# Superphysics 2021/2022

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## Contents

# 1 SESSION 01

## 1.1 Newton's $2^{\text{nd}}$ law

Newton's second law for a point mass[1] $m$ is a vector equation and we can write it as

$$\mathbf{F} = m\mathbf{a}.$$

We use boldface fonts for vector quantities; here $\mathbf{F}$ and $\mathbf{a}$ denote the net force acting on the particle and its acceleration.

In 1d and using IB notation this is

$$F = ma = m\frac{\Delta v}{\Delta t} = m\frac{\Delta(\Delta x/\Delta t)}{\Delta t}, \tag{1}$$

where $x$ is the position of the mass in some inertial reference frame and $v$ its velocity.

The message of the second law is this: if we know the forces acting on the mass *now* we can work out where the mass will move *next*. Solving the equation means finding the particle position at any time, i.e. the function $x(t)$. This can be done using calculus as we will see in a few weeks. For now notice that we should write the second law properly as

$$F = m\frac{dv}{dt} = m\frac{d(dx/dt)}{dt} \equiv m\frac{d^2x}{dt^2}, \tag{2}$$

where the last line is just notation to indicate the rate of change of the rate of change of position, i.e. the second derivative of position with respect to time.

The difference between $\Delta t$ and $dt$ is:

$$\begin{aligned} \Delta t &= \text{some small but fixed and finite time span} \\ dt &= \text{an infinitesimally small time span, i.e. ever increasingly smaller and smaller} \end{aligned}$$

Equation (2) is exact but equation (1) is not because the symbol $\Delta$ indicates we are just looking at the average velocity $\Delta x/\Delta t$ and average acceleration $\Delta v/\Delta t$ over some finite time span $\Delta t$. Instead, the second law is about the *instantaneous* acceleration $a(t)$ being related to the net force $F$: as long as we are just looking at average values, we will not be able to find an exact solution. That said, we can still obtain approximate solutions that can give us a very good sense of how the point mass $m$ is going to move and this is what we are going to look at now.

## 1.2 Problem 1: mass within a uniform gravitational field with drag (1d)

To see how this works, let's consider a point mass $m$ under the action of a uniform (constant) gravitational field $g$ and air drag. For simplicity we will use a linear model for drag, i.e. $f = -\alpha v$, where we will refer to $\alpha$ as the drag constant.

[1] By *point mass* or *particle* we mean an object whose detailed shape and internal structure can be ignored in the particular problem under study. For example, if we are interested in the earth's orbital motion around the sun we can treat it as a point mass. However, if the problem was that of explaining the tides we could not ignore the fact that the earth is roughly spherical.

If we choose a vertical axis pointing downwards the net force is

$$F = mg - \alpha v.$$

Let's now choose the *initial conditions* to be

$$\begin{aligned} x_0 &= 0 \\ v_0 &= 0, \end{aligned}$$

corresponding to a mass falling from rest.

Using the expression for the total force and the second law we can easily find the initial value of the acceleration $a(0)$:

$$F = ma(0) = mg - \alpha v_0 = mg \Rightarrow a(0) = g.$$

The mass will start falling with an acceleration $g$, the speed will increase and as a result drag will kick in, causing the net force and the acceleration to decrease until a terminal speed is achieved.

The bottom line is: acceleration does not stay constant. *However* we can ask where the mass will go after a *very short* time $\Delta t$. By *short* we mean that during the time interval $\Delta t$ the change of velocity is so small that we can effectively neglect the variation of the acceleration and treat it as if it stays constant.

Within this approximation we can find the new position and the new velocity at the time $t = \Delta t$:

$$\begin{aligned} x(t = \Delta t) &= \frac{1}{2} a \Delta t^2 \\ v(t = \Delta t) &= a \Delta t \end{aligned}$$

We now have a new pair of values for position and velocity. These are like new initial conditions and we can repeat the process: find the net force and the acceleration and then work out the new position and velocity after another $\Delta t$ time interval. And so on. This process can be iterated until we obtain an array of positions and velocities at many different times set apart by the interval $\Delta t$. All of this can be formalized through an algorithm, i.e. a well defined list of instructions which can then be computed numerically by a computer. We will now turn to this task.

### 1.2.1 Euler's method applied to Newton's 2$^{\text{nd}}$ law

This, $x^3 - 3x + 1 = 0$, is an example of algebraic equation: the unknown $x$ is a *number* and solving the equation means finding the numerical values of $x$ that make it true.

Newton's second law belongs to a different type of equations known as *differential equations*. The unknown is not a number but rather a *function* $x(t)$ and the equation is a relationship involving the function itself and its rates of change, specifically the first and second derivatives:

$$F(x, dx/dt) = m\frac{d^2x}{dt^2}.$$

Here $F(x, dx/dt)$ is the net force acting on the particle. In general it is represented mathematically by a function of both position $x$ and velocity $dx/dt$. Solving the differential equation means finding the functions $x(t)$ that make it true.

We will soon learn how to solve some differential equations using calculus; for now, following the discussion of the previous paragraph, we can write an *algorithm* to find approximate solutions. What we will use is known as Euler's method and it was originally devised by the mathematician Leonhard Euler (1707 - 1783) as a general way to solve differential equations numerically.

Assigning a specific 1d problem means knowing how the net force $F$ depends on the point mass position and velocity. This is in general expressed by some function of two variables $F(x, v)$. For example in the case of a point mass within a uniform gravitational field and moving through a viscous fluid we have,

$$F(x, v) = mg - \alpha v.$$

where in this case the force depends only on the velocity but not on the position.

Euler's algorithm is this:

1. Choose a short time interval $\Delta t$
2. Choose initial conditions $x_0$, $v_0$ at $t_0 = 0$
3. Find the initial net force $F_0 = F(x_0, v_0)$
4. Use Newton's equation to find the initial acceleration $a_0 = F_0/m$
5. Treating the accelearation as constant, find the new position at the time $t_1 = t_0 + \Delta t$
$$x_1 = x_0 + v_0\Delta t + \frac{1}{2}a_0\Delta t^2$$
6. Find the new velocity at the time $t_1$
$$v_1 = v_0 + a_0\Delta t$$
7. Repeat steps 2 to 6 in loop using as initial conditions the values of position and velocity found at the end of the previous step

The result of this process will be two long lists of numbers:

$$\{x_0, x_1, x_2, x_3, \ldots\}$$

$$\{v_0, v_1, v_2, v_3, \ldots\}$$

yielding the positions and velocities at times $\{t_0, t_1, t_2, t_3, \ldots\}$. It is important to stress that these values will not be exact. However the approximation will be better and better as the time increment $\Delta t$ is chosen to be smaller and smaller. The result can then be plotted, e.g. in a $v$ vs $t$ graph:

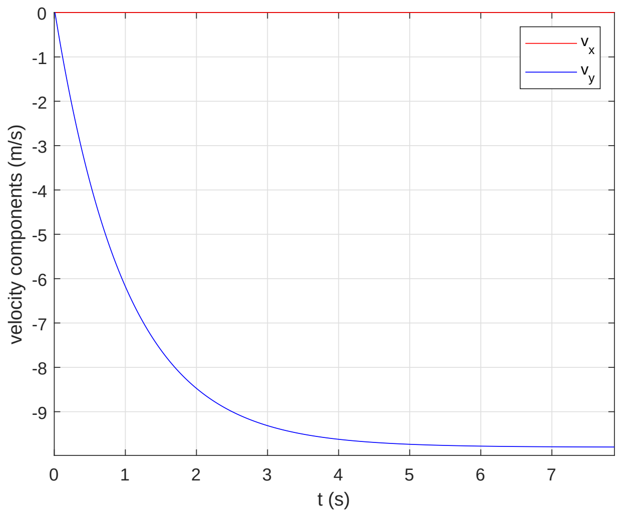



Figure 1: Euler's method applied to a mass falling from rest: $v_y$ vs $t$ graph showing terminal velocity.

# 2 SESSION 02

## 2.1 Problem 2: mass within a uniform gravitational field with drag (2d)

We will now re-consider free fall with drag but allowing for more general initial conditions that may result in a curved 2d path. We start by writing Newton's $2^{\text{nd}}$ law in vector form

$$\begin{pmatrix} F_x \\ F_y \end{pmatrix} = m \begin{pmatrix} a_x \\ a_y \end{pmatrix}.$$

This time we choose the $y$-axis directed upwards and we have

$$\begin{pmatrix} -\alpha v_x \\ -mg - \alpha v_y \end{pmatrix} = m \begin{pmatrix} a_x \\ a_y \end{pmatrix}.$$

The acceleration components follow as:

$$\begin{aligned} a_x &= -\frac{\alpha}{m} v_x \\ a_y &= -g - \frac{\alpha}{m} v_y. \end{aligned}$$

Euler's algorithm proceeds as in 1d except that now we have to consider both the $x$ and $y$ components:

1. Choose a short time interval $\Delta t$
2. Choose initial conditions $(x_0, y_0)$, $(v_{x0}, v_{y_0})$ at $t_0 = 0$
3. Find the initial net force components $(F_{x0}, F_{y_0})$
4. Use Newton's equation to find the initial acceleration components $(a_{x0}, a_{y_0}) = (1/m)(F_{x0}, F_{y_0})$
5. Find the new position $x$ and $y$ coordinates at the time $t_1 = t_0 + \Delta t$

$$\begin{aligned} x_1 &= x_0 + v_{x0}\Delta t + \frac{1}{2} a_{x0} \Delta t^2 \\ y_1 &= y_0 + v_{y_0}\Delta t + \frac{1}{2} a_{y_0} \Delta t^2 \end{aligned}$$

6. Find the new velocity components at the time $t_1$

$$\begin{aligned} v_{x1} &= v_{x0} + a_{x0}\Delta t \\ v_{y_1} &= v_{y_0} + a_{y_0}\Delta t \end{aligned}$$

7. Repeat steps 2 to 6 in loop using as initial conditions the values of position and velocity found at the end of the previous step

# 3 SESSION 03

## 3.1 Implementing Euler's algorithm in the 2-dimensional case

We can implement the algorithm outlined in the previous session in Excel. This is not the ideal tool because we need many iterations that will result in a very long data table. The advantage for now is that coding in a spreadsheet is a very interactive process and it allows to visualize everything on the fly as you go. Figure 2 shows the first 16 iterations but the full data table is over 1000 rows long.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| g | | -9.8 | | | | | | |
| m | | 1 | | | | | | |
| alpha | | 4 | | | | | | |
| Delta t | | 0.01 | | | | | | |
| | | | | | | | | |
| t | x | y | vx | vy | Fx | Fy | ax | ay |
| 0.00 | 0.000000 | 0.000000 | 10.000000 | -10.000000 | -40.000000 | 30.200000 | -40.000000 | 30.200000 |
| 0.01 | 0.098000 | -0.098490 | 9.600000 | -9.698000 | -38.400000 | 28.992000 | -38.400000 | 28.992000 |
| 0.02 | 0.192080 | -0.194020 | 9.216000 | -9.408080 | -36.864000 | 27.832320 | -36.864000 | 27.832320 |
| 0.03 | 0.282397 | -0.286710 | 8.847360 | -9.129757 | -35.389440 | 26.719027 | -35.389440 | 26.719027 |
| 0.04 | 0.369101 | -0.376671 | 8.493466 | -8.862567 | -33.973862 | 25.650266 | -33.973862 | 25.650266 |
| 0.05 | 0.452337 | -0.464014 | 8.153727 | -8.606064 | -32.614908 | 24.624255 | -32.614908 | 24.624255 |
| 0.06 | 0.532243 | -0.548844 | 7.827578 | -8.359821 | -31.310312 | 23.639285 | -31.310312 | 23.639285 |
| 0.07 | 0.608954 | -0.631260 | 7.514475 | -8.123428 | -30.057899 | 22.693714 | -30.057899 | 22.693714 |
| 0.08 | 0.682596 | -0.711360 | 7.213896 | -7.896491 | -28.855583 | 21.785965 | -28.855583 | 21.785965 |
| 0.09 | 0.753292 | -0.789235 | 6.925340 | -7.678632 | -27.701360 | 20.914527 | -27.701360 | 20.914527 |
| 0.10 | 0.821160 | -0.864976 | 6.648326 | -7.469486 | -26.593305 | 20.077946 | -26.593305 | 20.077946 |
| 0.11 | 0.886314 | -0.938667 | 6.382393 | -7.268707 | -25.529573 | 19.274828 | -25.529573 | 19.274828 |
| 0.12 | 0.948861 | -1.010390 | 6.127098 | -7.075959 | -24.508390 | 18.503835 | -24.508390 | 18.503835 |
| 0.13 | 1.008907 | -1.080225 | 5.882014 | -6.890920 | -23.528055 | 17.763681 | -23.528055 | 17.763681 |
| 0.14 | 1.066550 | -1.148246 | 5.646733 | -6.713284 | -22.586932 | 17.053134 | -22.586932 | 17.053134 |
| 0.15 | 1.121888 | -1.214526 | 5.420864 | -6.542752 | -21.683455 | 16.371009 | -21.683455 | 16.371009 |

Figure 2: Sample of the data generated in Excel: the parameters are $m = 1\,\text{kg}$ and $\alpha = 4\,\text{kg/s}$.

When working numerically it is preferable to visualize the solution graphically. A particular solution is shown in Figures 3 and 4, which have been produced with Matlab. The initial conditions were chosen so that the mass is shot at a height of $1000\,\text{m}$, with an angle of $45°$ degrees and an initial speed of $30\,\text{m/s}$. It is worth remarking how the horizontal component of the velocity eventually approaches zero. This also reflects in the path of the particle which tends to become vertical.

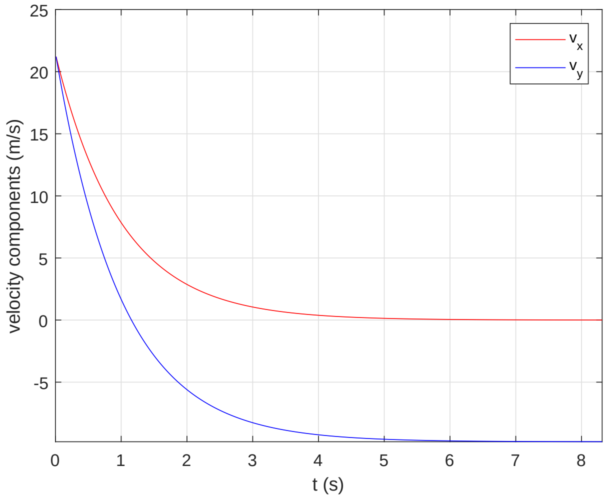



Figure 3: Horizontal and vertical velocity components vs time.

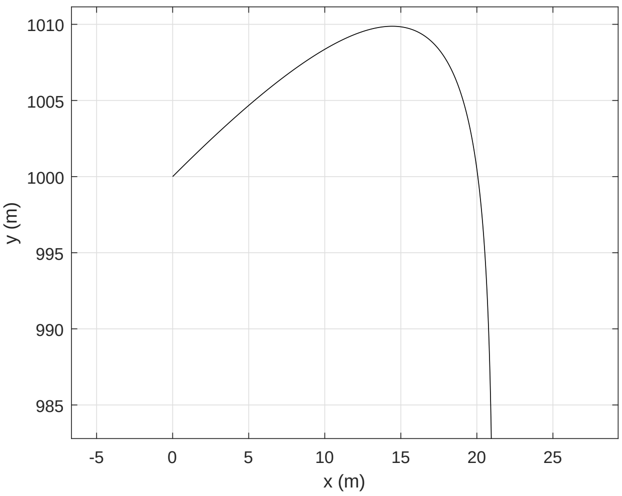



Figure 4: Path of the particle in the $x$-$y$ plane.

### 3.2 Problem 3: mass moving within a fixed radial gravitational field

Euler's algorithm can easily be applied to a different problem. An interesting area, which led to many important developments in both physics and mathematics, is that of celestial mechanics. In particular, the problem of two or more masses interacting gravitationally is very interesting.

Let's start by considering two interacting masses where one of them, $M$, is much larger than the other, $m$. A radial gravitational field is created by the point mass $M$ which is fixed at the origin of the coordinates. The field's magnitude at point $P(x, y)$ is given by

$$g = \frac{GM}{r^2},$$

where $r = |\mathbf{OP}| = \sqrt{x^2 + y^2}$. The second point mass $m$ has initial conditions such that its initial velocity vector lies in the $x$-$y$ plane and we want to find its path. We will do this in the next session.

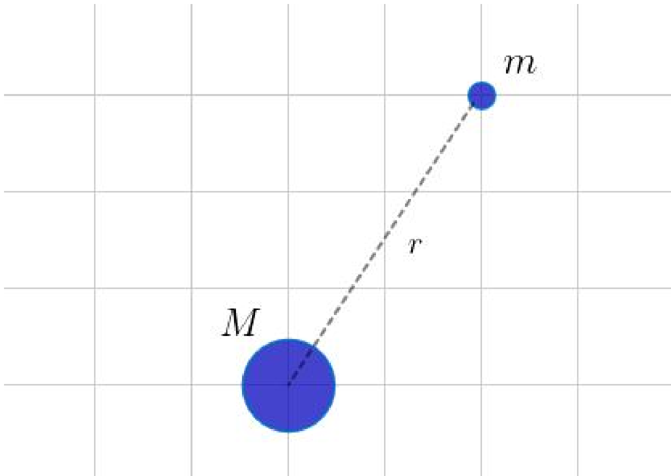



Figure 5: The point mass $M$ is the source of a static, radial gravitational field; the point mass $m$ is moving under the action of the field.

# 4 SESSION 04

## 4.1 Mass moving within a radial gravitational field

### 4.1.1 Force components

With reference to Figure 5 we want to study the motion of point mass $m$ within the gravitational field created by the point mass $M$.

You have to be aware that in reality, in the case of two interacting stars, *both* masses will move because they are mutually acting on each other. However, if one mass is much larger than the other (as in the case of the sun and the earth), then it is safe to neglect its movement. This follows from Newton's second and third laws. According to these we can write down the accelerations $a_m$ and $a_M$ the two masses will have due to their mutual gravitational attraction:

$$\begin{aligned} a_m &= \frac{F}{m} \\ a_M &= -\frac{F}{M}, \end{aligned}$$

where $F$ is the magnitude of the gravitational force acting on each mass. You can see that if $M \gg m$ then $a_M \ll a_m$, meaning that we can neglect the acceleration of the larger mass $M$ and consider it at rest. In what follows we will indeed assume that $M \gg m$ and approximate that $M$ will stay still at the origin of the coordinates.

Figure 6 shows the reference frame that we will use to solve the problem.

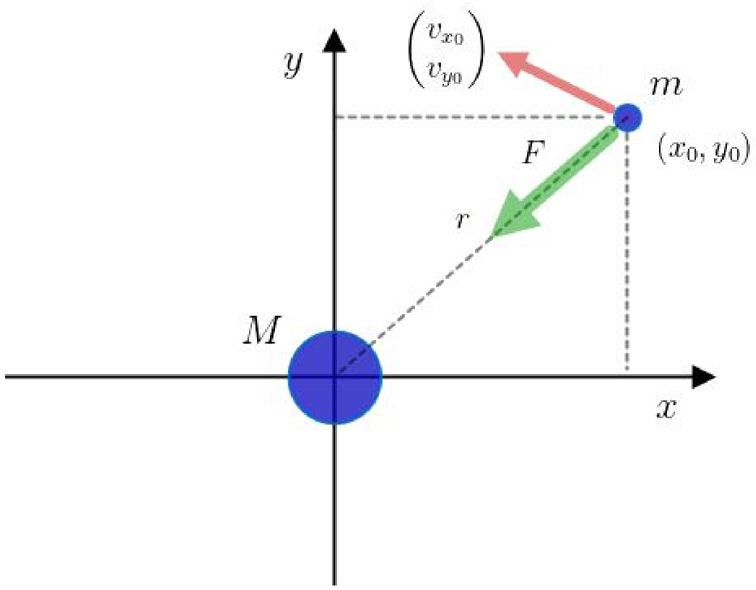



Figure 6: The axes are chosen in such a way that the initial velocity (red vector) is on the $x$-$y$ plane and the mass $M$ is at the origin.

The distance between the two masses is

$$r = \sqrt{x^2 + y^2}, \tag{3}$$

hence the magnitude of the gravitational force is

$$F = \frac{GMm}{r^2}.$$

Finding the force components is very easy if we refer to the following diagram:

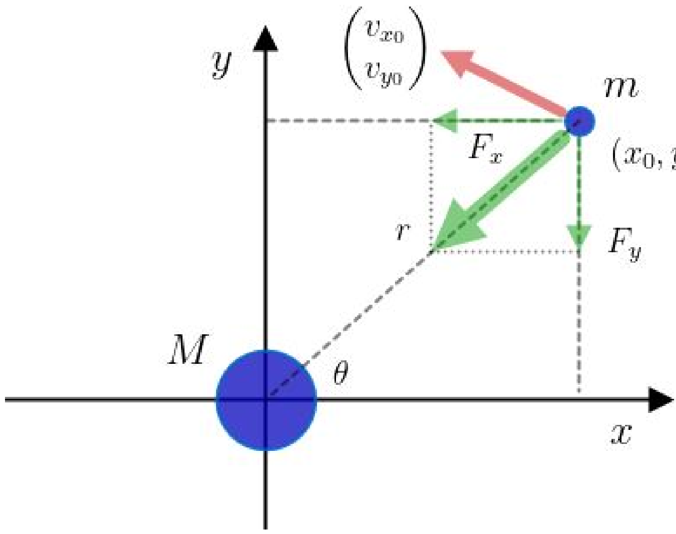



Figure 7: The gravitational force components are shown, as well as the angle $\theta$ between the $x$ axis and the position vector of mass $m$.

You can see that

$$F_x = -F\cos\theta = -F\frac{x}{r} = -\frac{GMmx}{r^3},$$
$$F_y = -F\sin\theta = -F\frac{y}{r} = -\frac{GMmy}{r^3},$$

where we used the fact that $\cos\theta = x/r$ and $\sin\theta = y/r$. Finally, using equation (3), we have that

$$F_x = -\frac{GMmx}{(x^2 + y^2)^{\frac{3}{2}}}$$
$$F_y = -\frac{GMmy}{(x^2 + y^2)^{\frac{3}{2}}}$$

and, from Newton's second law, we find the acceleration components which, of course, do not depend on the mass $m$:

$$a_x = -\frac{GMx}{(x^2 + y^2)^{\frac{3}{2}}}$$
$$a_y = -\frac{GMy}{(x^2 + y^2)^{\frac{3}{2}}}.$$

### 4.1.2 Implementing the solution on Excel

Now we can implement Euler's method and obtain an approximate solution of Newton's equation numerically on Excel. We already have the code from the previous session, all we have to do is input the new expressions for the force components:

SUM | fx =-$C$1*$C$2*$C$3*B7/(B7^2+C7^2)^1.5

| | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|
| 1 | G | | 56 | gravit. const. | | | | | |
| 2 | M | | 50 | gravit. source | | | | | |
| 3 | m | | 1 | mass | | | | | |
| 4 | Delta t | | 0.001 | time step | | | | | |
| 5 | | | | | | | | | |
| 6 | t | x | y | vx | vy | Fx | Fy | ax | ay |
| 7 | 0.000 | 10.000000 | 0.000000 | 0.000000 | 19.000000 | =-$C$1*$C$2*$( | 0.000000 | -28.000000 | 0.000000 |
| 8 | 0.001 | 9.999986 | 0.019000 | -0.028000 | 19.000000 | -27.999927 | -0.053200 | -27.999927 | -0.053200 |
| 9 | 0.002 | 9.999944 | 0.038000 | -0.056000 | 18.999947 | -27.999707 | -0.106399 | -27.999707 | -0.106399 |
| 10 | 0.003 | 9.999874 | 0.057000 | -0.084000 | 18.999840 | -27.999341 | -0.159598 | -27.999341 | -0.159598 |
| 11 | 0.004 | 9.999776 | 0.076000 | -0.111999 | 18.999681 | -27.998828 | -0.212795 | -27.998828 | -0.212795 |
| 12 | 0.005 | 9.999650 | 0.094999 | -0.139998 | 18.999468 | -27.998170 | -0.265990 | -27.998170 | -0.265990 |

Figure 8: First six rows in the Excel code; the full table on Excel is over 13,000 rows long.

Notice that we have set the gravitational constant to $G = 56$ in arbitrary units. This is because the actual value is of the order of $10^{-11}$ in IS units: using this would imply working with large numbers on the computer and we want to avoid this. Put it simply, $G = 56$ is the numerical value the gravitational constant takes in some -*unspecified* i.e. arbitrary- set of units. Similarly, within the same set of units, planetary or stellar masses will also be expressed by small numbers, e.g. $M = 50$ and $m = 1$ which complies with $m \ll M$.

As an example we can consider the initial conditions (Figure 9):

$$\begin{aligned} x_0 &= 10, \\ y_0 &= 0, \\ v_{x0} &= 0, \\ v_{y0} &= 20. \end{aligned}$$

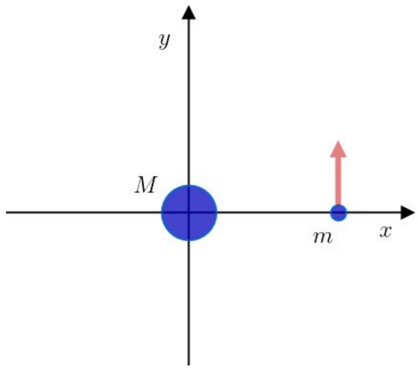



Figure 9: An example of possible initial conditions.

The result we obtain from the Excel code is this:

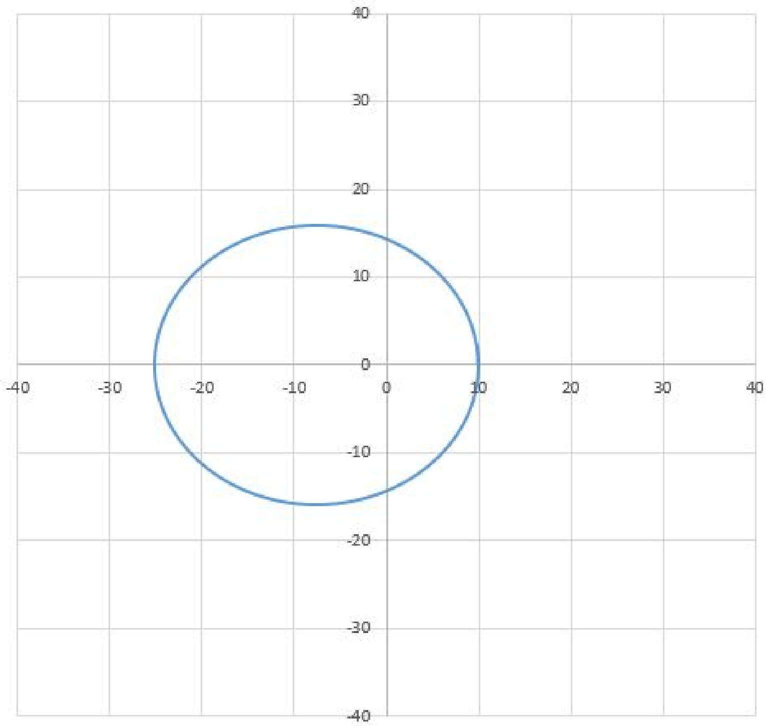



Figure 10: Orbit resulting from the initial conditions above.

Changing initial conditions will result in different paths. For example, Figure 11 shows a non bounded path:

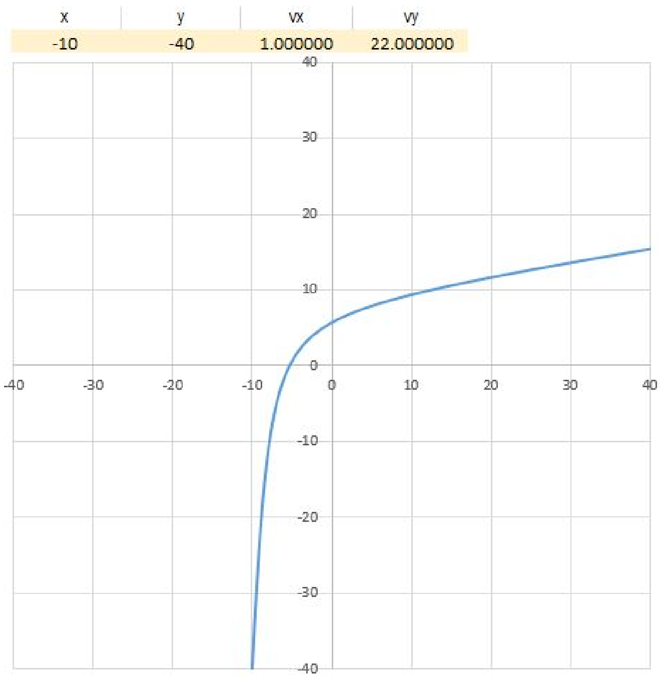


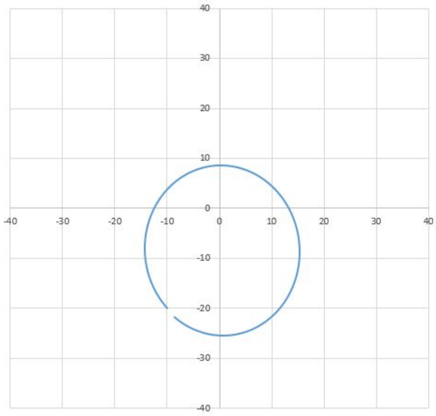

Figure 11: The mass $m$ starts from the bottom of the graph, it is deflected by the gravitational field and then travels away to infinity.

#### 4.1.3 Approximate nature of the numerical solutions

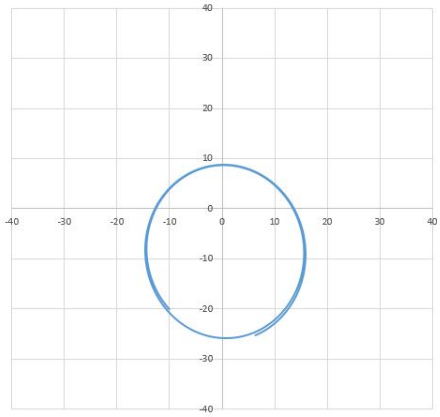
Finally we can check the impact that a specific choice of $\Delta t$ has on the solution. Figure 12 shows what happens when, given the same initial conditions, $\Delta t$ is progressively increased.

The orbit should be closed but you can see how the computational error increases with increasing $\Delta t$. Because of this if we want to obtain good numerical solutions we are better off using a more powerful coding tool other than Excel. In the next sessions we will do this using Python. This will allow for smaller values of $\Delta t$, faster computing times and it will be easier to write a general code that can be used for a wide variety of different physical systems.

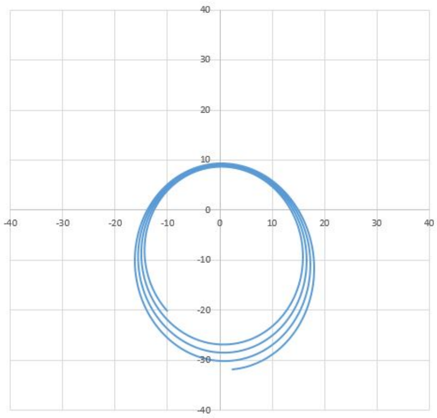
That said, the only way to obtain exact solutions will be to use calculus and we will do this soon. However, be aware that while some problems (e.g. two masses interacting gravitationally) can be solved exactly and analytically using calculus, many other problems exist (in fact most problems, e.g. three masses interacting gravitationally) for which it is known that it is not possible to find a general analytical solutions. For these kind of problems numerical approaches will still be invaluable.



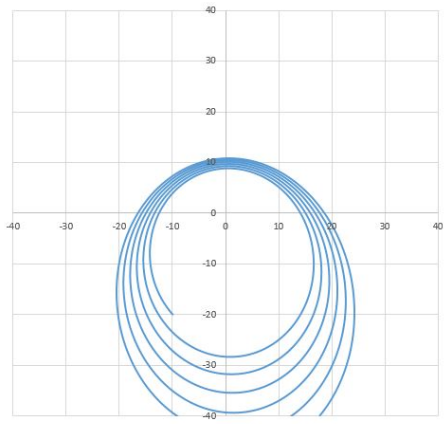

(a) $\Delta t = 0.0011$s




(b) $\Delta t = 0.0020$s




(c) $\Delta t = 0.0050$s




(d) $\Delta t = 0.0100$s

Figure 12: Effect of increasing $\Delta t$: the orbit should be closed, like in (a), but you can see that increasing the time step causes the solution to be less and less accurate.

# 5 SESSION 05

## 5.1 Problem 4: two interacting masses

We now turn to the problem of two masses, $m_1$ and $m_2$, which are interacting gravitationally. We want to consider their mutual interaction without assuming that the heavier mass is staying still. Figure 13 shows what is going on: $\mathbf{r}_1$ and $\mathbf{r}_2$ are the position vectors of the two masses, $\mathbf{f}_{12}$ is the force on $m_1$ due to $m_2$ and, similarly, $\mathbf{f}_{21}$ is the force on $m_2$ due to $m_1$.

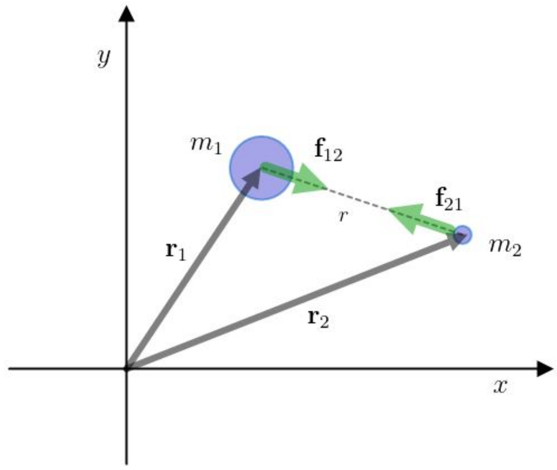



Figure 13: The two masses lie in the $x$-$y$ plane and suffer mutual gravitational interaction.

### 5.1.1 Newton's laws and center of mass

We can easily write down Newton's second law for each mass:

$$\begin{aligned} \mathbf{f}_{12} &= m_1\mathbf{a}_1 \\ \mathbf{f}_{21} &= m_2\mathbf{a}_2. \end{aligned} \tag{4}$$

Newton's third law ensures that $\mathbf{f}_{12} = -\mathbf{f}_{21}$, hence we can add the previous two equations to obtain

$$m_1\mathbf{a}_1 + m_2\mathbf{a}_2 = 0. \tag{5}$$

This equation has profound implications which can be revealed by first writing it as

$$m_1\frac{d^2\mathbf{r}_1}{dt^2} + m_2\frac{d^2\mathbf{r}_2}{dt^2} = 0,$$

where we have used $\mathbf{a}_1 = d^2\mathbf{r}_1/dt^2$ and $\mathbf{a}_2 = d^2\mathbf{r}_2/dt^2$ to express the accelerations in terms of the second derivative of the position vectors.[2] Equation (5) can be written as

$$\frac{d^2(m_1\mathbf{r}_1 + m_2\mathbf{r}_2)}{dt^2} = 0.$$

Finally we can divide by $m_1 + m_2$ to obtain

$$\frac{d^2}{dt^2}\left(\frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}\right) = 0.$$

Defining

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \tag{6}$$

the previous equation becomes

$$\frac{d^2\mathbf{R}}{dt^2} = 0. \tag{7}$$

To understand what equation (7) means we observe that since the masses are moving, their position vectors depend on time, i.e. they define two functions $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$. This implies that the vector $\mathbf{R}(t)$, which has units of meters, describes the position of a point in space which moves with zero acceleration, i.e. it moves with a constant velocity along a straight line.

By close inspection of equation (6) we can see that $\mathbf{R}$ is the weighted average of the two masses positions. It is quite easy to convince yourself that $\mathbf{R}$ defines the *center of mass* (CM) of the system.[3]

---

[2]If you are wondering how to find the second derivative of a vector the answer is easy, you just take the second derivative of its components, like this:

$$\frac{d^2}{dt^2}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} d^2x/dt^2 \\ d^2x/dt^2 \end{pmatrix}$$

[3]As an exercise you can try to verify that:

1. the masses positions and the CM, corresponding to the tips of the vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{R}$, are always aligned (Figure 14);
2. if $m_1 = m_2$ the CM is half way in between $m_1$ and $m_2$;
3. in general, the CM is closer to the heavier mass.

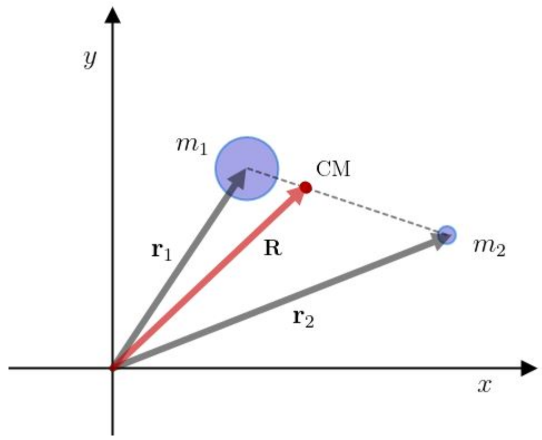



Figure 14: The CM (red circle) is on the line connecting the two masses and it is closer to the heavier one.

To summarize we have discovered an important fact: no matter how the two masses are moving, their center of mass has no acceleration and is either at rest or moving at a constant speed along a straight line. Figure 15 attempts to visualize this idea by showing the CM at three, equally spaced, instants of time.

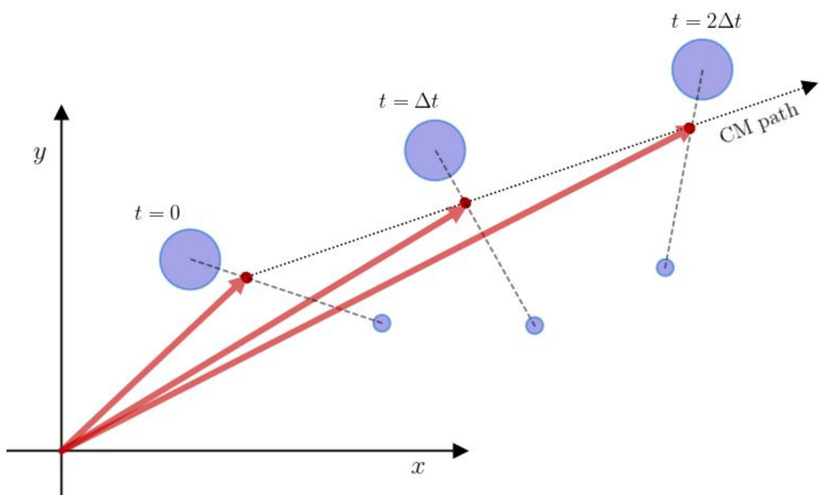



Figure 15: The CM moves at a constant speed along a straight line. Notice how the two masses and the CM are always aligned.

This nice behavior of the center of mass is a general property of all *isolated systems*, that is systems where the only forces in action are the internal ones acting between its constituents parts. In other words no external forces act on an isolated system.[4]

[4]As an exercise you can try to verify that the center of mass of an arbitrary isolated system comprised by $n$ particles always moves at a constant velocity.

#### 5.1.2 Center of mass frame

The specific velocity of the center of mass entirely depends on the frame of reference from which we are observing it: different inertial[5] observers will see the center of mass drifting at different speeds along different directions.

In particular there must be a special reference frame where the CM is simply at rest at all times: we call this the *center of mass frame*. In this frame it is always $\mathbf{R}(t) = 0$. The dynamics shown in Figure 15 will look different when observed from the center of mass frame. This is illustrated in Figure 16.

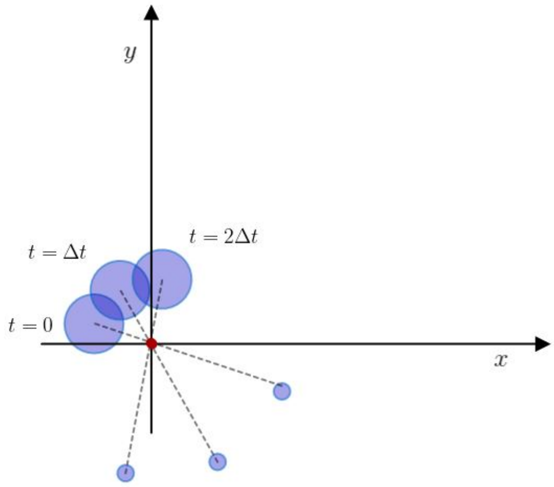



Figure 16: In the center of mass reference frame the CM is at rest in the origin while the two masses move around it.

#### 5.1.3 Solving Newton's second law

We are in the position to solve Newton's second law. We start by re-writing equations (4) as

$$\begin{aligned} \mathbf{f}(|\mathbf{r}_1 - \mathbf{r}_2|) &= m_1 \tfrac{d^2\mathbf{r}_1}{dt^2} \\ -\mathbf{f}(|\mathbf{r}_1 - \mathbf{r}_2|) &= m_2 \tfrac{d^2\mathbf{r}_2}{dt^2}, \end{aligned} \tag{8}$$

where $\mathbf{f}(|\mathbf{r}_1 - \mathbf{r}_2|) = \mathbf{f}_{12}(|\mathbf{r}_1 - \mathbf{r}_2|)$ shows explicitly that the mutual force between the two masses just depends on their distance.

Defining the relative position vector (of $m_1$ respect to $m_2$)

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

[5] An inertial observer is someone who is using an inertial frame of reference. The word *inertial* refers to the fact that Newton's laws will be valid for such an observer. In particular, an object acted by a zero net force will either remain at rest or move at a constant velocity.

the previous equations become

$$\begin{aligned} \mathbf{f}(r) &= m_1 \tfrac{d^2\mathbf{r}_1}{dt^2} \\ -\mathbf{f}(r) &= m_2 \tfrac{d^2\mathbf{r}_2}{dt^2}, \end{aligned} \tag{9}$$

where $r = |\mathbf{r}|$ is the distance between the two masses. This is illustrated in Figure 17.

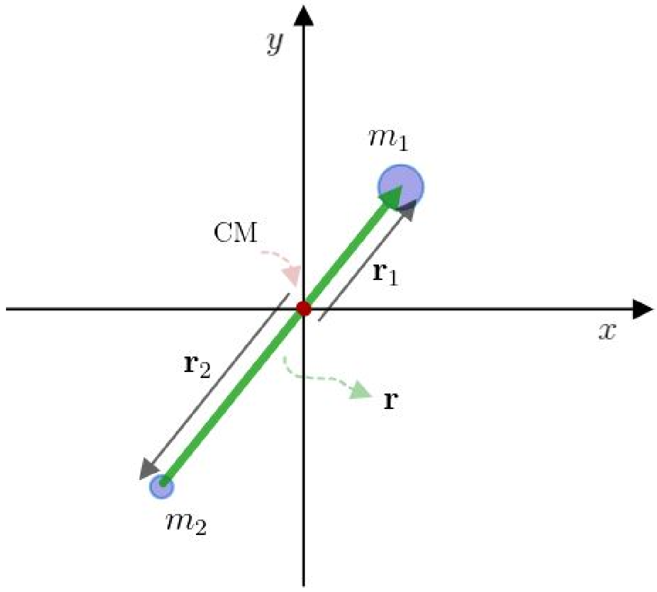



Figure 17: The relative position vector $\mathbf{r}$ (in green) measures the position of $m_1$ relative to $m_2$. This is shown in the center of mass reference frame.

Our goal now is to manipulate equations (9) in such way that they only contain the relative position $\mathbf{r}$. The key move to do this is working in the center of mass frame. In this frame $\mathbf{R} = 0$ at all times and equation (6) becomes

$$\frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} = 0.$$

This implies that

$$\mathbf{r}_2 = -\frac{m_1}{m_2}\mathbf{r}_1$$

and, using $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, we obtain

$$\mathbf{r} = \frac{m_1 + m_2}{m_2}\mathbf{r}_1.$$

We can now substitute this into the first of equations (9) and we get

$$\mathbf{f}(r) = \frac{m_1 m_2}{m_1 + m_2}\frac{d^2\mathbf{r}}{dt^2}.$$

### 5.1.4 Reduced mass of the system

Defining the *reduced mass* of the system as

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \tag{10}$$

we finally obtain

$$\mathbf{f}(r) = \mu \frac{d^2\mathbf{r}}{dt^2}. \tag{11}$$

This is an extremely interesting result because it shows that the dynamics of the system is reduced to that of a point mass $\mu$ moving within a radial gravitational field. Equation (11) can be solved to yield $\mathbf{r}(t)$ by proceeding as we did in Section 4. Once this is known the position vectors of the two masses easily follow from

$$\mathbf{r}_1 = \frac{m_2}{m_1 + m_2}\mathbf{r} = \frac{\mu}{m_1}\mathbf{r}$$

and

$$\mathbf{r}_2 = -\frac{m_1}{m_1 + m_2}\mathbf{r} = -\frac{\mu}{m_2}\mathbf{r}$$

To appreciate the meaning of the reduced mass we can consider some interesting special cases:

[$m_1 = m_2$] if the masses are both equal to some mass $m$ the reduced mass becomes $\mu = m/2$. In this case we have

$$\mathbf{r}_1 = \frac{1}{2}\mathbf{r}$$

and

$$\mathbf{r}_2 = -\frac{1}{2}\mathbf{r},$$

showing the the center of mass stays in between the two masses at all times.

[$m_2 \gg m_1$] if one mass is much larger than the other the reduced mass is approximatively $\mu \approx m_1$. Moreover

$$\mathbf{r}_1 \approx \mathbf{r}$$

and

$$\mathbf{r}_2 = -\frac{m_1}{m_2}\mathbf{r} \approx 0.$$

This applies for example to the system sun-earth. We can see that in this situation the reduced mass essentially coincides with the smaller mass. Moreover the larger mass stays very close to the center of mass at all times and its movement can be neglected.

# 6 SESSION 06

## 6.1 Using Python to solve Problem 4

The results of the previous session can be coded in Python. We will look at how this is done in detail during our next session. For now we can have a look at some results.

For example, we can choose the parameters in arbitrary units as:

$$G = 56$$
$$m_1 = 50$$
$$m_2 = 25.$$

The reduced mass follows as

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = 16.67$$

and, as we studied in the previous session, the problem is now reduced to that of a mass $\mu$ moving in a static gravitational field with a fixed source at the origin. Choosing initial conditions as

$$\mathbf{r}(0) = \begin{pmatrix} -15 \\ -5 \end{pmatrix}$$

and

$$\mathbf{v}(0) = \begin{pmatrix} -14 \\ 7 \end{pmatrix}$$

the Python code shows (figure 18) that the relative position vector $\mathbf{r}$ follows a closed orbit

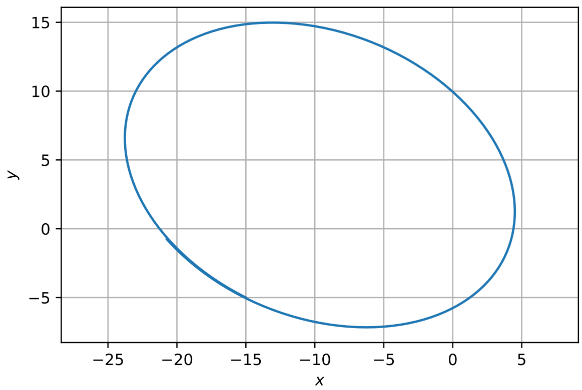



Figure 18: The relative position vector $\mathbf{r}$, measuring the position of $m_1$ relative to $m_2$, follows a closed elliptical orbit. The time step has been set to $\Delta t = 0.0001$ and the dynamics was computed with 60000 iterations.

Finally the position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ of the two masses can be found using equation (5.1.4) and equation (5.1.4). Figure 19 shows that $m_1$ and $m_2$ follow elliptical orbit around their common center of mass.

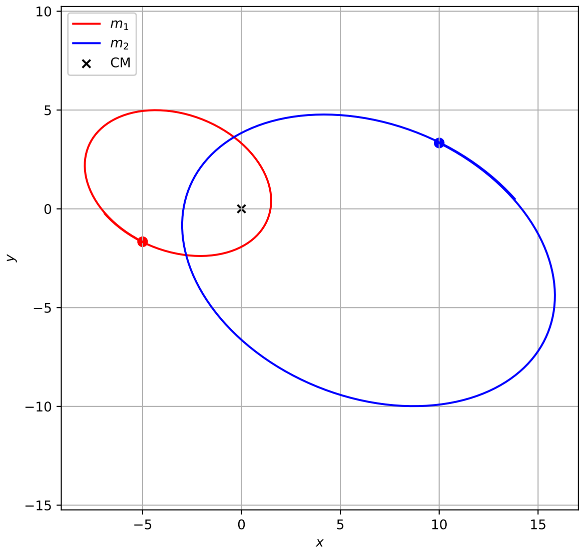



Figure 19: The masses $m_1$ (red) and $m_2$ (blue) orbit their common center of mass (black cross). The masses initial positions are shown as red and blue dots. Notice how the initial position of $m_1$ relative to $m_2$ correctly has components equal to $-15$ and $-5$.