}) = S(\vec{v})$ for all $\vec{v} \in \mathbb{F}^h$

QED

Q4. (a) False, disproof by counterexample
Let $\vec{v} \in \mathbb{R}^n$, pick $\vec{v} = -\vec{u}$ for $\vec{u} \in \mathbb{R}^n \Rightarrow T(\vec{v}) = T(-\vec{u}) = \|\vec{u}\|$ However,

$$-T(\vec{u}) = -\|\vec{u}\| \neq \|\vec{u}\|$$

Hence, the statement is false

(b) Direct Proof
Let $\vec{u}, \vec{v} \in \mathbb{R}^n$

$$\begin{aligned} \Rightarrow T(\vec{u}+\vec{v}) &= \vec{x}(\vec{u}+\vec{v}) \\ &= \vec{u}\vec{x} + \vec{v}\vec{x} \\ &= T(\vec{u}) + T(\vec{v}) \\ \Rightarrow T(c\vec{v}) &= c\vec{v} \cdot \vec{x} \\ &= cT(\vec{v}) \end{aligned}$$

Therefore, T satisfies definition of linear transfornmtion.
QED

(c) From Q2.c, we know A is invertible
By invertibility criteria, T_A is one to one which implies that whenever

$$\begin{aligned} &T_A(\vec{x}) = T_A(\vec{y}) \\ &\Rightarrow \vec{x} = \vec{y} \end{aligned}$$

which shows uniqueness.
By invertibility criteria, T_A is also onto,
$\Rightarrow$ Range $(T) = \mathbb{F}^n$ and $\ddot{y} \in \mathbb{F}^n$ which shows existence.
QED

(d) Let $A^{-1} \in M_{u\times n}$ be inverse of A

$$\begin{aligned} AB &= A^\top \\ A^{-1}AB &= A^{-1}A^\top \\ I_nB &= A^{-1}A^\top \\ B &= A^{-1}A^\top \\ \Rightarrow B^{-1} &= \left(A^\top\right)^{-1} A \end{aligned}$$

We know $\left(A^\top\right)^{-1}$ exists since it is just $\left(A^{-1}\right)^\top$

$$\begin{aligned} B^{-1}B &= \left(A^\top\right)^{-1} AA^{-1}A^\top \\ &= I_n \\ BB^{-1} &= A^{-1}A^\top \left(A^\top\right)^{-1} A \\ &= I_n \end{aligned}$$

so B is invertable.
QED