STA360 Homework 1 (Ken Ye)

1. Probability review:
   - Exercise 2.2 from book

   a.

$$E[a_1 Y_1 + a_2 Y_2] = E[a_1 Y_1] + E[a_2 Y_2] = a_1 E[Y_1] + a_2 E[Y_2] = a_1 \mu_1 + a_2 \mu_2$$
$$Var[a_1 Y_1 + a_2 Y_2] = Var[a_1 Y_1] + Var[a_2 Y_2] = a_1^2 Var[Y_1] + a_2^2 Var[Y_2] = a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2$$

   b.

$$E[a_1 Y_1 - a_2 Y_2] = E[a_1 Y_1] - E[a_2 Y_2] = a_1 E[Y_1] - a_2 E[Y_2] = a_1 \mu_1 - a_2 \mu_2$$
$$Var[a_1 Y_1 - a_2 Y_2] = Var[a_1 Y_1] + Var[a_2 Y_2] = a_1^2 Var[Y_1] + a_2^2 Var[Y_2] = a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2$$

   - Exercise 2.4 from book

   a.

$$Pr(H_j|E)Pr(E) = Pr(H_j \cap E) = Pr(E|H_j)Pr(H_j)$$

   (by axiom P3)

   b.

$$Pr(E) = Pr(E \cap H_1) + Pr(E \cap H_2) + \ldots + Pr(E \cap H_k) = Pr(E \cap H_1) + Pr(E \cap \{\cup_{k=2}^{K} H_k\})$$

   (since $\{H_1, \ldots, H_k\}$ is a partition of H)

   c.

$$Pr(E) = Pr(E \cap H_1) + Pr(E \cap H_2) + \ldots + Pr(E \cap H_k) = \sum_{k=1}^{K} Pr(E \cap H_k)$$

   (since $\{H_1, \ldots, H_k\}$ is a partition of H)

   d.

$$\begin{aligned} Pr(H_j|E)Pr(E) &= Pr(E|H_j)Pr(H_j) \quad \text{by a} \\ Pr(H_j|E) &= \frac{Pr(E|H_j)Pr(H_j)}{Pr(E)} \\ Pr(H_j|E) &= \frac{Pr(E|H_j)Pr(H_j)}{\sum_{k=1}^{K} Pr(E \cap H_k)} \quad \text{by c} \\ Pr(H_j|E) &= \frac{Pr(E|H_j)Pr(H_j)}{\sum_{k=1}^{K} Pr(E|H_k)Pr(H_k)} \end{aligned}$$

2. Sensitivity and Specificity:

   a.

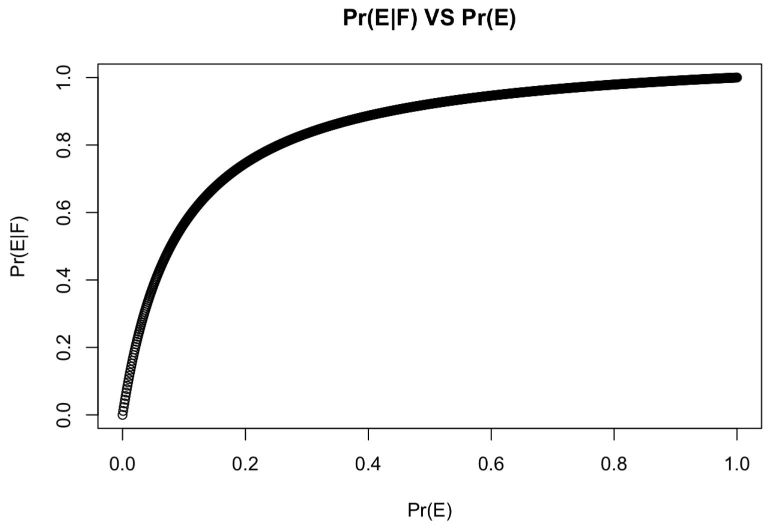
$$Pr(F) = Pr(E) \times Pr(F|E) + Pr(F) \times Pr(F|notE) = 0.15 \times 0.94 + 0.85 \times 0.08 = 0.209$$
$$Pr(E|F) = \frac{Pr(F|E)Pr(E)}{Pr(F)} = \frac{0.94 \times 0.15}{0.209} \approx 0.67464$$

   b.

```r
interval <- seq(0, 1, length.out = 1000)

# e is Pr(E)
f <- function(e){
  return (0.94*e / (0.94*e + 0.08*(1-e)))
}

plot(interval, f(interval),
     main = 'Pr(E|F) VS Pr(E)',
     xlab = 'Pr(E)',
     ylab = 'Pr(E|F)')
```



3. Joint distributions:

   a.

   $h(y, \theta)$ is nonnegative because both $f(y, \theta)$ and $g(\theta)$ are nonnegative everywhere by defination as they are valid pdfs, implying their product $h(y, \theta)$ must be nonnegative everywhere as well.

$$\int_{\Theta} \sum_{y \in} h(y, \theta) d\theta = \int_{\Theta} \sum_{y \in} f(y, \theta) g(\theta) d\theta = \int_{\Theta} 1 \times g(\theta) d\theta = 1$$

   Therefore, $h(y, \theta)$ is a valid joint pdf on $\times \Theta$.

   b.

$$p(y) = \int_0^{\infty} p(y, \theta) d\theta = \int_0^{\infty} f(y, \theta) g(\theta) d\theta$$

$$p(\theta) = \sum_{y=0}^{\infty} p(y, \theta) = \sum_{y=0}^{\infty} f(y, \theta) g(\theta) = 1 \times g(\theta) = g(\theta)$$

$$p(y|\theta) = \frac{p(y, \theta)}{p(\theta)} = \frac{f(y, \theta) g(\theta)}{\sum_{y=0}^{\infty} f(y, \theta) g(\theta)} = \frac{f(y, \theta)}{\sum_{y=0}^{\infty} f(y, \theta)} = \frac{f(y, \theta)}{1} = f(y, \theta)$$

$$p(\theta|y) = \frac{p(y, \theta)}{p(y)} = \frac{f(y, \theta) g(\theta)}{\int_0^{\infty} f(y, \theta) g(\theta) d\theta}$$

4. Posterior inference:

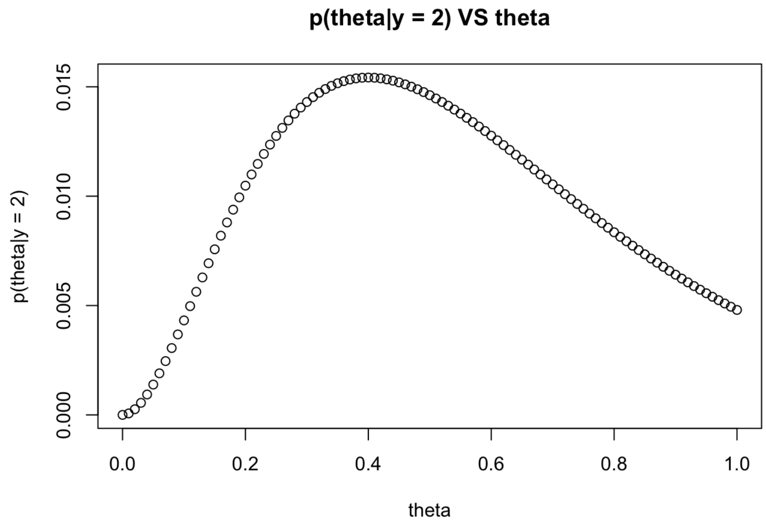
   a.

$$p(\theta|y = 2) = \frac{p(y = 2|\theta) p(\theta)}{p(y = 2)} = \frac{(5\theta)^2 e^{-5\theta}/2! \times \frac{1}{101}}{\sum_{\theta=0}^{1} (5\theta)^2 e^{-5\theta}/2! \times \frac{1}{101}} = \frac{(5\theta)^2 e^{-5\theta}}{\sum_{\theta=0}^{1} (5\theta)^2 e^{-5\theta}}$$

```
interval2 <- seq(0, 1, by = 1/100)

f2 <- function(theta){
  nom <- ((5*theta)^2)*exp(-5*theta)
  denom <- sum(nom)
  return (nom/denom)
}

plot(interval2, f2(interval2),
     main = 'p(theta|y = 2) VS theta',
     xlab = 'theta',
     ylab = 'p(theta|y = 2)')
```



b.

Under $p(\theta)$, median[θ] = 50/100 = 1/2, mode[θ] = {0, 1/100, 2/100, … , 1}

```
vals <- f2(interval2)
median_index <- which(cumsum(vals)>= 0.5)[1]
median <- interval2[median_index]
median
```

```
## [1] 0.49
```

```
mode <- interval2[which.max(vals)]
mode
```

```
## [1] 0.4
```

Under $p(\theta|y = 2)$, median[θ] = 0.2, mode[θ] = 0.4