Machine Learning
(機器學習)

Lecture 07: Combatting Overfitting

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Roadmap

1. When Can Machines Learn?
2. Why Can Machines Learn?
3. How Can Machines Learn?
4. How Can Machines Learn **Better**?

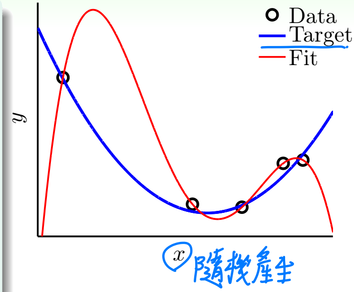
Lecture 07: Combatting Overfitting

- What is Overfitting?
- The Role of Noise and Data Size
- Deterministic Noise
- Dealing with Overfitting
- Regularized Hypothesis Set
- Weight Decay Regularization
- Regularization and VC Theory
- General Regularizers

Bad Generalization

- regression for $x \in \mathbb{R}$ with $N = 5$ examples
- target $f(x)$ = 2nd order polynomial
- label $y_n = f(x_n) +$ very small noise
- linear regression in $\mathcal{Z}$-space + Φ = 4th order polynomial
- unique solution passing all examples $\Longrightarrow E_{\text{in}}(g) = 0$
- $E_{\text{out}}(g)$ **huge**

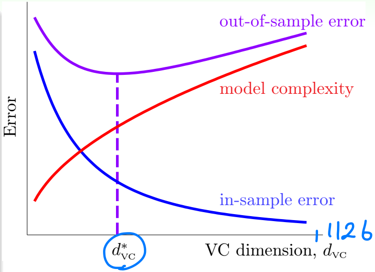
Ein, Eout 差很多



bad generalization: low E_{in}, high E_{out}

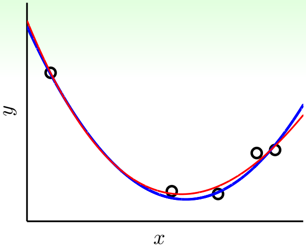
Bad Generalization and Overfitting

- take $d_{VC} = 1126$ for learning: bad generalization —(E_{out} - E_{in}) large 很大的數字.
- switch from $d_{VC} = d^*_{VC}$ to $d_{VC} = 1126$: **overfitting** —$E_{in} \downarrow$, $E_{out} \uparrow$ ← overfitting 發生
- switch from $d_{VC} = d^*_{VC}$ to $d_{VC} = 1$: **underfitting** —$E_{in} \uparrow$, $E_{out} \uparrow$ ← 做不夠好.

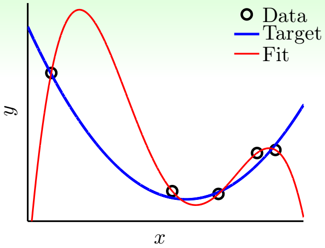


bad generalization: low E_{in}, high E_{out};
overfitting: low**er** E_{in}, high**er** E_{out}

Cause of Overfitting: A Driving Analogy



'good fit' $\Longrightarrow$



overfit

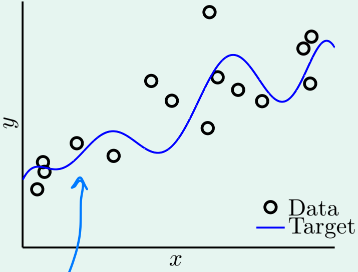
learning	driving
overfit mode complexity 太大	commit a car accident
use excessive d_{VC}	'drive too fast'
noise 資料量太小	bumpy road
limited data size N	limited observations about road condition

next: how does **noise** & **data size** affect overfitting?

Questions?

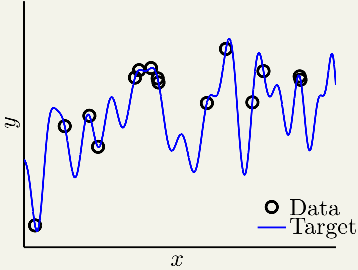
Case Study (1/2)

10-th order target function + **noise**



data 有 noise

50-th order target function **noiselessly**



沒有noise.

2次多項式 $\in H_2$

10次多項式 $\in H_{10}$

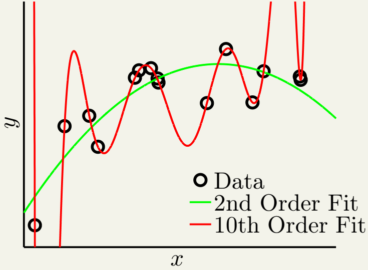
overfitting from best $g_2 \in \mathcal{H}_2$ to best $g_{10} \in \mathcal{H}_{10}$?

Case Study (2/2)

10-th order target function **+ noise**

	$g_2 \in \mathcal{H}_2$	$g_{10} \in \mathcal{H}_{10}$
E_{in}	0.050	0.034 較小
E_{out}	0.127	**9.00** over fit!

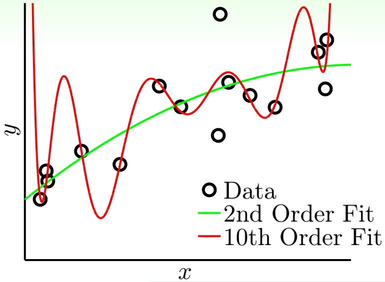
50-th order target function **noiselessly**

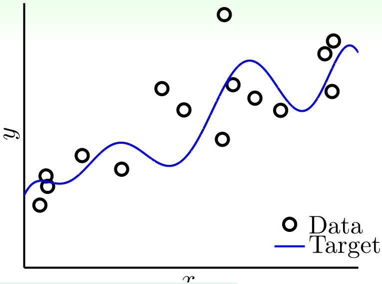


	$g_2 \in \mathcal{H}_2$	$g_{10} \in \mathcal{H}_{10}$
E_{in}	0.029	0.00001
E_{out}	0.120	**7680** overfit

overfitting from g_2 to g_{10}? **both yes!**

Irony of Two Learners





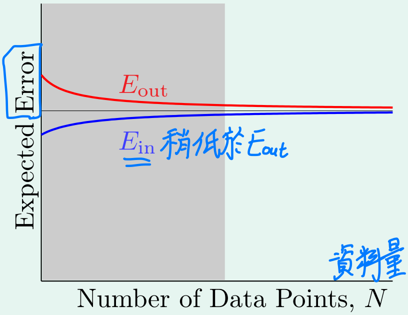
- learner *Overfit*: pick $g_{10} \in \mathcal{H}_{10}$
- learner *Restrict*: pick $g_2 \in \mathcal{H}_2$
- when both **know that target = 10th** —*R* 'gives up' ability to fit

弱学生做的比較好.

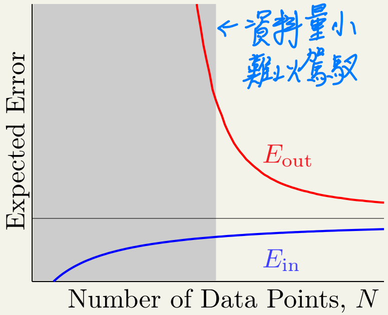
but *R* **wins in** E_{out} a lot! 以退為進.
philosophy: concession for **advantage**? **:-)**

Learning Curves Revisited

$\mathcal{H}_2$



$\mathcal{H}_{10}$

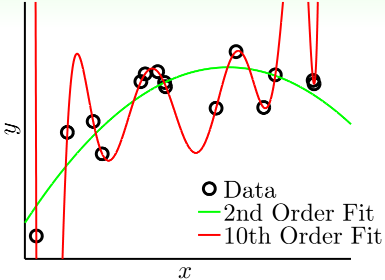


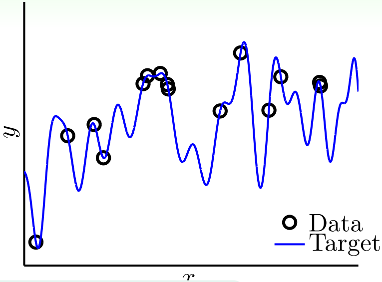
- $\mathcal{H}_{10}$: lower $\overline{E_{out}}$ when $N \to \infty$,
  but much larger generalization error for small N
- gray area: O overfits! ($\overline{E_{in}} \downarrow$, $\overline{E_{out}} \uparrow$)

R always **wins in** $\overline{E_{out}}$ if N small!

The 'No Noise' Case

沒有 noise $\mathcal{H}_2$ 还是做不好?





- learner *Overfit*: pick $g_{10} \in \mathcal{H}_{10}$
- learner *Restrict*: pick $g_2 \in \mathcal{H}_2$
- when both **know that there is no noise** —*R* still wins

is there really **no noise**?
'target complexity' acts like noise

Target 太複雜

Questions?

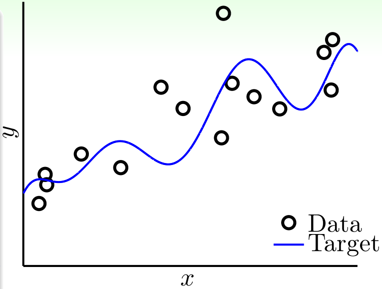
A Detailed Experiment

gaussian noise.

$$y = f(x) + \epsilon \sim \text{Gaussian}\Big(\underbrace{\sum_{q=0}^{Q_f} \alpha_q x^q}_{f(x)}, \sigma^2\Big)$$

noise level

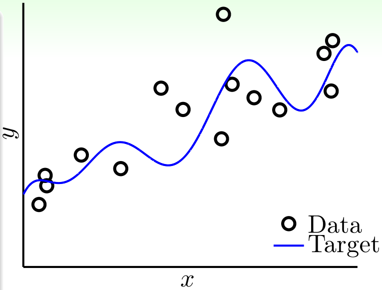
- Gaussian iid noise ϵ with level σ^2
- some 'uniform' distribution on $f(x)$ with complexity level $\boxed{Q_f}$ 某次方的 target function
- data size N



σ^2 与 Q_f 对 overfit 的影响

goal: **'overfit level'** for different (N, σ^2) and (N, Q_f)?

The Overfit Measure



- $g_2 \in \mathcal{H}_2$
- $g_{10} \in \mathcal{H}_{10}$
- $E_{\text{in}}(g_{10}) \le E_{\text{in}}(g_2)$ for sure

量測 overfit

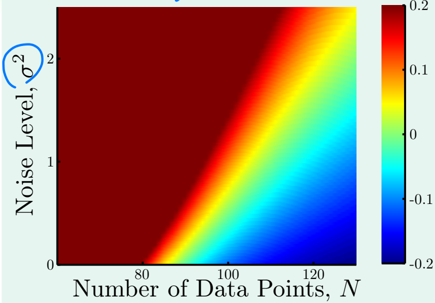
↓

overfit measure $E_{\text{out}}(g_{10}) - E_{\text{out}}(g_2)$

紅色overfit

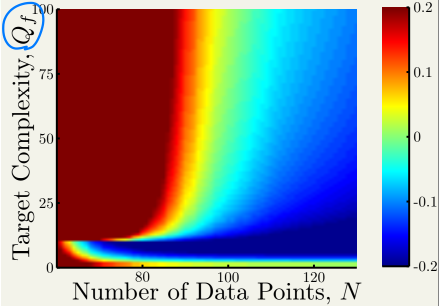
The Results

impact of σ^2 versus N



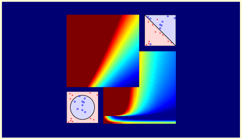
fixed $Q_f = 20$

impact of Q_f versus N



fixed $\sigma^2 = 0.1$

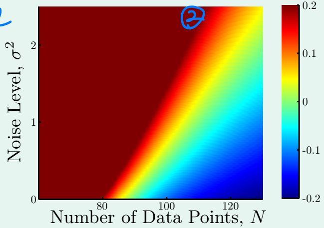
ring a bell? :-)



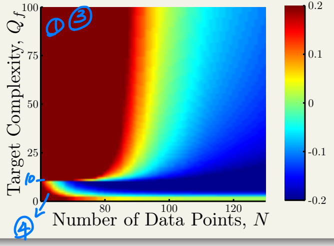
Impact of Noise and Data Size

impact of σ^2 versus N: **stochastic noise**

随机



impact of Q_f versus N: **deterministic noise**



four reasons of serious overfitting:

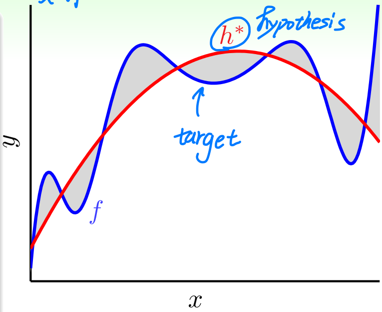
1 data size $N \downarrow$	overfit $\uparrow$
2 stochastic noise $\uparrow$	overfit $\uparrow$
3 deterministic noise $\uparrow$	overfit $\uparrow$
4 excessive power $\uparrow$	overfit $\uparrow$

overfitting 'easily' happens

Deterministic Noise

*target function 太複雜跟 noise 沒啥2樣.

- if $f \notin \mathcal{H}$: something of f cannot be captured by $\mathcal{H}$
- $\boxed{\text{deterministic noise}}$: difference between best $h^* \in \mathcal{H}$ and f
- acts like 'stochastic noise'—not new to CS: pseudo-random generator
- difference to stochastic noise:
  - depends on $\mathcal{H}$
  - fixed for a given $\mathbf{x}$



philosophy: when teaching a kid,
perhaps better not to use examples
from a complicated target function? :-)

Questions?

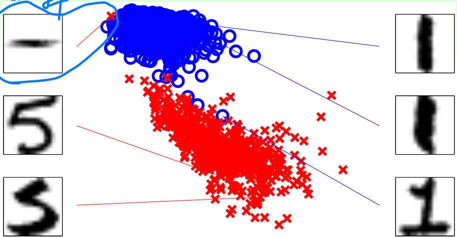
Driving Analogy Revisited

learning	driving
overfit	commit a car accident
use excessive d_{VC}	'drive too fast'
noise	bumpy road
limited data size N	limited observations about road condition
start from simple model	drive slowly
data cleaning/pruning 整理data	use more accurate road information
data hinting	exploit more road information
✓ **regularization**	put the brakes
validation	monitor the dashboard

all very **practical** techniques
to combat overfitting

Data Cleaning/Pruning

應該 prune 掉



- if 'detect' the outlier 5 at the top by
  - too close to other ∘, or too far from other ×
  - wrong by current classifier
  - ...

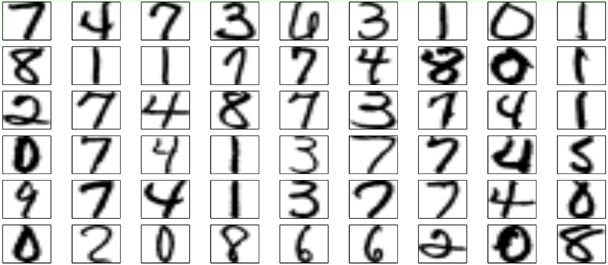
eg. daytime → nighttime

- possible action 1: correct the label (**data cleaning**)
- possible action 2: remove the example (**data pruning**)

possibly helps, but **effect varies**

Data Hinting

data augmentation



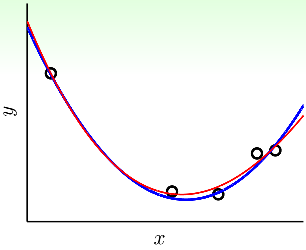
- slightly shifted/rotated digits carry the same meaning
- possible action: add **virtual examples** by shifting/rotating the given digits (**data hinting**, **data augmentation**)

不是同个 distribution !?

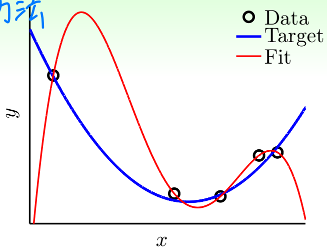
possibly helps, but **watch out**
—**virtual example not** $\overset{iid}{\sim} P(\mathbf{x}, y)$!

Regularization: The Magic of ‘Brake’

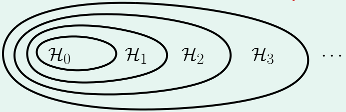
(对付overfitting的方法



‘regularized fit’ $\Longleftarrow$ overfit



- idea: ‘step back’ from $\mathcal{H}_{10}$ to $\mathcal{H}_2$ 高次走回低次. ⇒ regularization

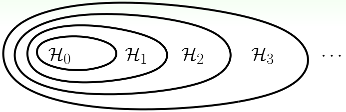


- name history: function approximation for **ill-posed problems**

how to step back?

Questions?

Stepping Back as Constraint



Q-th order polynomial transform for $x \in \mathbb{R}$:

$$\Phi_Q(x) \quad = \quad \left(1, x, x^2, \ldots, x^Q\right)$$

+ linear regression, denote $\tilde{\mathbf{w}}$ by $\mathbf{w}$

11个weight

hypothesis $\mathbf{w}$ in $\mathcal{H}_{10}$: $\quad w_0 + w_1 x + w_2 x^2 + w_3 x^3 + \ldots + w_{10} x^{10}$

hypothesis $\mathbf{w}$ in $\mathcal{H}_2$: $\quad w_0 + w_1 x + w_2 x^2$

that is, $\mathcal{H}_2 = \mathcal{H}_{10}$ AND 'constraint that $w_3 = w_4 = \ldots = w_{10} = 0$'

H_{10} 退化成 H_2

step back = **constraint**

Regression with Constraint

$$\mathcal{H}_{10} \equiv \left\{ \mathbf{w} \in \mathbb{R}^{10+1} \right\}$$

regression with $\mathcal{H}_{10}$:

$$\min_{\mathbf{w} \in \mathbb{R}^{10+1}} E_{\text{in}}(\mathbf{w})$$

$$\mathcal{H}_2 \equiv \left\{ \mathbf{w} \in \mathbb{R}^{10+1} \right.$$

constraint

$$\left. \text{while } w_3 = w_4 = \ldots = w_{10} = 0 \right\}$$

regression with $\mathcal{H}_2$:

$$\min_{\mathbf{w} \in \mathbb{R}^{10+1}} E_{\text{in}}(\mathbf{w}) \quad \text{s.t.} \quad w_3 = w_4 = \ldots = w_{10} = 0$$

step back = constrained optimization of E_{in}

why don't you just use $\mathbf{w} \in \mathbb{R}^{2+1}$? :-)

Regression with Looser Constraint

放宽

$$\mathcal{H}_2 \equiv \left\{ \mathbf{w} \in \mathbb{R}^{10+1} \text{ while } w_3 = \ldots = w_{10} = 0 \right\}$$

regression with $\mathcal{H}_2$:

$$\min_{\mathbf{w} \in \mathbb{R}^{10+1}} E_{\text{in}}(\mathbf{w}) \quad \text{s.t.} \quad w_3 = \ldots = w_{10} = 0$$

$$\mathcal{H}'_2 \equiv \left\{ \mathbf{w} \in \mathbb{R}^{10+1} \text{ while } \geq 8 \text{ of } w_q = 0 \right\}$$

只要有8个w是0啊

regression with $\mathcal{H}'_2$:

$$\min_{\mathbf{w} \in \mathbb{R}^{10+1}} E_{\text{in}}(\mathbf{w}) \quad \text{s.t.} \quad \sum_{q=0}^{10} [\![w_q \neq 0]\!] \leq 3$$

有3个係数不是零

- more flexible than $\mathcal{H}_2$: $\mathcal{H}_2 \subset \mathcal{H}'_2$
- less risky than $\mathcal{H}_{10}$: $\mathcal{H}'_2 \subset \mathcal{H}_{10}$

bad news for sparse hypothesis set $\mathcal{H}'_2$: ← 只有少数係数不是零

NP-hard to solve :-(

Regression with Softer Constraint

可是不好解

$$\mathcal{H}_2' \equiv \left\{ \mathbf{w} \in \mathbb{R}^{10+1} \quad \text{while } \geq 8 \text{ of } w_q = 0 \right\}$$

regression with $\mathcal{H}_2'$:

$$\min_{\mathbf{w} \in \mathbb{R}^{10+1}} E_{\text{in}}(\mathbf{w}) \text{ s.t. } \sum_{q=0}^{10} [\![w_q \neq 0]\!] \leq 3$$

$$\mathcal{H}(C) \equiv \left\{ \mathbf{w} \in \mathbb{R}^{10+1} \quad \text{while } \|\mathbf{w}\|^2 \leq C \right\}$$

regression with $\mathcal{H}(C)$:

(W的長度)$^2 \leq C$

$$\min_{\mathbf{w} \in \mathbb{R}^{10+1}} E_{\text{in}}(\mathbf{w}) \text{ s.t. } \sum_{q=0}^{10} w_q^2 \leq C$$

人為設定

- $\mathcal{H}(C)$: overlaps but not exactly the same as $\mathcal{H}_2'$
- soft and smooth structure over $C \geq 0$:
  $\mathcal{H}(0) \subset \mathcal{H}(1.126) \subset \ldots \subset \mathcal{H}(1126) \subset \ldots \subset \mathcal{H}(\infty) = \mathcal{H}_{10}$

用C来調節constraint的強度.

沒有constraint.

regularized hypothesis $\mathbf{w}_{\text{REG}}$:
optimal solution from
regularized hypothesis set $\mathcal{H}(C)$

Regularised.

Questions?

Matrix Form of Regularized Regression Problem

$$\min_{\mathbf{w} \in \mathbb{R}^{Q+1}} \quad E_{\text{in}}(\mathbf{w}) = \frac{1}{N} \underbrace{\sum_{n=1}^{N} (\mathbf{w}^T \mathbf{z}_n - y_n)^2}_{(\mathrm{Z}\mathbf{w}-\mathbf{y})^T(\mathrm{Z}\mathbf{w}-\mathbf{y})} \leftarrow \text{matrix form}$$

$$\text{s.t.} \quad \underbrace{\sum_{q=0}^{Q} w_q^2}_{\mathbf{w}^T\mathbf{w}} \leq C \leftarrow \text{加上constraint 的 regression}$$

- $\sum_n \ldots = (\mathrm{Z}\mathbf{w} - \mathbf{y})^T(\mathrm{Z}\mathbf{w} - \mathbf{y})$, **remember? :-)**
- $\mathbf{w}^T\mathbf{w} \leq C$: feasible $\mathbf{w}$ within a radius-$\sqrt{C}$ hypersphere

1个半径是$\sqrt{C}$的球体

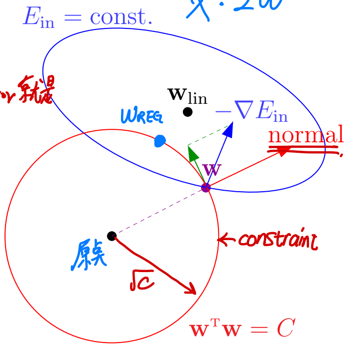
how to solve
constrained optimization problem?

The Lagrange Multiplier

$$\underline{\min_{\mathbf{w}\in\mathbb{R}^{Q+1}} \quad E_{\text{in}}(\mathbf{w}) = \frac{1}{N}(\mathrm{Z}\mathbf{w}-\mathbf{y})^T(\mathrm{Z}\mathbf{w}-\mathbf{y}) \text{ s.t. } \mathbf{w}^T\mathbf{w} \le C}$$

$\frac{\partial}{\partial w}$ ↓ $\frac{2}{N}(Zw-y)\cdot Z$ $\lambda \cdot 2W$

- decreasing direction: $\underline{-\nabla E_{\text{in}}(\mathbf{w})}$, **remember? :-)**
  負gradient
- 法向量 normal vector of $\underline{\mathbf{w}^T\mathbf{w} = C}$: $\mathbf{w}$. normal vector 就是 W本身
- if $-\nabla E_{\text{in}}(\mathbf{w})$ and $\mathbf{w}$ not parallel: can **decrease $E_{\text{in}}(\mathbf{w})$ without violating the constraint** *找-∇Ein(W)中垂直於W的分量
- at optimal solution $\mathbf{w}_{\text{REG}}$, $-\nabla E_{\text{in}}(\mathbf{w}_{\text{REG}}) \propto \boxed{\mathbf{w}_{\text{REG}}}$ ← 最後gradient与W平行



want: find Lagrange multiplier $\lambda > 0$ and $\mathbf{w}_{\text{REG}}$ such that $\underline{\nabla E_{\text{in}}(\mathbf{w}_{\text{REG}}) + \frac{2\lambda}{N}\boxed{\mathbf{w}_{\text{REG}}}} = \mathbf{0}$

Augmented Error

- if oracle tells you $\lambda > 0$, then

Find.

solving $$\underbrace{\nabla E_{\text{in}}(\mathbf{w}_{\text{REG}})}_{\downarrow\text{微分}} + \frac{2\lambda}{N}\boxed{\mathbf{w}_{\text{REG}}} = \mathbf{0}$$

$$\underbrace{\frac{2}{N}\left(\mathrm{Z}^T\mathrm{Z}\mathbf{w}_{\text{REG}} - \mathrm{Z}^T\mathbf{y}\right)}_{\downarrow\text{化簡}} + \frac{2\lambda}{N}\boxed{\mathbf{w}_{\text{REG}}} = \mathbf{0}$$

- optimal solution:

λ>0 則 inverse 存在

$$\mathbf{w}_{\text{REG}} \leftarrow \underline{(\mathrm{Z}^T\mathrm{Z} + \lambda\mathrm{I})}^{-1}\mathrm{Z}^T\mathbf{y}$$

$(1+\lambda)^{-1}y = w$

—called ridge regression in Statistics

linear regression 的進階版.

minimizing unconstrained E_{aug} effectively minimizes some C-constrained E_{in}

Augmented Error

- if oracle tells you $\lambda > 0$, then

$$\text{solving} \quad \nabla E_{\text{in}}(\mathbf{w}_{\text{REG}}) + \frac{2\lambda}{N}\boxed{\mathbf{w}_{\text{REG}}} = \mathbf{0}$$

积分

$$\text{equivalent to minimizing} \quad \underbrace{E_{\text{in}}(\mathbf{w}) + \frac{\lambda}{N}\overbrace{\mathbf{w}^T\mathbf{w}}^{\textbf{regularizer}}}_{\textbf{augmented error } E_{\text{aug}}(\mathbf{w})}$$

←最小化

多一項

- regularization with augmented error instead of constrained E_{in}

直接解 $E_{aug}(w)$

$$\mathbf{w}_{\text{REG}} \leftarrow \underset{\mathbf{w}}{\text{argmin}}\, E_{\text{aug}}(\mathbf{w}) \text{ for given } \lambda > 0 \text{ or } \lambda = 0$$

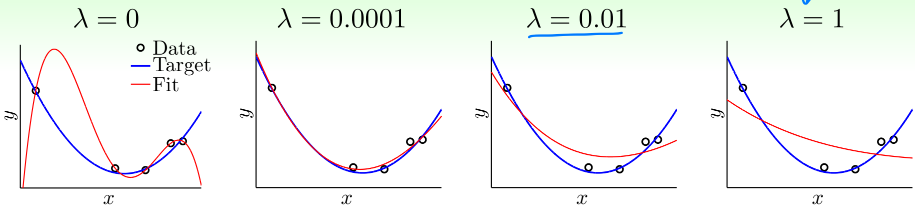
⇒ 沒有constraint

*但要先給定λ

minimizing unconstrained E_{aug} effectively
minimizes some C-constrained E_{in}

The Results

constraint 太強



overfitting $\Longrightarrow$ $\Longrightarrow$ $\Longrightarrow$ underfitting

philosophy: *a little **regularization** goes a long way!*

call '$+\frac{\lambda}{N}\mathbf{w}^T\mathbf{w}$' **weight-decay** regularization:

長度

larger λ ← 懲罰 long W

$\Longleftrightarrow$ prefer shorter $\mathbf{w}$

$\Longleftrightarrow$ effectively smaller C

—go with 'any' transform + linear model

Questions?

Regularization and VC Theory

Regularization by Constrained-Minimizing E_{in}

$$\min_{\mathbf{w}} E_{\text{in}}(\mathbf{w}) \text{ s.t. } \mathbf{w}^T\mathbf{w} \le C$$

$\rightarrow$

VC Guarantee of Constrained-Minimizing E_{in}

$$E_{\text{out}}(\mathbf{w}) \le E_{\text{in}}(\mathbf{w}) + \Omega(\mathcal{H}(C))$$

?

$\Updownarrow$ C equivalent to some λ

給定C等於給λ

Regularization by Minimizing E_{aug}

$$\min_{\mathbf{w}} E_{\text{aug}}(\mathbf{w}) = E_{\text{in}}(\mathbf{w}) + \frac{\lambda}{N}\mathbf{w}^T\mathbf{w}$$

因為是unconstraint,所以還是考慮了所有w 只是不會去選它們

minimizing E_{aug}: indirectly getting VC guarantee **without confining to $\mathcal{H}(C)$**

Another View of Augmented Error

complexity penalty

Augmented Error

$$E_{\text{aug}}(\mathbf{w}) = E_{\text{in}}(\mathbf{w}) + \frac{\lambda}{N}\mathbf{w}^T\mathbf{w}$$

VC Bound

$$E_{\text{out}}(\mathbf{w}) \le E_{\text{in}}(\mathbf{w}) + \boxed{\Omega(\mathcal{H})}$$

- regularizer $\mathbf{w}^T\mathbf{w}$: complexity of a single hypothesis
- generalization price $\Omega(\mathcal{H})$: complexity of a hypothesis set
- if $\frac{\lambda}{N}\Omega(\mathbf{w})$ 'represents' $\Omega(\mathcal{H})$ well, 因為 E_{aug}是等式，但E_{out}是不等式
  E_{aug} is a better $\boxed{\text{proxy}}$ of E_{out} than E_{in}

$\vec{w}$ 的complexity

代理人

minimizing E_{aug}:

不是數學

(heuristically) operating with the better proxy;
(technically) enjoying flexibility of whole $\mathcal{H}$

Effective VC Dimension

$$\min_{\mathbf{w}\in\mathbb{R}^{\tilde{d}+1}} E_{\text{aug}}(\mathbf{w}) = E_{\text{in}}(\mathbf{w}) + \frac{\lambda}{N}\Omega(\mathbf{w})$$

- model complexity? ← 没有变
  $d_{\text{VC}}(\mathcal{H}) = \tilde{d}+1$, because $\{\mathbf{w}\}$ '**all considered**' during minimization
- $\{\mathbf{w}\}$ '**actually needed**': $\mathcal{H}(C)$, with some C equivalent to λ
  H(C)内的w才是真的会被考虑的.
- $d_{\text{VC}}(\mathcal{H}(C))$:
  effective VC dimension $d_{\text{EFF}}(\mathcal{H}, \underbrace{\mathcal{A}}_{\min E_{\text{aug}}})$ algorithm
  等效VC dimension.

explanation of regularization:
$d_{\text{VC}}(\mathcal{H})$ large,
while $d_{\text{EFF}}(\mathcal{H}, \mathcal{A})$ small if $\mathcal{A}$ regularized

Questions?

General Regularizers $\Omega(\mathbf{w})$

want: constraint in the **'direction' of target function**

- target-dependent: some properties of target, if known
  - symmetry regularizer: $\sum [\![q \text{ is odd}]\!] \, w_q^2$ ← 对稱: 讓奇数次方項变小
- plausible: direction towards smoother or simpler

  stochastic/deterministic noise both **non-smooth**

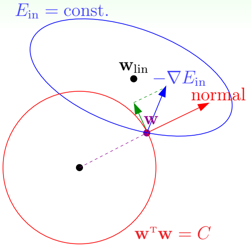
  - sparsity (L1) regularizer: $\sum |w_q|$ (next slide) ← 找出簡单的
- friendly: easy to optimize
  - weight-decay (L2) regularizer: $\sum w_q^2$ ← 容易 Optimize.
- bad? **:-)**: no worries, $\boxed{\text{guard by } \lambda}$

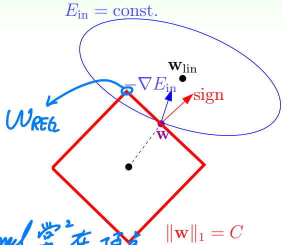
augmented error = error $\widehat{\text{err}}$ + regularizer Ω

regularizer: target-dependent, plausible, or friendly

domain knowledge

L2 and L1 Regularizer





Optimal 常常在頂点

L2 Regularizer

$$\Omega(\mathbf{w}) = \sum_{q=0}^{Q} w_q^2 = \|\mathbf{w}\|_2^2$$

- convex, differentiable everywhere
- easy to optimize

L1 Regularizer

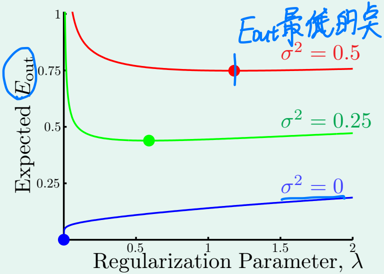
$$\Omega(\mathbf{w}) = \sum_{q=0}^{Q} |w_q| = \|\mathbf{w}\|_1$$

- convex, **not** differentiable everywhere 大部份 weight 都是零
- **sparsity** in solution

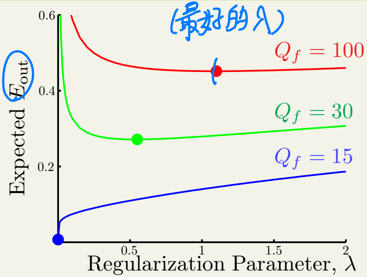
L1 useful if needing **sparse solution** ⇒省計算

The Optimal λ

stochastic noise



deterministic noise



- more noise $\Longleftrightarrow$ more regularization needed
  —more bumpy road $\Longleftrightarrow$ putting brakes more
- noise **unknown**—important to **make proper choices**

how to choose?
stay tuned for the next lecture! :-)

Questions?

Summary

1. How Can Machines Learn?

   Lecture 06: Beyond Basic Linear Models

2. How Can Machines Learn **Better**?

   Lecture 07: Combatting Overfitting

   - What is Overfitting?
     lower E_{in} but higher E_{out}
   - The Role of Noise and Data Size
     overfitting 'easily' happens!
   - Deterministic Noise
     what $\mathcal{H}$ cannot capture acts like noise
   - Dealing with Overfitting
     data cleaning/pruning/hinting & regularization
   - Regularized Hypothesis Set
     original $\mathcal{H}$ + constraint
   - Weight Decay Regularization
     add $\frac{\lambda}{N}\mathbf{w}^T\mathbf{w}$ in E_{aug}
   - Regularization and VC Theory
     regularization decreases d_{EFF}
   - General Regularizers
     target-dependent, [plausible], or [friendly]

- **next: choosing from the so-many models/parameters**