

MEC6602E: Transonic Aerodynamics

HOMEWORK 1

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Due date :
24 september 2024



1 Question 1: Analytical Calculation of Order of Convergence, Stability and Consistency

This section will analyse some finite difference algorithms of the following equation:

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0 \quad (1)$$

This will be done by 3 ways:

- Analysing convergence with Taylor
- Stability with Von Neumann
- Consistency with the limit of the error term $\Delta x, \Delta t \rightarrow 0$

a1. Lets analyse the backward algorithm:

$$u_j^{n+1} = u_j^n - \frac{c\Delta t}{\Delta x} (u_j^n - u_{j-1}^n) \quad (2)$$

This Algorithm is:

- Stable
- Convergent on order 1 in both time and space
- Consistent

a2. Lets analyse the forward algorithm:

$$u_j^{n+1} = u_j^n - \frac{c\Delta t}{\Delta x} (u_{j+1}^n - u_j^n) \quad (3)$$

This Algorithm is:

- Unstable
- Convergent on order 1 in both time and space
- Consistent

b. Lets analyse the centered algorithm:

$$u_j^{n+1} = u_j^n - \frac{c\Delta t}{\Delta x} (u_{j+1}^n - u_{j-1}^n) \quad (4)$$

This Algorithm is:

- UnStable
- Convergent on ordre 2 in space and ordre 1 in time
- Consistent

c. Lets analyse the leap-frog algorithm:

$$u_j^{n+1} = u_j^n - \frac{c\Delta t}{\Delta x} (u_{j+1}^n - u_{j-1}^n) \quad (5)$$

This Algorithm is:

- Metastable $|G| = 1$
- Convergent on ordre 2 in space and ordre 2 in time
- Consistent

d. Lets analyse the Lax-Wendroff algorithm:

$$u_j^{n+1} = u_j^n - \frac{c\Delta t}{2\Delta x} (u_{j+1}^n - u_{j-1}^n) + \frac{1}{2} \left(\frac{c\Delta t}{\Delta x} \right)^2 (u_{j+1}^n - 2u_j^n + u_{j-1}^n) \quad (6)$$

This Algorithm is:

- Stable $CFL < 1$
- Convergent on ordre 2 in space and ordre 1 in time
- Consistent

e. Lets analyse the Lax algorithm:

$$u_j^{n+1} = u_j^n - \frac{c\Delta t}{2\Delta x} (u_{j+1}^n - u_{j-1}^n) + \frac{1}{2} \left(\frac{c\Delta t}{\Delta x} \right)^2 (u_{j+1}^n - 2u_j^n + u_{j-1}^n) \quad (7)$$

This Algorithm is:

- Stable $CFL < 1$
- Bad convergence and not consistent

f. Lets analyse the Hybrid algorithm:

For $\theta = 0$ This Algorithm is:

- the same as the b, please refer to it

For $\theta = 1$ This Algorithm is:

- Always Stable
- first ordre in time, 2nd order in space
- consistent

For $\theta = 0.5$ This Algorithm is:

- Metastable $|G| = 1$
- first ordre in time, 2nd order in space
- consistent

g. Lets Analyse our chosen algorithm:

Here are the 2th order in space discretisation and the 4th order in time chosen:

$$\frac{\partial u}{\partial x} \approx \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \quad (8)$$

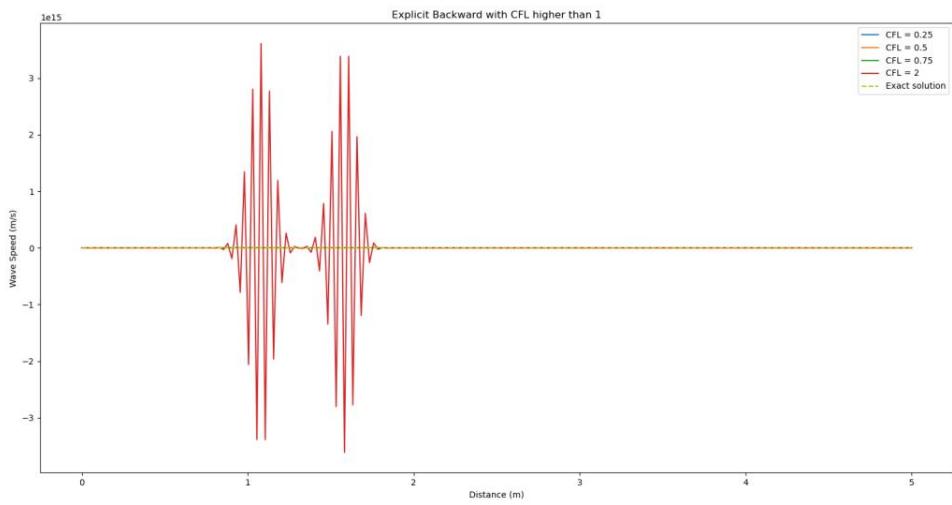
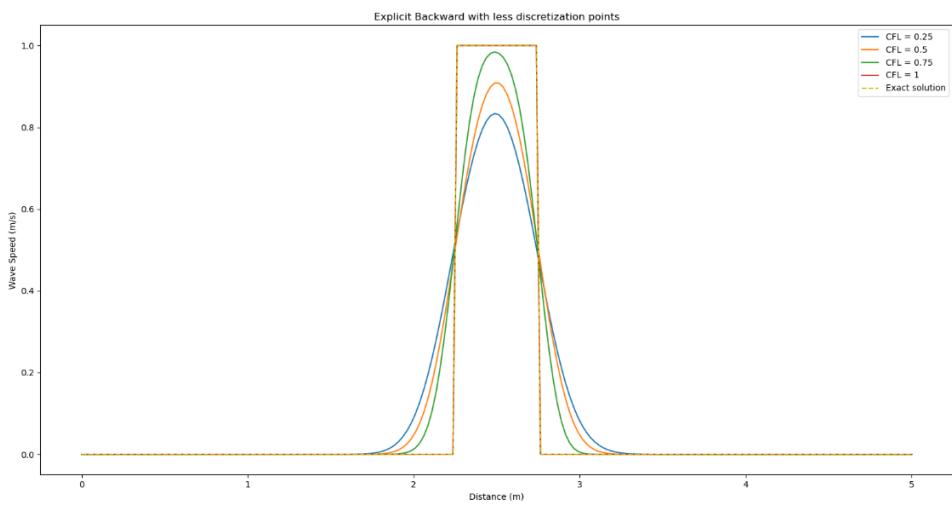
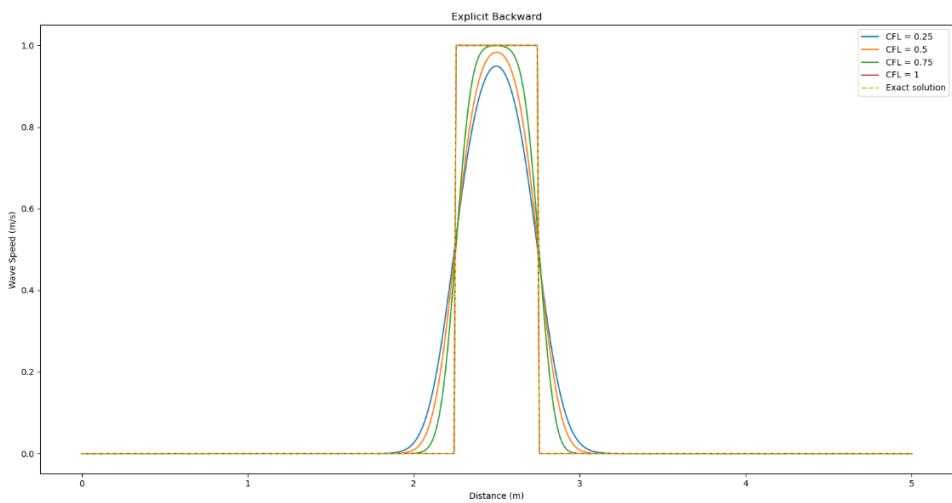
$$\frac{\partial u}{\partial t} \approx \frac{-u_j^{n+2} + 8u_j^{n+1} - 8u_j^{n-1} + u_j^{n-2}}{12\Delta t} \quad (9)$$

$$u_j^{n+1} = \frac{1}{8}u_j^{n+2} + u_j^{n-1} - \frac{1}{8}u_j^{n-2} - \frac{3c\Delta t}{4\Delta x} (u_{j+1}^n - u_{j-1}^n) \quad (10)$$

- Unstable
- 4nd order in time, 2nd order in space
- Consistent

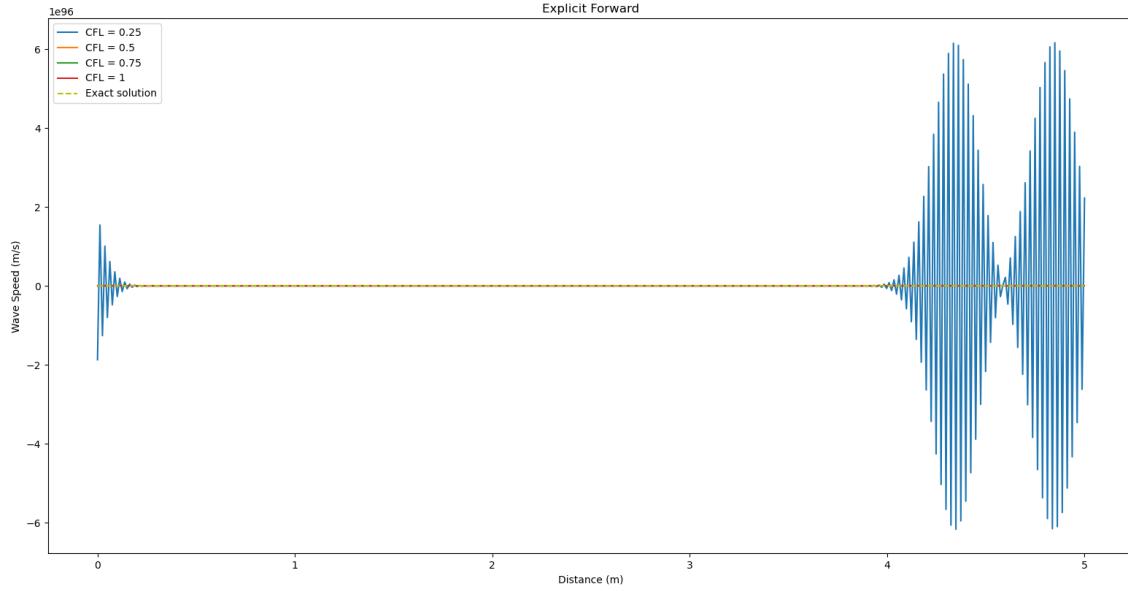
Of course, every calculation will be found on the annexe 1

2 NUMERICAL ANALYSIS

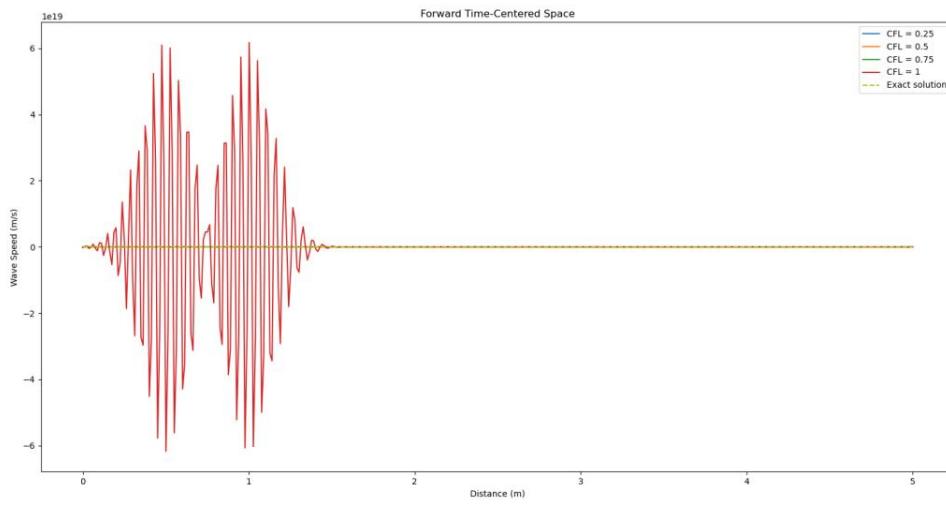


With the Explicit Backward scheme, we have some diffusion when the CFL number tends to 0. However, when the CFL number is equal to 1, the solution is near an exact solution, without any diffusion, dispersion, or loss. We will take this result as our “Exact solution” in the following figures.

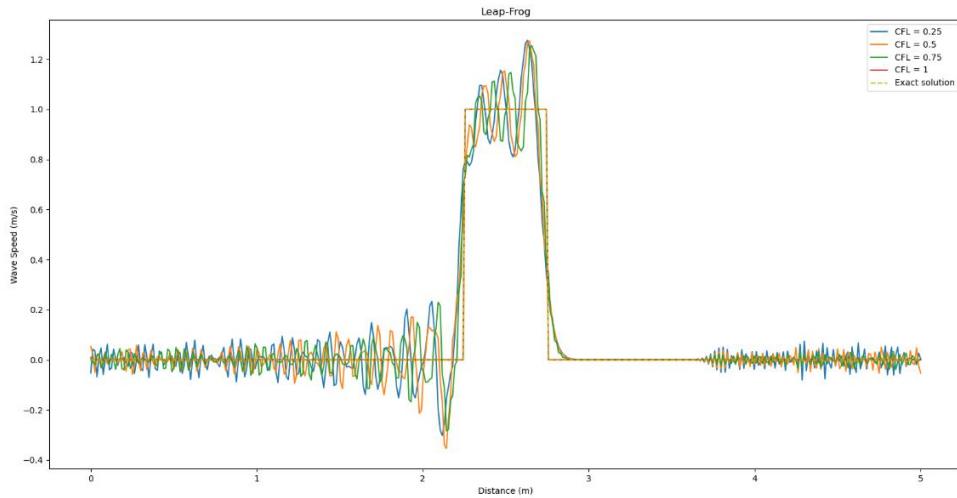
The accuracy increases with the number of discretization and the scheme is stable for CFL less than 1.



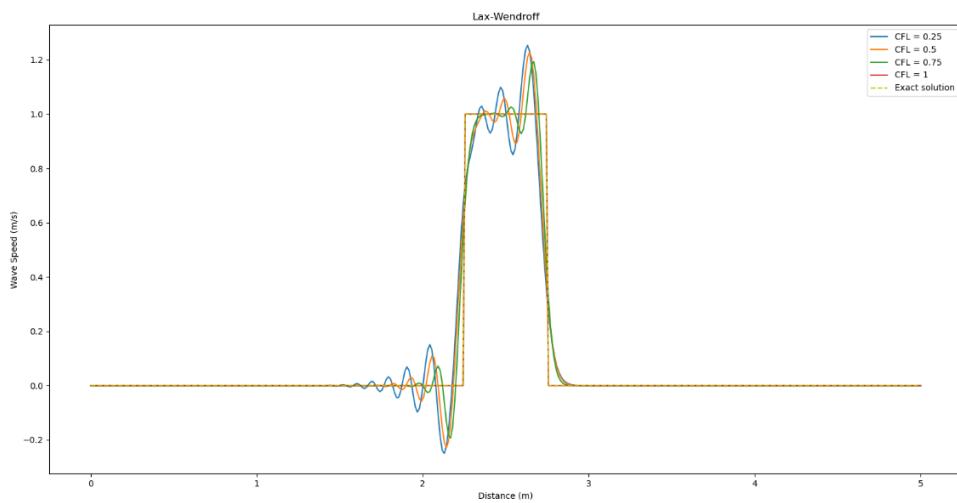
The Explicit Forward scheme is always unstable, so we cannot make any comment on the accuracy with a numerical analysis.



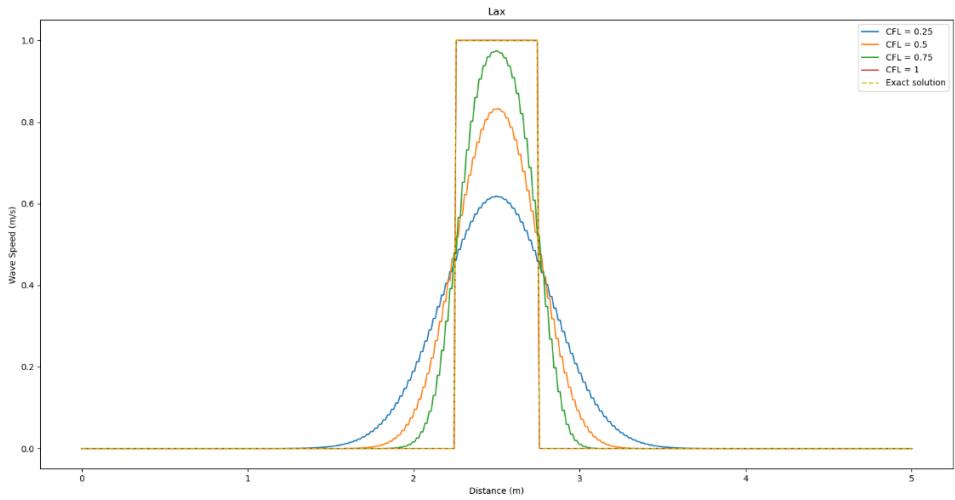
As for the Explicit Forward scheme, the Forward Time-Centered Space is always unstable and we cannot make any comment on the accuracy with a numerical analysis.



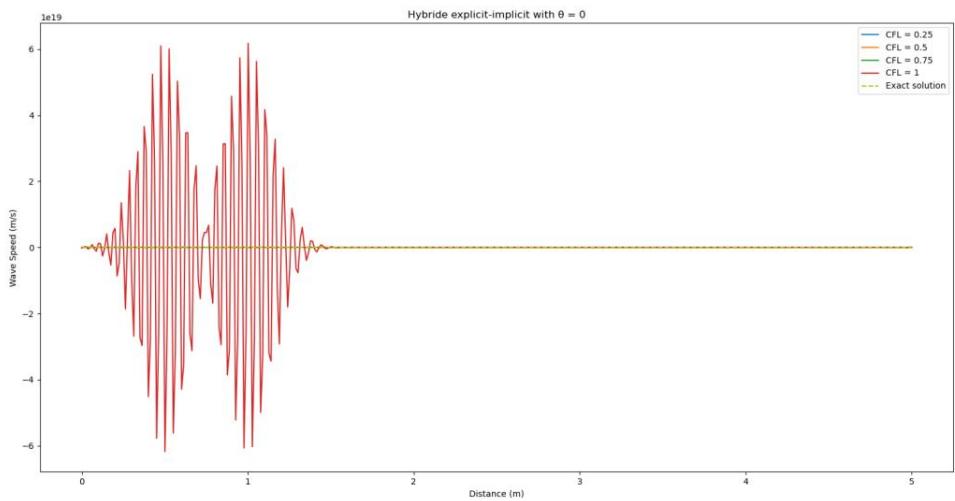
The Leap-Frog scheme has a lot of dispersion in its numerical solutions, resulting in a lot of oscillations. But with CFL equal one, we get the “Exact solution”. This scheme is stable for CFL less than 1.



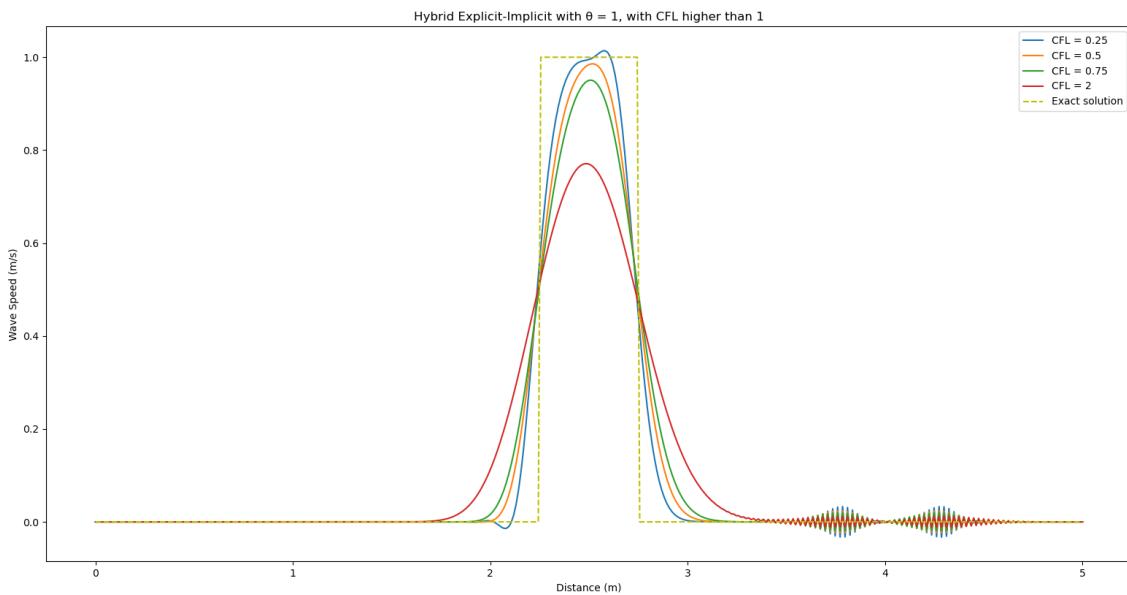
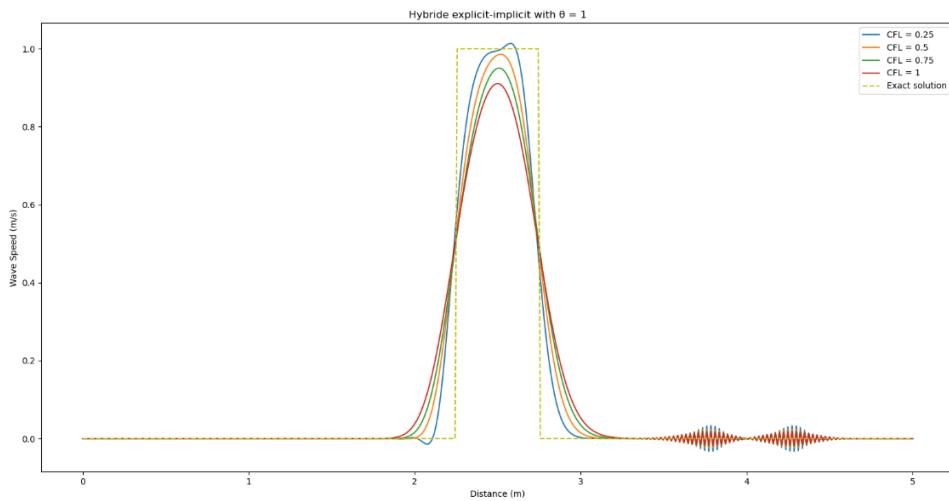
Like the previous scheme, Lax-Wendroff triggers some oscillations (dispersion), but they are more concentrated on the edges of the wave, resulting in a solution more accurate. It is also unstable for CFL higher than 1.



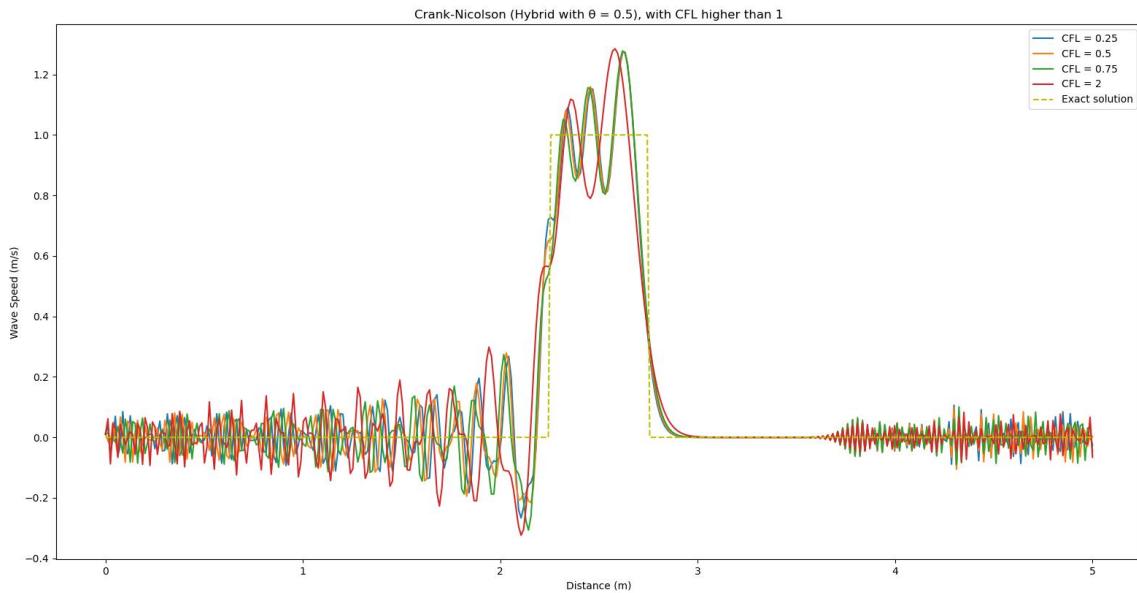
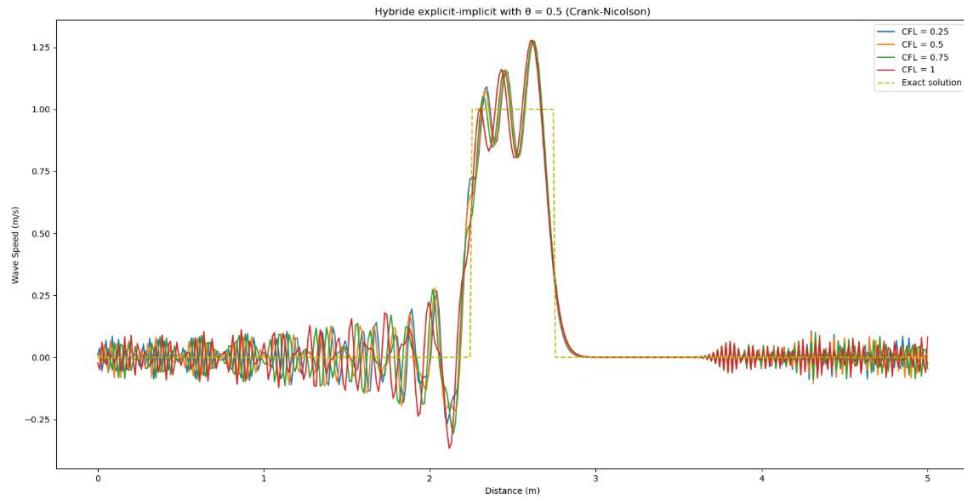
The Lax scheme is closer to the first scheme studied, the Explicit Backward. Indeed, there are some diffusions in the results, but they are larger than the Explicit Backward scheme. It is also stable for CFL less than 1.



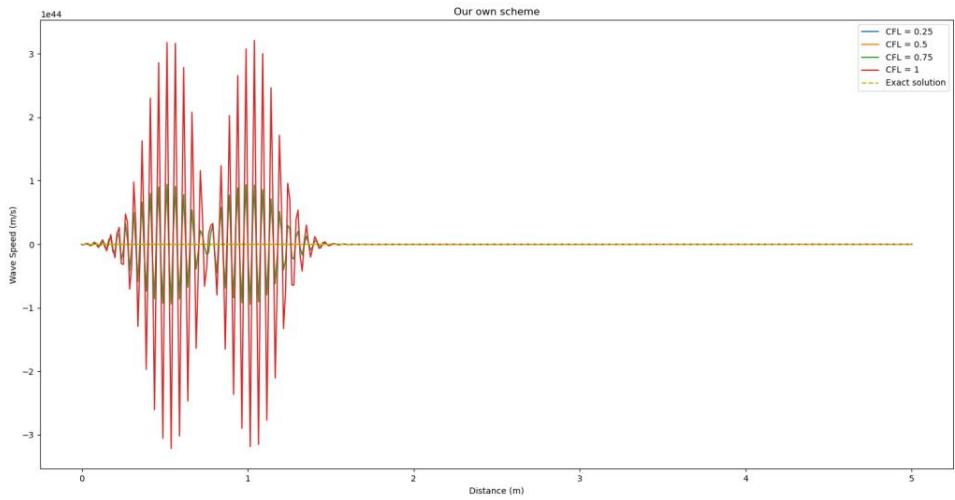
The Hybrid Explicit-Implicit with θ equal to 0 scheme is always unstable, as it is the same scheme as the Explicit Forward one.



The Hybrid Explicit-Implicit with θ equal to 1 scheme is always stable. This scheme has some diffusion and some dispersion (after the wave we can see some oscillations). The accuracy is better near CFL equal 1.



Crank-Nicolson is stable for all CFL. Is it not what we have found in our derivation (we found a growth-rate always equal to 1). There are a lot of dispersion in the results and changing the CFL number seems to have no effect on the results.



The scheme we studied (4^{th} order in time and 2^{nd} order in space) is unstable for all CFL number. Therefore we cannot analyse the accuracy of the scheme with numerical analysis.

3 ANNEXE 1

QUESTION 7)

$$a) v_j^{n+1} = v_j^n - c \frac{\Delta t}{\Delta x} (v_j^n - v_{j-1}^n)$$

STABILITY

$$T_k = \bar{T}_k + \epsilon_k$$

$$\Rightarrow v_j^{n+1} = v_j^n - c \frac{\Delta t}{\Delta x} (v_j^n - v_{j-1}^n)$$

$$\Rightarrow \bar{v}_j^{n+1} + \epsilon_j^{n+1} = \bar{v}_j^n + \epsilon_j^n - c \frac{\Delta t}{\Delta x} (\bar{v}_j^n + \epsilon_j^n - \bar{v}_{j-1}^n - \epsilon_{j-1}^n)$$

$$\Rightarrow \epsilon_j^{n+1} = \epsilon_j^n - c \frac{\Delta t}{\Delta x} (\epsilon_j^n - \epsilon_{j-1}^n)$$

$$\Rightarrow \epsilon_j^{n+1} = \epsilon_j^n - c \frac{\Delta t}{\Delta x} \epsilon_j^n + \frac{c \Delta t}{\Delta x} \epsilon_{j-1}^n$$

$$\Rightarrow \frac{\epsilon_j^{n+1}}{\epsilon_j^n} = 1 - \frac{c \Delta t}{\Delta x} + \frac{c \Delta t}{\Delta x} \frac{\epsilon_{j-1}^n}{\epsilon_j^n}$$

$$\epsilon = e^{at} e^{ik_n x} \Rightarrow \epsilon_j^n = e^{at} e^{ik_n x}; \epsilon_{j-1}^n = e^{at} e^{ik_n (x-\Delta x)}$$

$$\Rightarrow \frac{\epsilon_j^{n+1}}{\epsilon_j^n} = 1 - \frac{c \Delta t}{\Delta x} + \frac{c \Delta t}{\Delta x} \left(\frac{e^{at} e^{ik_n (x-\Delta x)}}{e^{at} e^{ik_n x}} \right)$$

$$\Rightarrow \frac{\epsilon_j^{n+1}}{\epsilon_j^n} = 1 - \frac{c \Delta t}{\Delta x} + \frac{c \Delta t}{\Delta x} (e^{-ik_n \Delta x}) = 1 - \frac{c \Delta t}{\Delta x} (1 - e^{-ik_n \Delta x}) = G$$

$$e^{ik_n \Delta x} = \cos(k_n \Delta x) - i \sin(k_n \Delta x); \frac{c \Delta t}{\Delta x} = \sigma$$

$$\Rightarrow |G| = 1 - \sigma (1 - \cos(k_n \Delta x) + i \sin(k_n \Delta x))$$

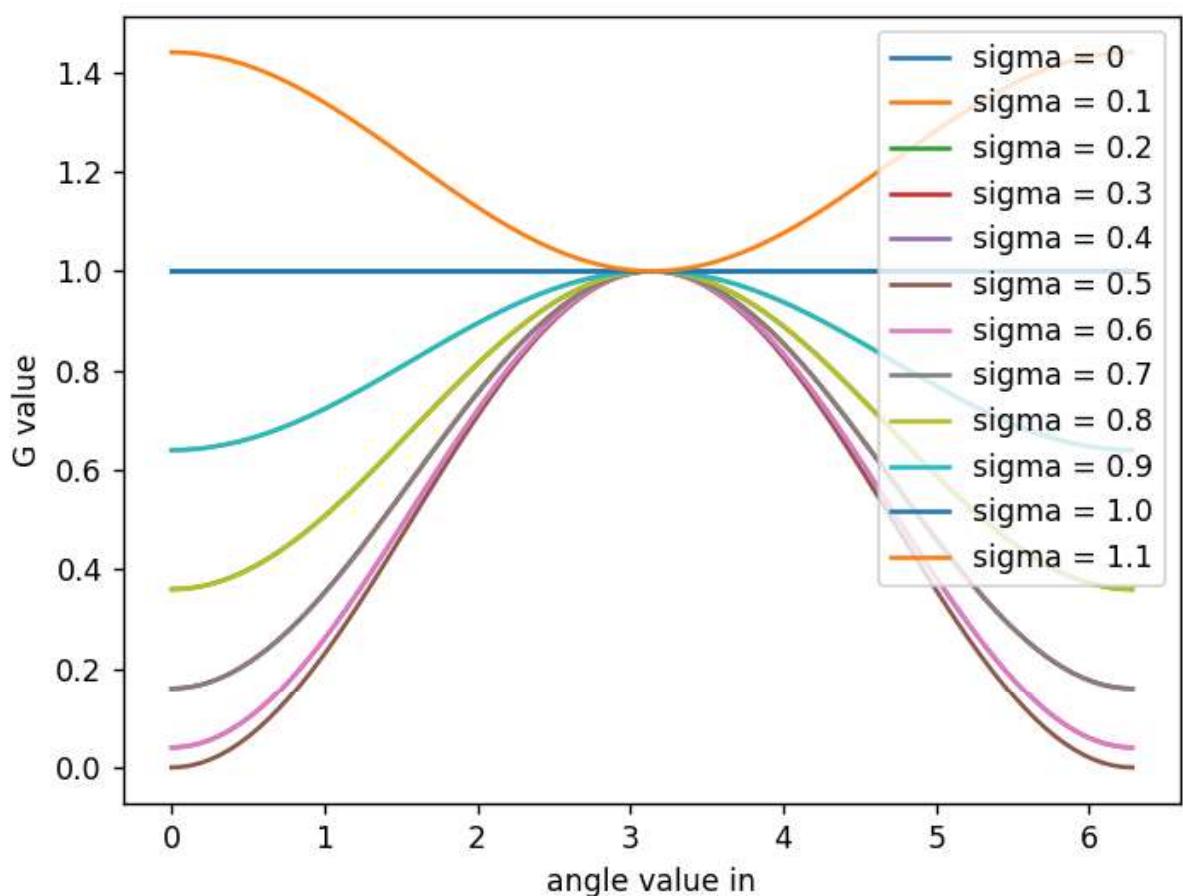
$$\Rightarrow |G|^2 = (1 - \sigma - \sigma \cos(k_n \Delta x))^2 + (i \sigma \sin(k_n \Delta x))^2$$

$$= 1 - \underline{\sigma} - \underline{\sigma \cos(k_n \Delta x)} - \underline{\sigma} + \underline{\sigma^2} + \underline{\sigma^2 \cos^2(k_n \Delta x)} - \underline{\sigma \cos(k_n \Delta x)} + \underline{\sigma^2 \cos^2(k_n \Delta x)} + \underline{\sigma^2 \sin^2(k_n \Delta x)} + \underline{\sigma^2 \sin^2(k_n \Delta x)}$$

$$\Rightarrow 1 - 2\sigma - 2\sigma \cos(k_n \Delta x) + \sigma^2 + 2\sigma^2 \cos^2(k_n \Delta x) + \sigma^2 \cos^2(k_n \Delta x) + \sigma^2 \sin^2(k_n \Delta x)$$

$$= (-2\omega - 2\omega \cos(k_m \Delta x) + 2\omega^2 + 2\omega^2 \cos(k_m \Delta x)) = |\omega|^2 < 1$$

We will show with a plot that $\sigma \leq 1$ to be stable when $k_m \Delta x \in [0, 2\pi]$:



ACCURACY

We have to develop into Taylor

$$U_j^{n+1} = U_j^n - \frac{c \Delta t}{\Delta x} (U_j^n - U_{j-1}^n) \quad (1)$$

$$U_j^n = U_j^M \quad (2)$$

$$U_j^{n+1} = U_j^n + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2} \frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6} \frac{\partial^3 u}{\partial t^3} + \dots = \sum_{k=0}^{\infty} \frac{\Delta t^k}{k!} \frac{\partial^k u}{\partial t^k} \quad (3)$$

$$U_{j-1}^n = U_j^n - \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2} \frac{\partial^2 u}{\partial x^2} - \frac{\Delta x^3}{6} \frac{\partial^3 u}{\partial x^3} + \dots = \sum_{k=0}^{\infty} \frac{(-\Delta x)^k}{k!} \frac{\partial^k u}{\partial x^k} \quad (4)$$

We put (2), (3) and (4) into (1)

$$\Rightarrow U_j^n + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2} \frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6} \frac{\partial^3 u}{\partial t^3} = U_j^M - \frac{c \Delta t}{\Delta x} \left(U_j^n - U_{j-1}^M + \Delta x \frac{\partial u}{\partial x} - \frac{\Delta x^2}{2} \frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^3}{6} \frac{\partial^3 u}{\partial x^3} \right)$$

We put (2), (3) and (4) into (1)

$$\Rightarrow \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2} \frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6} \frac{\partial^3 u}{\partial t^3} = -c \Delta t \left(\Delta x \frac{\partial u}{\partial x} - \frac{\Delta x^2}{2} \frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^3}{6} \frac{\partial^3 u}{\partial x^3} \right)$$

$$\Rightarrow \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2} \frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6} \frac{\partial^3 u}{\partial t^3} = -c \Delta t \left(\frac{\partial u}{\partial x} - \frac{\Delta x}{2} \frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^2}{6} \frac{\partial^3 u}{\partial x^3} \right)$$

$$\Rightarrow \frac{\partial u}{\partial t} = -c \Delta t \left(\frac{\partial u}{\partial x} - \frac{\Delta x}{2} \frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^2}{6} \frac{\partial^3 u}{\partial x^3} \right) - \frac{\Delta t^2}{2} \frac{\partial^2 u}{\partial t^2} - \frac{\Delta t^3}{6} \frac{\partial^3 u}{\partial t^3}$$

Δt

$$\Rightarrow \frac{\partial u}{\partial t} - c \frac{\partial u}{\partial x} = \frac{c \Delta x}{2} \frac{\partial^2 u}{\partial x^2} + \frac{c \Delta x^2}{6} \frac{\partial^3 u}{\partial x^3} - \frac{\Delta t}{2} \frac{\partial^2 u}{\partial t^2} - \frac{\Delta t^2}{6} \frac{\partial^3 u}{\partial t^3}$$

The order of convergence for both time and space are 1

$$\mathcal{O}(\Delta t) + \mathcal{O}(\Delta x)$$

CONSISTENCY

The error term is: $\frac{c \Delta x}{2} \frac{\partial^2 v}{\partial x^2} + \frac{c \Delta x^2}{6} \frac{\partial^3 v}{\partial x^3} - \frac{\Delta t}{2} \frac{\partial^2 v}{\partial x^2} - \frac{\Delta t^2}{6} \frac{\partial^3 v}{\partial t^2}$

$$\lim_{\Delta x \rightarrow 0} \frac{c \Delta x}{2} \frac{\partial^2 v}{\partial x^2} + \frac{c \Delta x^2}{6} \frac{\partial^3 v}{\partial x^3} - \frac{\Delta t}{2} \frac{\partial^2 v}{\partial x^2} - \frac{\Delta t^2}{6} \frac{\partial^3 v}{\partial t^2} = 0$$

Thus, the backward scheme is consistent

QUESTION 1 c)

$$\text{STABILITY: } v_j^{n+1} = v_j^{n-1} - c \frac{\Delta t}{\Delta x} (v_{j+1}^n - v_{j-1}^n)$$

$$v_j^n = \bar{v}_j^n + \epsilon_j^n$$

$$\Rightarrow \bar{v}_j^{n+1} + \epsilon_j^{n+1} = \bar{v}_j^{n-1} + \epsilon_j^{n-1} - c \frac{\Delta t}{\Delta x} \left(\bar{v}_{j+1}^n + \epsilon_{j+1}^n - \bar{v}_{j-1}^n + \epsilon_{j-1}^n \right)$$

$$\Rightarrow \epsilon_j^{n+1} = \epsilon_j^{n-1} - c \frac{\Delta t}{\Delta x} \left(\epsilon_{j+1}^n - \epsilon_{j-1}^n \right)$$

$$\Rightarrow G = \frac{\epsilon_j^{n+1}}{\epsilon_j^n} = \frac{\epsilon_j^{n-1}}{\epsilon_j^n} - c \frac{\Delta t}{\Delta x} \left(\frac{\epsilon_{j+1}^n}{\epsilon_j^n} - \frac{\epsilon_{j-1}^n}{\epsilon_j^n} \right)$$

$$\epsilon_j^n = e^{at} e^{ik_n \Delta x}$$

$$\Rightarrow G = \frac{e^{at - \Delta t} e^{ik_n \Delta x}}{e^{at} e^{ik_n \Delta x}} - \sigma \left(\frac{e^{at} e^{ik_n(x+\Delta x)}}{e^{at} e^{ik_n x}} - \frac{e^{at} e^{ik_n(x-\Delta x)}}{e^{at} e^{ik_n x}} \right)$$

$$e^{-a \Delta t} - \sigma \left(e^{ik_n \Delta x} - e^{-ik_n \Delta x} \right)$$

$$\Rightarrow G = \frac{1}{G} - \sigma (2i \sin(k_n \Delta x))$$

We Multiply by G on each side

$$\Rightarrow G^2 = 1 - \sigma G (2i \sin(k_n \Delta x))$$

$$\Rightarrow G^2 + \sigma G (2i \sin(k_n \Delta x)) - 1$$

$$G_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$b = 2i \sigma \sin(kn\Delta x) \Rightarrow b^2 = -4 \sigma^2 \sin^2(kn\Delta x)$$

$$\Rightarrow b^2 - 4ac = -4 \sigma^2 \sin^2(kn\Delta x) + 4$$

$$\Rightarrow G = \frac{-i2\sigma \sin(kn\Delta x) \pm \sqrt{4(1 - \sigma^2 \sin^2(kn\Delta x))}}{2}$$

$$\Rightarrow |G| = \sqrt{|-i\sigma \sin(kn\Delta x) \pm \sqrt{1 - \sigma^2 \sin^2(kn\Delta x)}}| \leq 1$$

$$\Rightarrow |G|^2 = \sigma^2 \sin^2(kn\Delta x) + 1 - \sigma^2 \sin^2(kn\Delta x) = 1$$

$$\Rightarrow |G|^2 = 1 = |G| \quad \text{THE CONDITION } |G| \leq 1 \text{ IS RESPECTED
HOWEVER}$$

We NEED TO VERIFY THE $\Delta = 1 - \sigma^2 \sin^2(kn\Delta x) > 0$

$$\Rightarrow \sigma^2 \sin^2(kn\Delta x) \leq 1$$

$$\Rightarrow \sigma^2 \leq 1 \quad \text{WORST CASE } \sin^2(kn\Delta x) = 1$$

$$\Rightarrow G = \frac{c\Delta t}{\Delta x} \leq 1$$

If $\Delta = b^2 - 4ac \leq 0$ THERE WOULD BE

An imaginary part that would add to $-i\sigma \sin(kn\Delta x)$

ACCURACY

LET'S DEVELOP IN TAYLOR

$$v_j^{n+1} = v_j^n - \frac{c \Delta t}{\Delta x} (v_{j+1}^n - v_{j-1}^n) \Rightarrow v_j^{n+1} - v_j^n = \frac{c \Delta t}{\Delta x} (v_{j+1}^n - v_{j-1}^n) \quad (1)$$

$$v_j^{n+1} = v_j^n + \Delta t \frac{\partial v}{\partial t} + \Delta t^2 \frac{\partial^2 v}{\partial t^2} + \frac{\Delta t^3}{6} \frac{\partial^3 v}{\partial t^3} + \dots$$

$$v_j^n = v_j^n - \Delta t \frac{\partial v}{\partial t} + \frac{\Delta t^2}{2} \frac{\partial^2 v}{\partial t^2} - \frac{\Delta t^3}{6} \frac{\partial^3 v}{\partial t^3} + \dots$$

$$v_j^{n+1} - v_j^n = 2 \Delta t \frac{\partial v}{\partial t} + 2 \frac{\Delta t^3}{6} \frac{\partial^3 v}{\partial t^3} + 2 \frac{\Delta t^5}{120} \frac{\partial^5 v}{\partial t^5} + \dots \quad (2)$$

$$v_{j+1}^n - v_{j-1}^n = 2 \Delta x \frac{\partial v}{\partial x} + 2 \frac{\Delta x^3}{6} \frac{\partial^3 v}{\partial x^3} + 2 \frac{\Delta x^5}{120} \frac{\partial^5 v}{\partial x^5} + \dots \quad (3)$$

(2) into (1)

$$\Rightarrow 2 \Delta t \frac{\partial v}{\partial t} + 2 \frac{\Delta t^3}{6} \frac{\partial^3 v}{\partial t^3} + 2 \frac{\Delta t^5}{120} \frac{\partial^5 v}{\partial t^5} + \dots = - \frac{c \Delta t}{\Delta x} \left(2 \Delta x \frac{\partial v}{\partial x} + 2 \frac{\Delta x^3}{6} \frac{\partial^3 v}{\partial x^3} + 2 \frac{\Delta x^5}{120} \frac{\partial^5 v}{\partial x^5} + \dots \right)$$

$$\Rightarrow \frac{\Delta t \partial v}{\partial t} + \frac{\Delta t^3}{6} \frac{\partial^3 v}{\partial t^3} + \frac{\Delta t^5}{120} \frac{\partial^5 v}{\partial t^5} + \dots = - \frac{c \Delta t}{\Delta x} \left(\frac{\Delta x \partial v}{\partial x} + \frac{\Delta x^3}{6} \frac{\partial^3 v}{\partial x^3} + \frac{\Delta x^5}{120} \frac{\partial^5 v}{\partial x^5} + \dots \right)$$

$$\Rightarrow \frac{\partial v}{\partial t} + c \frac{\partial v}{\partial x} = \frac{\Delta x^2}{6} \frac{\partial^3 v}{\partial x^3} + \frac{\Delta x^4}{120} \frac{\partial^5 v}{\partial x^5} - \frac{\Delta t^2}{6} \frac{\partial^3 v}{\partial t^3} - \frac{\Delta t^4}{120} \frac{\partial^5 v}{\partial t^5}$$

$$\Rightarrow = \mathcal{O}(\Delta x^2) + \mathcal{O}(\Delta t^2)$$

CONSISTENCY

$$\lim_{\Delta t \rightarrow 0} \left(\frac{\Delta x^2}{6} \frac{\partial^3 u}{\partial x^3} + \frac{\Delta x^4}{120} \frac{\partial^5 u}{\partial x^5} - \frac{\Delta t^2}{6} \frac{\partial^3 u}{\partial t^3} - \frac{\Delta t^4}{120} \frac{\partial^5 u}{\partial t^5} \right) = 0$$

THIS ALGORITHM IS CONSISTENT

QUESTION 1a)

STABILITY

$$U_j^{n+1} = \left(\frac{U_{j+1}^n + U_{j-1}^n}{2} \right) - \frac{c \Delta t}{\Delta x} \left(U_{j+1}^n - U_{j-1}^n \right)$$

$$U^n = \bar{U}^n + \epsilon^n$$

$$\Rightarrow \bar{U}_j^{n+1} + \epsilon_j^{n+1} = \left(\frac{\bar{U}_{j+1}^n + \epsilon_{j+1}^n + \bar{U}_{j-1}^n + \epsilon_{j-1}^n}{2} \right) - \frac{c \Delta t}{2 \Delta x} \left(\bar{U}_{j+1}^n + \epsilon_{j+1}^n - \bar{U}_{j-1}^n - \epsilon_{j-1}^n \right)$$

$$\Rightarrow \epsilon_j^{n+1} = \left(\frac{\epsilon_{j+1}^n + \epsilon_{j-1}^n}{2} \right) - \frac{c \Delta t}{2 \Delta x} \left(\epsilon_{j+1}^n - \epsilon_{j-1}^n \right)$$

$$\epsilon_n^k = e^{\sigma k} e^{iknx} \quad ; \quad \frac{c \Delta t}{2 \Delta x} = \sigma$$

$$\Rightarrow G = \frac{\epsilon_j^{n+1}}{\epsilon_j^n} = \frac{1}{2} \left(\frac{e^{ikn(x+\Delta x)} + e^{-ikn(x+\Delta x)}}{e^{ikn(x)} + e^{-ikn(x)}} \right) - \sigma \left(\frac{e^{ikn(x+\Delta x)} - e^{-ikn(x+\Delta x)}}{e^{ikn(x)} + e^{-ikn(x)}} \right)$$

$$\Rightarrow G = \frac{1}{2} \left(e^{ikn \Delta x} + e^{-ikn \Delta x} \right) - \sigma \left(e^{ikn \Delta x} - e^{-ikn \Delta x} \right)$$

$$e^{ix} + e^{-ix} = 2 \cos(x) \quad ; \quad e^{ix} - e^{-ix} = 2i \sin(x) \quad ;$$

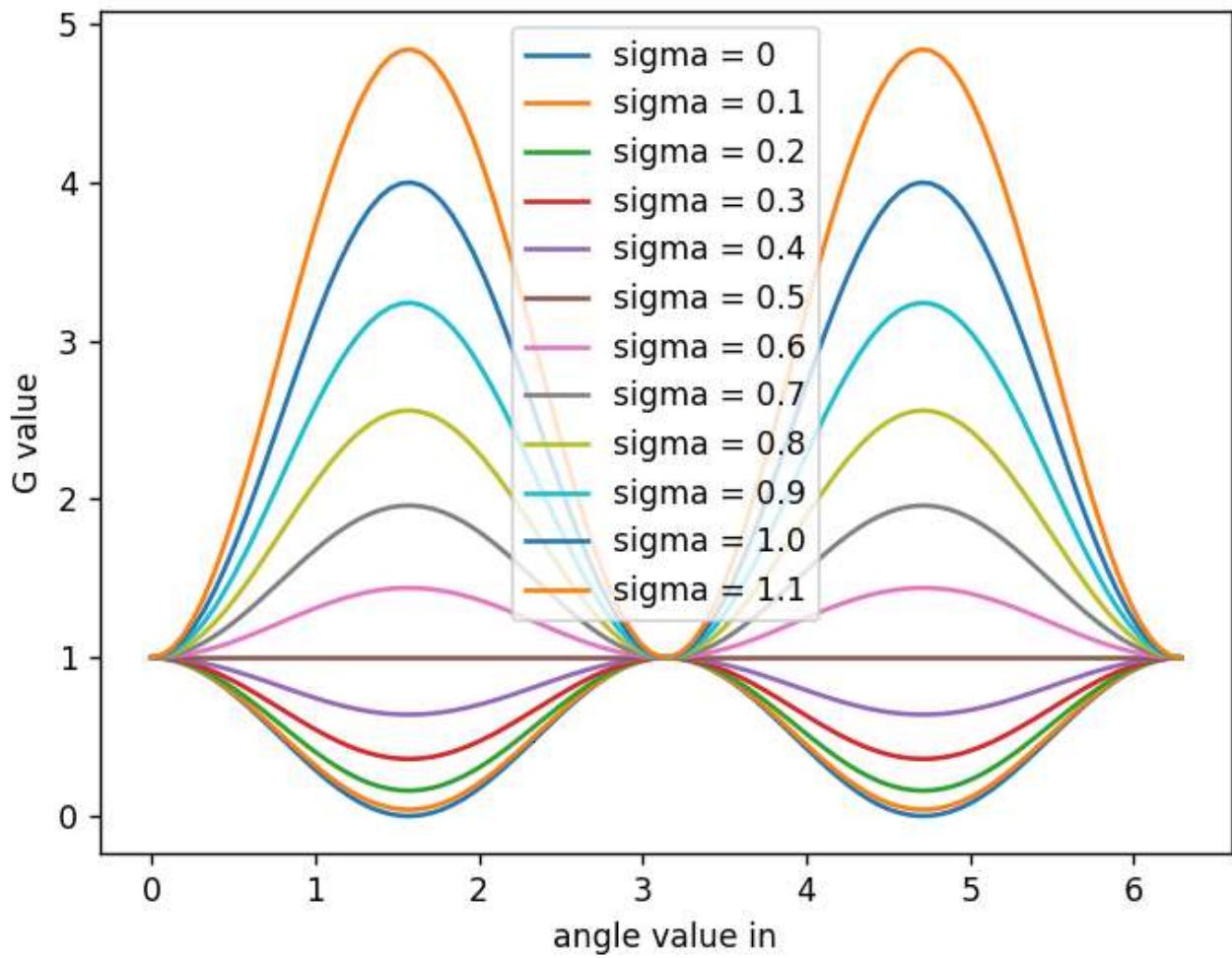
$$\Rightarrow G = \cos(kn \Delta x) + 2\sigma i \sin(kn \Delta x)$$

$$\Rightarrow |G|^2 = \cos^2(kn \Delta x) + 4\sigma^2 \sin^2(kn \Delta x) < 1$$

$$\Rightarrow \sigma^2 < \frac{-\cos^2(kn \Delta x)}{4 \sin^2(kn \Delta x)} + 1$$

WE SEE ON THE GRAPH THAT SIGMA SHOULD BE LOWER THAN 0.5.

$$\sigma = \frac{c \Delta t}{2 \Delta x} \Rightarrow \underbrace{\frac{c \Delta t}{\Delta x}}_{CFL} = CFL \leq 1$$



ACCURACY

$$v_j^{n+1} = \frac{(v_{j+1}^n + v_{j-1}^n)}{2} - \frac{c \Delta t}{2 \Delta x} (v_{j+1}^n - v_{j-1}^n) \quad (1)$$

$$v_j^{n+1} = v_j^n + \Delta t \frac{\partial v}{\partial x} + \frac{\Delta t^2}{2} \frac{\partial^2 v}{\partial x^2} + \frac{\Delta t^3}{6} \frac{\partial^3 v}{\partial x^3} + \dots \quad (2)$$

$$v_{j+1}^n = v_j^n + \frac{\Delta x}{2} \frac{\partial^2 v}{\partial x^2} + \frac{\Delta x^3}{6} \frac{\partial^3 v}{\partial x^3} + \dots$$

$$v_{j-1}^n = v_j^n - \frac{\Delta x}{2} \frac{\partial^2 v}{\partial x^2} - \frac{\Delta x^3}{6} \frac{\partial^3 v}{\partial x^3} + \dots$$

$$\Rightarrow v_{j+1}^n + v_{j-1}^n = 2v_j^n + 2 \frac{\Delta x^2}{2} \frac{\partial^2 v}{\partial x^2} + 2 \frac{\Delta x^4}{24} \frac{\partial^4 v}{\partial x^4} + \dots \quad (3)$$

$$\Rightarrow v_{j+1}^n - v_{j-1}^n = 2 \frac{\Delta x}{2} \frac{\partial^2 v}{\partial x^2} + 2 \frac{\Delta x^3}{6} \frac{\partial^3 v}{\partial x^3} + 2 \frac{\Delta x^5}{120} \frac{\partial^5 v}{\partial x^5} + \dots \quad (4)$$

(2)(3)(4) into (1)

$$\Rightarrow v_j^n + \Delta t \frac{\partial v}{\partial x} + \frac{\Delta t^2}{2} \frac{\partial^2 v}{\partial x^2} + \frac{\Delta t^3}{6} \frac{\partial^3 v}{\partial x^3} + \dots = \frac{1}{2} \left(2v_j^n + 2 \frac{\Delta x^2}{2} \frac{\partial^2 v}{\partial x^2} + 2 \frac{\Delta x^4}{24} \frac{\partial^4 v}{\partial x^4} + \dots \right) - \frac{c \Delta t}{2 \Delta x} \left(2 \frac{\Delta x}{2} \frac{\partial^2 v}{\partial x^2} + 2 \frac{\Delta x^3}{6} \frac{\partial^3 v}{\partial x^3} + 2 \frac{\Delta x^5}{120} \frac{\partial^5 v}{\partial x^5} + \dots \right)$$

$$\Rightarrow \frac{\partial v}{\partial t} = \frac{\Delta x^2}{2 \Delta t} \frac{\partial^2 v}{\partial x^2} + \frac{\Delta x^4}{24 \Delta t} \frac{\partial^4 v}{\partial x^4} + \dots - c \left(\frac{\Delta x}{2} \frac{\partial^2 v}{\partial x^2} + \frac{\Delta x^3}{6} \frac{\partial^3 v}{\partial x^3} + \frac{\Delta x^5}{120} \frac{\partial^5 v}{\partial x^5} + \dots \right) - \frac{\Delta t}{2} \frac{\partial^2 v}{\partial x^2} - \frac{\Delta t^3}{6} \frac{\partial^3 v}{\partial x^3}$$

$$\Rightarrow \frac{\partial v}{\partial t} + c \frac{\partial v}{\partial x} = \frac{\Delta x^2}{2 \Delta t} \frac{\partial^2 v}{\partial x^2} + \frac{\Delta x^4}{24 \Delta t} \frac{\partial^4 v}{\partial x^4} + \dots - c \left(\frac{\Delta x^2}{6} \frac{\partial^3 v}{\partial x^3} + \frac{\Delta x^4}{120} \frac{\partial^5 v}{\partial x^5} + \dots \right) - \frac{\Delta t}{2} \frac{\partial^2 v}{\partial x^2} - \frac{\Delta t^3}{6} \frac{\partial^3 v}{\partial x^3}$$

$$= \mathcal{O}\left(\frac{\Delta x^2}{\Delta t}\right) + \mathcal{O}(\Delta t) \quad ?$$

CONSISTENCY

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta x^2}{2\Delta t} \frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^4}{24\Delta t} \frac{\partial^4 u}{\partial x^4} + \dots - c \left(\frac{\Delta x^2}{6} \frac{\partial^3 u}{\partial x^3} + \frac{\Delta x^4}{120} \frac{\partial^5 u}{\partial x^5} + \dots \right) - \frac{\Delta t}{2} \frac{\partial^2 u}{\partial x^2} - \frac{\Delta t^2}{6} \frac{\partial^3 u}{\partial x^3} \neq 0$$

THE TERMS $\frac{\Delta x^m}{\Delta t}$ MAKES THIS ALGORITHM INCONSISTENT...

LETS CREATE OUR OWN ALGORITHM

$$\frac{\partial^2 u}{\partial t^2} + c \frac{\partial^2 u}{\partial x^2} = 0$$

SECOND ORDER SPACE

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{u_{j+1}^n - u_{j-1}^n}{2 \Delta x}$$

FOURTH ORDER IN TIME

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{-u_j^{n+2} + 8u_j^{n+1} - 8u_j^{n-1} + u_j^{n-2}}{12 \Delta t}$$

$$\Rightarrow \frac{-u_j^{n+2} + 8u_j^{n+1} - 8u_j^{n-1} + u_j^{n-2}}{12 \Delta t} + c \left(\frac{u_{j+1}^n - u_{j-1}^n}{2 \Delta x} \right) = 0$$

$$\Rightarrow \frac{-\frac{1}{12}u_j^{n+2} + \frac{2}{3}u_j^{n+1} - \frac{2}{3}u_j^{n-1} + \frac{1}{12}u_j^{n-2}}{\Delta t} + c \left(\frac{u_{j+1}^n - u_{j-1}^n}{2 \Delta x} \right) = 0$$

$$\Rightarrow u_j^{n+1} = \frac{1}{8}u_j^{n+2} + u_j^{n-1} - \frac{1}{8}u_j^{n-2} - 3 \frac{c \Delta t}{4 \Delta x} \left(u_{j+1}^n - u_{j-1}^n \right)$$

$$\Rightarrow \frac{1}{8}u_j^{n+2} - u_j^{n+1} + u_j^{n-1} - \frac{1}{8}u_j^{n-2} - 3 \frac{c \Delta t}{4 \Delta x} \left(u_{j+1}^n - u_{j-1}^n \right) = 0$$

$$u_j^{n+2} = u_j^n \pm 2 \Delta t \frac{\partial u}{\partial t} + \frac{4 \Delta t^2}{2} \frac{\partial^2 u}{\partial t^2} \pm \frac{8 \Delta t^3}{6} \frac{\partial^3 u}{\partial t^3} + \frac{16}{24} \frac{\Delta t^4 \partial^4 u}{\partial t^4} + \dots$$

$$u_j^{n+1} = u_j^n \pm \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2} \frac{\partial^2 u}{\partial t^2} \pm \frac{\Delta t^3}{6} \frac{\partial^3 u}{\partial t^3} + \frac{\Delta t^4 \partial^4 u}{24 \partial t^4} + \dots$$

$$u_{j+1}^n = u_j^n \pm \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2} \frac{\partial^2 u}{\partial x^2} \pm \frac{\Delta x^3}{6} \frac{\partial^3 u}{\partial x^3} - \frac{\Delta x^4 \partial^4 u}{24 \partial x^4} + \dots$$

We saw before ...

$$v_{j+1} - v_{j-1} = 2\Delta x \frac{\partial u}{\partial x} + \frac{1}{6} \Delta x^3 \frac{\partial^3 u}{\partial x^3} + \frac{1}{120} \Delta x^5 \frac{\partial^5 u}{\partial x^5} + \dots$$

$$\Rightarrow \frac{1}{8} (v_j^{n+2} - v_j^{n+1} + v_j^{n-1} - \frac{1}{8} v_j^{n-2}) = (v_j^{n-1} - v_j^{n+1}) + \frac{1}{8} (v_j^{n+2} - v_j^{n-2})$$

$$\Rightarrow v_j^{n-1} - v_j^{n+1} = -2 \left(\Delta t \frac{\partial u}{\partial t} + \frac{1}{6} \Delta t^3 \frac{\partial^3 u}{\partial t^3} + \frac{1}{120} \Delta t^5 \frac{\partial^5 u}{\partial t^5} + \dots \right)$$

$$\Rightarrow v_j^{n+2} - v_j^{n-2} = 2 \left(2\Delta t \frac{\partial u}{\partial t} + \frac{8\Delta t^3}{6} \frac{\partial^3 u}{\partial t^3} + \frac{32}{120} \Delta t^5 \frac{\partial^5 u}{\partial t^5} \right)$$

$$\Rightarrow 2 \cdot \frac{1}{6} \left(\left(2\Delta t \frac{\partial u}{\partial t} + \frac{8\Delta t^3}{6} \frac{\partial^3 u}{\partial t^3} + \frac{32}{120} \Delta t^5 \frac{\partial^5 u}{\partial t^5} \right) - \left(\Delta t \frac{\partial u}{\partial t} + \frac{1}{6} \Delta t^3 \frac{\partial^3 u}{\partial t^3} + \frac{1}{120} \Delta t^5 \frac{\partial^5 u}{\partial t^5} + \dots \right) \right) - \frac{3c\Delta t}{4} \left(2 \left(\Delta t \frac{\partial u}{\partial t} + \frac{1}{6} \Delta t^3 \frac{\partial^3 u}{\partial t^3} \right) + \frac{1}{120} \Delta x^5 \frac{\partial^5 u}{\partial x^5} \right) = 0$$

$$\Rightarrow 2 \cdot \left(\left(\Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^3}{6} \frac{\partial^3 u}{\partial t^3} + \frac{4}{120} \Delta t^5 \frac{\partial^5 u}{\partial t^5} \right) - \left(\Delta t \frac{\partial u}{\partial t} + \frac{1}{6} \Delta t^3 \frac{\partial^3 u}{\partial t^3} + \frac{1}{120} \Delta t^5 \frac{\partial^5 u}{\partial t^5} + \dots \right) \right) - \frac{3c\Delta t}{4} \left(2 \left(\Delta t \frac{\partial u}{\partial t} + \frac{1}{6} \Delta t^3 \frac{\partial^3 u}{\partial t^3} + \frac{1}{120} \Delta t^5 \frac{\partial^5 u}{\partial t^5} \right) + \frac{1}{120} \Delta x^5 \frac{\partial^5 u}{\partial x^5} \right) = 0$$

DIVIDE PER Δt ; TAKE NOTE THAT THE TERM Δt^3 IS GONE

$$\Rightarrow \frac{\partial u}{\partial t} + \frac{1}{40} \Delta t^4 \frac{\partial^5 u}{\partial t^5} - \frac{3c}{4} \left(\frac{\partial u}{\partial t} + \frac{1}{6} \Delta x^2 \frac{\partial^3 u}{\partial x^3} + \frac{1}{120} \Delta x^4 \frac{\partial^5 u}{\partial x^5} \right) = 0$$

$$\Rightarrow \frac{\partial u}{\partial t} - c \frac{\partial u}{\partial x} = -\frac{1}{40} \Delta t^4 \frac{\partial^5 u}{\partial t^5} + \frac{3c}{4} \left(\frac{1}{6} \Delta x^2 \frac{\partial^3 u}{\partial x^3} + \frac{1}{120} \Delta x^4 \frac{\partial^5 u}{\partial x^5} \right)$$

\in ERROR

We see that it's $O(\Delta t^4) + O(\Delta x^2)$

CONSISTENCY

$$\text{ERROR} = -\frac{1}{120} \frac{\partial^5 u}{\partial t} + \frac{3\varepsilon}{4} \left(\frac{1}{6} \frac{\Delta x^2}{\partial x^3} \frac{\partial^2 u}{\partial x^3} + \frac{1}{120} \Delta x^4 \frac{\partial^5 u}{\partial x} \right)$$

$$\lim_{\Delta x \rightarrow 0} \text{ERROR} = 0$$

$$\Delta t \rightarrow 0$$

THUS, THIS ALGORITHM IS CONSISTENT

STABILITY

$$\frac{1}{\Delta t} v_j^{n+2} - v_j^{n+1} + v_j^{n-1} - \frac{1}{8} v_j^{n-2} - \frac{3}{4} \frac{\Delta t}{\Delta x} \left(v_{j+1}^n - v_{j-1}^n \right) = 0$$

LINER EQUATION SO:

$$\Rightarrow \frac{1}{\Delta t} e_j^{n+2} - e_j^{n+1} + e_j^{n-1} - \frac{1}{8} e_j^{n-2} - \frac{3}{4} \frac{\Delta t}{\Delta x} \left(e_{j+1}^n - e_{j-1}^n \right) = 0$$

AFTER DIVISION BY $e_j^n = e^{at} e^{ikn\Delta x}$; $\sigma = \frac{3}{4} \frac{\Delta t}{\Delta x}$

$$\Rightarrow \frac{1}{\Delta t} e^{2adt} - e^{adt} + e^{-adt} - \frac{1}{8} e^{-2adt} - \sigma \left(e^{ikn\Delta x} - e^{-ikn\Delta x} \right) = 0$$

$$\Rightarrow \frac{1}{\Delta t} e^{2adt} - e^{adt} + e^{-adt} - \frac{1}{8} e^{-2adt} - \sigma \left(2 \sin(kn\Delta x) \right) = 0$$

$$\Rightarrow \frac{1}{\Delta t} (e^{adt})^2 - (e^{adt}) + (e^{-adt}) - \frac{1}{8} (e^{-2adt}) - \sigma (2 \sin(kn\Delta x)) = 0$$

We HAVE TO FIND $e^{adt} = G$

$$\Rightarrow \frac{1}{\Delta t} G^2 - G + \frac{1}{\Delta t} - \frac{1}{8} \frac{1}{G^2} - \sigma \left(2 \sin(kn\Delta x) \right) = 0$$

HARD EQUATION BUT... WE SEE THAT WHEN

$G = 0 \Rightarrow G = 1$. THUS IF $\sigma \neq 0$ THE SINUS

PART WILL JUST MAKES $|G|$ OSCILLATE AROUND

1 AND MOST IMPORTANTLY AT THE VALUES $\sigma > 1$ WHICH MAKES IT UNSTABLE...

WE DID NEWTON-RAPHSON ANALYSIS

TO PROVE OUR POINT

WE SEE IN THE PLOT THAT $\sigma \in [0, 2]$ WILL ALWAYS
MAKE $G \gg 1$ WHICH MAKES IT UNSTABLE

```

def f(x,C):
    if x == 0:
        return np.inf
    else:
        return 1/8 * x**2 - x + x**-1 - 1/8 * x **-2 - C

def fp(x):
    if x == 0:
        return np.inf
    else: return 1/4*x - 1 + -x**-2 - 1/4 * x**-3

def newton_raphson(C,x0, tol = 1e-6, max_iter = 400):
    iteration = 0
    while iteration < max_iter:
        fx = f(x0,C)
        fpx = fp(x0)
        if fpx == 0:
            print("you are cooked bruv")
            return None
        x1 = x0 - fx/fpx
        if all(abs(x1 - x0)) < tol:
            return x1
        x0 = x1
        iteration += 1
    print ("ur cooked")

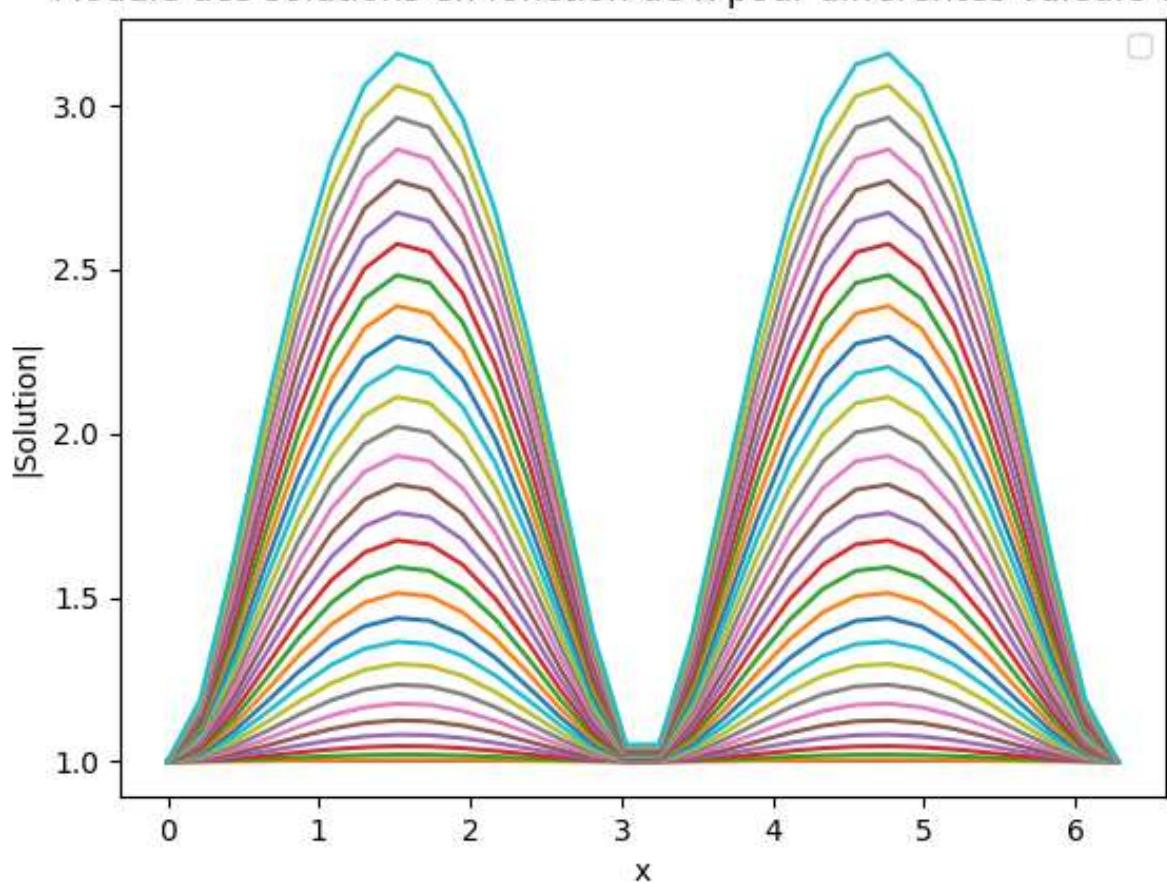
x = np.linspace(0,2*np.pi,30)
sigma_mat = np.linspace(0,3,30)
for sigma in sigma_mat:
    C = j*sigma*(2*np.sin(x))

    x0 = 1 + 0j
    sol = newton_raphson(C,x0, tol = 1e-6, max_iter = 100)
    print(abs(sol))
    plt.plot(x,abs(sol))

plt.xlabel('x')
plt.ylabel('|Solution|')
plt.title('Module des solutions en fonction de x pour différentes valeurs de σ')
plt.legend()
plt.show()

```

Module des solutions en fonction de x pour différentes valeurs de σ



1. b) Forward Time-Centered Space

$$u_j^{n+1} = u_j^n - \frac{c\Delta t}{2\Delta x} (u_{j+1}^n - u_{j-1}^n)$$

Stability: $\xi_j^{n+1} = \xi_j^n - \frac{c\Delta t}{2\Delta x} (\xi_{j+1}^n - \xi_{j-1}^n)$

$$\begin{aligned} \text{so } G &= \left| \frac{\xi_j^{n+1}}{\xi_j^n} \right| = \left| 1 - \frac{c\Delta t}{2\Delta x} \frac{\xi_{j+1}^n - \xi_{j-1}^n}{\xi_j^n} \right| \\ &= \left| 1 - \frac{c\Delta t}{2\Delta x} \frac{e^{at} e^{ik_m(x+\Delta x)} - e^{at} e^{ik_m(x-\Delta x)}}{e^{at} e^{-ik_m x}} \right| \\ &= \left| 1 - \frac{c\Delta t}{2\Delta x} (e^{ik_m \Delta x} - e^{-ik_m \Delta x}) \right| \\ &= \left| 1 - \frac{c\Delta t}{2\Delta x} (2i \sin(k_m \Delta x)) \right| \\ &= \sqrt{1^2 + \left(\frac{\Delta t c}{\Delta x} \right)^2 \sin^2(k_m \Delta x)} \end{aligned}$$

$$\text{So } G < 1 \text{ if } 1 + \left(\frac{c\Delta t}{\Delta x} \right)^2 < 1 \Leftrightarrow \frac{c\Delta t}{\Delta x} < 0$$

This scheme is unconditionally unstable.

$$\begin{aligned} \text{Accuracy: } u + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2} \frac{\partial^2 u}{\partial t^2} + \dots &= u - \frac{c\Delta t}{2\Delta x} \left[\left(u + \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2} \frac{\partial^2 u}{\partial x^2} \right. \right. \\ &\quad \left. \left. - \left(u - \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2} \frac{\partial^2 u}{\partial x^2} \dots \right) \right] \right] \end{aligned}$$

$$\Rightarrow \frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = - \frac{\Delta t}{2} \frac{\partial^2 u}{\partial t^2} - c \frac{\Delta x^2}{6} \frac{\partial^3 u}{\partial x^3} + \dots$$

The scheme is $\begin{cases} 1^{\text{st}} \text{ order in time} \\ 2^{\text{nd}} \text{ order in space} \end{cases}$ and is consistent.

d) Lax-Wendroff

$$u_j^{n+1} = u_j^n - \frac{c\Delta t}{2\Delta x} (u_{j+1}^n - u_{j-1}^n) + \frac{1}{2} \left(\frac{c\Delta t}{\Delta x} \right)^2 (u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

$$\text{Stability: } \epsilon_j^{n+1} = \epsilon_j^n - \frac{c\Delta t}{2\Delta x} (\epsilon_{j+1}^n - \epsilon_{j-1}^n) + \frac{1}{2} \left(\frac{c\Delta t}{\Delta x} \right)^2 (\epsilon_{j+1}^n - 2\epsilon_j^n + \epsilon_{j-1}^n)$$

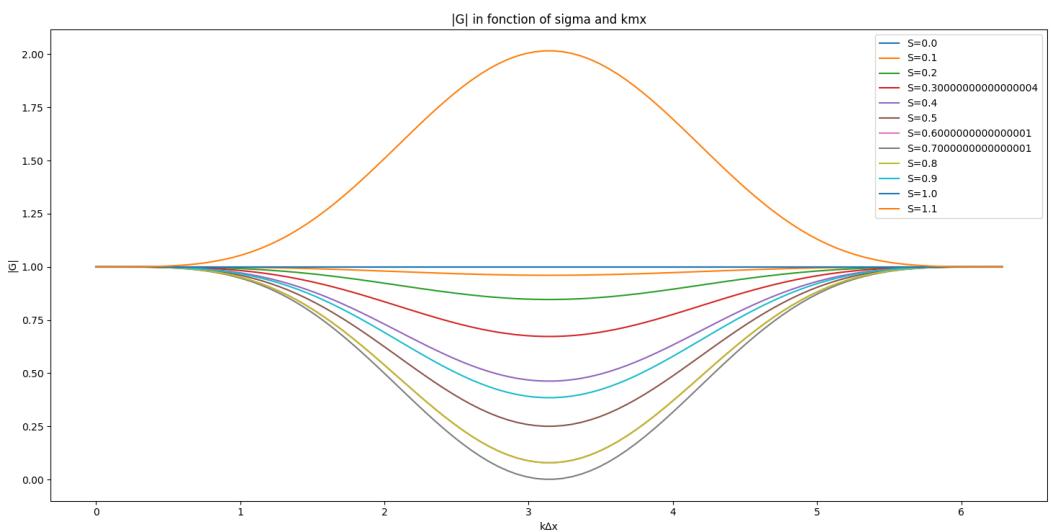
$$\begin{aligned} \text{so } G &= \left| 1 - \frac{c\Delta t}{2\Delta x} \frac{\epsilon_{j+1}^n - \epsilon_{j-1}^n}{\epsilon_j^n} + \frac{1}{2} \left(\frac{c\Delta t}{\Delta x} \right)^2 \frac{\epsilon_{j+1}^n - 2\epsilon_j^n + \epsilon_{j-1}^n}{\epsilon_j^n} \right| \\ &= \left| 1 - \frac{c\Delta t}{2\Delta x} \left[2 \cos(\frac{k_m \Delta x}{2}) \right] + \frac{1}{2} \left(\frac{c\Delta t}{\Delta x} \right)^2 \left[-2 + 2 \cos(\frac{k_m \Delta x}{2}) \right] \right| \\ &= \left| \left[1 + \left(\frac{c\Delta t}{\Delta x} \right)^2 (\cos(\frac{k_m \Delta x}{2}) - 1) \right] + \left[- \frac{c\Delta t}{\Delta x} i \sin(\frac{k_m \Delta x}{2}) \right] \right| \\ &= \left| \left[1 - \left(\frac{c\Delta t}{\Delta x} \right)^2 2 \sin^2 \left(\frac{k_m \Delta x}{2} \right) \right] + \left[- \frac{c\Delta t}{\Delta x} i \sin(\frac{k_m \Delta x}{2}) \right] \right| \\ &= \left| 1 - 4 \left(\frac{c\Delta t}{\Delta x} \right)^2 \sin^2 \left(\frac{k_m \Delta x}{2} \right) + 4 \left(\frac{c\Delta t}{\Delta x} \right)^4 \sin^4 \left(\frac{k_m \Delta x}{2} \right) + \left(\frac{c\Delta t}{\Delta x} \right)^2 \sin^2 \left(\frac{k_m \Delta x}{2} \right) \right|^{\frac{1}{2}} \end{aligned}$$

By numerical analysis (see figure), the scheme is stable when $CFL \leq 1$.

$$\begin{aligned} \text{Accuracy: } u + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2} \frac{\partial^2 u}{\partial t^2} &= u - \frac{c\Delta t}{2\Delta x} \left(2\Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{3} \frac{\partial^3 u}{\partial x^3} \right) + \frac{1}{2} \left(\frac{c^2 \Delta t^2}{\Delta x^2} \right) \\ &\quad \left(\cancel{\frac{\Delta x \frac{\partial u}{\partial x}}{\Delta x}} + \frac{\Delta x^2 \frac{\partial^2 u}{\partial x^2}}{2} + \dots - \cancel{\frac{\Delta x \frac{\partial u}{\partial x}}{\Delta x}} + \frac{\Delta x^2 \frac{\partial^2 u}{\partial x^2}}{2} - \dots \right) \end{aligned}$$

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = -\frac{\Delta t}{2} \frac{\partial^2 u}{\partial t^2} - c \frac{\Delta x c^2}{6} \frac{\partial^3 u}{\partial x^3} + c^2 \frac{\Delta t}{2} \frac{\partial^2 u}{\partial x^2} + \dots$$

The scheme is $\left\{ \begin{array}{l} 1^{\text{st}} \text{ order in time} \\ 2^{\text{nd}} \text{ order in space} \end{array} \right.$ and is consistent



g) Hybrid explicit-implicit

$$u_j^{n+1} + \Theta \left(\frac{CFL}{2} \right) (u_{j+1}^{n+1} - u_{j-1}^{n+1}) = u_j^n - (1-\Theta) \frac{CFL}{2} (u_{j+1}^n - u_{j-1}^n)$$

* with $\Theta = 0$, this scheme is equivalent of the one studied in 1. B) so unstable, 1st order in time, 2nd order in space, and consistent.

$$* \text{with } \Theta = 1: u_j^{n+1} + \left(\frac{CFL}{2} \right) (u_{j+1}^{n+1} - u_{j-1}^{n+1}) = u_j^n$$

Stability:

$$e^{\alpha \Delta t} + \frac{CFL}{2} (e^{\alpha \Delta t} e^{ik\Delta x} - e^{\alpha \Delta t} e^{-ik\Delta x}) = 1$$

$$e^{\alpha \Delta t} \left[1 + \frac{CFL}{2} (e^{ik\Delta x} - e^{-ik\Delta x}) \right] = 1$$

$$e^{\alpha \Delta t} = 1 / \left[1 + \frac{CFL}{2} (2 i \sin(k\Delta x)) \right]$$

see figure (always stable)

$$= 1 - \frac{CFL (2 i \sin(k\Delta x))}{1 + \cancel{CFL^2} \sin^2(k\Delta x)}$$

Accuracy

$$u + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2!} \frac{\partial^2 u}{\partial t^2} + \dots + \left(\frac{CFL}{2} \right) \left[\left(u + \Delta t \frac{\partial u}{\partial t} + \Delta x \frac{\partial u}{\partial x} + \dots \right) - u \right]$$

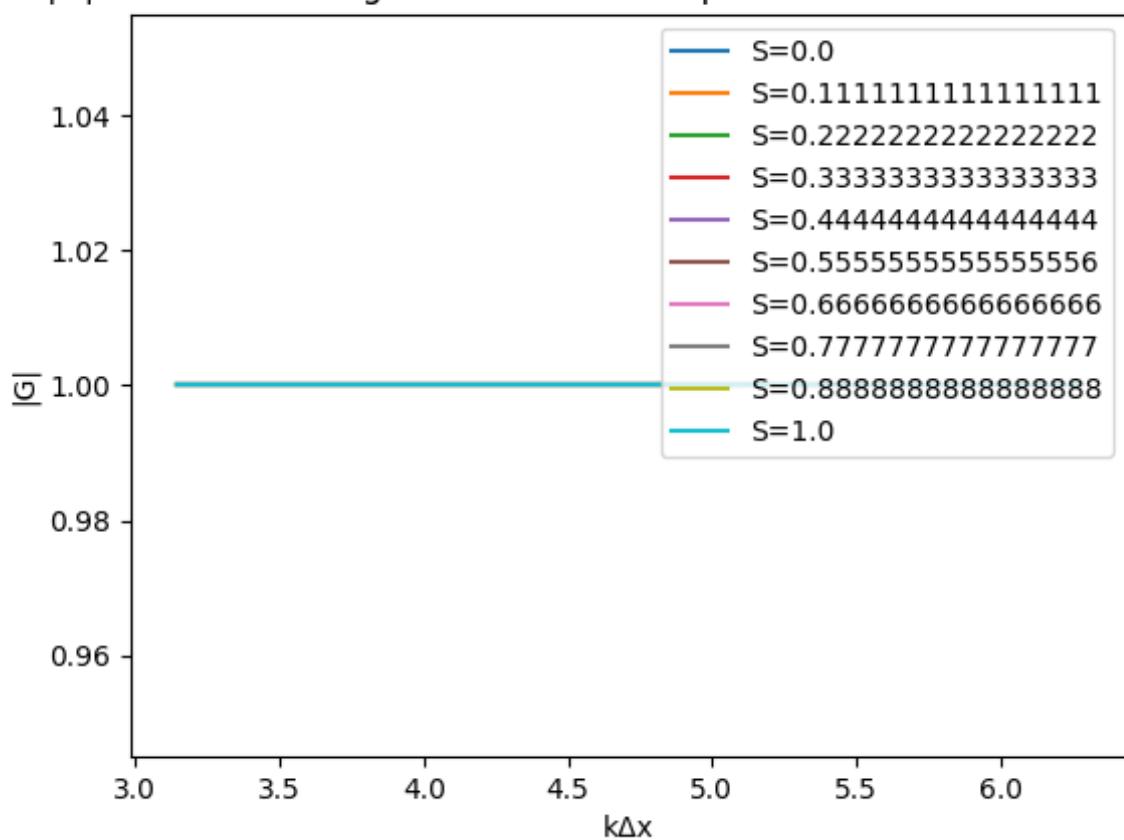
$$CFL = c \frac{\Delta t}{\Delta x} + \Delta t \frac{\partial u}{\partial t} - \Delta x \frac{\partial u}{\partial x} + \dots \Big] = 0$$

$$\Rightarrow \frac{\Delta t \frac{\partial u}{\partial t}}{\Delta t} + \frac{\Delta t^2 \frac{\partial^2 u}{\partial t^2}}{2!} + \dots + \left(\frac{CFL}{2} \right) \left[2 \frac{\Delta x \frac{\partial u}{\partial x}}{\Delta t} + \dots + 2 \frac{\Delta x^2 \frac{\partial^2 u}{\partial x^2}}{2!} + \dots \right] = 1$$

$$\Rightarrow \frac{\Delta t \frac{\partial u}{\partial t}}{\Delta t} + \frac{\Delta t^2 \frac{\partial^2 u}{\partial t^2}}{2!} + \dots + \frac{CFL}{2} \left[2 \frac{\Delta x \frac{\partial u}{\partial x}}{\Delta t} \right] = 1 \Delta t$$

$$\Rightarrow \frac{\Delta u}{\Delta t} + c \frac{\partial u}{\partial x} = \frac{\Delta u}{2 \Delta t^2} + \frac{CFL}{2} \left[2 \frac{\Delta x^2 \frac{\partial^2 u}{\partial x^2}}{6} \right] \rightarrow - \frac{c \Delta x^2 \frac{\partial^2 u}{\partial x^2}}{12}$$

$|G|$ in fonction of sigma and $k\Delta x$ for implicite scheme when theta = 1



$$u_j^{n+1} + \theta \left(\frac{CFL}{2} \right) (u_{j+1}^{n+1} - u_{j-1}^{n+1}) = u_j^n - (1-\theta) \frac{CFL}{2} (u_{j+1}^n - u_{j-1}^n)$$

$$\theta = 0.5$$

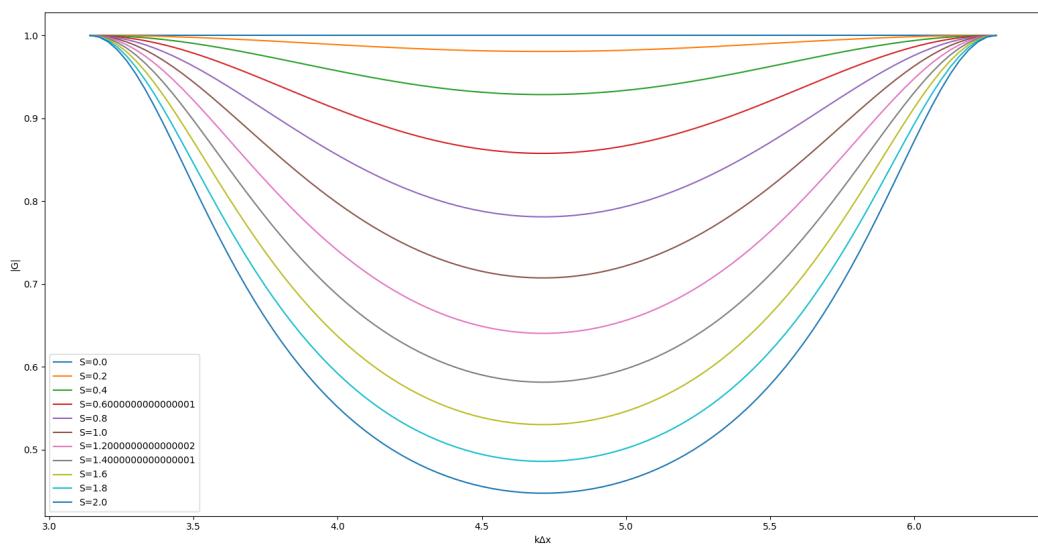
$$u_j^{n+1} + \frac{CFL}{4} (u_{j+1}^{n+1} - u_{j-1}^{n+1}) = u_j^n - \frac{CFL}{4} (u_{j+1}^n - u_{j-1}^n)$$

Stability: $e^{a\Delta t} + \frac{CFL}{4} (e^{a\Delta t} e^{ik\Delta x} - e^{a\Delta t} e^{-ik\Delta x}) = 1 - \frac{CFL}{4} (e^{ik\Delta x} - e^{-ik\Delta x})$

$$\begin{aligned} G = |e^{a\Delta t}| &= \sqrt{1-a^2} / \sqrt{1+a^2} \quad \text{with } a = \frac{CFL}{4} (2i \sin(k\Delta x)) \\ &= \frac{(1-a)^2}{1-a^2} = \frac{1 - 4i \sin(k\Delta x) \frac{CFL}{4} - 4 \sin^2(k\Delta x) \frac{CFL^2}{16}}{1 + \frac{CFL^2}{16} \sin^2(k\Delta x)} \end{aligned}$$

By numerical analysis the scheme is unconditionally unstable ($G = 1$ for all CFL).

$$\begin{aligned} \text{Accuracy} &\cancel{u + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2} \frac{\partial^2 u}{\partial t^2} + \dots + \frac{CFL}{4} \left[\left(u + \cancel{\Delta t \frac{\partial u}{\partial t}} + \cancel{\Delta x \frac{\partial u}{\partial x}} + \frac{\Delta t^2}{2} \frac{\partial^2 u}{\partial t^2} + \frac{\Delta x^2 \frac{\partial^2 u}{\partial x^2}}{2} + \dots \right) - \left(u + \cancel{\Delta t \frac{\partial u}{\partial t}} - \cancel{\Delta x \frac{\partial u}{\partial x}} + \frac{\Delta t^2}{2} \frac{\partial^2 u}{\partial t^2} + \frac{\Delta x^2 \frac{\partial^2 u}{\partial x^2}}{2} - \dots \right) \right]} \\ &= \cancel{u} - \frac{CFL}{4} \left[\left(u + \cancel{\Delta x \frac{\partial u}{\partial x}} + \frac{\Delta x^2 \frac{\partial^2 u}{\partial x^2}}{2} + \dots \right) - \left(u - \cancel{\Delta x \frac{\partial u}{\partial x}} + \frac{\Delta x^2 \frac{\partial^2 u}{\partial x^2}}{2} - \dots \right) \right] \\ &\Rightarrow \cancel{\Delta t \frac{\partial u}{\partial t}} + \frac{\Delta t^2}{2} \frac{\partial^2 u}{\partial t^2} + \dots + \frac{c\Delta t}{4\Delta x} \left(2\cancel{\Delta x \frac{\partial u}{\partial x}} + 2 \frac{\Delta x^2}{6} \frac{\partial^3 u}{\partial x^3} + \dots \right) = - \frac{c\Delta t}{4\Delta x} \left(2\cancel{\Delta x \frac{\partial u}{\partial x}} + c \frac{\Delta x^2}{6} \frac{\partial^3 u}{\partial x^3} + \dots \right) \\ &\Rightarrow \frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = - \frac{\Delta t}{2} \frac{\partial^2 u}{\partial t^2} - \frac{\Delta x^2}{6} \frac{\partial^3 u}{\partial x^3} - \dots \quad \text{Space: 2nd, Time: 1st} \\ &\qquad\qquad\qquad \text{Consistent} \end{aligned}$$



4 ANNEXE 2: GITHUB LINK

Cliquez !!! Lien github !!!