依旧设 $\alpha_1=\alpha_2=\alpha$，则 $\mu_{1i}(t) = C(Q_i(t)\, Z_{0i}(t))^{\alpha} \geq \mu_i$ 时，matching happens.

即 $Q_i(t)\, Z_{0i}(t) \geq \frac{1}{\alpha} \log(\mu_i/C) = \bar{\mu}_i$ 时.

$$
\begin{cases}
Q_1(t) = Q_1(0) + A_1(t) - R_{01}(t) - M_1(t) & \\
Q_2(t) = Q_2(0) + A_2(t) - R_{02}(t) - M_2(t) & (1) \\
Z_0(t) = Z_0(0) + R_{11}(t) + R_{12}(t) + D_{21}(t) + D_{22}(t) - M_1(t) - M_2(t) & (2) \\
Z_{1i}(t) = Z_{1i}(0) + M_i(t) - D_{1i}(t) - R_{1i}(t), \quad i=1,2 & (3) \\
Z_{2i}(t) = Z_{2i}(0) + D_{1i}(t) - D_{2i}(t), \quad i=1,2 & (4)
\end{cases}
$$

假定 $Z_{0i}(t)\, Q_i(t) \leq \bar{\mu}_{1i}$，改写 Matching equation $M_i(t)$:

$$
\boxed{M(t) = \int_0^t \mathbb{1}_{\{(Q(s-)+1)Z_0(s-) \geq \bar{\mu}_1\}}\, dA(s) + \int_0^t \mathbb{1}_{\{Q(s-)(Z_0(s-)+1) \geq \bar{\mu}_1\}}\, d(R_1(s) + D_2(s))}
$$

$$
M_1(t) = \int_0^t \mathbb{1}\{(Q_1(s-)+1)\, Z_0(s-1) \geq \bar{\mu}_1\}\, dA_1(s) + \int_0^t \left[I_1(s-)(1 - I_2(s-)) + I_1(s-) I_2(s-) \frac{Q_1(s-)}{Q_1(s-)+Q_2(s-)} \right] d\, (R_{11}(s) + R_{12}(s) + D_{21}(s) + D_{22}(s))
$$

$$
I_1(s-) = \mathbb{1}\{Q(s-)(Z_0(s-)+1) \geq \bar{\mu}_{11}\}
$$

1. $X_1(t) = Q(0) + A_1(t) - R_{01}(t)$

   $Y(t) = Z_0(0) + R_{11}(t) + R_{12}(t) + D_{21}(t) + D_{22}(t)$

2. $M_1(t) + M_2(t) = M(t) = \int_0^t \mathbb{1}\{(Q_1(s-)+1)\, Z_0(s-) \geq \bar{\mu}_{11}\}\, dA_1(s) + \int_0^t \mathbb{1}\{(Q_2(s-)+1)\, Z_0(s-) \geq \bar{\mu}_{12}\}\, dA_2(s) + \int_0^t \mathbb{1}\{Q(s-)(Z_0(s-)+1) \geq \min\{\bar{\mu}_{11}, \bar{\mu}_{12}\}\}\, d(R_{1i}(s) + D_{2i}(s))$

$$
= \left[\sup_{0 \leq s \leq t} \left[\frac{X(s) + Y(s) - \sqrt{(X(s) - Y(s))^2 + 4\bar{\mu}_1}}{2} \right]^+ \right]
$$

$X_1(t) = Q_1(0) + A_1(t) - R_{01}(t)$, $X_2(t) = Q_2(0) + A_2(t) - R_{02}(t)$, $Y(t) = Z_0(0) + \sum_{i=1}^{2} (R_{1i}(t) + D_{2i}(t))$

$X(t) = X_1(t) + X_2(t)$.

t_k: kth matching happens at t_k with $0 < t_1 < t_2 < \cdots < t_k < \cdots$.

$$
\begin{cases}
(X_1(t) - x_1)(Y(t) - x_1 - x_2) = \bar{\mu}_{11} \\
(X_2(t) - x_2)(Y(t) - x_1 - x_2) = \mu_{12}
\end{cases}
\Rightarrow \underbrace{(X_1(t) + X_2(t)}_{X(t)} - \underbrace{x_1 - x_2)}_{x}(Y(t) - \underbrace{x_1 - x_2}_{x}) = \bar{\mu}_{11} + \bar{\mu}_{12}
$$

$$
M(t) = M_1(t) + M_2(t)
$$

$$
\underbrace{(X_1(t) + X_2(t) - x)}_{= Q_1(t) + Q_2(t)} \underbrace{(Y(t) - x)}_{= Z_0(t)} = \bar{\mu}_{11} + \bar{\mu}_{12}.
$$

$$
\Rightarrow M(t) = M_1(t) + M_2(t) = \left[\sup_{0 \leq s \leq t} \left[\frac{X(t) + Y(s) - \sqrt{(X(t) - Y(s))^2 + 4(\bar{\mu}_{11} + \bar{\mu}_{12})}}{2} \right]^+ \right]
$$

$t \in [t_k, t_{k+1})$, $\quad X(t) - k = Q_1(t) + Q_2(t)$, $Y(t) - k = Z_0(t)$

$$
\Rightarrow \quad (X(t) - k + 1)(Y(t) - k + 1) > \bar{\mu}_{11} + \bar{\mu}_{12}. \quad \Rightarrow M(t) = k.
$$

$$
(X(t) - k)(Y(t) - k) \leq \bar{\mu}_{11} + \bar{\mu}_{12}
$$

3. System Dynamics:

$$
Q_i^n(t) = Q_i^n(0) + A_i^n(t) - R_{0i}^n(t) - M_i^n(t)
$$

$$
Z_0^n(t) = Z_0^n(0) + \sum_{i=1}^{2} (R_{1i}^n(t) + D_{2i}^n(t) - M_i^n(t))
$$

$$
Z_{1i}^n(t) = Z_{1i}^n(0) + M_i^n(t) - D_{1i}^n(t) - R_{1i}^n(t)
$$

$$
Z_{2i}^n(t) = Z_{2i}^n(0) + D_{1i}^n(t) - D_{2i}^n(t).
$$

4. Centering operation.

$\bar{Q}_i^n(t) = Q(t)/n$: Fluid-scaled process.

$$\bar{Q}_i^n(t) = (\bar{X}_i^n(t) - E[\bar{X}_i^n(t)]) - (\bar{M}_i^n(t) - E[\bar{M}_i^n(t)]) + E[\bar{X}_i^n(t) - \bar{M}_i^n(t)] \quad , i=1,2$$

$$\bar{Z}_0^n(t) = (\bar{Y}^n(t) - E[\bar{Y}^n(t)]) - \sum_{i=1}^{2} (\bar{M}_i^n(t) - E[\bar{M}_i^n(t)]) + E[\bar{Y}^n(t)] - \sum_{i=1}^{2} E[\bar{M}_i^n(t)]$$

$$\bar{Z}_{1i}^n(t) = \bar{Z}_{1i}^n(0) - (\bar{R}_{1i}^n(t) - \theta_{1i}\int_0^t \bar{Z}_{1i}^n(s)ds) - (\bar{D}_{1i}^n(t) - \mu_{1i}\int_0^t \bar{Z}_{1i}^n(s)ds) + (\bar{M}_i^n(t) - E[\bar{M}_i^n(t)]) - \theta_{1i}\int_0^t \bar{Z}_{1i}^n(s)ds - \mu_{1i}\int_0^t \bar{Z}_{1i}^n(s)ds + E[\bar{M}_i^n(t)]$$

$$\bar{Z}_{2i}^n(t) = \bar{Z}_{2i}^n(0) + (\bar{D}_{1i}^n(t) - \mu_{1i}\int_0^t \bar{Z}_{1i}^n(s)ds) - (\bar{D}_{2i}^n(t) - \mu_{2i}\int_0^t \bar{Z}_{2i}^n(s)ds) + \mu_{1i}\int_0^t \bar{Z}_{1i}^n(s)ds - \mu_{2i}\int_0^t \bar{Z}_{2i}^n(s)ds$$

where $E[\bar{X}_i^n(t)] = \bar{Q}_i^n(0) + \lambda t - \theta_{0i}\int_0^t \bar{Q}_i^n(s)ds$, $E[\bar{Y}^n(t)] = \bar{Z}_0^n(t) + \sum_{i=1}^{2}(\theta_{1i}\int_0^t \bar{Z}_{1i}^n(s)ds + \mu_{2i}\int_0^t \bar{Z}_{2i}^n(s)ds)$

$$E[\bar{M}^n(t)] = E[\bar{M}_1^n(t) + \bar{M}_2^n(t)] = \sup_{0\le s\le t}\left[\frac{E[\bar{X}^n(s)] + E[\bar{Y}^n(s)] - \sqrt{(E[\bar{X}^n(s)] - E[\bar{Y}^n(s)])^2 + 4(\bar{\mu}_1 + \bar{\mu}_2)}}{2}\right]^+$$

Proposition EC.3

$$\|\phi(Q_i, z_0, z_{1i}, z_{2i}) - \phi(Q_i', z_0', z_{1i}', z_{2i}')\|_b \le \sum_{i=1}^{2}\left\{\theta_{0i}\int_0^b |Q_i(s) - Q_i'(s)|ds + 2(\theta_{1i}+\mu_{1i})\int_0^b |z_{1i}(s) - z_{1i}'(s)|ds + 2\mu_{2i}\int_0^b |z_{2i}(s) - z_{2i}'(s)|ds\right\}$$
$$+ 3\sup_{0\le s\le b}\{|M_{11}(s) - M_{11}'(s)| + |M_{21}(s) - M_{21}'(s)|\}$$