Beta-Binomial

$$y \mid \theta \sim \text{Binomial}(n, \theta)$$
$$\theta \sim \text{Beta}(\alpha, \beta) \qquad E[\theta] = \frac{\alpha}{\alpha+\beta}$$

$$p(\theta \mid y) = \frac{p(y \mid \theta)\, p(\theta)}{p(y)} \propto p(y \mid \theta)\, p(\theta)$$

$$p(y \mid \theta) = \binom{n}{y} \theta^{y} (1-\theta)^{n-y} \propto_{\theta} \theta^{y} (1-\theta)^{n-y}$$

$$p(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1} \propto_{\theta} \theta^{\alpha-1} (1-\theta)^{\beta-1}$$

$$p(\theta \mid y) \propto \theta^{y} (1-\theta)^{n-y} \times \theta^{\alpha-1} (1-\theta)^{\beta-1}$$
$$\propto \theta^{y+\alpha-1} (1-\theta)^{n-y+\beta-1}$$
$$\overset{d}{=} \text{Beta}(\alpha+y, \beta+n-y)$$

$$E[\theta \mid y] = \frac{\alpha+y}{\alpha+\beta+n}$$

Poisson-Gamma

$$y_1, \ldots, y_n \mid \lambda \overset{iid}{\sim} \text{Po}(\lambda)$$
$$\lambda \sim \text{Ga}(a, b) \qquad E[\lambda] = \frac{a}{b}$$

$$p(\lambda \mid y_1, \ldots, y_n) \propto p(y_1, \ldots, y_n \mid \lambda)\, p(\lambda)$$

$$p(y_1, \ldots, y_n \mid \lambda) = \prod_{i=1}^{n} p(y_i \mid \lambda)$$
$$= \prod_{i=1}^{n} \frac{\lambda^{y_i}}{y_i!} e^{-\lambda}$$
$$= \left(\frac{1}{\prod_{i=1}^{n} y_i!} \right) \lambda^{\sum_{i=1}^{n} y_i} e^{-n\lambda} \propto_{\lambda} \lambda^{\sum y_i} e^{-n\lambda}$$

$$p(\lambda) = \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b\lambda} \propto \lambda^{a-1} e^{-b\lambda}$$

$$p(\lambda \mid y_1, \dots, y_n) \propto \lambda^{\Sigma y_i} e^{-n\lambda} \times \lambda^{a-1} e^{-b\lambda}$$

$$\propto \lambda^{\Sigma y_i + a - 1} e^{-\lambda(n+b)}$$

$$\overset{d}{=} Ga\left(a + \sum_{i=1}^{n} y_i,\ b+n\right)$$

$$E[\lambda \mid y_1, \dots, y_n] = \frac{a + \sum_{i=1}^{n} y_i}{b+n}$$

Normal – Inv. Gamma

$$y_1, \dots, y_n \mid \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$$

$$\theta \sim N(\theta_0, \tau^2)$$

$$\sigma^2 \sim \text{Inv. Gamma}\left(\frac{\nu_0}{2}, \frac{\nu_0}{2}\sigma_0^2\right)$$

$E[1/\sigma^2] = 1/\sigma_0^2$

$var(\sigma^2)$ decreasing in ν_0

$$p(\theta \mid y_1, \dots, y_n, \sigma^2) \propto p(y_1, \dots, y_n \mid \theta, \sigma^2)\ p(\theta)\ \cancel{p(\sigma^2)}$$

$$p(y_1, \dots, y_n \mid \theta, \sigma^2) = \prod_{i=1}^{n} p(y_i \mid \theta, \sigma^2)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y_i - \theta)^2}$$

$$= (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \theta)^2}$$

$$p(\theta) = (2\pi\tau^2)^{-1/2} e^{-\frac{1}{2\tau^2}(\theta - \theta_0)^2}$$

$$p(\theta|y,\sigma^2) \propto \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\theta)^2\right)\exp\left(-\frac{1}{2\tau^2}(\theta-\theta_0)^2\right)$$

$$\propto \exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i-\theta)^2 + \frac{1}{\tau^2}(\theta-\theta_0)^2\right)\right\}$$

$$\sum_{i=1}^{n}(y_i-\theta)^2 = \sum_{i=1}^{n}\left[y_i^2 - 2y_i\theta + \theta^2\right]$$

$$= \sum_{i=1}^{n} y_i^2 - 2\theta\sum_{i=1}^{n} y_i + n\theta^2$$

$$= \sum_{i=1}^{n} y_i^2 - 2\theta n\bar{y} + n\theta^2$$

$$(\theta-\theta_0)^2 = \theta^2 - 2\theta\theta_0 + \theta_0^2$$

$$p(\theta|y,\sigma^2) \propto \exp\left\{-\frac{1}{2}\left(\frac{n\theta^2}{\sigma^2} - \frac{2\theta n\bar{y}}{\sigma^2} + \frac{1}{\tau^2}\theta^2 - 2\theta\frac{\theta_0}{\tau^2}\right)\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left(\theta^2\left(\frac{n}{\sigma^2}+\frac{1}{\tau^2}\right) - 2\theta\left(\frac{n}{\sigma^2}\bar{y} + \frac{1}{\tau^2}\theta_0\right)\right)\right\}$$

$$p(\theta|y,\sigma^2) \propto \exp\left\{-\frac{1}{2}(a\theta^2 - 2b\theta)\right\}$$

$$\theta|y,\sigma^2 \sim N(b/a, 1/a)$$

$$\theta|y_1,\ldots,y_n,\sigma^2 \sim N(\mu_n, \sigma_n^2)$$

$$\sigma_n^2 = \left(\frac{n}{\sigma^2}+\frac{1}{\tau^2}\right)^{-1}$$

$$\mu_n = \sigma_n^2\left(\frac{n}{\sigma^2}\bar{y} + \frac{1}{\tau^2}\theta_0\right)$$

$$\sigma^2 \sim \text{Inv. Gamma}\left(\frac{\nu_0}{2}, \frac{\nu_0}{2}\sigma_0^2\right)$$

$$p(\sigma^2) \propto (\sigma^2)^{-\frac{\nu_0}{2}-1} e^{-\frac{\nu_0\sigma_0^2}{2\sigma^2}}$$

$$p(\sigma^2|y_1,\ldots,y_n,\theta) \propto (\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\theta)^2}$$

$$\times (\sigma^2)^{-\frac{\nu_0}{2}-1} e^{-\frac{\nu_0\sigma_0^2}{2\sigma^2}}$$

$$\propto (\sigma^2)^{-\frac{(n+\nu_0)}{2}-1} e^{-\frac{1}{\sigma^2}\left(\frac{1}{2}\Sigma(y_i-\theta)^2+\frac{\nu_0\sigma_0^2}{2}\right)}$$

$$\overset{d}{=} \text{Inv. Gamma}\left(\frac{n+\nu_0}{2}, \frac{1}{2}\left(\sum_{i=1}^{n}(y_i-\theta)^2+\nu_0\sigma_0^2\right)\right)$$

Multivariate Normal

$y_i \in \mathbb{R}^p$, $\mu \in \mathbb{R}^p$, $\Sigma \in \mathbb{R}^{p\times p}$

$$y_1,\dots,y_n \mid \mu,\Sigma \overset{iid}{\sim} N_p(\mu,\Sigma)$$

$$\mu \sim N_p(\mu_0, \Lambda_0)$$

$$\Sigma \sim \text{Inv. Wishart}(\nu_0, S_0)$$

$$p(\mu \mid y_1,\dots,y_n,\Sigma) \propto p(y_1,\dots,y_n \mid \mu,\Sigma)\, p(\mu)$$

$$p(y_1,\dots,y_n \mid \mu,\Sigma) = \prod_{i=1}^{n} p(y_i \mid \mu,\Sigma)$$

$$= \prod_{i=1}^{n} (2\pi)^{-p/2} |\Sigma|^{-1/2} e^{-\frac{1}{2}(y_i-\mu)^T\Sigma^{-1}(y_i-\mu)}$$

$$= (2\pi)^{-\frac{np}{2}} |\Sigma|^{-\frac{n}{2}} e^{-\frac{1}{2}\sum_{i=1}^{n}(y_i-\mu)^T\Sigma^{-1}(y_i-\mu)}$$

$$p(\mu) = (2\pi)^{-p/2} |\Lambda_0|^{-1/2} e^{-\frac{1}{2}(\mu-\mu_0)^T\Lambda_0^{-1}(\mu-\mu_0)}$$

$$p(\mu \mid y_1,\dots,y_n,\Sigma) \propto \exp\left(-\frac{1}{2}\sum_{i=1}^{n}(y_i-\mu)^T\Sigma^{-1}(y_i-\mu) - \frac{1}{2}(\mu-\mu_0)^T\Lambda_0^{-1}(\mu-\mu_0)\right)$$

$$\sum_{i=1}^{n}(y_i-\mu)^T\Sigma^{-1}(y_i-\mu) = \sum_{i=1}^{n}\left[(y_i-\mu)^T\Sigma^{-1}y_i - (y_i-\mu)^T\Sigma^{-1}\mu\right]$$

$$= \sum_{i=1}^{n}\left[y_i^T\Sigma^{-1}y_i - \mu^T\Sigma^{-1}y_i - y_i^T\Sigma^{-1}\mu + \mu^T\Sigma^{-1}\mu\right]$$

$$= \sum_{i=1}^{n} y_i^T\Sigma^{-1}y_i - 2\mu^T\Sigma^{-1}\sum_{i=1}^{n}y_i + \sum_{i=1}^{n}\mu^T\Sigma^{-1}\mu$$

$$\propto \mu^T(n\Sigma^{-1})\mu - 2\mu^T(n\Sigma^{-1})\bar{y}$$

$$(\mu-\mu_0)^T\Lambda_0^{-1}(\mu-\mu_0) = \mu^T\Lambda_0^{-1}\mu - 2\mu^T\Lambda_0^{-1}\mu_0 + \mu_0^T\Lambda_0^{-1}\mu_0$$
$$\propto \mu^T\Lambda_0^{-1}\mu - 2\mu^T\Lambda_0^{-1}\mu_0$$

$p(\mu \mid y_1,\dots,y_n,\Sigma)$
$$\propto \exp\left\{-\frac{1}{2}\left(\mu^T(\Lambda_0^{-1}+n\Sigma^{-1})\mu - 2\mu^T(\Lambda_0^{-1}\mu_0 + n\Sigma^{-1}\bar{y})\right)\right\}$$

$$p(\mu) \propto \exp\left\{-\frac{1}{2}(\mu^TA\mu - 2\mu^Tb)\right\}$$
$$\mu \sim N_p(A^{-1}b, A^{-1})$$

$$\mu \mid y_1,\dots,y_n,\Sigma \sim N_p(\mu_n, \Sigma_n)$$
$$\Sigma_n = (\Lambda_0^{-1} + n\Sigma^{-1})^{-1}$$
$$\mu_n = \Sigma_n(\Lambda_0^{-1}\mu_0 + n\Sigma^{-1}\bar{y})$$

$\Sigma \sim$ Inv. Wishart (ν_0, S_0)
$$p(\Sigma) \propto |\Sigma|^{-\frac{(\nu_0+p+1)}{2}} e^{-\frac{1}{2}tr(S_0\Sigma^{-1})}$$

$$a \in \mathbb{R} \Rightarrow a = tr(a)$$
$$tr(A+B) = tr(A) + tr(B)$$
$$tr(ABC) = tr(CAB) = tr(BCA)$$

$p(\Sigma \mid y_1,\dots,y_n,\mu) \propto$
$$|\Sigma|^{-n/2} e^{-\frac{1}{2}\sum_{i=1}^{n}(y_i-\mu)^T\Sigma^{-1}(y_i-\mu)} \times |\Sigma|^{-\frac{(\nu_0+p+1)}{2}} e^{-\frac{1}{2}tr(S_0\Sigma^{-1})}$$
$$\propto |\Sigma|^{-\frac{1}{2}(n+\nu_0+p+1)} \exp\left(-\frac{1}{2}\left(tr(S_0\Sigma^{-1}) + \sum_{i=1}^{n}(y_i-\mu)^T\Sigma^{-1}(y_i-\mu)\right)\right)$$

$$\sum_{i=1}^{n}(y_i-\mu)^T\Sigma^{-1}(y_i-\mu) = tr\left(\sum_{i=1}^{n}(y_i-\mu)^T\Sigma^{-1}(y_i-\mu)\right)$$

$$= \sum_{i=1}^{n} tr\left((y_i-\mu)^T\Sigma^{-1}(y_i-\mu)\right)$$

$$= \sum_{i=1}^{n} tr\left((y_i-\mu)(y_i-\mu)^T\Sigma^{-1}\right)$$

$$= tr\left(\sum_{i=1}^{n}(y_i-\mu)(y_i-\mu)^T\Sigma^{-1}\right)$$

$$S_\theta = \sum_{i=1}^{n}(y_i-\mu)(y_i-\mu)^T \qquad = tr\left(S_\theta\Sigma^{-1}\right)$$

$$p(\Sigma \mid y_1,\dots,y_n,\mu) \propto |\Sigma|^{-\frac{(n+\nu_0+p+1)}{2}} e^{-\frac{1}{2}\left(tr(S_0\Sigma^{-1})+tr(S_\theta\Sigma^{-1})\right)}$$

$$\propto |\Sigma|^{-\frac{(n+\nu_0+p+1)}{2}} e^{-\frac{1}{2}tr\left\{(S_0+S_\theta)\Sigma^{-1}\right\}}$$

$$\stackrel{d}{=} \text{Inv. Wishart}\left(\nu_0+n,\ S_0+S_\theta\right)$$