Amortized Analysis of a Pointer–Jumping Solution to *Daily Temperatures*

Model. We process days right-to-left. On iteration k we may follow a sequence of *pointer jumps* along a chain of previously created nodes until we find an appropriate attachment point, then we append a new node as the new head. Let h_k be the number of pointer jumps performed on iteration k. All other work per iteration is bounded by a constant A. Let $b \geq 1$ be the constant cost of one jump. Thus the actual cost satisfies

$$c_k \;\leq\; b\,h_k + A.$$

Potential. Let C_k be the active chain after iteration k, and define the potential

$$\Phi_k \;=\; |C_k| \qquad (k \geq 0),$$

with $\Phi_0 = 0$. Clearly $\Phi_k \geq 0$ for all k.

Lemma 1 (Chain Shortening)**.** *For every $k \geq 1$,*

$$|C_k| \;\leq\; |C_{k-1}| \;-\; h_k \;+\; 1, \qquad \textit{equivalently} \qquad \Delta\Phi_k \;\leq\; 1 - h_k.$$

Proof. During iteration k we traverse h_k pointers from the head of C_{k-1}. Those h_k head nodes are no longer on the path from the new head to the tail in C_k. We then attach the new node as the head, increasing the path length by 1. No cycles are created and no other nodes are added or removed from the path. Hence the inequality. □

Theorem 1 (Amortized $O(1)$ time per iteration)**.** *Define the amortized cost $\widehat{c}_k = c_k + \Delta\Phi_k$. Then $\widehat{c}_k \leq A + (b-1)$ for all k, and the total actual cost over n iterations satisfies*

$$\sum_{k=1}^{n} c_k \;\leq\; \big(A + (b-1)\big)\, n.$$

Proof. By the lemma, $\Delta\Phi_k \le 1 - h_k$, hence

$$\widehat{c}_k \;=\; c_k + \Delta\Phi_k \;\le\; (b\,h_k + A) + (1 - h_k) \;=\; A + (b-1).$$

Summing and using telescoping of the potential gives

$$\sum_{k=1}^{n} c_k \;=\; \sum_{k=1}^{n} \widehat{c}_k - (\Phi_n - \Phi_0) \;\le\; \big(A + (b-1)\big)\, n - \Phi_n \;\le\; \big(A + (b-1)\big)\, n. \quad \square$$

Theorem 2 (Bound on pointer jumps)**.** *The total number of pointer jumps satisfies* $\sum_{k=1}^{n} h_k \le n$.

Proof. From $\Delta\Phi_k \le 1 - h_k$ we have $h_k \le 1 - \Delta\Phi_k$. Summing and telescoping yields

$$\sum_{k=1}^{n} h_k \;\le\; n - \sum_{k=1}^{n} \Delta\Phi_k = n - (\Phi_n - \Phi_0) \le n.$$

$\square$