EC.6. Stochastic model converges to fluid model

We establish the convergence of stochastic model to fluid model when $\alpha_1 = \alpha_2 = \alpha$. Recall that we apply a threshold rule to control the matching process, i.e., a matching happens if and only if $\mu_1(t) \geq \mu_1$ for some threshold value μ_1. In view of (1), when $\alpha_1 = \alpha_2$, a matching happens if and only if $Z_0(t)Q(t) \geq \bar{\mu}_1$ for some $\bar{\mu}_1$. Assume that $Z_0(t)Q(t) \leq \bar{\mu}_1$, we can describe the system dynamics of the stochastic model underlying the ride-hailing systems using (4)–(7), as well as the following equivalent form of the matching equation (9),

$$M(t) = \int_0^t \mathbf{1}_{\{(Q(s-)+1)Z_0(s-) \geq \bar{\mu}_1\}} dA(s) + \int_0^t \mathbf{1}_{\{Q(s-)(Z_0(s-)+1) \geq \bar{\mu}_1\}} d(R_1(s) + D_2(s)), \tag{EC.33}$$

where the first term means that a passenger arrives and the matching happens, and the second term is that a car becomes empty and the matching happens.

Before stepping into detailed proof, we provide a roadmap to facilitate understanding of the big picture. As a first step, we need to characterize the state evolution dynamics of the stochastic model using the system input processes and parameters. The main difficulty lies in the nontrivial dependence of the matching process $M(t)$ on the system inputs, which is tackled by Lemma 2. Having established the dynamic equations, we observe that the stochastic and fluid model both satisfy differential equations of form (EC.39)-(EC.42), with difference in residual terms $B_i(t)$, $i = 1, 2, 3, 4$. For given $B_i(t)$, $i = 1, 2, 3, 4$, we first show that (EC.39)-(EC.42) can be regarded as a contraction mapping (Proposition 6), thus validating the exsitence and uniqueness of the stochastic model. Next, we establish using the Law of Large Numbers that the difference in residual terms $B_i(t)$, $i = 1, 2, 3, 4$, is asymptotically negligible (Lemma 3). Thus, combined with the fact that (EC.39)-(EC.42) is Lipschitz continuous, we complete the proof of convergence (Theorem 6).

We begin with the following lemma, which is a result of threshold control policy and the proof is done by induction. Here $\lceil x \rceil$ denotes the smallest integer greater or equal to x.

Lemma EC.1. *Let $X(t) = Q(0) + A(t) - R_0(t)$ and $Y(t) = Z_0(0) + R_1(t) + D_2(t)$, then*

$$M(t) = \left\lceil \sup_{0 \leq s \leq t} \left[\frac{X(s) + Y(s) - \sqrt{(X(s) - Y(s))^2 + 4\bar{\mu}_1}}{2} \right]^+ \right\rceil. \tag{EC.34}$$

Proof. We use sample path arguments as follows. Assume that the kth matching happens at time t_k with $0 < t_1 < t_2 < \cdots < t_k < \cdots$. For any $t \in [0, t_1)$, we have $X(t) = Q(t), Y(t) = Z_0(t)$, and $X(t)Y(t) \leq \bar{\mu}_1$ which implies that the right-hand side of (EC.34) is 0. When $t = t_1$, since the first matching happened, we have $X(t_1) = Q(t_1) + 1, Y(t_1) = Z_0(t_1) + 1, X(t_1)Y(t_1) > \bar{\mu}_1$, and $(X(t_1) - 1)(Y(t_1) - 1) \leq \bar{\mu}_1$. Since $\sup_{0 \leq s \leq t} \frac{X(s)+Y(s)-\sqrt{(X(s)-Y(s))^2+4\bar{\mu}_1}}{2}$ is the root to equation

$(X(t)-x)(Y(t)-x)=\bar{\mu}_1$, the right-hand side of (EC.34) is 1 when $t=t_1$. By induction, one can verify that when $t\in[t_k,t_{k+1})$, $X(t)-k=Q(t), Y(t)-k=Z_0(t), (X(t)-k+1)(Y(t)-k+1)>\bar{\mu}_1$, and $(X(t)-k)(Y(t)-k)\le\bar{\mu}_1$. These imply that the right-hand side of (EC.34) is k for $t\in[t_k,t_{k+1})$ and prove the result. □

To carry out our asymptotic analysis, we index the performance of the sequence of queues by a superscript n, denoted by $Q^n, Z_0^n, Z_1^n, Z_2^n,\ M^n, R_0^n, R_1^n, D_1^n, D_2^n, X^n,$ and Y^n, where the scaling is defined in the beginning of Section 4. According to (EC.34), for the nth system, the system dynamics can be characterized as follows,

$$Q^n(t)=Q^n(0)+A^n(t)-R_0^n(t)-M^n(t), \tag{EC.35}$$

$$Z_0^n(t)=Z_0^n(0)+R_1^n(t)+D_2^n(t)-M^n(t), \tag{EC.36}$$

$$Z_1^n(t)=Z_1^n(0)+M^n(t)-D_1^n(t)-R_1^n(t), \tag{EC.37}$$

$$Z_2^n(t)=Z_2^n(0)+D_1^n(t)-D_2^n(t). \tag{EC.38}$$

We add a bar to denote quantities of fluid-scaled processes, e.g., $\bar{Q}^n(t)=Q^n(t)/n$. By using centering operation, the fluid-scaled dynamics can be represented as follows

$$\begin{aligned}
\bar{Q}^n(t)&=(\bar{X}^n(t)-\bar{X}_1^n(t))-(\bar{M}^n(t)-\bar{M}_1^n(t))+\bar{X}_1^n(t)-\bar{M}_1^n(t),\\
\bar{Z}_0^n(t)&=(\bar{Y}^n(t)-\bar{Y}_1^n(t))-(\bar{M}^n(t)-\bar{M}_1^n(t))+\bar{Y}_1^n(t)-\bar{M}_1^n(t),\\
\bar{Z}_1^n(t)&=\bar{Z}_1^n(0)-(\bar{R}_1^n(t)-\theta_1\int_0^t\bar{Z}_1^n(s)ds)-(\bar{D}_1^n(t)-\mu_1\int_0^t\bar{Z}_1^n(s)ds)+(\bar{M}^n(t)-\bar{M}_1^n(t))\\
&\quad-\theta_1\int_0^t\bar{Z}_1^n(s)ds-\mu_1\int_0^t\bar{Z}_1^n(s)ds)+\bar{M}_1^n(t)\\
\bar{Z}_2^n(t)&=\bar{Z}_2^n(0)+(\bar{D}_1^n(t)-\mu_1\int_0^t\bar{Z}_1^n(s)ds)-(\bar{D}_2^n(t)-\mu_1\int_0^t\bar{Z}_2^n(s)ds)+\mu_1\int_0^t\bar{Z}_1^n(s)ds-\mu_2\int_0^t\bar{Z}_2^n(s)ds,
\end{aligned}$$

where $\bar{M}_1^n(t)=\sup_{0\le s\le t}\left[\frac{\bar{X}_1^n(s)+\bar{Y}_1^n(s)-\sqrt{(\bar{X}_1^n(s)-\bar{Y}_1^n(s))^2+4\bar{\mu}_1}}{2}\right]^+$, $\bar{X}_1^n(t)=\bar{Q}^n(0)+\lambda t-\theta_0\int_0^t\bar{Q}^n(s)ds$, and $\bar{Y}_1^n(t)=\bar{Z}_0^n(t)+\theta_1\int_0^t\bar{Z}_1^n(s)ds+\mu_2\int_0^t\bar{Z}_2^n(s)ds$. Note that $\bar{M}_1^n(t)$ is centering process of matching process $\bar{M}^n(t)$. We have the following result.

Proposition EC.3. *For any given $(B_1,B_2,B_3,B_4)\in D([0,T],\mathbb{R}^4)$, there exists a unique $(Q,Z_0,Z_1,Z_2)\in D([0,T],\mathbb{R}^4)$ satisfying*

$$Q(t)=X_1(t)+B_1(t)-M_1(t), \tag{EC.39}$$

$$Z_0(t)=Y_1(t)+B_2(t)-M_1(t), \tag{EC.40}$$

$$Z_1(t)=Z_1(0)+B_3(t)-\theta_1\int_0^t Z_1(s)ds-\mu_1\int_0^t Z_1(s)ds+M_1(t) \tag{EC.41}$$

$$Z_2(t)=Z_2(0)+B_4(t)+\mu_1\int_0^t Z_1(s)ds-\mu_2\int_0^t Z_2(s)ds, \tag{EC.42}$$

where $M_1(t) = \sup_{0\le s\le t}\left[\frac{X(s)+Y(s)-\sqrt{(X(s)-Y(s))^2+4\bar{\mu}_1}}{2}\right]^+$, $X_1(t) = Q(0) + \lambda t - \theta_0\int_0^t Q(s)ds$, *and* $Y_1(t) = Z_0(t) + \theta_1\int_0^t Z_1(s)ds + \mu_2\int_0^t Z_2(s)ds$. *Thus, these representations constitute a mapping* f *from* $(B_1, B_2, B_3, B_4) \in D([0,T],\mathbb{R}^4)$ *to* $(Q, Z_0, Z_1, Z_2) \in D([0,T],\mathbb{R}^4)$. *In addition, the mapping* f *is Lipschitz continuous under the uniform norm.*

Proof. For any $(Q, Z_0, Z_1, Z_2) \in D([0,b],\mathbb{R}^4)$ and $b < \frac{1}{4\theta_0+5\theta_1+2\mu_1+5\mu_2}$, define a mapping Φ from $D([0,b],\mathbb{R}^4)$ to $D([0,b],\mathbb{R}^4)$ given by (EC.39)-(EC.42). We prove that Φ is a contraction mapping. For any $(Q, Z_0, Z_1, Z_2) \in D([0,b],\mathbb{R}^4)$ and $(Q', Z_0', Z_1', Z_2') \in D([0,b],\mathbb{R}^4)$, we have

$$\begin{aligned}||\Phi(Q,Z_0,Z_1,Z_2)-\Phi(Q',Z_0',Z_1',Z_2')||_b \le \theta_0\int_0^b |Q(s)-Q(s')|ds + 2(\theta_1+\mu_1)\int_0^b |Z_1(s)-Z_1(s')|ds\\ + 2\mu_2\int_0^b |Z_2(s)-Z_2(s')|ds + 3\sup_{0\le s\le b}|M_1(s)-M_1'(s)|.\end{aligned} \tag{EC.43}$$

For any $s\in[0,b]$, we have

$$\begin{aligned}
&|M_1(s)-M_1'(s)|\\
&\le \sup_{0\le r\le s}\left|\left[\frac{X_1(t)+Y_1(t)-\sqrt{(X_1(t)-Y_1(t))^2+4\bar{\mu}_1}}{2}\right]^+ - \left[\frac{X_1'(t)+Y_1'(t)-\sqrt{(X_1'(t)-Y_1'(t))^2+4\bar{\mu}_1}}{2}\right]^+\right|\\
&\le \sup_{0\le r\le s}\left|\frac{X_1(t)+Y_1(t)-\sqrt{(X_1(t)-Y_1(t))^2+4\bar{\mu}_1}}{2} - \frac{X_1'(t)+Y_1'(t)-\sqrt{(X_1'(t)-Y_1'(t))^2+4\bar{\mu}_1}}{2}\right|\\
&\le \sup_{0\le r\le s}\left|\frac{X_1(r)-X_1'(r)+Y_1(r)-Y_1'(r)}{2} + \frac{[X_1(r)-Y_1(r)-X_1'(r)+Y_1'(r)][X_1(r)-Y_1(r)+X_1'(r)-Y_1'(r)]}{2[\sqrt{(X_1(t)-Y_1(t))^2+4\bar{\mu}_1}+\sqrt{(X_1'(t)-Y_1'(t))^2+4\bar{\mu}_1}]}\right|\\
&\le \sup_{0\le r\le s}|X_1(r)-X_1'(r)|+|Y_1(r)-Y_1'(r)|\\
&\le \theta_0\int_0^s |Q(r)-Q'(r)|dr + \theta_1\int_0^s |Z_1(r)-Z_1'(r)|dr + \mu_2\int_0^s |Z_2(r)-Z_2'(r)|dr\\
&\le \theta_0\int_0^b |Q(r)-Q'(r)|dr + \theta_1\int_0^b |Z_1(r)-Z_1'(r)|dr + \mu_2\int_0^b |Z_2(r)-Z_2'(r)|dr.
\end{aligned}$$

Thus we have

$$\sup_{0\le s\le b}|M_1(s)-M_1'(s)| \le \theta_0\int_0^b |Q(r)-Q'(r)|dr + \theta_1\int_0^b |Z_1(r)-Z_1'(r)|dr + \mu_2\int_0^b |Z_2(r)-Z_2'(r)|dr,$$

and by plugging in (EC.43), we get

$$\begin{aligned}
&||\Phi(Q,Z_0,Z_1,Z_2)-\Phi(Q',Z_0',Z_1',Z_2')||_b\\
&\le 4\theta_0\int_0^b |Q(s)-Q'(s)|ds + (5\theta_1+2\bar{\mu}_1)\int_0^b |Z_1(s)-Z_1'(s)| + 5\mu_2\int_0^b\int_0^b |Z_2(s)-Z_2'(s)|\\
&\le (4\theta_0+5\theta_1+2\mu_1+5\mu_2)b||(Q,Z_0,Z_1,Z_2)-(Q',Z_0',Z_1',Z_2')||_b\\
&< ||(Q,Z_0,Z_1,Z_2)-(Q',Z_0',Z_1',Z_2')||_b.
\end{aligned}$$

Thus, the mapping Φ is a contraction. By Contraction Mapping theorem, there exists a unique solution to (EC.39)-(EC.42) on $[0,b]$. It now remains to extend the existence and uniqueness result from $[0,b]$ to $[0,T]$. Replacing $(Q(t),Z_0(t),Z_1(t),Z_2(t))$ by $(Q(b+t),Z_0(b+t),Z_1(b+t),Z_2(b+t))$, and following the previous argument, we obtain a unique solution for (EC.39)-(EC.42) on $[0,2b]$. Repeating this approach for $\lceil T/b\rceil$ times gives a unique solution on the interval $[0,T]$.

Next we prove that the mapping f is Lipschitz continuous under the uniform norm. For any $||(B_1,B_2,B_3,B_4)-(B_1',B_2',B_3',B_4')||_T<\epsilon$, following the similar steps of previous proof, we have

$$\begin{aligned}
&||f(B_1,B_2,B_3,B_4)-f(B_1',B_2',B_3',B_4')||_T\\
\le\,& 4||(B_1,B_2,B_3,B_4)-(B_1',B_2',B_3',B_4')||_T+4\theta_0\int_0^T|Q(s)-Q'(s)|ds\\
&+(5\theta_1+2\mu_1)\int_0^T|Z_1(s)-Z_1'(s)|ds+5\mu_2\int_0^T|Z_2(s)-Z_2'(s)|ds\\
\le\,& 4\epsilon+(4\theta_0+5\theta_1+2\mu_1+5\mu_2)\int_0^T||f(B_1,B_2,B_3,B_4)-f(B_1',B_2',B_3',B_4')||_s ds.
\end{aligned}$$

Thus by Gronwall's inequality, we have

$$||f(B_1,B_2,B_3,B_4)-f(B_1',B_2',B_3',B_4')||_T\le 4\epsilon e^{(4\theta_0+5\theta_1+2\mu_1+5\mu_2)T},$$

which implies that the mapping f is Lipschitz continuous under the uniform norm. $\square$

When $n\to\infty$, the weak convergence of $\bar{A}^n(t)-\lambda t$, $\bar{R}_0^n(t)-\theta_0\int_0^t\bar{Q}^n(s)ds$, $\bar{R}_1^n(t)-\theta_1\int_0^t\bar{Z}_1^n(s)ds$, $\bar{R}_2^n(t)-\mu_2\int_0^t\bar{Z}_2^n(s)ds$ are from the functional strong law of large numbers. We further prove that $\bar{M}^n(t)-\bar{M}_1^n(t)$ and $\bar{D}_1^n(t)-\mu_1\int_0^t\bar{Z}_1^n(s)ds$ converge to 0 when $n\to\infty$.

LEMMA EC.2. *For any $\epsilon>0$ and any given T, we have*

$$\lim_{n\to\infty}\mathbb{P}\left(\sup_{t\le T}|\bar{M}^n(t)-\bar{M}_1^n(t)|>\epsilon\right)=0, \tag{EC.44}$$

and

$$\lim_{n\to\infty}\mathbb{P}\left(\sup_{t\le T}|\bar{D}_1^n(t)-\mu_1\int_0^t\bar{Z}_1^n(s)ds|>\epsilon\right)=0. \tag{EC.45}$$

Proof. For any $t<T$, since

$$\left|\bar{M}^n(t)-\sup_{0\le s\le t}\left[\frac{\bar{X}_1^n(s)+\bar{Y}_1^n(s)-\sqrt{(\bar{X}_1^n(s)-\bar{Y}_1^n(s))^2+4\bar{\mu}_1}}{2}\right]^+\right|\le 1/n,$$

we have

$$|\bar{M}^n(t)-\bar{M}_1^n(t)|\le\frac{1}{n}+\left|\sup_{0\le s\le t}\left[\frac{\bar{X}_1^n(s)+\bar{Y}_1^n(s)-\sqrt{(\bar{X}_1^n(s)-\bar{Y}_1^n(s))^2+4\bar{\mu}_1}}{2}\right]^+-\bar{M}_1^n(s)\right|$$

$$\leq \frac{1}{n} + \sup_{0\leq s\leq t} |\bar{X}^n(t) - \bar{X}_1^n(t)| + |\bar{Y}^n(t) - \bar{Y}_1^n(t)|$$
$$\leq \frac{1}{n} + \sup_{0\leq s\leq t} |\bar{A}^n(s) - \lambda s| + \left|\bar{R}_0^n(t) - \theta_0 \int_0^t \bar{Q}^n(s)ds\right|$$
$$+ \left|\bar{R}_1^n(t) - \theta_1 \int_0^t \bar{Z}_1^n(s)ds\right| + \left|\bar{D}_2^n(t) - \mu_2 \int_0^t \bar{Z}_2^n(s)ds\right|.$$

The convergence of $\bar{A}^n(s) - \lambda s$, $\bar{R}_0^n(t) - \theta_0 \int_0^t \bar{Q}^n(s)ds$, $\bar{R}_1^n(t) - \theta_1 \int_0^t \bar{Z}_1^n(s)ds$, $\bar{D}_2^n(t) - \mu_2 \int_0^t \bar{Z}_2^n(s)ds$ imply the convergence of $\bar{M}^n(t) - \bar{M}_1^n(t)$.

Next, we prove the convergence of $\bar{D}_1^n(t) - \mu_1 \int_0^t \bar{Z}_1^n(s)ds$. We first show that, the pick up rates in $[0,T]$ converge to μ_1 in probability when $n \to \infty$, i.e., for any $\epsilon_1 > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{t\leq T} \frac{Q^n(t-)+1}{n}\frac{Z_0^n(t-)}{n} \geq \bar{\mu}_1 + \epsilon_1\right) + \mathbb{P}\left(\sup_{t\leq T} \frac{Q^n(t-)}{n}\frac{Z_0^n(t-)+1}{n} \geq \bar{\mu}_1 + \epsilon_1\right) = 0. \quad \text{(EC.46)}$$

These two terms denote two cases of large pick up rate for matching, arrival of a rider or arrival an empty car. Since $\frac{Q^n(t-)}{n}\frac{Z_0^n(t-)}{n} < \bar{\mu}_1$, the first case implies that $Z_0^n(t-) > \epsilon_1 n^2$. This contradicts with $Z_0^n(t-) \leq n$ when n is large enough. Similarly, the second case implies $Q^n(t-) > \epsilon_1 n^2$. Since $Q^n(t-) \leq A^n(t-) + Q^n(0)$, we have $A^n(t-) > \epsilon_1 n^2 - \bar{Q}(0)n$. Thus we have

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{t\leq T} \frac{Q^n(t-)}{n}\frac{Z_0^n(t-)+1}{n} \geq \bar{\mu}_1 + \epsilon_1\right) \leq \lim_{n\to\infty} \mathbb{P}(A^n(t-) > \epsilon_1 n^2 - \bar{Q}(0)n) = 0.$$

Combing two cases we prove (EC.46). □

When $n \to \infty$, by the coupling method in view of (EC.46), the number of pick ups $D_1^n(t)$ is lower bounded by a nonhomogeneous Poisson process with rate $\mu_1 Z_1^n(t)$ and upper bounded by a non-homogeneous Poisson process with rate $(\mu_1 + \epsilon_1)Z_1^n(t)$ with probability one. Thus, when $\epsilon_1 \to 0$, we obtain the convergence of $\bar{D}_1^n(t)$ to $\mu_1 \int_0^t \bar{Z}_1^n(s)ds$. Now we are ready to state our result for fluid limit.

Theorem EC.4. *Suppose the initial condition holds. Then as $n \to \infty$, $(\bar{Q}^n, \bar{Z}_0^n, \bar{Z}_1^n, \bar{Z}_2^n)$ converges to $(\bar{Q}, \bar{Z}_0, \bar{Z}_1, \bar{Z}_2)$ in probability, where*

$$\bar{Q}(t) = \bar{X}_1(t) - \bar{M}_1(t),$$
$$\bar{Z}_0(t) = \bar{Y}_1(t) - \bar{M}_1(t),$$
$$\bar{Z}_1(t) = \bar{Z}_1(0) - \theta_1 \int_0^t \bar{Z}_1(s)ds - \mu_1 \int_0^t \bar{Z}_1(s)ds + \bar{M}_1(t)$$
$$\bar{Z}_2(t) = \bar{Z}_2(0) + \mu_1 \int_0^t \bar{Z}_1(s)ds - \mu_2 \int_0^t \bar{Z}_2(s)ds,$$

where $\bar{M}_1(t) = \sup_{0\leq s\leq t}\left[\frac{\bar{X}_1(s)+\bar{Y}_1(s)-\sqrt{(\bar{X}_1(s)-\bar{Y}_1(s))^2+4\bar{\mu}_1}}{2}\right]^+$, $\bar{X}_1(t) = \bar{Q}(0) + \lambda t - \theta_0 \int_0^t \bar{Q}(s)ds$, *and* $\bar{Y}_1(t) = \bar{Z}_0(t) + \theta_1 \int_0^t \bar{Z}_1(s)ds + \mu_2 \int_0^t \bar{Z}_2(s)ds$.

Proof. For each n, by choosing

$$
\begin{aligned}
\bar{B}_1^n(t) &= (\bar{A}^n(t)-\lambda t) - (\bar{R}_0^n(t) - \theta_0 \int_0^t \bar{Q}^n(s)ds) - (\bar{M}^n(t) - \bar{M}_1^n(t)),\\
\bar{B}_2^n(t) &= (\bar{R}_1^n(t) - \theta_1 \int_0^t \bar{Z}_1^n(s)ds) + (\bar{D}_2^n(t) - \mu_2 \int_0^t \bar{Z}_2^n(s)ds) - (\bar{M}^n(t) - \bar{M}_1^n(t)),\\
\bar{B}_3^n(t) &= -(\bar{R}_1^n(t) - \theta_1 \int_0^t \bar{Z}_1^n(s)ds) - (\bar{D}_1^n(t) - \mu_1 \int_0^t \bar{Z}_1^n(s)ds) + (\bar{M}^n(t) - \bar{M}_1^n(t)),\\
\bar{B}_4^n(t) &= (\bar{D}_1^n(t) - \mu_1 \int_0^t \bar{Z}_1^n(s)ds) - (\bar{D}_2^n(t) - \mu_2 \int_0^t \bar{Z}_2^n(s)ds),
\end{aligned}
$$

we have that $(\bar{B}_1^n, \bar{B}_2^n, \bar{B}_3^n, \bar{B}_4^n)$ and $(\bar{Q}^n, \bar{Z}_0^n, \bar{Z}_1^n, \bar{Z}_2^n)$ jointly satisfy (EC.39)-(EC.42). Since $(\bar{B}_1^n, \bar{B}_2^n, \bar{B}_3^n, \bar{B}_4^n)$ converges to 0 in probability and the mapping defined by (EC.39)-(EC.42) is Lipschitz continuous, we obtain the convergence result of $(\bar{Q}^n, \bar{Z}_0^n, \bar{Z}_1^n, \bar{Z}_2^n)$. □

EC.7. Justification of the (approximate) upper bound

We first restate the core optimzation problem (27) as follows:

$$
\begin{aligned}
\max_{\mu_1}\ & z_2, && \text{(EC.47)}\\
\text{s.t. } & \lambda = q\theta_0 + z_1\theta_1 + z_2\mu_2, && \text{(EC.48)}\\
& z_1\mu_1 = z_2\mu_2, && \text{(EC.49)}\\
& 1 = z_1 + z_2 + z_0, && \text{(EC.50)}\\
& \mu_1 = C(q)^{\alpha_1}(z_0)^{\alpha_2}. && \text{(EC.51)}\\
& q, z_0, z_1, z_2 \geq 0.
\end{aligned}
$$

We denote the optimal value of Problem (EC.47) as $z_2^*(\lambda, \mu_2, C)$, reflecting the dependence on λ, μ_2, C. Next we establish a sensitivity result regarding $z_2^*(\lambda, \mu_2, C)$, which is central to our analysis later.

LEMMA EC.3. *Suppose Problem* (EC.47) *has feasible solutions under two sets of parameters,* (λ, μ_2, C) *and* (λ', μ_2', C'). *Then*

$$
z_2^*(\lambda', \mu_2', C') \to z_2^*(\lambda, \mu_2, C), \qquad \text{(EC.52)}
$$

when $(\lambda', \mu_2', C') \to (\lambda, \mu_2, C)$.

Proof. For parameters (λ, μ_2, C), denote the optimal solution of Problem (EC.47) by $(\mu_1^*, q^*, z_0^*, z_1^*, z_2^*)$. We aim to find a feasible solution of Problem (EC.47) that is "close" to $(\mu_1^*, q^*, z_0^*, z_1^*, z_2^*)$ when the parameter set is changed to (λ', μ_2', C').