#2

b) $P(\theta|y_1,\dots y_n) = \dfrac{P(y_1,\dots,y_n|\theta)p(\theta)}{P(y_1,\dots,y_n)}$

$\propto P(y_1,\dots,y_n)p(\theta)$

$= \left(\prod_{i=1}^{n} P(y_i|\theta)\right)\cdot P(\theta)$

$= \left(\prod_{i=1}^{n} \theta^{y_i}(1-\theta)^{1-y_i}\right) p(\theta)$

$= \theta^{\Sigma y_i}(1-\theta)^{n-\Sigma y_i} \cdot \dfrac{\Gamma(10)}{4\Gamma(2)\Gamma(8)}\left[3\theta(1-\theta)^7 + \theta^7(1-\theta)\right]$

$\propto \theta^{\Sigma y_i}(1-\theta)^{n-\Sigma y_i} \cdot \left[3\theta(1-\theta)^7 + \theta^7(1-\theta)\right]$

$= 3\cdot\theta^{\Sigma y_i+1}(1-\theta)^{n+7-\Sigma y_i} + \theta^{\Sigma y_i+7}(1-\theta)^{n+1-\Sigma y_i}$

$\Downarrow$

$\int_0^1 3\theta^{\Sigma y_i+1}(1-\theta)^{n+7-\Sigma y_i} + \theta^{\Sigma y_i+7}(1-\theta)^{n+1-\Sigma y_i}\, d\theta = \frac{1}{c}$ (find normalizing constant)

$3B(2+\Sigma y_i, 8+n-\Sigma y_i) + B(8+\Sigma y_i, 2+n-\Sigma y_i) = \frac{1}{c}$ $\left(B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right)$

$\Downarrow$

$P(\theta|y_1,\dots y_n) = \dfrac{\theta^{\Sigma y_i+1}(1-\theta)^{n+7-\Sigma y_i}}{\frac{\Gamma(2+\Sigma y_i)\Gamma(8+n-\Sigma y_i)}{\Gamma(2+\Sigma y_i+8+n-\Sigma y_i)}} + \dfrac{\theta^{\Sigma y_i+7}(1-\theta)^{n+1-\Sigma y_i}}{\frac{\Gamma(8+\Sigma y_i)\Gamma(2+n-\Sigma y_i)}{\Gamma(8+\Sigma y_i+2+n-\Sigma y_i)}} = Beta(2+\Sigma y_i, 8+n-\Sigma y_i) + Beta(8+\Sigma y_i, 2+n-\Sigma y_i)$

d) $p(\theta|y_1,\dots y_n) \sim Beta(2+\Sigma y_i, 8+n-\Sigma y_i) + Beta(8+\Sigma y_i, 2+n-\Sigma y_i)$

distribution 1

distribution 2

#3

a) $P(Y_1, \dots, Y_n|\theta) = \prod_{i=1}^{n} \left(\frac{2}{\Gamma(a)} \theta^{2a} y_i^{2a-1} e^{-\theta^2 y_i^2} \right)$

$$= \frac{2^n}{(\Gamma(a))^n} \theta^{2an} \prod_{i=1}^{n} y_i^{2a-1} e^{-\theta^2 \sum_{i=1}^{n} y_i^2}$$

b) prior: $p(\theta) \propto \theta^{\alpha} e^{-\beta\theta^2}$

posterior: $p(\theta|y_1, \dots y_n) \underset{\theta}{\propto} p(y_1, \dots, y_n|\theta) p(\theta)$

$$\underset{\theta}{\propto} \theta^{2an} e^{-\theta^2 \sum y_i^2} \theta^{\alpha} e^{-\beta\theta^2}$$

$$= \theta^{2an+\alpha} e^{-(\beta+\sum y_i^2)\theta^2}$$

c) since $p(y|\theta, a) \underset{y}{\propto} y^{2a-1} e^{-\theta^2 y^2} \Rightarrow \mathbb{E}[y|\theta, a) = \frac{\Gamma(a+\frac{1}{2})}{\theta\Gamma(a)}$

Let $2an+\alpha = 2a'-1$, and $-(\beta+\sum y_i^2)\theta^2 = -\theta'^2 y^2$

$2an+\alpha+1 = 2a'$ $\qquad \frac{(\beta+\sum y_i^2)\theta^2}{y^2} = \theta'^2$

$\frac{2an+\alpha+1}{2} = a'$. $\qquad \frac{(\beta+\sum y_i^2)^{\frac{1}{2}}\theta}{y} = \theta'$

Therefore, $\mathbb{E}[\theta|y_1, \dots y_n] = \frac{\Gamma(a'+\frac{1}{2})}{\theta'\Gamma(a')}$, where $a' = \frac{2an+\alpha+1}{2}$ and

$$\theta' = \frac{(\beta+\sum y_i^2)^{\frac{1}{2}}\theta}{y}$$

```
library(latex2exp)
set.seed(0)
```

#1

a.

In deciding Monte Carlo sample size, to make sure it is sufficient so that the answers are correct to within three decimal places with 95% probability, the following must be true:

$$2\sqrt{\sigma^2/S} <= 0.001$$

```
# store data and parameters
y_a <- c(12, 9, 12, 14, 13, 13, 15, 8, 15, 6)
y_b <- c(11, 11, 10, 9, 9, 8, 7, 10, 6, 8, 8, 9, 7)
a_a <- 120
b_a <- 10
a_b <- 12
b_b <- 1
n_a <- length(y_a)
sy_a <- sum(y_a)
n_b <- length(y_b)
sy_b <- sum(y_b)
```

```
smc <- 10000000 # 10 million
theta_a_mc <- rgamma(smc, a_a + sy_a, b_a + n_a)
y_a_predmc <- rpois(smc,theta_a_mc)

theta_b_mc <- rgamma(smc, a_b + sy_b, b_b + n_b)
y_b_predmc <- rpois(smc,theta_b_mc)

p1 <- mean(theta_a_mc > theta_b_mc)
p2 <- mean(y_a_predmc > y_b_predmc)

print(paste0('The Monte Carlo sample size is ', smc))
```

```
## [1] "The Monte Carlo sample size is 1e+07"
```

```
print(paste0('The probability that theta_A > theta_B given y_A and y_B is ', format(roun
d(p1, 3))))
```

```
## [1] "The probability that theta_A > theta_B given y_A and y_B is 0.995"
```

```
print(paste0('The probability that predicted Y_A > predicted Y_A given y_A and y_B is ',
format(round(p2, 3))))
```

```
## [1] "The probability that predicted Y_A > predicted Y_A given y_A and y_B is 0.698"
```

```
# Monte Carlo sample size
var <- var(theta_a_mc > theta_b_mc)
min_sample_size <- var / ((0.001/2)^2)
print(min_sample_size)
```

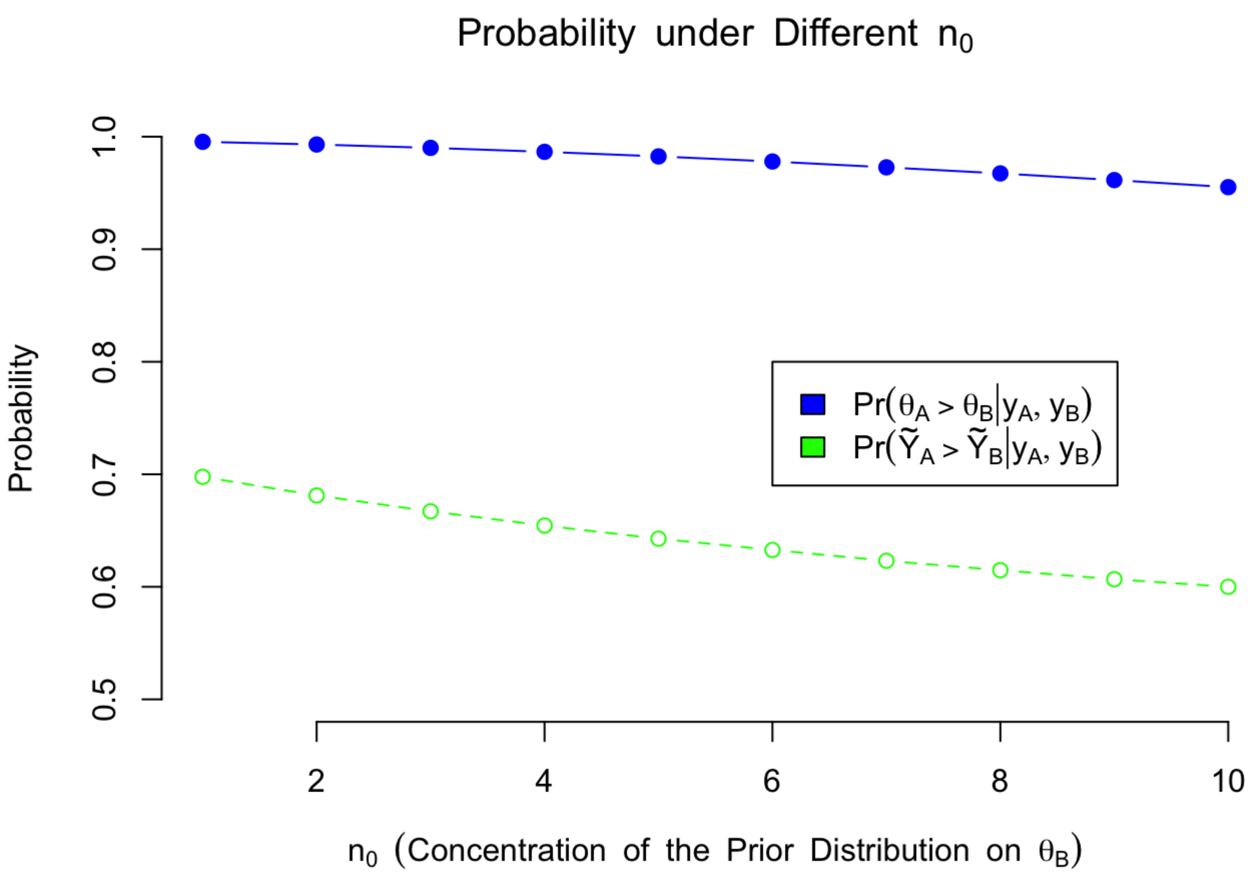
```
## [1] 18342.31
```

Therefore, the selection of 10 million as the Monte Carlo sample size was sufficient to make sure that the answers are correct to within three decimal places with 95% probability.

b.

```
prob_theta <- list()
prob_pred <- list()
n_0_seq <- seq(1:10)

for (n in n_0_seq){
  theta_a_mc <- rgamma(smc, a_a + sy_a, b_a + n_a)
  y_a_predmc <- rpois(smc,theta_a_mc)

  theta_b_mc <- rgamma(smc, a_b*n + sy_b, n + n_b)
  y_b_predmc <- rpois(smc,theta_b_mc)

  p1 <- mean(theta_a_mc > theta_b_mc)
  p2 <- mean(y_a_predmc > y_b_predmc)

  prob_theta <- append(prob_theta, p1)
  prob_pred <- append(prob_pred, p2)
}
```

```
plot(n_0_seq, prob_theta,
     col = 'blue',
     type = "b",
     frame = FALSE,
     pch = 19,
     main = TeX('$Probability \\ under \\ Different \\ n_0$'),
     xlab = TeX('$n_0 \\ (Concentration \\ of \\ the \\ Prior \\ Distribution \\ on \\
\\theta_B)$'),
     ylab = TeX('$Probability$'),
     ylim = c(0.5, 1))

points(n_0_seq, prob_pred,
      col = 'green',
      type = "b",
      lty = 2)

legend(6,0.8,
       legend = c(TeX('$Pr(\\theta_A > \\theta_B | y_A, y_B)$'), TeX('$Pr(\\tilde{Y}_A >
\\tilde{Y}_B | y_A, y_B)$')),
       fill = c('blue', 'green'))
```




```
print(prob_theta[1])
```

```
## [[1]]
## [1] 0.9954453
```

```
print(prob_theta[10])
```

```
## [[1]]
## [1] 0.9551444
```

```
print(prob_pred[1])
```

```
## [[1]]
## [1] 0.6977145
```

```
print(prob_pred[10])
```

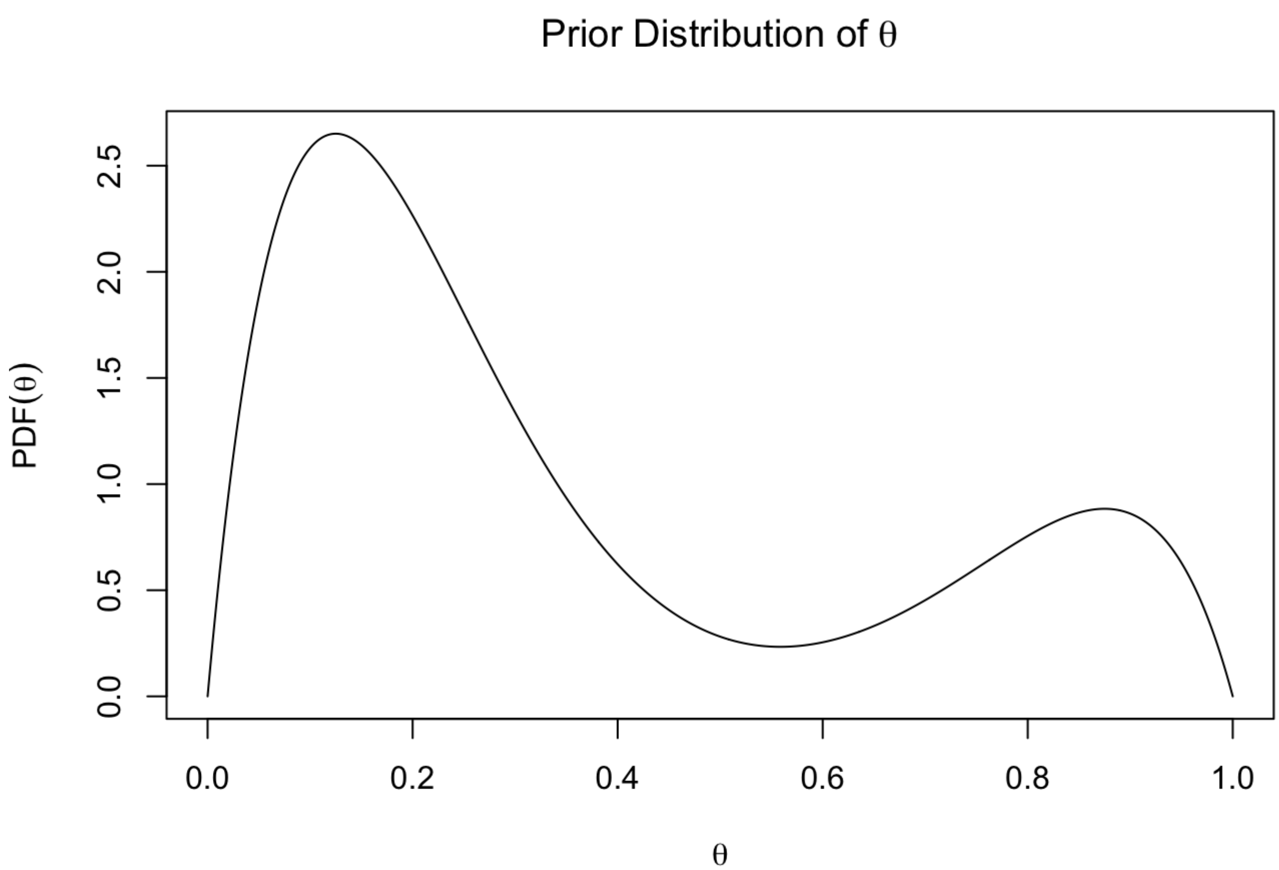
```
## [[1]]
## [1] 0.599989
```

We know that as n_0 increases, the concentration of the prior distribution$\theta_B \sim \mathrm{gamma}(12 \times n_0, n_0)$ would increase (higher peak and lower variance). According to the graph, we see that both $\Pr(\theta_A > \theta_B | y_A, y_B)$ and $\Pr(\tilde{Y_A} > \tilde{Y_B} | y_A, y_B)$ decreases when n_0 increases from 1 to 10. However, $\Pr(\tilde{Y_A} > \tilde{Y_B} | y_A, y_B)$ decreases by more (from 0.698 to 0.600) than $\Pr(\theta_A > \theta_B | y_A, y_B)$ (from 0.995 to 0.955). Therefore, $\Pr(\tilde{Y_A} > \tilde{Y_B} | y_A, y_B)$ is more sensitive to the concentration of the prior distribution than $\Pr(\theta_A > \theta_B | y_A, y_B)$.

#2

a.

```
# plot prior theta distribution
thetas <- seq(0, 1, length.out = 1000)
p_theta <- (1/4) * (gamma(10) / (gamma(2) * gamma(8))) * (3 * thetas * (1 - thetas)^7 +
thetas^7 * (1-thetas))

plot(thetas, p_theta,
     type = "l",
     main = TeX("Prior Distribution of \\theta"),
     xlab = TeX("\\theta"),
     ylab = TeX('$PDF(\\theta)$'))
```



As we can observe from the graph, the prior distribution of θ is bimodal with peaks at around $\theta = 0.1$ and $\theta = 0.9$. This makes sense because the scientists have the prior knowledge that an experimental machine in a lab is either fine, in which case it would have a low failure rate (small θ), or comes from a bad batch of machines, in which case it would have a high failure rate (large θ). Therefore, their beliefs center around a comparatively small θ at around 0.1, and a comparatively large θ at around 0.9.

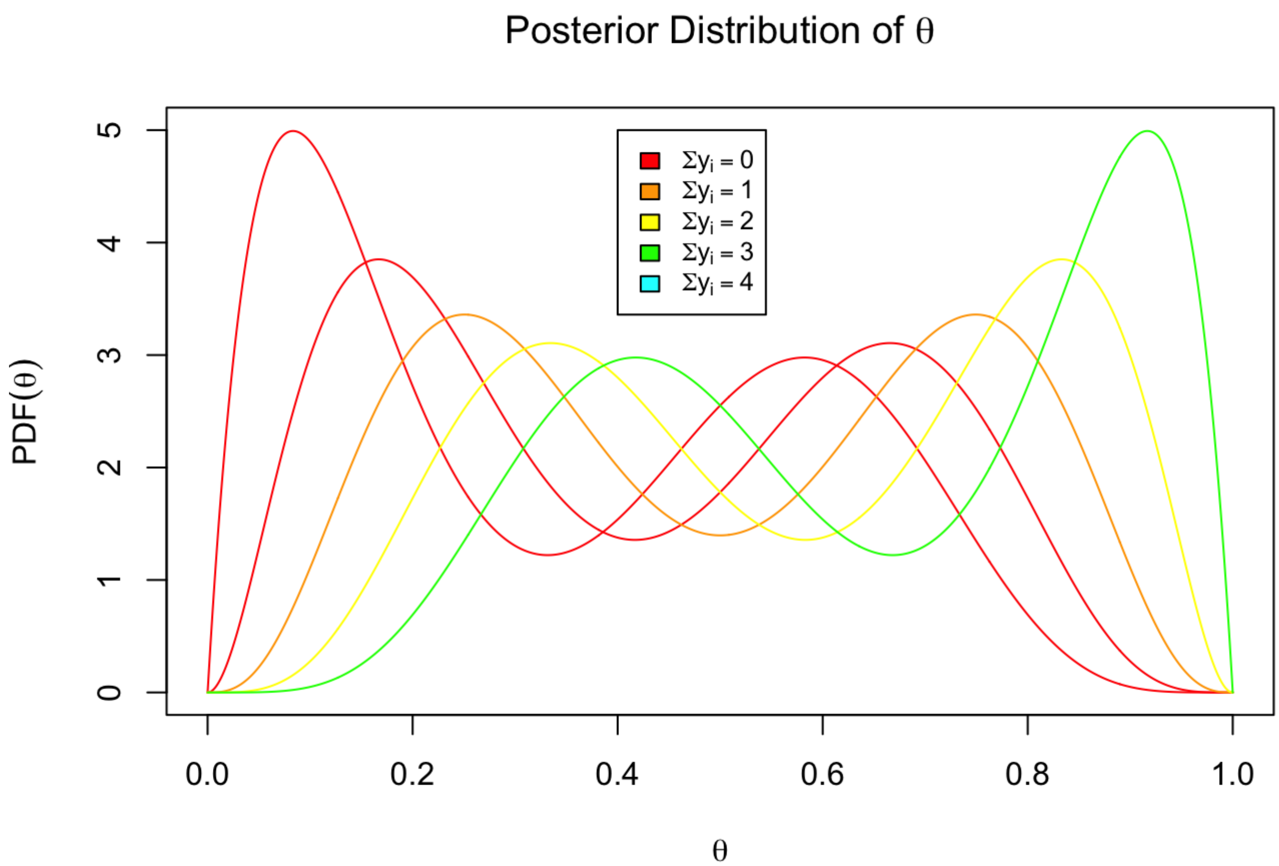
```
# find mode
thetas[which.max(p_theta)]
```

```
## [1] 0.1251251
```

The scientists think it is more likely that their machine is fine, which is revealed by the fact that the mode (highest peak) of the θ distribution is at 0.125, corresponding to the belief that the experimental machine in a lab is fine, in which case it would have a low failure rate (small θ).

c.

```
# plot posterior theta distribution
n <- 4
y_i <- 0:4
colors <- c('red', 'orange','yellow', 'green', 'cyan')

for (y in y_i){
  post_p_theta <- thetas^(y + 1) * (1 - thetas)^(7 + n - y) / beta(2 + y, 8 + n - y)  +
thetas^(y + 7) * (1 - thetas)^(1 + n -y) / (beta(8 + y, 2 + n - y))

  if (y == 0){
    plot(thetas, post_p_theta,
     type = "l",
     col = colors[1],
     main = TeX("Posterior Distribution of \\theta"),
     xlab = TeX("\\theta"),
     ylab = TeX('$PDF(\\theta)$'),
     ylim = c(0, 5))
  }
  else{
    lines(thetas, post_p_theta, col = colors[y])
  }
}

legend(0.4, 5,
       legend = c(TeX('$\\Sigma y_i = 0$'),
                  TeX('$\\Sigma y_i = 1$'),
                  TeX('$\\Sigma y_i = 2$'),
                  TeX('$\\Sigma y_i = 3$'),
                  TeX('$\\Sigma y_i = 4$')),
       fill = colors,
       cex =.8)
```


Posterior Distribution of θ
Σyᵢ = 0
Σyᵢ = 1
Σyᵢ = 2
Σyᵢ = 3
Σyᵢ = 4
PDF(θ)
0
1
2
3
4
5
0.0
0.2
0.4
0.6
0.8
1.0
θ