CS 350: Homework #2 Solutions

February 10, 2016

1 Problem 1

The availability of a web server is rated at 96%. Assuming the connections are independent.

a. This is a **Binomial Distribution**. $\binom{20}{0}(1-0.96)^0\,(0.96)^{(20-0)} \approx 0.442$.

b. $\binom{20}{1}(1-0.96)^1(0.96)^{20-1} \approx 0.368$.

c. $\binom{20}{0}(1-0.96)^0(0.96)^{(20-0)} + \binom{20}{1}(1-0.96)^1(0.96)^{(20-1)} + \binom{20}{2}(1-0.96)^2(0.96)^{(20-2)}$
$+ \binom{20}{3}(1-0.96)^3(0.96)^{(20-3)} \approx 0.993$

d. This is a **Geometric Distribution**. $\sum_{i=4}^{\infty} 0.96^i(1-0.96) \approx 0.96^4 \approx 0.849$. In particular, assuming the 4 (or more) consecutive requests can start at any point of the request consequence is also acceptable way.

e. The average number of requests that will be successfully served before a connection is refused is $\frac{0.96}{1-0.96} = 24$.

f. The average number of requests that will be successfully served out of a set of 50 requests for connections is $50 \times 0.96 = 48$.

g. For part (g), we need to use **CLT**. Basically, let the value $X_n = x_1+x_2+\ldots+x_n$ be the number of successful requests. Let's define another random variable $Z = \frac{(\Sigma_{i=1}^n x_i) - n\mu}{\sigma\sqrt{n}} = \frac{\frac{(\Sigma_{i=1}^n x_i)}{n} - \mu}{\frac{\sigma}{\sqrt{n}}}$, which follows a standard normal distribution as n grows large. Our goal is to make sure that at least 95% of the requests will be processed successfully, that is, $\frac{(\Sigma_{i=1}^n x_i)}{n} \geq 0.95$, and the corresponding probability we need to find is

$$P\left(\frac{(\Sigma_{i=1}^n x_i)}{n} \geq 0.95\right) = P\left(\frac{\frac{(\Sigma_{i=1}^n x_i)}{n} - \mu}{\frac{\sigma}{\sqrt{n}}} \geq \frac{0.95-\mu}{\frac{\sigma}{\sqrt{n}}}\right)$$

$$= P\left(Z \geq \frac{0.95-0.96}{\frac{\sqrt{0.96*0.04}}{\sqrt{n}}}\right) \approx P(Z \geq -0.051\sqrt{n})$$

Clearly, when n approaches infinity, the probability goes to 1. For instance, let's assume that $n = 3600$, then $P(Z \geq -0.051\sqrt{n}) = P(Z < 3.06) = 0.9989$.

h. Referring to the Lecture Note 3, the `reliability` of an on-line solution is a measure of the probability that the system functions correctly (provides services) continuously for *a specified time period*, i.e., it is a measure of continuity of service, which is what part (*d*) is asking about.

2 Problem 2: part 1

a. The arrival rate of packets is described by a Poisson process, where $\lambda = 200$ *packets/second*. Given that the rate of arrival per interval h is λh, the probability of x arrivals per interval h is $f(x) = \left(\frac{(\lambda h)^x}{x!}\right) e^{-\lambda h}, x = 0, 1, 2, \ldots$; We know that the time interval is $h = 10$ *milliseconds* $=$ 0.01 *second*, therefore, $\lambda h = 200 \times 0.01 = 2$, and $f(x) = \frac{2^x}{e^2(x!)}, x = 0, 1, 2, \ldots$

b. The probability that more than 3 packet will be received within a 10-millisecond period of time: $P(x > 3) = 1 - P(x \leq 3) = 1 - P(x = 0) - P(x = 1) - P(x = 2) - P(x = 3) = 1 - \frac{19}{3e^2} \approx 0.143.$

c. The inter-arrival time of packets is exponential distribution, the probability distribution is $f(x) = \lambda_1 e^{-\lambda_1 x}$, where $\lambda_1 = 200 \times 0.001 = 0.2$.

d. An important characteristic of the exponential distribution is that it is "memoryless", we know that $P(X > s + t | X > t) = P(X > s)$. Therefore, the probability that two packets will be separated by more than 8 milliseconds is $P(x > 8) = 1 - P(x \leq 8) = 1 - (1 - e^{-0.2\times 8}) = e^{-1.6} \approx$ 0.202.

e. The standard deviation of the inter-arrival time of packets is equal to $\frac{1}{\lambda_1} = 5$ *milliseconds* $=$ 0.005 *second*

3 Problem 2: part 2

This is an $M/M/1$ queuing system with $\lambda = 200$ *packets/second* and $T_s = 4.8$ *milliseconds/packet* $=$ 0.0048 *second/packet*.

a. The capacity of the Ethernet adapter $\mu = \frac{1}{T_s} \approx 208.33$ *packets/second*.

b. The probability of having j packets in the system is $P(S_j) = (1 - \rho)\rho^j$, where $j = 0, 1, 2, \ldots$ and $\rho = \frac{\lambda}{\mu} \approx 0.96$.

c. The average number of packets waiting in the buffer of the adapter at any point in time is $w = \frac{\rho^2}{1 - \rho} \approx \frac{0.96^2}{1 - 0.96} = 23.04$ *packets*.

d. The average waiting time in the buffer is $T_w = \frac{\rho}{\mu(1 - \rho)} \approx 0.115$ *seconds*.

e. The slowdown caused by queuing at the Ethernet adapter is $Slowdown = \frac{1}{1 - \rho} = 25$.

4 Problem 3

Write a Java function that returns a random value that is distributed according to an exponential distribution with a mean of T=1/Lambda.
Note that, we can generate a random variable x that follows an exponential distribution by equating the cumulative distribution function for the exponential distribution, namely $F(x) = 1 - e^{-\lambda x}$, to Y and solving for x. Thus, we can obtain that $x = -\dfrac{\ln(1-Y)}{\lambda}$.

a. Java Source Code

Listing 1: Generate 100 random values

```
import java.util.*;
import java.lang.*;

public class hw0203 {
    public static double ExpDistributionGen(double Lambda) {
        Random RandomGenerator = new Random();
        double Y = RandomGenerator.nextDouble();
        double x = (- Math.log(1.0-Y))/Lambda;
        return x; }
        public static void GenExpDist100() {
            for (int i = 0; i < 100; i++) {
                System.out.println(ExpDistributionGen(4));
            }
        }
        public static void main (String[] args) {
            GenExpDist100();
        }
}
```

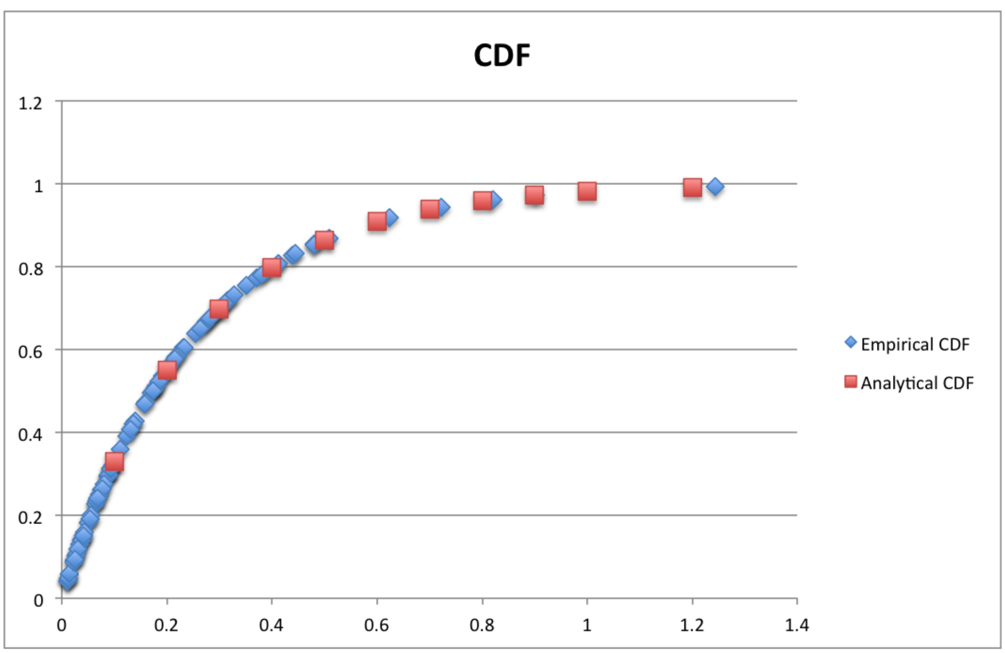

Figure 1: Comparison with analytical CDF

From Figure 1, we can see that the empirically and analytically obtained CDFs almost match. Since we are generating the random variable that follows an exponential distribution using a random variable with uniform distribution, and the correctness of the transformation can be found in the lecture notes.

b. Inter-arrival Times

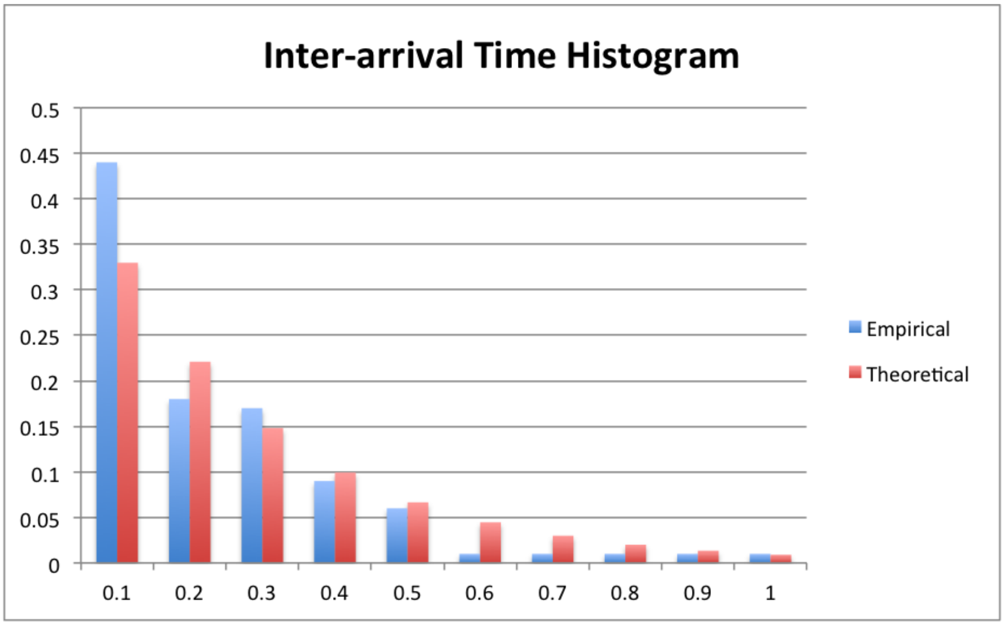

Figure 2: Distribution of the inter-arrival times

Figure 2 is a normalized histogram showing the relative frequency (on the Y axis) of the various number of queries observed over 1-second intervals (on the X axis). The distribution of the inter-arrival times is $f(x) = 4e^{-4x}$.

5 Problem 4

a. The probability mass function (PMF) for the random variable t: $Pr(t = 1) = 50\%$, $Pr(t = 3) = 30\%$, $Pr(t = 10) = 14\%$, $Pr(t = 30) = 6\%$,

b. The mean of the random variable t is: $Pr(t = 1) \times 1 + Pr(t = 3) \times 3 + Pr(t = 10) \times 10 + Pr(t = 30) \times 30 = 4.6$.

c. The standard deviation of the random variable t is $sqrt((1 - 4.6^2 \times 0.5 + (3 - 4.6)^2 \times .3 + (10 - 4.6)^2 \times 0.14 + (30 - 4.6)^2 \times 0.06)) \approx 7.07$

d. Write a Java function that returns a random value that is distributed according to the PMF you obtained for part (a).

Listing 2: Generate 100 random values according to (a)

```
import java.util.*;
import java.lang.*;

public class hw0204 {
    public static double RandomGen() {
        Random RandomGenerator = new Random();
        double RandomVal = RandomGenerator.nextDouble();
```

```
            if (RandomVal >= 0 && RandomVal < 0.5 ){
                return 1;
            } else if (RandomVal >= 0.5 && RandomVal < 0.8 ){
                return 3;
            } else if (RandomVal >= 0.8 && RandomVal < 0.94 ) {
                return 10;
            } else {
                return 30;
            }
        }
        public static void main (String[] args) {
            System.out.println(RandomGen());
        }
    }
```

6 Problem 5

Write a Java function – say Zrand() – that returns a random value that is distributed according to a standard normal distribution.

a. Standard-normal Random Variable

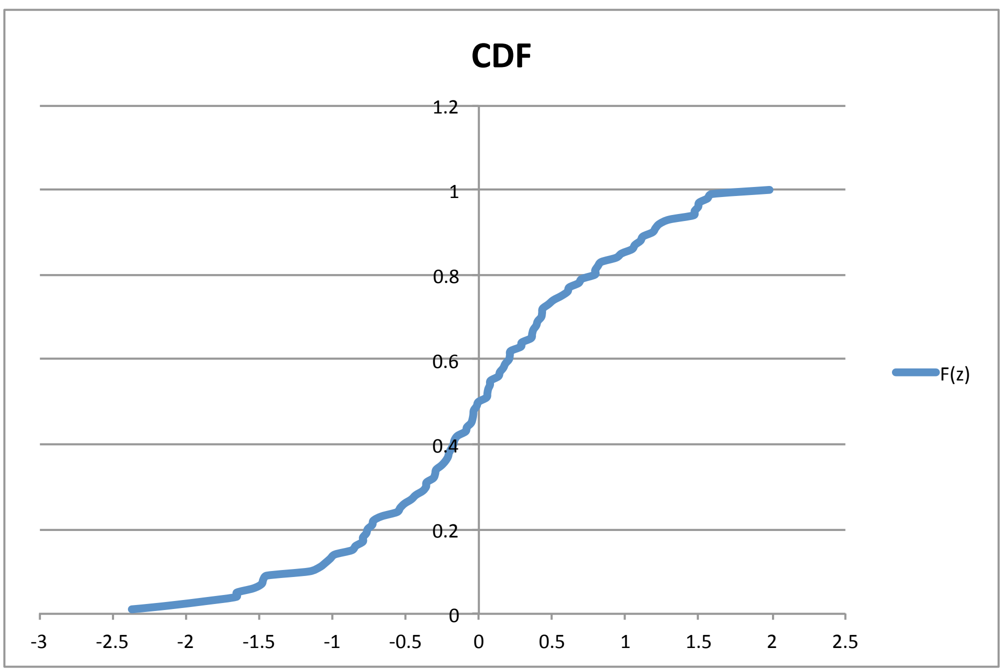

Figure 3: CDF of the standard normal distribution

From Figure 3, we can see that $P(x \leq 0) \approx 0.5$, $P(x \leq 1) \approx 0.8$, $P(x \leq 2) \approx 0.98$, $P(x \leq 3) \approx 1$, and $P(x \leq 4) \approx 1$. Therefore, the CDF of the random variable that I have obtained empirically almost match the CDF given in the standard normal distribution table. The reason is what the **Central Limit Theorem (CLT)** states.

b. Normally-distributed Random Variable

For the normally-distributed random variable z' with a mean = 72 and standard deviation = 16, we can compute the probability that this random variable is between 66 and 80 by looking up its Z-value table. For 66, its corresponding Z-value is $\frac{66-72}{16} = -0.375$; For 80, its corresponding Z-value is $\frac{80-72}{16} = 0.5$. Therefore, this probability can be computed as $F(0.5) - F(-0.375) = F(0.5) + F(0.375) - 1 \approx 0.338$. In my program, I set the loop as 1000000, which is large enough to get an relatively accurate result. The results I got empirically are around 0.3355.

The Java source code for this problem is as follows:

Listing 3: Normally-distributed Random Variable

```
package hw2;

public class normalDistribution
{

    public static double Zrand()
    {
        double sum = 0;
        double N = 30;

        for(int i =0; i< N; i++)
        {
            sum += Math.random();
        }

        return (sum - 15.0)/1.58;
    }

    public static void StandardNormalDistributionGen()
    {
        for(int i = 0; i < 100; i++)
        {
           System.out.println(Zrand());
        }
    }

    public static void NormallyDistributed()
    {
        int hit = 0;

        for(int i = 0; i < 1000000; i++)
        {
            double zValue = Zrand();
            double x = zValue * 16 + 72;

            if (x >= 66 && x <=80)
            {
              hit ++;
            }

        }

        System.out.println("Hit probability:" + (hit / 1000000.0));
    }

    public static void main(String [] args)
```

```
    {
      StandardNormalDistributionGen();
        NormallyDistributed();
    }
}
```