QF620 Stochastic Modelling in Finance

Section: G2

Prepared For: Prof Tee Chyng Wen

Team Member	Student ID
Balina Kirti Kumar	01357389
Chia Jun Xian Edmund	01356812
Kendrick Winata	01496603
Lee Zong Kang	01236468
Najwa Jia Hui Rujok	01319968
Yao MingYu	01275953

Part I (Analytical Option Formulae)

Black-Scholes model

SDE Derivation	$dS_t = rS_t dt + \sigma S_t dW_t$ $f(S_t) = X_t = \log(S_t)$ **Using Ito's Formula,** $dX_t = \left(r - \frac{\sigma^2}{2}\right)dt + \sigma dW_t$ **Integrating both sides,** $\int_0^T dX_t = \left(r - \frac{\sigma^2}{2}\right)\int_0^T dt + \sigma\int_0^T dW_t$ $S_T = S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T + \sigma W_T}$
Valuation	$V_c = e^{-rT}\mathbb{E}[(S_T - K)^+]$ $S_T - K > 0$ $x > \frac{\log\left(\frac{K}{S_0}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = x^*$ $V_c = S_0\Phi\left(\frac{\log\left(\frac{S_0}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\log\left(\frac{S_0}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$ It is common to let $d_1 = \frac{\log\left(\frac{S_0}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$ Hence, $V_c = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)$

Option	Valuation Formula
Vanilla Call	$V_c = S_0\,\Phi(d_1) - Ke^{-rt}\Phi(d_2)$
Vanilla Put	$V_p = Ke^{-rt}\Phi(-d_2) - S_0\,\Phi(-d_1)$
Digital cash-or-nothing Call	$V_c = e^{-rt}\Phi(d_2)$
Digital cash-or-nothing Put	$V_p = e^{-rt}\Phi(-d_2)$
Digital asset-or-nothing Call	$V_c = S_0\,\Phi(d_1)$
Digital asset-or-nothing Put	$V_p = S_0\,\Phi(-d_1)$

Bachelier model

SDE Derivation	$dS_t = \sigma dW_t$ **Integrating both sides,** $\int_0^T dS_t = \int_0^T \sigma dW_t$ $S_t = S_0 + \sigma W_t$
Valuation	$V_c = e^{-rT}\mathbb{E}[(S_T - K)^+]$ $S_0 + \sigma\sqrt{T}x - K > 0$ $x > \frac{K - S_0}{\sigma\sqrt{T}} = x^*$ $V_c = e^{-rT}\left[(S_0 - K)\Phi\left(\frac{S_0-K}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}\phi\left(\frac{S_0-K}{\sigma\sqrt{T}}\right)\right]$ It is common to let $d_1 = \frac{S_0 - K}{\sigma\sqrt{T}}$. Hence, $V_c = e^{-rT}\left[S_0\,\Phi(d_1) - \sigma\sqrt{T}\phi(d_1)\right]$

Option	Valuation Formula
Vanilla Call	$V_c = e^{-rt}(S_0 - \mathrm{K})\Phi(d_1) + \sigma\sqrt{T}\phi(d_2)$
Vanilla Put	$V_p = e^{-rt}(K - S_0)\Phi(-d_1) + \sigma\sqrt{T}\phi(-d_1)$
Digital cash-or-nothing Call	$V_c = e^{-rt}\Phi(d_1)$
Digital cash-or-nothing Put	$V_p = e^{-rt}\Phi(-d_1)$
Digital asset-or-nothing Call	$V_c = e^{-rt}\left(S_0\,\Phi(d_1) + \sigma\sqrt{T}\phi(d_1)\right)$
Digital asset-or-nothing Put	$V_p = e^{-rt}\left(S_0\,\Phi(-d_1) + \sigma\sqrt{T}\phi(-d_1)\right)$

Black76 model

SDE Derivation	$dF_0 = \sigma F_0 dW_t$ $f(F_t) = X_t = \log(F_t)$ **Using Ito's Formula,** $dX_t = \left(r - \frac{\sigma^2}{2}\right)dt + \sigma dW_t$

	Integrating both sides, $\int_0^T dX_t = \left(r - \frac{\sigma^2}{2}\right) \int_0^T dt + \sigma \int_0^T dW_t$ $F_t = F_0 e^{\left[\left(-\frac{\sigma^2}{2}\right)T + \sigma W_T\right]}$
Valuation	$V_c = e^{-rT}\mathbb{E}[(F_T - K)^+]$ $F_T - K > 0$ $x > \frac{\log\left(\frac{K}{F_0}\right)+\left(\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = x^*$ $V_c = e^{-rT}\left[F_0\Phi\left(-\frac{\log\left(\frac{F_0}{K}\right)+\left(\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - K\Phi\left(-\frac{\log\left(\frac{F_0}{K}\right)-\left(\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)\right]$ It is common to let $d_1 = \frac{\log\left(\frac{F_0}{K}\right)+\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$ Hence, $V_c = F_0 e^{-rT}\Phi(d_1) - Ke^{-rT}\Phi(d_2)$

Option	Valuation Formula
Vanilla Call	$V_c = e^{-rt}(F_0\Phi(d_1) - K\Phi(d_2))$
Vanilla Put	$V_p = e^{-rt}\left(K\Phi(-d_2) - F_0\Phi(-d_1)\right)$
Digital cash-or-nothing Call	$V_c = e^{-rt}\Phi(d_2)$
Digital cash-or-nothing Put	$V_p = e^{-rt}\Phi(-d_2)$
Digital asset-or-nothing Call	$V_c = F_0 e^{-rt}\Phi(d_1)$
Digital asset-or-nothing Put	$V_c = F_0 e^{-rt}\Phi(-d_1)$

Displaced-Diffusion model

SDE Derivation	$dF_t = \sigma[\beta F_t + (1-\beta)F_0]dW_t$ $F_t = X_t = \log(\beta F_t + (1-\beta)F_0)$ **Using Ito's Formula,** $dX_t = \beta\sigma dW_t - \frac{1}{2}\beta^2\sigma^2 dt$ **Integrating both sides,** $\int_0^T dX_t = \int_0^T \beta\sigma dW_t - \int_0^T \frac{1}{2}\beta^2\sigma^2 dt$ $F_T = \frac{F_0}{\beta} e^{\beta\sigma W_T - \frac{1}{2}\beta^2\sigma^2 T} - \frac{1-\beta}{\beta}F_0$
Valuation	$V_c = e^{-rT}E[(F_T - K)^+]$ $F_T - K > 0$ $x > \frac{\log\left(\frac{K+\frac{1-\beta}{\beta}F_0}{\frac{F_0}{\beta}}\right)+\left(\frac{(\sigma\beta)^2}{2}\right)T}{\beta\sigma\sqrt{T}} = x^*$ $V_c = e^{-rT}\left[\frac{F_0}{\beta}\Phi\left(\frac{\log\left(\frac{K+\frac{1-\beta}{\beta}F_0}{\frac{F_0}{\beta}}\right)+\left(\frac{(\sigma\beta)^2}{2}\right)T}{\beta\sigma\sqrt{T}}\right) - \left(K + \frac{1-\beta}{\beta}F_0\right)\Phi\left(-\frac{\log\left(\frac{K+\frac{1-\beta}{\beta}F_0}{\frac{F_0}{\beta}}\right)-\left(\frac{(\sigma\beta)^2}{2}\right)T}{\beta\sigma\sqrt{T}}\right)\right]$ $V_c = e^{-rT}\left[\frac{F_0}{\beta}\Phi\left(\frac{\log\left(\frac{F_0}{F_0+\beta(K-F_0)}\right)+\left(\frac{(\sigma\beta)^2}{2}\right)T}{\beta\sigma\sqrt{T}}\right) - \left(K + \frac{1-\beta}{\beta}F_0\right)\Phi\left(-\frac{\log\left(\frac{F_0}{F_0+\beta(K-F_0)}\right)-\left(\frac{(\sigma\beta)^2}{2}\right)T}{\beta\sigma\sqrt{T}}\right)\right]$ It is common to let $d_1 = \frac{\log\left(\frac{F_0}{F_0+\beta(K-F_0)}\right)+\left(\frac{(\sigma\beta)^2}{2}\right)T}{\beta\sigma\sqrt{T}}$ and $d_2 = d_1 - \beta\sigma\sqrt{T}$ Hence, $V_c = e^{-rT}[\frac{F_0}{\beta}\Phi(d_1) - \left(K + \frac{1-\beta}{\beta}F_0\right)\Phi(d_2)]$

Option	Valuation Formula
Vanilla Call	$V_c = e^{-rT}\left[\frac{F_0}{\beta}\Phi(d_1) - \left(K + \frac{1-\beta}{\beta}F_0\right)\Phi(d_2)\right]$
Vanilla Put	$V_p = e^{-rT}\left[\left(K + \frac{1-\beta}{\beta}F_0\right)\Phi(-d_2) - \frac{F_0}{\beta}\Phi(-d_1)\right]$
Digital cash-or-nothing Call	$V_c = e^{-rt}\Phi(d_2)$
Digital cash-or-nothing Put	$V_p = e^{-rt}\Phi(-d_2)$
Digital asset-or-nothing Call	$V_c = e^{-rT}\left[\frac{F_0}{\beta}\Phi(d_1) - \left(\frac{1-\beta}{\beta}F_0\right)\Phi(d_2)\right]$
Digital asset-or-nothing Put	$V_p = e^{-rT}\left[\left(\frac{F_0}{\beta}\right)\Phi(-d_1) - \left(\frac{1-\beta}{\beta}F_0\right)\Phi(-d_2)\right]$

Part II (Model Calibration)

The calibration of implied volatility using the Displaced Diffusion (DD) and SABR models involves aligning the initial implied volatility with the Black-Scholes model to ensure an effective calibration process. The approximation functions used are:

<u>SABR model (Hagan et al., 2014, Eq. (14)):</u>

$$\sigma_{\text{BS-SABR}} = \frac{\alpha}{(F_0K)^{(1-\beta)/2}\left\{1+\frac{(1-\beta)^2}{24}\log^2\left(\frac{F_0}{K}\right)+\frac{(1-\beta)^4}{1920}\log^4\left(\frac{F_0}{K}\right)+\cdots\right\}}$$
$$\times\frac{z}{x(z)}\times\left\{1+\left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(F_0K)^{(1-\beta)/2}}+\frac{1}{4}\frac{\rho\beta\nu\alpha}{(F_0K)^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}\nu^2\right]T+\cdots\right\}$$

Where: $z=\frac{\nu}{\alpha}(F_0K)^{(1-\beta)/2}\log\left(\frac{F_0}{K}\right), x(z)=\log\left[\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right]$

<u>Displaced-Diffusion model (Rubinstein, M. (1983)):</u>

$$F_T=\frac{F_0}{\beta}e^{-\frac{\beta^2\sigma^2T}{2}+\beta\sigma W_T^*}-\frac{1-\beta}{\beta}F_0$$

Where: Displaced-Diffusion $=$ Black $\left(\frac{F_0}{\beta}, K+\frac{1-\beta}{\beta}F_0, \sigma\beta, T\right)$

The target implied volatility is attained via the Black-Scholes reporting model. For American options, the binomial tree model is used to consider the early exercise possibility.

The key difference between a European and American option is the ability to exercise. An American option can be exercised any time before or on the expiration date while a European option can only be exercised on the expiration date. The following outlines the valuation of both option types.

For European options, the risk-neutral expectation is given by:

$$V_n^E=\frac{1}{1+r}\mathbb{E}_n^*[V_{n+1}]=\frac{1}{1+r}[p^*\times V_{n+1}^u+q^*\times V_{n+1}^d]$$

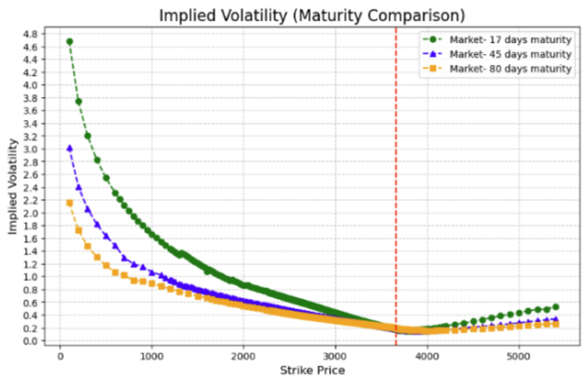
For American options, the risk-neutral expectation should become:

$$V_n^A=\max\left\{\frac{1}{1+r}[p^*\times V_{n+1}^u+q^*\times V_{n+1}^d],(K-S_n)^+\right\}$$

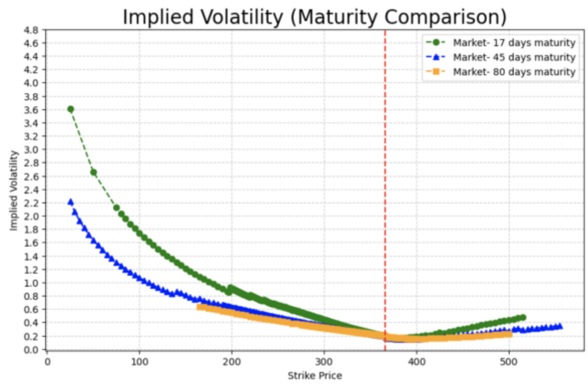
<u>Market Implied Volatility</u>

Implied volatility from market prices was computed using the Black-Scholes model based on the 3 option maturity dates. They are 2020-12-18 (17 days), 2021-01-15 (45 days), and 2021-02-19 (80 days).

SPX



SPY



Based on the implied volatility graphs, options with shorter maturity dates exhibit a more prominent implied volatility smile compared to options with longer maturity dates. The reason can be attributed to several factors:

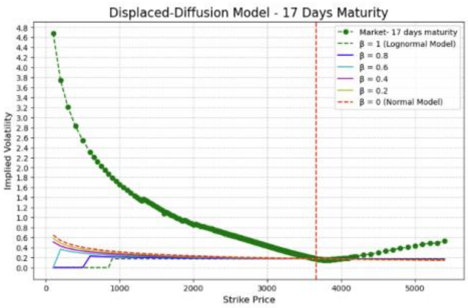
1. <u>Uncertainty and Short-Term Events</u>
   Shorter maturities are more vulnerable to uncertainties. Limited timeframes make shorter-term options riskier, which is reflected in higher implied volatility.
2. <u>Time Decay</u>
   Investors/traders may demand higher implied volatility to accommodate potential price swings before expiration.
3. <u>Market Expectations</u>
   Shorter-term options are more influenced by current market sentiment and expectations.

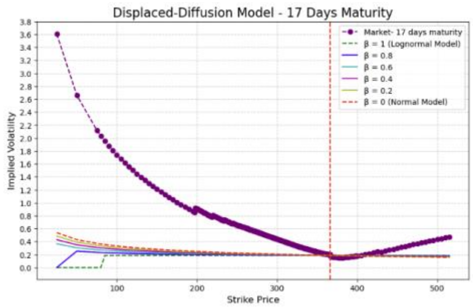
Model Calibration using Displaced-Diffusion ("DD") Model

To calibrate the Displaced Diffusion (DD) implied volatility, the initial step involves computing the lognormal volatility (σLN) using the Black-Scholes formula for At-The-Money (ATM) options at a specified expiration date. σLN is determined based on the mid-price between call and put option ATM prices. Using various displacement parameters (beta) from 0 to 1 (e.g., 0.2, 0.4, 0.6, and 0.8), the DD call and put formulas are applied to obtain implied volatility. The resulting plot illustrates the relationship between DD implied volatility and strike price for each maturity date.

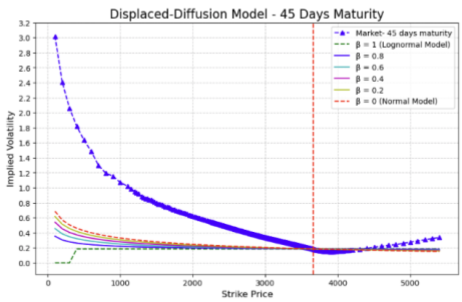
Maturity **SPX** **SPY**

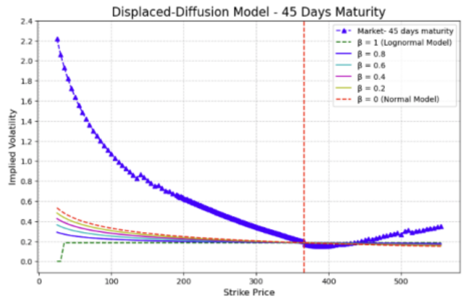
17 Days



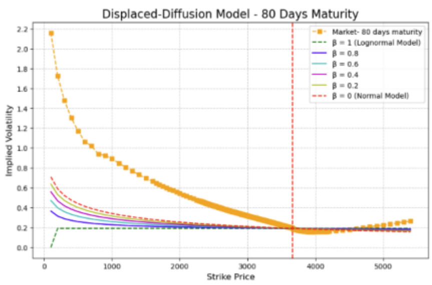


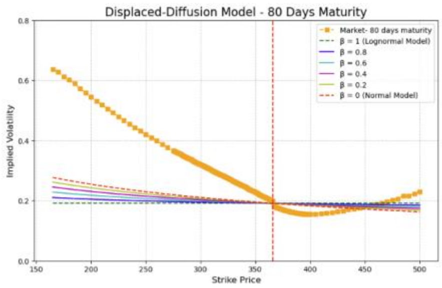
45 Days





80 Days





From the graph, the calibration of the DD model deviates significantly from the market implied volatility, particularly for short-term options. The poor fit of the DD model can be attributed to its inability to generate sufficient skew in the implied volatility surface. This depicts a recognized limitation of the DD model.

While the DD model effectively matches ATM strikes, it falls short when it comes to deriving implied volatilities for strikes outside the ATM price, particularly at the extreme ends. The inherent challenge lies in the model's inability to alter the skewness and kurtosis of the curve without impacting the implied volatility of strikes near both ends of the curve. The primary reason for this lies in the market pricing having considerably higher implied volatility for in-the-money and out-of-the-money strikes. In cases of deep-in-the-money or deep-out-of-the-money options, despite their low probability, investors tend to overvalue or undervalue options. Therefore, a model capable of incorporating these additional factors is necessary, leading to the introduction of the SABR model.

Model Calibration using SABR Model

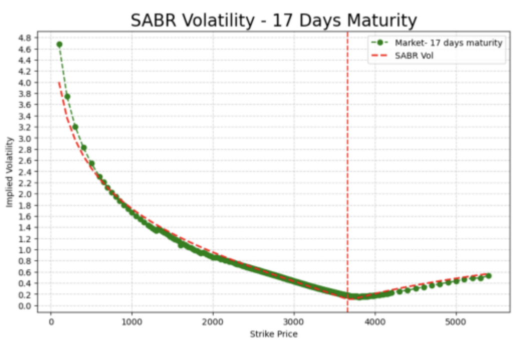
Unlike the DD model, which relies solely on the Beta parameter, the SABR model employs a more comprehensive approach, incorporating multiple parameters to accurately capture the intricacies of the observed volatility smile in derivatives markets. The SABR model was calibrated using the least squares method, where key parameters—Alpha (α), Rho (ρ), Nu (ν), and Beta (β)—were determined, with Beta held constant at 0.7. The inclusion of these additional parameters significantly enhances the SABR model's ability to closely fit the market's implied volatility.

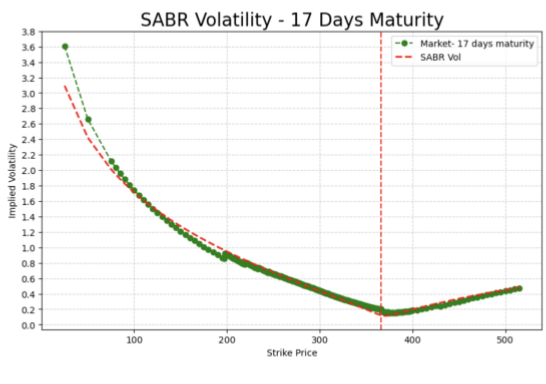
Alpha (α) denotes the initial volatility level in the SABR model, controlling the level of the volatility graph and contributing to the overall shape of the volatility smile. Rho (ρ) represents the correlation between the Brownian motions of asset price and volatility. This parameter measures the degree to which asset price and volatility move in tandem or in opposite directions, impacting the slope of the volatility smile. Nu (ν) characterizes the volatility of volatility, the stochastic nature of volatility. It influences the curvature of the volatility smile, depicting the sensitivity of volatility to changes in the underlying asset's price. Beta (β) is introduced in the SABR model to facilitate adjustments to the tail behavior of the volatility smile which influences the skewness of the graph.

Collectively, these parameters provide a comprehensive framework to capture the entire options market, resulting in a perfect fitting capability and allowing for a nuanced simulation of the volatility curve.

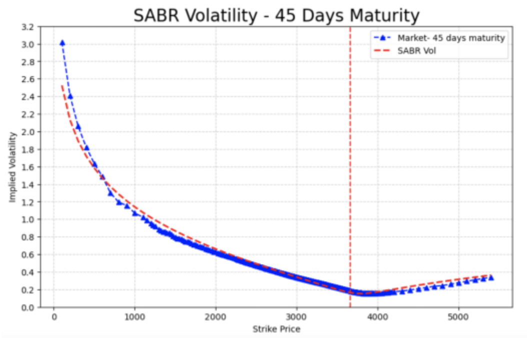
Maturity **SPX** **SPY**

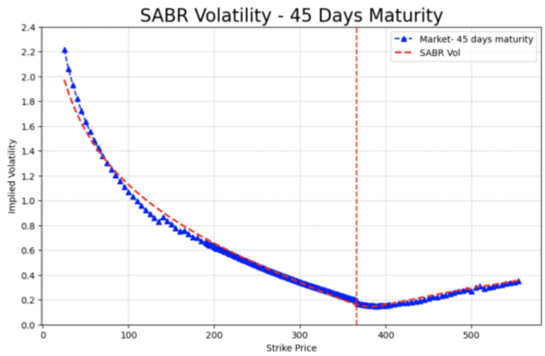
17 Days



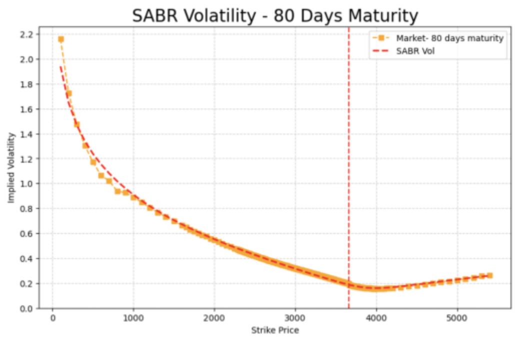


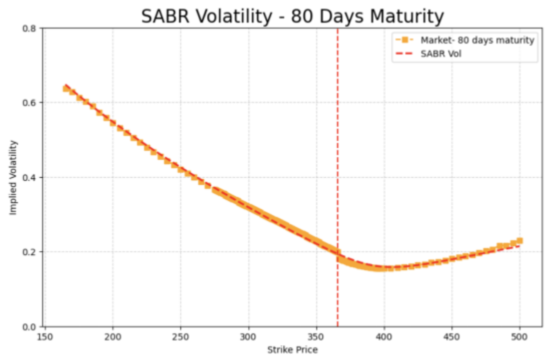
45 Days





80 Days





Maturity	Parameters	SPX	SPY
17 Days	Alpha (α)	1.211893	0.664322
	Rho (ρ)	-0.300260	-0.411423
	Nu (ν)	5.460390	5.252549
45 Days	Alpha (α)	1.816001	0.907834
	Rho (ρ)	-0.401415	-0.487295
	Nu (ν)	2.791031	2.728857
80 Days	Alpha (α)	2.139349	1.120977
	Rho (ρ)	-0.570732	-0.629672
	Nu (ν)	1.842179	1.740378

Part III (Static Replication)

Payoff 1

Given that the current date is 1st December 2020, To evaluate an exotic European derivative expiring on 15th January 2021 (45 days to maturity) which pays: $S_t^{\frac{1}{3}} + 1.5 * \log(S_t) + 10.0$

We determine the price of the contract using 3 models – Black-Scholes Model, Bachelier Model, and Static-replication of European payoffs.

Black-Scholes

Under Black-Scholes model, stock price process follows the stochastic differential equation:

$$dS_t = rS_t dt + \sigma S_t dW_t$$

Solving the SDE using Itô's formula as derived in part I, we arrive at: $S_t = S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_t\right]$

Deriving the option pricing formula for the exotic European derivative:

$$V_0 = e^{-rT} E\left[S_t^{\frac{1}{3}} + 1.5 * \log(S_t) + 10.0\right]$$

$$V_0 = e^{-rT} E\left[\left\{S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_t\right]\right\}^{\frac{1}{3}} + 1.5 * \log\left(S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_t\right]\right) + 10.0\right]$$

$$V_0 = e^{-rT} E\left[S_0^{\frac{1}{3}} \exp\left[\frac{1}{3}\left(r - \frac{\sigma^2}{2}\right)T + \frac{1}{3}\sigma W_t\right] + 1.5 * \log\left(S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_t\right]\right) + 10.0\right]$$

$$V_0 = e^{-rT}\left[S_0^{\frac{1}{3}} \exp\left(\frac{1}{3}(rT) - \frac{1}{9}\sigma^2 T\right) + 1.5 * \log(S_0) + \left(r - \frac{\sigma^2}{2}\right)T + 10.0\right]$$

To determine the appropriate σ, we fit the Black-Scholes model to the ATM option price for both SPX (strike: 3660) and SPY (strike: 366). This yields us σ of 0.1854 and 0.1848 for SPX and SPY respectively. With σ, the price of the derivative for SPX is 37.7048, and that for SPY price is 25.9952.

Bachelier

Under Bachelier model, stock price process follows the stochastic differential equation: $dS_t = \sigma dW_t$

Integrating this stochastic equation, we can show that: $S_t = S_0 + \sigma W_t$

Deriving the option pricing formula for the exotic European derivative:

$$V_0 = e^{-rT} E\left[S_t^{\frac{1}{3}} + 1.5 * \log(S_t) + 10.0\right]$$

$$V_0 = e^{-rT} E\left[(S_0 + \sigma W_t)^{\frac{1}{3}} + 1.5 * \log(S_0 + \sigma W_t) + 10.0\right]$$

$$V_0 = \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (S_0 + \sigma\sqrt{T}x)^{\frac{1}{3}} e^{-\frac{x^2}{2}} dx + \frac{1.5e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} log(S_0 + \sigma\sqrt{T}x)\, e^{-\frac{x^2}{2}} dx + 10.0\, e^{-rT}$$

To determine the value of σ, we fit the Bachelier model to the ATM option price for both SPX (strike price 3660) and SPY (strike price 366). This yields us σ of 0.1853 and 0.1848 for SPX and SPY respectively. With σ, the price of the derivative for SPX is 37.7136, and that for SPY price is 25.0007.

Static Replication

Carr and Madan showed in 1998 that any European payoff can be replicated using a portfolio of cash, forward contracts, and European call & put options.

With the given payoff function: $h(S_t) = S_t^{\frac{1}{3}} + 1.5 * \log(S_t) + 10.0, h'(S_t) = \frac{1}{3} S_t^{-\frac{2}{3}} + \frac{1.5}{S_t}, h''(S_t) = -\frac{2}{9} S_t^{-\frac{5}{3}} - \frac{1.5}{S_t^2}$

Hence, the twice differentiable payoff can be static replicated with a portfolio of options as follows:

$$V_0 = e^{-rT} h(F) + \int_0^F h''(K)P(K)dK + \int_F^\infty h''(K)C(K)dK$$

$$V_0 = e^{-rT}\left(S_t^{\frac{1}{3}} + 1.5 * \log(S_t) + 10.0\right) + \int_0^F \left(-\frac{2}{9}K^{-\frac{5}{3}} - \frac{1.5}{K^2}\right)P(K)dK + \int_F^\infty \left(-\frac{2}{9}K^{-\frac{5}{3}} - \frac{1.5}{K^2}\right)C(K)dK$$

$$V_0 = e^{-rT}\left(S_0^{\frac{1}{3}} e^{\frac{1}{3}rT} + 1.5 * \log(S_0 e^{rT}) + 10.0\right) + \int_0^F \left(-\frac{2}{9}K^{-\frac{5}{3}} - \frac{1.5}{K^2}\right)P(K)dK + \int_F^\infty \left(-\frac{2}{9}K^{-\frac{5}{3}} - \frac{1.5}{K^2}\right)C(K)dK$$

Using the SABR parameters computed in part II to calculate the SABR implied volatility, the SPX static replication price is 37.7004, the SPY static replication price is 25.9927.

	Payoff Function 1	
	SPX	**SPY**
Black-Scholes	Price = 37.70484553986259 $\sigma = 0.1853718842874737$	Price = 25.995155797750314 $\sigma = 0.1848115441954425$
Bachelier	Price = 37.71359681715785 $\sigma = 0.1853093750058496$	Price = 26.000676619258684 $\sigma = 0.18479898144523038$
Static Replication	Price = 37.700414693 SABR params: Alpha (α) = 1.81600099 Beta (β) = 0.7 Rho (ρ) = -0.40141522 Nu (ν) = 2.79103083	Price = 25.992666574 SABR params: Alpha (α) = 0.90783421 Beta (β) = 0.7 Rho (ρ) = -0.48729513 Nu (ν) = 2.72885668

Payoff 2

Payoff 2 is the pricing of variance swaps. Variance swaps allow for the ability to gain explicit volatility and variance exposure, reducing the need for delta or gamma hedging if we were using vanilla options to gain volatility exposure.

$$\sigma_{MF}^2 T = E\left[\int_0^T \sigma_t^2 dt\right]$$

The final pricing formula of the variance swap is: $E\left[\int_0^T \sigma_t^2 dt\right] = 2e^{rT}\int_0^F \frac{P(K)}{K^2}dK + 2e^{rT}\int_F^\infty \frac{C(K)}{K^2}dK$

Using the same implied volatility for the Black-Scholes and Bachelier Models described in Payoff 1 as well as the SABR parameters, the price of the 'Model-Free' integrated variance payoff is:

	Payoff Function 2	
	SPX	**SPY**
Black-Scholes	Price = 0.004236501 $\sigma = 0.1853718842874737$	Price = 0.004210928 $\sigma = 0.1848115441954425$
Bachelier	Price = 0.004260983 $\sigma = 0.1853093750058496$	Price = 0.004237392 $\sigma = 0.18479898144523038$
Static Replication	Price = 0.006333770 SABR params: Alpha (α) = 1.81600099 Beta (β) = 0.7 Rho (ρ) = -0.40141522 Nu (ν) = 2.79103083	Price = 0.006013238 SABR params: Alpha (α) = 0.90783421 Beta (β) = 0.7 Rho (ρ) = -0.48729513 Nu (ν) = 2.72885668

Part IV (Dynamic Hedging)

A natural result of the Black-Scholes model is that the fair value of the option should equal the cost of replication if all assumptions hold.

This exercise is aimed to simulate the effects of breaking the continuous hedging assumption made by Black & Scholes.

With given variables S_0 = \$100, σ = 0.2, r = 5%, T = $\frac{1}{12}$ years and K = \$100, we generated a Brownian motion series which is then used to simulate 50,000 different stock prices based on the Black-Scholes model. We will then attempt to delta hedge N = 21 and N = 84 times throughout the life of the option using stock and risk-free bond.

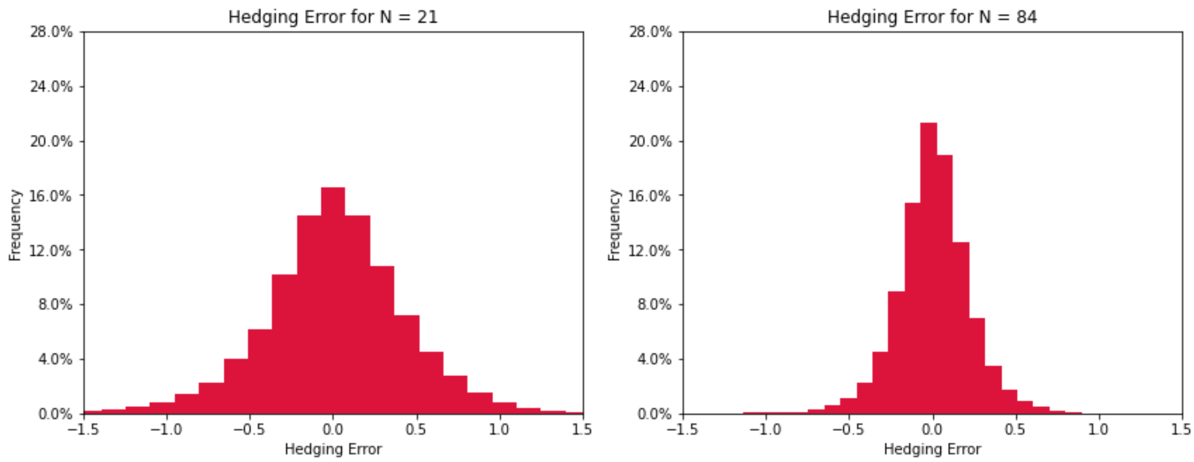
The dynamic hedging strategy for this option is:

$$C_t = \phi_t S_t - \psi_t B_t$$

and hedging error will be defined as:

$$Hedging\ Error = (\phi_t S_t - \psi_t B_t) - max\ \{S_t - K, 0\}$$

We then obtain the following statistics from our simulation:



N	Mean Error	Standard Deviation of Error
21	0.00744	0.42
84	0.00577	0.22

As we increase the number of hedges in discrete time, we approach the BS model's assumption of continuous hedging and obtain lower mean hedging errors and lower standard deviation as N increases.