

LETS CREATE OUR OWN ALGORITHM

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0$$

SECOND ORDER IN SPACE

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{u_{j+1}^n - u_{j-1}^n}{2 \Delta x}$$

FOURTH ORDER IN TIME

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{-u_j^{n+2} + 8u_j^{n+1} - 8u_j^{n-1} + u_j^{n-2}}{12 \Delta t}$$

$$\Rightarrow \frac{-u_j^{n+2} + 8u_j^{n+1} - 8u_j^{n-1} + u_j^{n-2}}{12 \Delta t} + c \left(\frac{u_{j+1}^n - u_{j-1}^n}{2 \Delta x} \right) = 0$$

$$\Rightarrow \frac{-\frac{1}{12}u_j^{n+2} + \frac{2}{3}u_j^{n+1} - \frac{2}{3}u_j^{n-1} + \frac{1}{12}u_j^{n-2}}{\Delta t} + c \left(\frac{u_{j+1}^n - u_{j-1}^n}{2 \Delta x} \right) = 0$$

$$\Rightarrow u_j^{n+1} = \frac{1}{8}u_j^{n+2} + u_j^{n-1} - \frac{1}{8}u_j^{n-2} - 3 \frac{c \Delta t}{4 \Delta x} (u_{j+1}^n - u_{j-1}^n)$$

$$\Rightarrow \frac{1}{8}u_j^{n+2} - u_j^{n+1} + u_j^{n-1} - \frac{1}{8}u_j^{n-2} - 3 \frac{c \Delta t}{4 \Delta x} (u_{j+1}^n - u_{j-1}^n) = 0$$

$$u_j^{n+2} = u_j^n \pm 2 \Delta t \frac{\partial u}{\partial t} + \frac{4 \Delta t^2}{2} \frac{\partial^2 u}{\partial t^2} \pm \frac{8 \Delta t^3}{6} \frac{\partial^3 u}{\partial t^3} + \frac{16}{24} \frac{\Delta t^4 \partial^4 u}{\partial t^4} + \dots$$

$$u_j^{n+1} = u_j^n \pm \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2} \frac{\partial^2 u}{\partial t^2} \pm \frac{\Delta t^3}{6} \frac{\partial^3 u}{\partial t^3} + \frac{\Delta t^4 \partial^4 u}{24 \partial t^4} + \dots$$

$$u_{j+1}^n = u_j^n \pm \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2} \frac{\partial^2 u}{\partial x^2} \pm \frac{\Delta x^3}{6} \frac{\partial^3 u}{\partial x^3} - \frac{\Delta x^4 \partial^4 u}{24 \partial x^4} + \dots$$

We saw before...

$$v_{j+1} - v_{j-1} = 2\Delta x \frac{\partial v}{\partial x} + \frac{1}{6} \Delta x^2 \frac{\partial^2 v}{\partial x^2} + \frac{1}{120} \Delta x^4 \frac{\partial^4 v}{\partial x^4} + \dots$$

$$\Rightarrow \frac{1}{8} (v_j^{n+2} - v_0^{n+1} + v_j^{n-1} - \frac{1}{8} v_0^{n-2}) = (v_j^{n-1} - v_j^{n+1}) + \frac{1}{8} (v_j^{n+2} - v_j^{n-2})$$

$$\Rightarrow v_j^{n-1} - v_j^{n+1} = -2 \left(\Delta t \frac{\partial v}{\partial t} + \frac{1}{6} \Delta t^2 \frac{\partial^2 v}{\partial t^2} + \frac{1}{120} \Delta t^4 \frac{\partial^4 v}{\partial t^4} + \dots \right)$$

$$\Rightarrow v_j^{n+2} - v_j^{n-2} = 2 \left(2\Delta t \frac{\partial v}{\partial t} + \frac{1}{6} \Delta t^3 \frac{\partial^2 v}{\partial t^2} + \frac{32}{120} \Delta t^5 \frac{\partial^4 v}{\partial t^4} \right)$$

$$\Rightarrow 2 \cdot \frac{1}{8} \left(\left(2\Delta t \frac{\partial v}{\partial t} + \frac{1}{6} \Delta t^3 \frac{\partial^2 v}{\partial t^2} + \frac{32}{120} \Delta t^5 \frac{\partial^4 v}{\partial t^4} \right) - \left(\Delta t \frac{\partial v}{\partial t} + \frac{1}{6} \Delta t^2 \frac{\partial^2 v}{\partial t^2} + \frac{1}{120} \Delta t^4 \frac{\partial^4 v}{\partial t^4} + \dots \right) \right) - \frac{3}{4} \cancel{c} \left(2 \left(\Delta x \frac{\partial v}{\partial x} + \frac{1}{6} \Delta x^2 \frac{\partial^2 v}{\partial x^2} + \frac{1}{120} \Delta x^4 \frac{\partial^4 v}{\partial x^4} \right) \right)$$

$$\Rightarrow 2 \cdot \left(\left(2\Delta t \frac{\partial v}{\partial t} + \frac{1}{6} \Delta t^3 \frac{\partial^2 v}{\partial t^2} + \frac{4}{120} \Delta t^5 \frac{\partial^4 v}{\partial t^4} \right) - \left(\Delta t \frac{\partial v}{\partial t} + \frac{1}{6} \Delta t^2 \frac{\partial^2 v}{\partial t^2} + \frac{1}{120} \Delta t^4 \frac{\partial^4 v}{\partial t^4} + \dots \right) \right) - \frac{3}{4} \cancel{c} \left(2 \left(\Delta x \frac{\partial v}{\partial x} + \frac{1}{6} \Delta x^2 \frac{\partial^2 v}{\partial x^2} + \frac{1}{120} \Delta x^4 \frac{\partial^4 v}{\partial x^4} \right) \right)$$

DIVIDE PER Δt ; TAKE NOTE THAT THE TERM Δt^2 IS GONE

$$\Rightarrow \frac{\partial v}{\partial t} + \frac{1}{40} \Delta t^4 \frac{\partial^5 v}{\partial t^5} - \frac{3}{4} c \left(\frac{\partial v}{\partial x} + \frac{1}{6} \Delta x^2 \frac{\partial^2 v}{\partial x^2} + \frac{1}{120} \Delta x^4 \frac{\partial^4 v}{\partial x^4} \right) = 0$$

$$\Rightarrow \frac{\partial v}{\partial t} - c \frac{\partial v}{\partial x} = -\frac{1}{40} \cancel{\Delta t^4} \frac{\partial^5 v}{\partial t^5} + \frac{3}{4} c \left(\frac{1}{6} \cancel{\Delta x^2} \frac{\partial^2 v}{\partial x^2} + \frac{1}{120} \Delta x^4 \frac{\partial^4 v}{\partial x^4} \right)$$

ERROR

We see that it's $O(\Delta t^4) + O(\Delta x^2)$

CONSISTENCY

$$\epsilon_{ROR} = -\frac{1}{4} \gamma_0 \Delta t^4 \frac{\partial^5 u}{\partial t^5} + \frac{3\zeta}{4} \left(\frac{1}{6} \Delta x^2 \frac{\partial^3 u}{\partial x^3} + \frac{1}{120} \Delta x^4 \frac{\partial^5 u}{\partial x^5} \right)$$

$$\lim_{\Delta x \rightarrow 0, \Delta t \rightarrow 0} \epsilon_{ROR} = 0$$

THUS, THIS ALGORITHM IS CONSISTENT

STABILITY

$$\frac{1}{8} v_j^{n+2} - v_j^{n+1} + v_j^{n-1} - \frac{1}{8} v_j^{n-2} - \frac{3\gamma c \Delta t}{\Delta x} (v_{j+1}^n - v_{j-1}^n) = 0$$

LINEAR EQUATION SO:

$$\Rightarrow \frac{1}{8} e_j^{n+2} - e_j^{n+1} + e_j^{n-1} - \frac{1}{8} e_j^{n-2} - \frac{3\gamma c \Delta t}{\Delta x} (e_{j+1}^n - e_{j-1}^n) = 0$$

AFTER DIVISION BY $e_j^n = e^{at} e^{ik_n \Delta x}$; $\sigma = \frac{3\gamma c \Delta t}{\Delta x}$

$$\Rightarrow \frac{1}{8} e^{2at} - e^{at} + e^{-at} - \frac{1}{8} e^{-2at} - \sigma (e^{ik_n \Delta x} - e^{-ik_n \Delta x}) = 0$$

$$\Rightarrow \frac{1}{8} e^{2at} - e^{at} + e^{-at} - \frac{1}{8} e^{-2at} - \sigma (2 i \sin(k_n \Delta x)) = 0$$

$$\Rightarrow \frac{1}{8} (e^{at})^2 - (e^{at}) + (e^{-at}) - \frac{1}{8} (e^{-at})^2 - \sigma (2 i \sin(k_n \Delta x)) = 0$$

We HAVE TO FIND $e^{at} = G$

$$\Rightarrow \frac{1}{8} G^2 - G + \frac{1}{8} G - \frac{1}{8} G^2 - \sigma (2 i \sin(k_n \Delta x)) = 0$$

HARD EQUATION BUT... WE SEE THAT WHEN $G = 0 \Rightarrow G = 1$. THUS IF $\sigma \neq 0$ THE SINUS

PART WILL JUST MAKES $|G|$ OSCILLATE AROUND 1 AND MOST IMPORTANTLY AT THE VALUES WHICH WILL MAKE IT UNSTABLE...

WHICH WILL MAKE IT UNSTABLE $\sigma > 1$

WE DID NEWTON-RAPHSON ANALYSIS

TO PROVE OUR POINT

WE SEE IN THE PLOT THAT $\sigma \in [0, 2]$ WILL ALWAYS MAKE $G > 1$ WHICH MAKES IT UNSTABLE

```

def f(x,C):
    if x == 0:
        return np.inf
    else:
        return 1/8 * x**2 - x + x**-1 - 1/8 * x **-2 - C

def fp(x):
    if x == 0:
        return np.inf
    else: return 1/4*x - 1 + -x**-2 - 1/4 * x**-3

def newton_raphson(C,x0, tol = 1e-6, max_iter = 400):
    iteration = 0
    while iteration < max_iter:
        fx = f(x0,C)
        fpx = fp(x0)
        if fpx == 0:
            print("you are cooked bruv")
            return None

        x1 = x0 - fx/fpx
        if all(abs(x1 - x0)) < tol:
            return x1
        x0 = x1
        iteration += 1

    print ("ur cooked")

x = np.linspace(0,2*np.pi,30)
sigma_mat = np.linspace(0,3,30)
for sigma in sigma_mat:
    C = j*sigma*(2*np.sin(x))

    x0 = 1 + 0j
    sol = newton_raphson(C,x0, tol = 1e-6, max_iter = 100)
    print(abs(sol))
    plt.plot(x,abs(sol))

plt.xlabel('x')
plt.ylabel('|Solution|')
plt.title('Module des solutions en fonction de x pour différentes valeurs de σ')
plt.legend()
plt.show()

```

Module des solutions en fonction de x pour différentes valeurs de σ

