# MEC6602E: Transonic Aerodynamics

## Projet de remise

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# QUESTION 1)

a) $u_j^{n+1} = u_j^n - c\frac{\Delta t}{\Delta x}\left(u_j^n - u_{j-1}^n\right)$

STABILITY

$T_k = \bar{T}_k + \varepsilon_k$

$\Rightarrow u_j^{n+1} = u_j^n - c\frac{\Delta t}{\Delta x}\left(u_j^n - u_{j-1}^n\right)$

$\Rightarrow \cancel{\bar{u}_j^{n+1}} + \epsilon_j^{n+1} = \cancel{\bar{u}_j^n} + \epsilon_j^n - c\frac{\Delta t}{\Delta x}\left(\cancel{\bar{u}_j^n} + \epsilon_j^n - \cancel{\bar{u}_{j-1}^n} - \bar{\epsilon}_{j-1}^n\right)$

$\Rightarrow \epsilon_j^{n+1} = \epsilon_j^n - \frac{c\Delta t}{\Delta x}\left(\epsilon_j^n - \epsilon_{j-1}^n\right)$

$\Rightarrow \epsilon_j^{n+1} = \epsilon_j^n - c\frac{\Delta t}{\Delta x}\epsilon_j^n + \frac{c\Delta t}{\Delta x}\epsilon_{j-1}^n$

$\Rightarrow \frac{\epsilon_j^{n+1}}{\epsilon_j^n} = 1 - \frac{c\Delta t}{\Delta x} + \frac{c\Delta t}{\Delta x}\frac{\epsilon_{j-1}^n}{\epsilon_j^n}$

$\epsilon = e^{at}e^{ik_m x} \Rightarrow \epsilon_j^n = e^{at}e^{ik_m x}$; $\epsilon_{j-1}^n = e^{at}e^{ik_m(x-\Delta x)}$

$\Rightarrow \frac{\epsilon_j^{n+1}}{\epsilon_j^n} = 1 - \frac{c\Delta t}{\Delta x} + \frac{c\Delta t}{\Delta x}\left(\frac{\cancel{e^{at}}e^{ik_m(\cancel{x}-\Delta x)}}{\cancel{e^{at}}e^{ik_m(\cancel{x})}}\right)$

$\Rightarrow \frac{\epsilon_j^{n+1}}{\epsilon_j^n} = 1 - \frac{c\Delta t}{\Delta x} + \frac{c\Delta t}{\Delta x}\left(e^{-ik_m(\Delta x)}\right) = 1 - \frac{c\Delta t}{\Delta x}\left(1 - e^{-ik_m\Delta x}\right) = |G|$

$e^{ik_m\Delta x} = \cos(k_m\Delta x) - i\sin(k_m\Delta x)$ ; $\frac{c\Delta t}{\Delta x} = \sigma$

$\Rightarrow |G| = 1 - \sigma\left(1 - \cos(k_m\Delta x) + i\sin(k_m\Delta x)\right)$

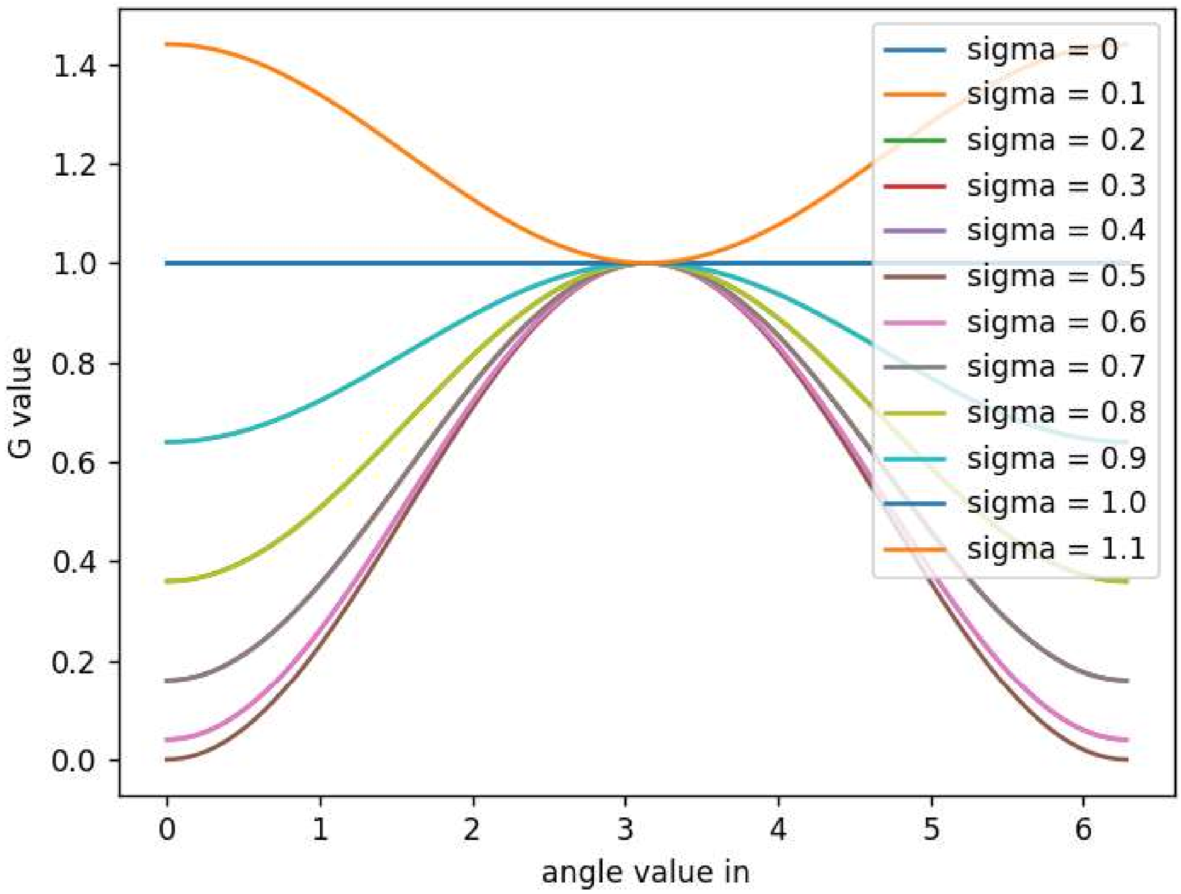
$\Rightarrow |G|^2 = \left(1 - \sigma - \sigma\cos(k_m\Delta x)\right)^2 + \left(i\sigma\sin(k_m\Delta x)\right)^2$

$= 1 - \sigma - \sigma\cos(k_m\Delta x) - \sigma + \sigma^2 + \sigma^2\cos(k_m\Delta x) - \sigma\cos(k_m\Delta x)$
$+ \sigma^2\cos(k_m\Delta x) + \sigma^2\cos^2(k_m\Delta x) + \sigma^2\sin^2(k_m\Delta x)$

$\Rightarrow 1 - 2\sigma - 2\sigma\cos(k_m\Delta x) + \sigma^2 + 2\sigma^2\cos(k_m\Delta x) +$
$\sigma^2\cos(k_m\Delta) + \sigma^2\sin(k_m\Delta x)$

$= 1 - 2\sigma - 2\sigma \cos(K_m \Delta x) + 2\sigma^2 + 2\sigma^2 \cos(K_m \Delta x) = |G|^2 \leq 1$

We will show WITH a plot that $\sigma \leq 1$ to be stable whe $K_m \Delta x \in [0, 2\pi]$:



# ACCURACY

We have to develop into taylor

$$U_j^{n+1} = U_j^n - \frac{c\Delta t}{\Delta x}\left(U_j^n - U_{j-1}^n\right) \quad (1)$$

$$U_j^n = U_j^n \quad (2)$$

$$U_j^{n+1} = U_j^n + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} + \dots = \sum_{k=0}^{\infty} \frac{\Delta t^k}{k!}\frac{\partial^{(k)} u}{\partial t^{(k)}} \quad (3)$$

$$U_{j-1}^n = U_j^n - \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} - \Delta x^3 \frac{\partial^3 u}{\partial x^3} + \dots = \sum_{k=0}^{\infty} \frac{(-\Delta x)^k}{k!}\frac{\partial^{(k)} u}{\partial x^{(k)}} \quad (4)$$

We put (2), (3) and (4) into (1)

$$\Rightarrow \cancel{U_j^n} + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} = \cancel{U_j^n} - \frac{c\Delta t}{\Delta x}\left(\cancel{U_j^n} - \cancel{U_j^n} + \Delta x \frac{\partial u}{\partial x} - \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3}\right)$$

We put (2), (3) and (4) into (1)

$$\Rightarrow \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} = -\frac{c\Delta t}{\cancel{\Delta x}}\left(\Delta x \frac{\partial u}{\partial x} - \frac{\Delta x^{2}}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^{3}}{6}\frac{\partial^3 u}{\partial x^3}\right)$$

$$\Rightarrow \Delta t \frac{\partial u}{\partial t} + \Delta t^2 \frac{\partial^2 u}{\partial t} + \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} = -c\Delta t\left(\frac{\partial u}{\partial x} - \frac{\Delta x}{2}\frac{\partial^2 u}{\partial x} + \frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3}\right)$$

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{-c\cancel{\Delta t}\left(\frac{\partial u}{\partial x} - \frac{\Delta x}{2}\frac{\partial^2 u}{\partial x} + \frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3}\right) - \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t} - \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3}}{\cancel{\Delta t}}$$

$$\Rightarrow \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = \frac{c\Delta x}{2}\frac{\partial^2 u}{\partial x} + \frac{c\Delta x^2}{6}\frac{\partial^3 u}{\partial x} - \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t} - \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t}$$

The order of convergeng for both time and space are 1

$$O(\Delta t) + O(\Delta x)$$

# CONSISTENCY

The error term is: $\frac{c\Delta x}{2}\frac{\partial^2 u}{\partial x} + \frac{c\Delta x^2}{6}\frac{\partial^3 u}{\partial x} - \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t} - \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t}$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \frac{c\overset{\to 0}{\Delta x}}{2}\frac{\partial^2 u}{\partial x} + \frac{c\overset{\to 0}{\Delta x^2}}{6}\frac{\partial^3 u}{\partial x} - \frac{\overset{\to 0}{\Delta t}}{2}\frac{\partial^2 u}{\partial t} - \frac{\overset{\to 0}{\Delta t^2}}{6}\frac{\partial^3 u}{\partial t} = 0$$

Thus, the backward scheme is consistent

## QUESTION 1 c)

STABILITY : $U_j^{n+1} = U_j^{n-1} - \frac{c\Delta t}{\Delta x}\left(U_{j+1}^{n} - U_{j-1}^{n}\right)$

$U_m^k = \bar{U}_m^k + \epsilon_m^k$

$$\Rightarrow \cancel{\bar{U}_j^{n+1}} + \epsilon_j^{n+1} = \cancel{\bar{U}_j^{n-1}} + \epsilon_j^{n-1} - \frac{c\Delta t}{\Delta x}\left(\cancel{\bar{U}_{j+1}^{n}} + \epsilon_{j+1}^{n} - \cancel{\bar{U}_{j-1}^{n}} + \epsilon_{j-1}^{n}\right)$$

$$\Rightarrow \epsilon_j^{n+1} = \epsilon_j^{n-1} - \frac{c\Delta t}{\Delta x}\left(\epsilon_{j+1}^{n} - \epsilon_{j-1}^{n}\right)$$

$$\Rightarrow G = \frac{\epsilon_j^{n+1}}{\epsilon_j^n} = \frac{\epsilon_j^{n-1}}{\epsilon_j^n} - \frac{c\Delta t}{\Delta x}\left(\frac{\epsilon_{j+1}^n}{\epsilon_j^n} - \frac{\epsilon_{j-1}^n}{\epsilon_j^n}\right)$$

$\epsilon_j^n = e^{at} e^{ik_n \Delta t}$ ; $\frac{c\Delta t}{\Delta x}$

$$\Rightarrow G = \frac{e^{a(t-\Delta t)} e^{ik_n x}}{e^{at} e^{ik_n x}} - \sigma\left(\frac{e^{at} e^{ik_n (x+\Delta x)}}{e^{at} e^{ik_n x}} - \frac{e^{at} e^{ik_n (x-\Delta x)}}{e^{at} e^{ik_n x}}\right)$$

$$e^{-a\Delta t} - \sigma\left(e^{ik_n \Delta x} - e^{-ik_n \Delta x}\right)$$

$$\Rightarrow G = 1/G - \sigma\left(2i \sin(k_n \Delta x)\right)$$

We Multiply by $G$ on each side

$$\Rightarrow G^2 = 1 - \sigma G\left(2i \sin(k_n \Delta x)\right)$$

$$\Rightarrow G^2 + \sigma G\left(2i \sin(k_n \Delta x)\right) - 1$$

$$G_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$b = 2i\,\sigma \sin(k_m \Delta x) \Rightarrow b^2 = -4\sigma^2 \sin^2(k_m \Delta x)$$

$$\Rightarrow b^2 - 4ac = -4\sigma^2 \sin^2(k_m \Delta x) + 4$$

$$\Rightarrow G = \frac{-i\cancel{2}\sigma \sin(k_m \Delta x) \pm \sqrt{\cancel{4}\left(1 - \sigma^2 \sin^2(k_m \Delta x)\right)}}{\cancel{2}}$$

$$\Rightarrow |G| = \left| -i\sigma \sin(k_m \Delta x) \pm \sqrt{1 - \sigma^2 \sin^2(k_m \Delta x)} \right| \leq 1$$

$$\Rightarrow |G|^2 = \sigma^2 \sin^2(k_m \Delta x) + 1 - \sigma^2 \sin^2(k_m \Delta x) = 1$$

$$\Rightarrow |G|^2 = 1 = |G|$$

THE CONDITION $|G| \leq 1$ IS RESPECTED
HOWEVER

WE NEED TO VERIFY THE $\Delta = 1 - \sigma^2 \sin^2(k_m \Delta x) \geq 0$

$$\Rightarrow \sigma^2 \sin^2(k_m \Delta x) \leq 1$$

WORST CASE $\sin^2(k_m \Delta x) = 1$

$$\Rightarrow \sigma^2 \leq 1$$

$$\Rightarrow \sigma = \frac{c\Delta t}{\Delta x} \leq 1$$

if $\Delta = b^2 - 4ac \leq 0$ THERE WOULD BE
An imaginary part that would add to $-i\sigma \sin(k_m \Delta x)$

# ACCURACY

LETS DEVELLOP IN TAYLOR

$$U_j^{n+1} = U_j^{n-1} - \frac{c\Delta t}{\Delta x}\left(U_{j+1}^n - U_{j-1}^n\right) \Rightarrow U_j^{n+1} - U_j^{n-1} = -\frac{c\Delta t}{\Delta x}\left(U_{j+1}^n - U_{j-1}^n\right) \quad (1)$$

$$U_j^{n+1} = U_j^n + \Delta t\frac{\partial u}{\partial t} + \Delta t^2\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t} + \dots$$

$$U_j^{n-1} = U_j^n - \Delta t\frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} - \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} + \dots$$

0 2x 0 2x 0 2x ...

$$U_j^{n+1} - U_j^{n-1} = 2\Delta t\frac{\partial u}{\partial t} + \frac{2\Delta t^3}{6}\frac{\partial^3 u}{\partial t} + \frac{2\Delta t^5}{120}\frac{\partial^5 u}{\partial t^5} + \dots \quad (2)$$

$$U_{j+1}^n - U_{j-1}^n = 2\Delta x\frac{\partial u}{\partial x} + \frac{2\Delta x^3}{6}\frac{\partial^3 u}{\partial x} + \frac{2\Delta x^5}{120}\frac{\partial^5 u}{\partial x^5} + \dots \quad (3)$$

(2) (3) into (1)

$$\Rightarrow 2\Delta t\frac{\partial u}{\partial t} + \frac{2\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} + \frac{2\Delta t^5}{120}\frac{\partial^5 u}{\partial t^5} + \dots = -\frac{c\Delta t}{\Delta x}\left(2\Delta x\frac{\partial u}{\partial x} + \frac{2\Delta x^3}{6}\frac{\partial^3 u}{\partial x} + \frac{2\Delta x^5}{120}\frac{\partial^5 u}{\partial x^5} + \dots\right)$$

$$\Rightarrow \cancel{\Delta t}\frac{\partial u}{\partial t} + \frac{\Delta t^{\cancel{3}\,2}}{6}\frac{\partial^3 u}{\partial t^3} + \frac{\Delta t^{\cancel{5}\,4}}{120}\frac{\partial^5 u}{\partial t^5} + \dots = -\frac{c\cancel{\Delta t}}{\cancel{\Delta x}}\left(\cancel{\Delta x}\frac{\partial u}{\partial x} + \frac{\Delta x^{\cancel{3}\,2}}{6}\frac{\partial^3 u}{\partial x} + \frac{\Delta x^{\cancel{5}\,4}}{120}\frac{\partial^5 u}{\partial x^5} + \dots\right)$$

$$\Rightarrow \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3} + \frac{\Delta x^4}{120}\frac{\partial^5 u}{\partial x^5} - \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^3} - \frac{\Delta t^4}{120}\frac{\partial^5 u}{\partial t^5}$$

$$\Rightarrow \qquad = O(\Delta x^2) + O(\Delta t^2)$$

CONSISTENCY

$$\lim_{\substack{\Delta t \to 0 \\ \Delta x \to 0}} \left( \overset{\to 0}{\frac{\Delta x^2}{6}} \frac{\partial^3 u}{\partial x^3} + \overset{\to 0}{\frac{\Delta x^4}{120}} \frac{\partial^3 u}{\partial x^5} - \overset{\to 0}{\frac{\Delta t^2}{6}} \frac{\partial^3 u}{\partial t^3} - \overset{\to 0}{\frac{\Delta t^4}{120}} \frac{\partial^5 u}{\partial t^5} \right) = 0$$

<u>THIS ALGORITHM IS CONSISTENT</u>

QUESTION 1a)

STABILITY

$$u_j^{n+1} = \left(\frac{u_{j+1}^n + u_{j-1}^n}{2}\right) - \frac{c\Delta t}{\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right)$$

$U_n^k = \bar{U}_n^k + \epsilon_n^k$

$$\Rightarrow \cancel{\bar{u}_j^{n+1}} + \epsilon_j^{n+1} = \left(\frac{\cancel{\bar{u}_{j+1}^n} + \epsilon_{j+1}^n + \cancel{\bar{u}_{j-1}^n} + \epsilon_{j-1}^n}{2}\right) - \frac{c\Delta t}{2\Delta x}\left(\cancel{\bar{u}_{j+1}^n} + \epsilon_{j+1}^n - \cancel{\bar{u}_{j-1}^n} - \epsilon_{j-1}^n\right)$$

$$\Rightarrow \epsilon_j^{n+1} = \left(\frac{\epsilon_{j+1}^n + \epsilon_{j-1}^n}{2}\right) - \frac{c\Delta t}{2\Delta x}\left(\epsilon_{j+1}^n - \epsilon_{j-1}^n\right)$$

$\epsilon_n^k = e^{a^k} e^{ik_m n}$ ; $\frac{c\Delta t}{2\Delta x} = \sigma$

$$\Rightarrow G = \frac{\epsilon_j^{n+1}}{\epsilon_j^n} = \frac{1}{2}\left(\frac{e^{at}e^{ik_n(x+\Delta x)}}{e^{at}e^{ik_n x}} + \frac{e^{at}e^{ik_n(x-\Delta x)}}{e^{at}e^{ik_n x}}\right) - \sigma\left(\frac{e^{at}e^{ik_n(x+\Delta x)}}{e^{at}e^{ik_n x}} - \frac{e^{at}e^{ik_n(x-\Delta x)}}{e^{at}e^{ik_n x}}\right)$$

$$\Rightarrow G = \frac{1}{2}\left(e^{ik_m\Delta x} + e^{-ik_n\Delta x}\right) - \sigma\left(e^{ik_n\Delta x} - e^{-ik_m\Delta x}\right)$$

$e^{ix} + e^{-ix} = 2\cos(x)$ ; $e^{ix} - e^{-ix} = -2i\sin(x)$ ;

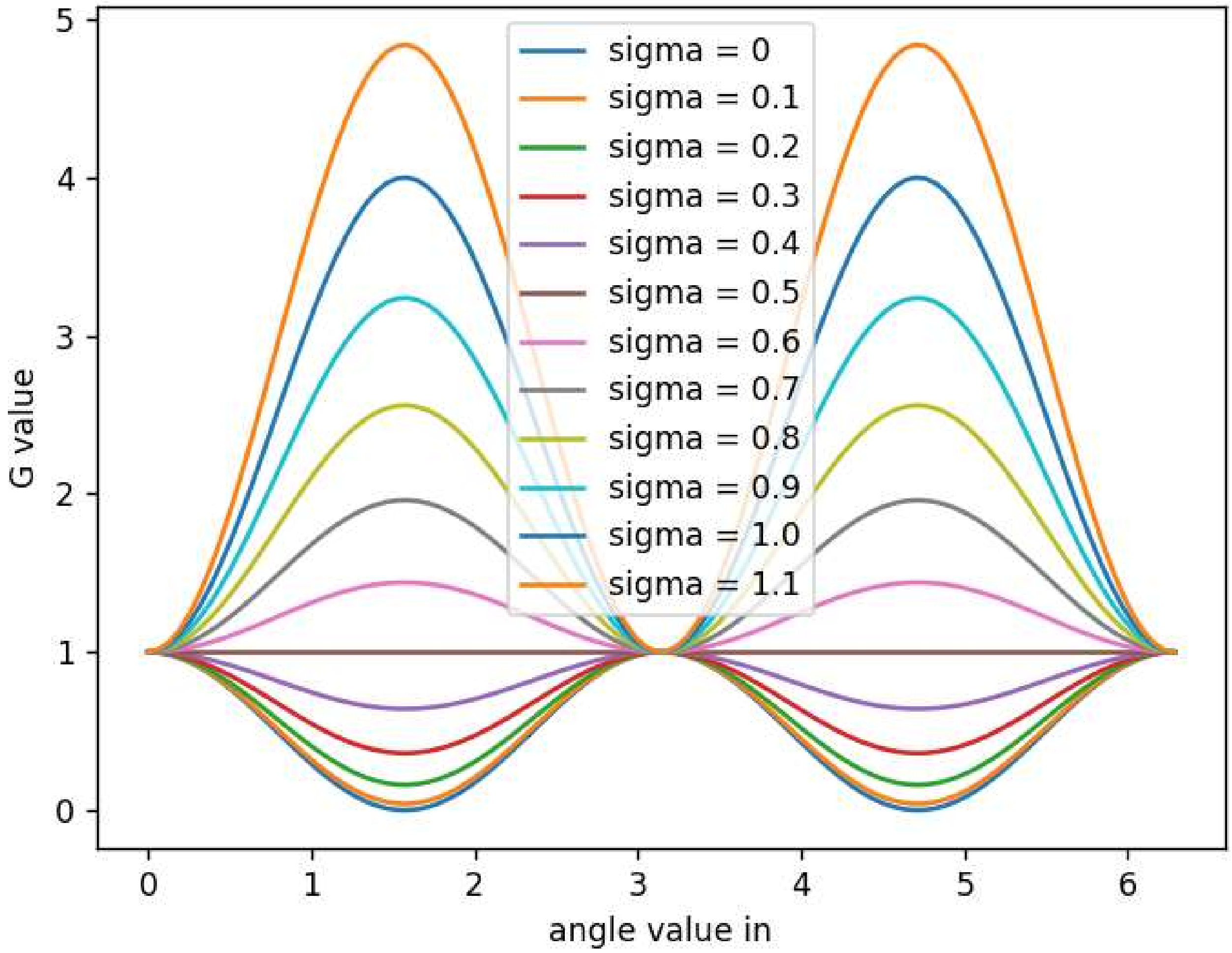
$$\Rightarrow G = \cos(k_n\Delta x) + 2\sigma\, i\sin(k_n\Delta x)$$

$$\Rightarrow |G|^2 = \cos^2(k_n\Delta x) + 4\sigma^2\sin^2(k_n\Delta x) < 1$$

$$\Rightarrow \sigma^2 < \frac{-\cos^2(k_n\Delta x)}{4\sin^2(k_n\Delta x)} + 1$$

WE SEE ON THE GRAPH THAT SIGMA SHOULD B LOWER THAN 0,5.

$$\sigma = \frac{c\Delta t}{2\Delta X} \Rightarrow \underline{\frac{c\Delta t}{\Delta x} = CFL \leq 1}$$



sigma = 0
sigma = 0.1
sigma = 0.2
sigma = 0.3
sigma = 0.4
sigma = 0.5
sigma = 0.6
sigma = 0.7
sigma = 0.8
sigma = 0.9
sigma = 1.0
sigma = 1.1
G value
angle value in
0
1
2
3
4
5
6

# ACCURACY

$$u_j^{n+1} = \frac{(u_{j+1}^n + u_{j-1}^n)}{2} - \frac{c\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) \quad (1)$$

$$u_j^{n+1} = u_j^n + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} + \dots \quad (2)$$

$$u_{j+1}^n = u_j^n + \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3} + \dots$$

$$u_{j-1}^n = u_j^n - \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} - \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3} + \dots$$

$$\Rightarrow u_{j+1}^n + u_{j-1}^n = 2u_j^n + 2\frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} + 2\frac{\Delta x^4}{24}\frac{\partial^4 u}{\partial x^4} + \dots \quad (3)$$

$$\Rightarrow u_{j+1}^n - u_{j-1} = 2\Delta x\frac{\partial u}{\partial x} + 2\frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x} + 2\frac{\Delta x^5}{120}\frac{\partial^5 u}{\partial x^5} + \dots \quad (4)$$

(2) (3) (4) into (1)

$$\Rightarrow \cancel{u_j^n} + \Delta t\frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} + \dots = \frac{1}{\cancel{2}}\left(\cancel{2u_j^n} + \cancel{2}\frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} + \cancel{2}\frac{\Delta x^4}{24}\frac{\partial^4 u}{\partial x^4} + \dots\right) - \frac{c\Delta t}{\cancel{2\Delta x}}\left(\cancel{2\Delta x}\frac{\partial u}{\partial x} + \cancel{2}\frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x} + \cancel{2}\frac{\Delta x^5}{120}\frac{\partial^5 u}{\partial x^5} + \dots\right)$$

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{\Delta x^2}{2\Delta t}\frac{\partial^2 u}{\partial x} + \frac{\Delta x^4}{24\Delta t}\frac{\partial^4 u}{\partial x^4} + \dots - c\left(\frac{\partial u}{\partial x} + \frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x} + \frac{\Delta x^4}{120}\frac{\partial^5 u}{\partial x^5} + \dots\right) - \frac{\Delta t}{2}\frac{\partial^2 u}{\partial x} - \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \frac{\Delta x^2}{2\Delta t}\frac{\partial^2 u}{\partial x} + \frac{\Delta x^4}{24\Delta t}\frac{\partial^4 u}{\partial x^4} + \dots - c\left(\frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x} + \frac{\Delta x^4}{120}\frac{\partial^5 u}{\partial x^5} + \dots\right) - \frac{\Delta t}{2}\frac{\partial^2 u}{\partial x} - \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial x}$$

$$= O\left(\frac{\Delta x^2}{\Delta t}\right) + O(\Delta t) \quad ?$$

# CONSISTENCY

$$\lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \frac{\Delta x^2}{2\Delta t}\frac{\partial^2 u}{\partial x} + \frac{\Delta x^4}{24\Delta t}\frac{\partial^3 u}{\partial x^4} + \dots - c\left(\frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x} + \frac{\Delta x^4}{120}\frac{\partial^5 u}{\partial x^5} + \dots\right) - \frac{\Delta t}{2}\frac{\partial^2 u}{\partial x} - \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial x} \neq 0$$

THE TERMS $\frac{\Delta x^n}{\Delta t}$ MAKES THIS ALGORITHM INCONSISTENT...

LETS CREATE OUR OWN ALGORITHM

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

SECOND ORDER SPACE

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{u_{j+1}^{n} - u_{j-1}^{n}}{2\Delta x}$$

FOURTH ORDER IN TIME

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{-u_j^{n+2} + 8u_j^{n+1} - 8u_j^{n-1} + u_j^{n-2}}{12\Delta t}$$

$$\Rightarrow \frac{-u_j^{n+2} + 8u_j^{n+1} - 8u_j^{n-1} + u_j^{n-2}}{12\Delta t} + c\left(\frac{u_{j+1}^{n} - u_{j-1}^{n}}{2\Delta x}\right) = 0$$

$$\Rightarrow \Rightarrow \frac{-\frac{1}{12}u_j^{n+2} + \frac{2}{3}u_j^{n+1} - \frac{2}{3}u_j^{n-1} + \frac{1}{12}u_j^{n-2}}{\Delta t} + c\left(\frac{u_{j+1}^{n} - u_{j-1}^{n}}{2\Delta x}\right) = 0$$

$$\Rightarrow u_j^{n+1} = \frac{1}{8}u_j^{n+2} + u_j^{n-1} - \frac{1}{8}u_j^{n-2} - \frac{3c\Delta t}{4\Delta x}\left(u_{j+1}^{n} - u_{j-1}^{n}\right)$$

$$\Rightarrow \frac{1}{8}u_j^{n+2} - u_j^{n+1} + u_j^{n-1} - \frac{1}{8}u_j^{n-2} - 3\frac{c\Delta t}{4\Delta x}\left(u_{j+1}^{n} - u_{j-1}^{n}\right) = 0$$

$$u_j^{n\pm 2} = u_j^n \pm 2\Delta t\frac{\partial u}{\partial t} + \frac{4\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} \pm \frac{8\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} + \frac{16}{24}\Delta t^4\frac{\partial^4 u}{\partial t} + \dots$$

$$u_j^{n\pm 1} = u_j^n \pm \Delta t\frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} \pm \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} + \frac{\Delta t^4}{24}\frac{\partial^4 u}{\partial t} + \dots$$

$$u_{j\pm 1}^{n} = u_j^n \pm \Delta x\frac{\partial u}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} \pm \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3} - \frac{\Delta x^4}{24}\frac{\partial^4 u}{\partial x} + \dots$$

We SAW BEFORE....

$$u_{j+1} - u_{j-1} = 2\Delta x \frac{\partial u}{\partial x} + \frac{2}{6} \Delta x^3 \frac{\partial^3 u}{\partial x^3} + \frac{2}{120} \Delta x^5 \frac{\partial^5 u}{\partial x^5} + \dots$$

$$\Rightarrow \frac{1}{8} u_j^{n+2} - u_j^{n+1} + u_j^{n-1} - \frac{1}{8} u_j^{n-2} = \left(u_j^{n-1} - u_j^{n+1}\right) + \frac{1}{8}\left(u_j^{n+2} - u_j^{n-2}\right)$$

$$\Rightarrow u_j^{n-1} - u_j^{n+1} = -2\left(\Delta t \frac{\partial u}{\partial t} + \frac{1}{6} \Delta t^3 \frac{\partial^3 u}{\partial t^3} + \frac{1}{120} \Delta t^5 \frac{\partial^5 u}{\partial t^5} + \dots\right)$$

$$\Rightarrow u_j^{n+2} - u_j^{n-2} = 2\left(2\Delta t \frac{\partial u}{\partial t} + \frac{8\Delta t^3}{6} \frac{\partial u}{\partial t} + \frac{32}{120} \Delta t^5 \frac{\partial u}{\partial t^5}\right)$$

$$\Rightarrow 2 \cdot \frac{1}{8}\left(\left(2\Delta t \frac{\partial u}{\partial t} + \frac{8\Delta t^3}{6} \frac{\partial^3 u}{\partial t^3} + \frac{32}{120} \Delta t^5 \frac{\partial u}{\partial t^5}\right) - \left(\Delta t \frac{\partial u}{\partial t} + \frac{1}{6} \Delta t^3 \frac{\partial^3 u}{\partial t^3} + \frac{1}{120} \Delta t^5 \frac{\partial^5 u}{\partial t^5} + \dots\right)\right) - \frac{3}{4} \frac{c\Delta t}{\Delta x}\left(\cancel{2}\left(\Delta x \frac{\partial u}{\partial x} + \frac{1}{6} \Delta x^3 \frac{\partial^3 u}{\partial x^3} + \right) \frac{1}{120} \Delta x^5 \frac{\partial^5 u}{\partial x^5}\right) = 0$$

$$\Rightarrow 2 \cdot \left(\left(2\Delta t \frac{\partial u}{\partial t} + \cancel{\frac{\Delta t^3}{6} \frac{\partial^3 u}{\partial t^3}} + \frac{4}{120} \Delta t^5 \frac{\partial u}{\partial t^5}\right) - \left(\Delta t \frac{\partial u}{\partial t} + \cancel{\frac{1}{6} \Delta t^3 \frac{\partial^3 u}{\partial t^3}} + \frac{1}{120} \Delta t^5 \frac{\partial^5 u}{\partial t^5} + \dots\right)\right) - \frac{3}{4} \frac{c\Delta t}{\Delta x}\left(\cancel{2}\left(\Delta x \frac{\partial u}{\partial x} + \frac{1}{6} \Delta x^3 \frac{\partial^3 u}{\partial x^3} + \frac{1}{120} \Delta x^5 \frac{\partial^5 u}{\partial x^5}\right)\right)$$

$$\Rightarrow \Delta t \frac{\partial u}{\partial t} + \frac{1}{40} \Delta t^5 \frac{\partial^5 u}{\partial t^5} + \dots - \frac{c\Delta t}{\cancel{\Delta x}}\left(\cancel{\Delta x} \frac{\partial u}{\partial x} + \frac{1}{6} \Delta x^{\cancel{3}\,2} \frac{\partial^3 u}{\partial x^3} + \frac{1}{120} \Delta x^{\cancel{5}\,4} \frac{\partial^5 u}{\partial x^5}\right) = 0$$

DIVIDE PER $\Delta t$; TAKE NOTE THAT THE TERM $\Delta t^3$ IS GONE

$$\Rightarrow \frac{\partial u}{\partial t} + \frac{1}{40} \Delta t^4 \frac{\partial^5 u}{\partial t^5} - \frac{3}{4} c\left(\frac{\partial u}{\partial x} + \frac{1}{6} \Delta x^2 \frac{\partial^3 u}{\partial x^3} + \frac{1}{120} \Delta x^4 \frac{\partial^5 u}{\partial x}\right) = 0$$

$$\Rightarrow \frac{\partial u}{\partial t} - c \frac{\partial u}{\partial x} = \underbrace{-\frac{1}{40} \Delta t^4 \frac{\partial^5 u}{\partial t} + \frac{3}{4} c\left(\frac{1}{6} \Delta x^2 \frac{\partial^3 u}{\partial x^3} + \frac{1}{120} \Delta x^4 \frac{\partial^5 u}{\partial x}\right)}_{\text{ERROR}}$$

We see that it's $O(\Delta t^4) + O(\Delta x^2)$

CONSISTENCY

$$ERROR = -\frac{1}{40}\Delta t^4 \frac{\partial^5 u}{\partial t} + \frac{3c}{4}\left(\frac{1}{6}\Delta x^2 \frac{\partial^3 u}{\partial x^3} + \frac{1}{120}\Delta x^4 \frac{\partial^5 u}{\partial x}\right)$$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} ERROR = 0$$

THUS, THIS ALGORITHM IS CONSISTENT

# STABILITY

$$\frac{1}{8}U_j^{n+2}-U_j^{n+1}+U_j^{n-1}-\frac{1}{8}U_j^{n-2}-\frac{3}{4}\frac{c\Delta t}{\Delta x}\left(U_{j+1}^{n}-U_{j-1}^{n}\right)=0$$

LINEAR EQUATION SO:

$$\Rightarrow \frac{1}{8}\epsilon_j^{n+2}-\epsilon_j^{n+1}+\epsilon_j^{n-1}-\frac{1}{8}\epsilon_j^{n-2}-\frac{3}{4}\frac{c\Delta t}{\Delta x}\left(\epsilon_{j+1}^{n}-\epsilon_{j-1}^{n}\right)=0$$

AFTER DIVISION BY $\epsilon_j^n = e^{at}e^{ik_m\Delta x}$ ; $\sigma = \frac{3}{4}\frac{c\Delta t}{\Delta x}$

$$\Rightarrow \frac{1}{8}e^{2a\Delta t}-e^{a\Delta t}+e^{-a\Delta t}-\frac{1}{8}e^{-2a\Delta t}-\sigma\left(e^{ik_m\Delta x}-e^{-ik_m\Delta x}\right)=0$$

$$\Rightarrow \frac{1}{8}e^{2a\Delta t}-e^{a\Delta t}+e^{-a\Delta t}-\frac{1}{8}e^{-2a\Delta t}-\sigma\left(2i\sin(k_m\Delta x)\right)=0$$

$$\Rightarrow \frac{1}{8}(e^{a\Delta t})^2-(e^{a\Delta t})+(e^{a\Delta t})^{-1}-\frac{1}{8}(e^{a\Delta t})^{-2}-\sigma(2i\sin k_m\Delta x)=0$$

WE HAVE TO FIND $e^{a\Delta t} = G$

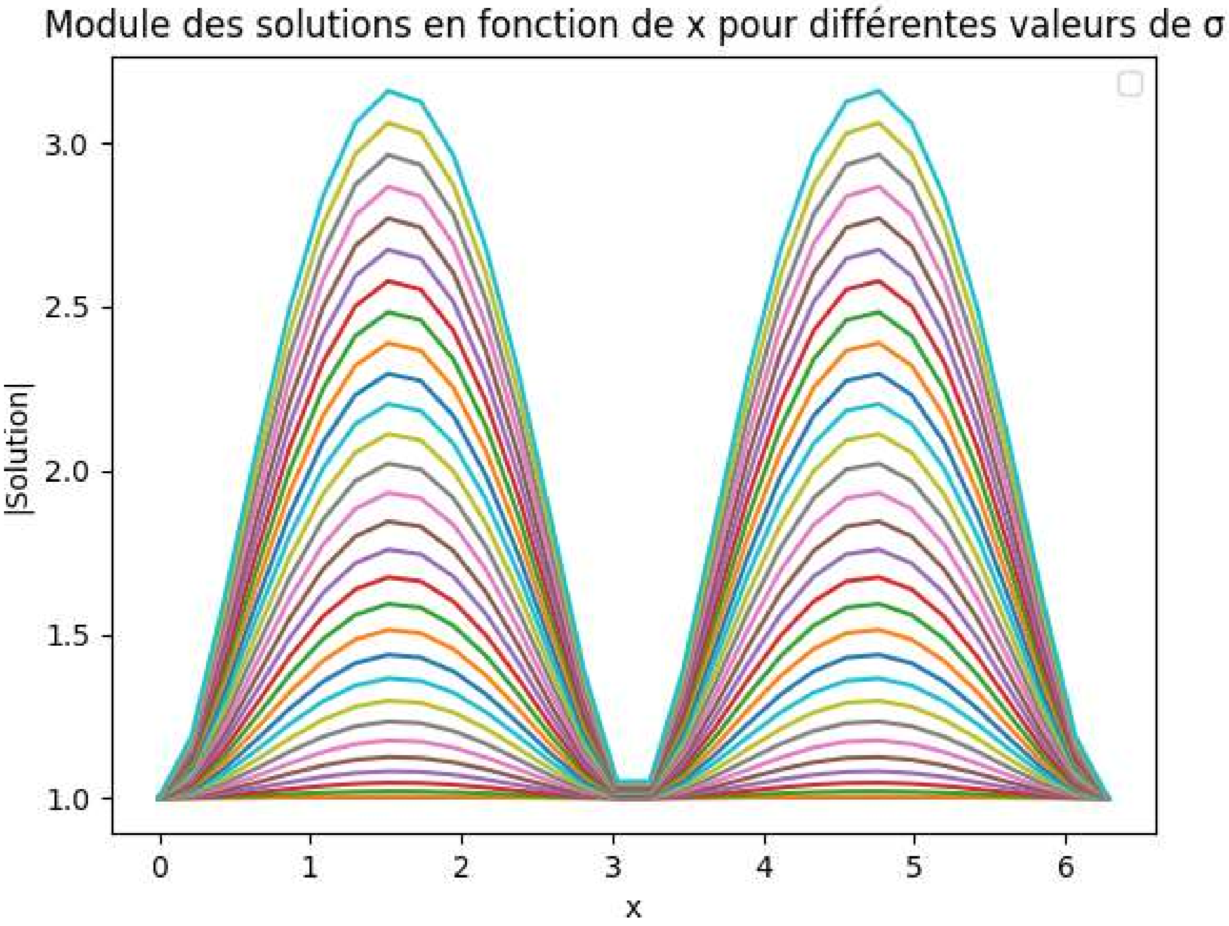
$$\Rightarrow \frac{1}{8}G^2-G+\frac{1}{G}-\frac{1}{8}\frac{1}{G^2}-\sigma\left(2i\sin(k_m\Delta x)\right)=0$$

HARD EQATION BUT... WE SEE THAT WHEN $\sigma=0 \Rightarrow G=1$. THUS IF $\sigma\neq 0$ THE SINUS PART WILL JUST MAKES $|G|$ OSCILATTE AROUND 1 ANS MOST IMPORTANTLY AT THE VALUES $>1$ WHICH WILL MAKE IT UNSTABLE...

WE DID NEWTON-RALPHSON ANALYSIS TO PROVE OUR POINT

WE SEE IN THE PLOT THAT $\sigma\in[0,2]$ WILL ALWAYS MAKE $G>1$ WHICH MAKES IT UNSTABLE

```
def f(x,C):
    if x == 0:
        return np.inf
    else:
        return 1/8 * x**2 - x + x**-1 - 1/8 * x **-2 - C

def fp(x):
    if x == 0:
        return np.inf
    else: return 1/4*x - 1 + -x**-2 - 1/4 * x**-3

def newton_raphson(C,x0, tol = 1e-6, max_iter = 400):
    iteration = 0
    while iteration < max_iter:
        fx = f(x0,C)
        fpx = fp(x0)
        if fpx == 0:
            print("you are cooked bruv")
            return None

        x1 = x0 - fx/fpx
        if all(abs (x1 - x0)) < tol:
            return x1
    x0 = x1
    iteration += 1

    print ("ur cooked")




x = np.linspace(0,2*np.pi,30)
sigma_mat = np.linspace(0,3,30)
for sigma in sigma_mat:
    C = j*sigma*(2*np.sin(x))

    x0 = 1 + 0j
    sol = newton_raphson(C,x0, tol = 1e-6, max_iter = 100)
    print(abs(sol))
    plt.plot(x,abs(sol))

plt.xlabel('x')
plt.ylabel('|Solution|')
plt.title('Module des solutions en fonction de x pour différentes valeurs de σ')
plt.legend()
plt.show()
```


Module des solutions en fonction de x pour différentes valeurs de σ
|Solution|
3.0
2.5
2.0
1.5
1.0
0
1
2
3
4
5
6
x