FINAL EXAM

PSTAT 171 – Fall 2016

By writing my name and signing below I swear that I have not engaged in any form of academic dishonesty or cheating.

Name:____

Signature:____

Section meeting date & time:___________________________________

Recall the midterm was worth 50 points, which is 5/12 of the 120 possible for this final. To replace your midterm score with 5/12 of your score on this final, check here: ☐

- Do not turn the page until you are directed to do so.
- Turn your cellphone/pager/ipod off during the exam period.
- You have 180 minutes to complete the exam.
- You may use one double-sided 8.5" x 11" sheet of notes, and your own calculator. You cannot borrow or lend anything during the exam period.
- You may only consult the supervisor to clarify the meaning of an exam question.
- This test has twelve problems, and is worth 120 points.
- Partial credit will be awarded for all significant work. Full credit will not be awarded if the grader cannot understand how you arrived at your final answer, even if the final answer is correct. $\boxed{BOX}$ or clearly MARK your final answers.
- Do not write anything in the boxes shown below. Good luck!

Problem	1	2	3	4	5
Score					
Problem	6	7	8	9	10
Score					
Problem	11	12			
Score					
FINAL TOTAL			MIDTERM (5/12 OF FINAL TOTAL)		

1. (10 Points) Cinderella has saved \$22,000 for a down payment and qualifies for a thirty-year mortgage with level monthly payments at an annual rate of 6% convertible monthly. In addition, Cinderella is opening a retirement account that has an annual rate of 3% payable monthly. During the next forty years, Cinderella will have \$2,600 at the end of each month to use for a combination of level mortgage payments and retirement contributions. Cinderella plans to retire at the end of forty years, at which point she needs to be able to withdraw \$4,100 at the beginning of each month for twenty-five years.

How expensive a house can she buy?

Solution: Let \$$X$ denote Cinderella's monthly mortgage payment. Her contributions to the retirement fund are \$(2,600 – X) at the end of each month for the first thirty years and \$2,600 per month during the next ten years. The accumulated value of these contributions at the end of 40 years equals

$$\begin{aligned} F.V. &= (1)\$(2{,}600 - X)\, s_{\overline{360}|0.0025}(1.0025)^{120} + \$2{,}600 s_{\overline{120}|0.0025} \\ &= \$(2{,}600 - X)\frac{(1.0025)^{360} - 1}{0.0025}(1.0025)^{120} + \$2{,}600\frac{(1.0025)^{120} - 1}{0.0025} \\ &= \$786.318082(2{,}600 - X) + 363{,}327.69. \end{aligned}$$

The present value of the withdrawals of \$4,100 made at the beginning of each month for twenty-five years, computed at the time of the first withdrawal, equals

$$\begin{aligned} P.V. &= \$4{,}100\ddot{a}_{\overline{300}|0.0025} \\ &= \$4{,}100\frac{1 - (1.0025)^{-300}}{0.0025}(1.0025) \\ &= \$866{,}754.94 \end{aligned}$$

Since the accumulated value of the retirement contributions must suffice to allow for withdrawals of \$4,100 for twenty-five years, we must have

$$\$786.318082(2{,}600 - X) + \$363{,}327.69 = \$866.754.94.$$

Solving this equation yields $X = 1{,}959.766404$. Therefore, the most expensive house Cinderella can buy today is worth

$$\begin{aligned} P.V. &= \$22{,}000 + Xa_{\overline{360}|0.005} \\ &= \$22{,}000 + 1{,}959.766404\frac{1 - (1.005)^{-360}}{0.005} \\ &= \$22{,}000 + \$326{,}872.60 \\ &= \boxed{\$348{,}872.60}. \end{aligned}$$

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2. (10 Points) A small business takes out a loan of \$12,000 at a nominal rate of 12%, compounded quarterly, to help finance its start-up fixed costs. Payments of \$750 are made at the end of every six months for as long as is necessary to pay back the loan, plus a smaller final payment (if necessary).

Three months before the 9th payment is due, the company refinances the loan at a nominal rate of 9%, compounded monthly. Under the refinanced loan, payments of \$*R* are to be made monthly, with the first monthly payment to be made at the same time that the 9^{th} payment under the old loan was to be made. A total of thirty monthly payments of \$*R* each will completely pay off the loan. Determine R.

Solution: The equivalent semiannual rate for the old nominal rate of 12%, compounded quarterly, is $j = (1.03)^2 - 1 = 6.09\%$. Immediately following the 8^{th} semiannual payment, the retrospective method can be used to compute the outstanding balance of the original loan as follows:

$$\begin{aligned} B_8 &= \$12{,}000(1.0609)^8 - \$750 s_{\overline{8}|0.0609} \\ &= \$12{,}000(1.0609)^8 - \$750\frac{(1.0609)^8 - 1}{0.0609} \\ &= \$19{,}256.48 - \$7{,}447.12 \\ &= \$11{,}809.35 \end{aligned}$$

The outstanding loan balance three months later on the refinancing date is

$$B_{8.5} = \$11{,}809.35(1.03) = \$12{,}163.63.$$

At this time, the rate of interest changes to $\frac{0.09}{12} = 0.0075$ effective monthly.
The value on the refinancing date of the thirty payments of \$*R* each, computed using the prospective method, is

$$B_{8.5} = Ra_{\overline{30}|0.0075}(1.0075)^{-2} = R\frac{1-(1.0075)^{-30}}{0.0075(1.0075)^2} = \$26.3779R.$$

Equate both expressions for $B_{8.5}$ and solve for R to obtain that

$$R = \frac{12{,}163.63}{26.3779} = \boxed{461.13}.$$

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3. (10 points) Mr. Marin purchases a perpetuity-immediate that makes monthly payments. The first payment is \$512, and each payment thereafter increases by \$6.

Mr. Chong purchases a perpetuity-due which makes monthly payments of \$1000.

Using the same annual effective rate $i > 0$, the present value of both perpetuities are equal.

Calculate i.

Solution: For the given annual effective rate $i > 0$, let j denote the equivalent monthly rate.

The present value of Mr. Marin's perpetuity is

$$P.V._I = \$\left[506a_{\overline{\infty}|j} + 6\left(Ia\right)_{\overline{\infty}|j}\right] = \$\left[\frac{506}{j} + \frac{6}{jd_j}\right] = \$\left[\frac{506}{j} + \frac{6(1+j)}{j^2}\right]$$

The present value of Mr. Chong's perpetuity is

$$P.V._C = \$1000\ddot{a}_{\overline{\infty}|j} = \$\frac{1000}{d_j} = \$\frac{1000(1+j)}{j}$$

Equating the present values and clearing the denominators by multiplying the resulting equation by j^2 we obtain

$$506j + 6(1 + j) = 1000j(1 + j).$$

By expanding and rearranging terms in this equation we get the quadratic equation

$$1000j^2 + 488j - 6 = 0.$$

The positive solution to this equation is $j = 1.2\%$ per month.

Therefore, the equivalent annual effective rate is

$$i = (1 + j)^{12} - 1 = (1 + 0.012)^{12} - 1 = \boxed{15.38946\%}$$

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4. (10 points) At the beginning of the year, an investment fund was established with an initial deposit of \$1000. A new deposit of \$1000 was made at the end of four months. Withdrawals of \$200 and \$500 were made at the end of 6 and 8 months, respectively. The amount in the fund at the end of the year is \$1560. Calculate the dollar-weighted yield rate earned by the fund during the year.

Solution: Interest earned during the year is

$$\begin{aligned} I &= B_1 + \text{Withdrawals} - \text{Deposits} - B_0 \\ &= \$1560 + \$200 + \$500 - \$1000 - \$1000 \\ &= \$260. \end{aligned}$$

The dollar-weighted yield rate earned by the fund is

$$\begin{aligned} i &= \frac{I}{B_0 + \sum_{0 \le t \le 1} (1-t) C_t} \\ &= \frac{260}{1000 + 1000\left(1 - \frac{4}{12}\right) - 200\left(1 - \frac{6}{12}\right) - 500\left(1 - \frac{8}{12}\right)} \\ &= \frac{260}{1400} \\ &= \boxed{18.5714\% \text{ per annum}} \end{aligned}$$

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5. (10 points) The following is a table of interest rates credited under the investment year and portfolio methods. Unfortunately some of the entries are missing.

Calendar Year of Investment	Investment Year Rates			Calendar Year of Portfolio Rate	Portfolio Rate
y	i_1^y	i_2^y	i_3^y	$y+3$	i^{y+3}
2005	5.2%		4.8%	2008	4.3%
2006	4.2%	4.7%	4.0%	2009	
2007		5.5%	5.7%	2010	5.1%
2008	6.0%	5.0%		2011	5.5%

Gimli invested \$1000 on January 1, 2005 for six years using the investment year method. The accumulated value on his investment was \$1247.33.

Legolas invested \$2000 on January 1, 2008 for three years using the portfolio method.

Gimli and Legolas earned the same average yield rate on their investments.

Calculate i_2^{2005}.

Solution: The average yield rate j on Gimli's investment satisfies the equation of value

$$1247.33 = 1000(1+j)^6 \Rightarrow j = 3.7521\%$$

Since this was also the average yield rate j on Legolas' investment, the accumulated value on his investment was

$$S_3 = 2000(1+j)^3 = 2000(1.037521)^3 = \$2233.68$$

The accumulated value on Legolas' investment can also be computed using the portfolio method as follows:

$$2233.68 = 2000(1.043)(1+i^{2009})(1.051) = 2192.386(1+i^{2009})$$

Solving this equation gives $i^{2009} = 1.8835\%$.

Similarly, the accumulated value on Gimli's investment was

$$\begin{aligned}1247.33 &= 1000(1.052)(1+i_2^{2005})(1.048)(1.043)(1.018835)(1.051)\\ &= 1231.3109(1+i_2^{2005})\end{aligned}$$

Solving this equation gives $i_2^{2005} = \boxed{1.3010\%}$.

6. (10 points) You are considering investing $1000 for a three-year period, beginning January 1, 2014 and ending December 31, 2016. The market offers only zero-coupon bonds maturing in one, two or three years. Looking into your crystal ball, you see the following term structure interest rates by date of purchase:

	Yield to Maturity		
Date of Purchase	One-year Bond	Two-year Bond	Three-year Bond
January 1, 2014	3.0%	4.0%	4.5%
January 1, 2015	4.0%	6.0%	7.0%
January 1, 2016	3.0%	5.0%	8.0%

What is the maximum accumulated value of your investment at the end of three years, assuming that you hold any bonds that you buy until maturity? You can buy (but not sell) as many bonds as you need at any times between January 1, 2014 and December 31, 2016.

Solution: There are four possibilities:

(a) Buy one-year bonds in 2014, one-year bonds in 2015, and one-year bonds in 2016. The accumulated value is

$$F.V. = \$1000(1.03)(1.04)(1.03) = \$1103.34.$$

(b) Buy one-year bonds in 2014, and two-year bonds in 2015. The accumulated value is

$$F.V. = \$1000(1.03)(1.06)^2 = \$1157.31.$$

(c) Buy two-year bonds in 2014, and one-year bonds in 2016. The accumulated value is

$$F.V. = \$1000(1.04)^2(1.03) = \$1114.05.$$

(d) Buy three-year bonds in 2014. The accumulated value is

$$F.V. = \$1000(1.045)^3 = \$1141.17.$$

The maximum accumulated value is $\boxed{\$1157.31}$.

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7. (10 points) A \$1000 par value 20-year bond with annual level coupons is bought to yield an annual effective rate of 5%.

The amount for accumulation of premium in the tenth year is \$29.23.

Calculate the book value of the bond at the end of the tenth year. Assume that the bond matures at par.

Solution: The principal adjustment at the end of the tenth year is

$$\$29.23 = \$C(g-i)v^{20-10+1} = \$1000(g-0.05)(1.05)^{-11},$$

which solves for g = 10%. Therefore, the book value of the bond at the end of the tenth year is

$$B_{10} = \$C\left[1+(g-i)a_{\overline{20-10}|i}\right] = \$1000\left[1+(0.10-0.05)\frac{1-(1.05)^{-10}}{0.05}\right] = \boxed{\$1,386.09}.$$

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8. (10 points) The real rate of interest is 4% per annum. The expected annual rate of inflation over the next two years is 5%. What is the net present value of the following cashflows?

Year	t = 0	t = 1	t = 2
Cashflow	-300	160	160

Solution: The nominal or market rate of interest is

$$i = (1 + \text{real rate})(1 + \text{inflation rate}) - 1 = (1.04)(1.05) - 1 = 9.20\%.$$

The net present value of the cashflows at the rate at the nominal rate of 9.20% per annum is

$$N.P.V. = -300 + \frac{160}{1.0920} + \frac{160}{(1.0920)^2} = \boxed{-\$19.30}.$$

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9. (10 points) A \$1000 par value bond with 7% annual coupons and maturing at par in twenty years sells at a price to yield 6%. Determine the Macaulay duration of the bond.

Solution: First, we compute the following auxiliary quantities:

$$a_{\overline{20}|0.06} = \frac{1-(1.06)^{-20}}{0.06} = 11.4699$$

$$\ddot{a}_{\overline{20}|0.06} = (1.06)a_{\overline{20}|0.06} = (1.06)(11.4699) = 12.1581$$

$$(Ia)_{\overline{20}|0.06} = \frac{\ddot{a}_{\overline{20}|0.06} - 20v^{20}}{0.06} = \frac{12.1581-(20)(1.06)^{-20}}{0.06} = 98.7004$$

Therefore, the Macaulay duration of the bond is

$$\begin{aligned}\bar{d} &= \frac{\sum_{k=1}^{20} 70kv^k + 1000(20)v^{20}}{\sum_{k=1}^{20} 70v^k + 1000(20)v^{20}} \\ &= \frac{70(Ia)_{\overline{20}|0.06} + 20{,}000(1.06)^{-20}}{70a_{\overline{20}|0.06} + 1000(1.06)^{-20}} \\ &= \frac{70(98.7004)+20000(0.3118)}{70(11.4699)+1000(0.3118)} \\ &= \boxed{11.7925 \text{ years}}.\end{aligned}$$

10. (10 points) Warren Buffett observes that the one-year, two-year and three-year spot rates are 1%, 2% and 3%, respectively. A $10,000 par value three-year 5% bond with annual coupons is redeemable at par and is currently yielding an annual effective rate of 2%. Compute the maximum arbitrage or risk-free gain that Mr. Buffett can obtain today by buying and selling bonds, under the following constraints:

(a) Today, Mr. Buffett can buy or sell at most one $10,000 par value three-year 5% bond to yield an annual effective rate of 2%. In addition, today he can buy or sell one-year, two-year and three-year zero-coupon bonds of any maturity values at the given spot rates.
(b) Mr. Buffett cannot use his own capital for investment: all money used to buy bonds must come from selling other bonds.
(c) There are no transaction fees, and the given rates can be used to buy or sell bonds.
(d) Mr. Buffett cannot incur any losses (including negative cashflows) during the three years of investment.
(e) All parties involved meet their contractual obligation.

Briefly describe the strategy that Mr. Buffett follows to make his riskless profit.

Solution: The coupon bond is selling for a price of

$$P = \$500 a_{\overline{3}|0.02} + \$\frac{10,000}{(1.02)^3} = \$500 \frac{1-(1.02)^{-3}}{0.02} + \$\frac{10,000}{(1.02)^3} = \$10,865.16.$$

This coupon bond can be seen as a portfolio consisting of one one-year zero-coupon bond with redemption value $500, one two-year zero-coupon bond with redemption value $500, and one three-year zero-coupon bond with redemption value $10,500. The price of this portfolio is

$$P = \$\frac{500}{1.01} + \$\frac{500}{(1.02)^2} + \$\frac{10,500}{(1.03)^3} = \$10,584.62.$$

Mr. Buffett can get a positive gain by taking advantage of this difference in prices, that is, today he can buy the portfolio of zero-coupon bonds for $10,584.62 and sell the coupon bond for $10,865.16. Therefore, the maximum arbitrage gain (per coupon bond sold) that Mr. Buffett can obtain today is $\boxed{\$280.54}$.

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11. (10 points) Megabucks Insurance, Inc. has precisely calculated its liabilities and will need to make payments of \$2,000 at the end of one year, \$4,000 at the end of two years, \$6,000 at the end of three years and \$5,000 at the end of four years.

The only investments available to Megabucks are bonds with annual coupons both priced and redeemable at par with the following coupon rates:

Term of Bond	Coupon Rate
1 year	7%
2 year	4%
3 year	5%
4 year	6%

Assuming that Megabucks can purchase these bonds at any face amounts what bonds should Megabucks purchase to immunize its possible exposure to interest rate changes using absolute matching?

Solution: The only option to fund the \$5,000 payment is with a 4-year bond. The final payment for that bond would be $C + 0.06C$ meaning Megabucks will need to purchase a 4-year bond with face of \$4,717 = \$5,000/1.06. Over the 4 years this bond will pay out:

Asset	Year 1	Year 2	Year 3	Year 4
4 year bond	\$283	\$283	\$283	\$5,000
3 year bond				
2 year bond				
1 year bond				
Required Total	\$2,000	\$4,000	\$6,000	\$5,000

Now given this the 3-year bond will need to have a payment of \$6,000 - \$283 = \$5,717 in the third year. This bond would have a final payment of $C + 0.05C$ meaning a required face value of \$5,445 = \$5,717/1.05. With this bond we then have the table

Asset	Year 1	Year 2	Year 3	Year 4
4 year bond	\$283	\$283	\$283	\$5,000
3 year bond	\$272	\$272	\$5,717	
2 year bond				
1 year bond				
Required Total	\$2,000	\$4,000	\$6,000	\$5,000

Moving along then, the 2-year bond will need to have a payment of \$4,000 - \$283 - \$272 = \$3,445 in the second year. This bond would have a final payment of $C + 0.04C$ meaning a required face value of \$3,313 = \$3,445/1.04. With this bond we then have the table

Asset	Year 1	Year 2	Year 3	Year 4
4 year bond	\$283	\$283	\$283	\$5,000

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3 year bond	\$272	\$272	\$5,717	
2 year bond	\$133	\$3,445		
1 year bond				
Required Total	\$2,000	\$4,000	\$6,000	\$5,000

Finally this leaves a total payment from the 1-year bond at the end of the first year of \$2,000 - \$283 - \$272 - \$133 = \$1,312. This bond will have a final payment of $C + 0.07C$ meaning a required face of \$1,226 = \$1,312/1.07.

Megabucks should purchase a \$1,226 1-year bond, a \$3,313 2-year bond, a \$5,445 3-year bond and a \$4,717 4-year bond for absolute matching.

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12. (10 points) Bill Gates bought a twelve-year $100 par value bond with 6% annual coupons that was callable at par any time on or after the tenth coupon payment. The bond was purchased at a price to yield at least 5% per annum. However one year later the company went bankrupt and Bill was repaid as if the bond was called immediately after the first coupon payment with a redemption value of $104. What was Bill's rate of return on his investment?

Solution: Since the bond will be purchased at a premium (yield < coupon rate) Bill used the Frank formula to compute the price of the bond, assuming the earliest possible redemption date, i.e. that $i = 0.05$, $n = 10$ and $C = F = \$100$.

$$P = \$6a_{\overline{10}|0.05} + \$100(1.05)^{-10} = \$6\frac{1-(1.05)^{-10}}{0.05} + \$100(1.05)^{-10} = \$107.72.$$

However, one year later, Bill received a coupon of $6 plus redemption value of $104, for a total of $110. Therefore, Bill's yield rate satisfies

$$\$110 = \$107.72(1+i) \Rightarrow i = \frac{110}{107.72} - 1 = \boxed{2.114954\%}.$$

END OF EXAM