MEC6602E: Transonic Aerodynamics

HOMEWORK 1

Prepared by:
Kamal Jemmali 2019339
Hugo Javourez 2420717

Submitted to :
M.Frédéric Plante. Ph.D, CPI

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POLYTECHNIQUE MONTRÉAL
UNIVERSITÉ D'INGÉNIERIE

1 Question 1: Analytical Calculation of Order of Convergence, Stability and Consistency

This section will analyse some finite difference algorithms of the following equation:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \tag{1}$$

This will be done by 3 ways:

- Analysing convergence with Taylor
- Stability with Von Neumann
- Consistensy with le limit of the error term $\Delta x, \Delta t \longrightarrow 0$

a1. Lets analyse the backward algorithm:

$$u_j^{n+1} = u_j^n - \frac{c\Delta t}{\Delta x}\left(u_j^n - u_{j-1}^n\right) \tag{2}$$

This Algorithm is:

- Stable
- Convergent on ordre 1 in both time and space
- Consistent

a2. Lets analyse the frontward algorithm:

$$u_j^{n+1} = u_j^n - \frac{c\Delta t}{\Delta x}\left(u_{j+1}^n - u_j^n\right) \tag{3}$$

This Algorithm is:

- UnStable
- Convergent on ordre 1 in both time and space
- Consistent

b. Lets analyse the centered algorithm:

$$u_j^{n+1} = u_j^n - \frac{c\Delta t}{\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) \tag{4}$$

This Algorithm is:

- UnStable
- Convergent on ordre 2 in space and ordre 1 in time
- Consistent

c. Lets analyse the leap-frog algorithm:

$$u_j^{n+1} = u_j^{n-1} - \frac{c\Delta t}{\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) \tag{5}$$

This Algorithm is:

- Metastable $|G| = 1$
- Convergent on ordre 2 in space and ordre 2 in time
- Consistent

d. Lets analyse the Lax-Wendroff algorithm:

$$u_j^{n+1} = u_j^n - \frac{c\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{1}{2}\left(\frac{c\Delta t}{\Delta x}\right)^2\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) \tag{6}$$

This Algorithm is:

- Stable $CFL < 1$
- Convergent on ordre 2 in space and ordre 1 in time
- Consistent

e. Lets analyse the Lax algorithm:

$$u_j^{n+1} = u_j^n - \frac{c\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{1}{2}\left(\frac{c\Delta t}{\Delta x}\right)^2\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) \tag{7}$$

This Algorithm is:

- Stable $CFL < 1$
- Bad convergence and not consistent

f. Lets analyse the Hybrid algorithm:
For $\theta = 0$ This Algorithm is:

- the same as the b, please refer to it

For $\theta = 1$ This Algorithm is:

- Always Stable
- first ordre in time, 2nd order in space
- consistent

For $\theta = 0.5$ This Algorithm is:

- Metastable $|G| = 1$
- first ordre in time, 2nd order in space
- consistent

g. Lets Analyse our chosen algorithm:
Here are the 2th order in space discretisation and the 4th order in time chosen:

$$\frac{\partial u}{\partial x} \approx \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \tag{8}$$

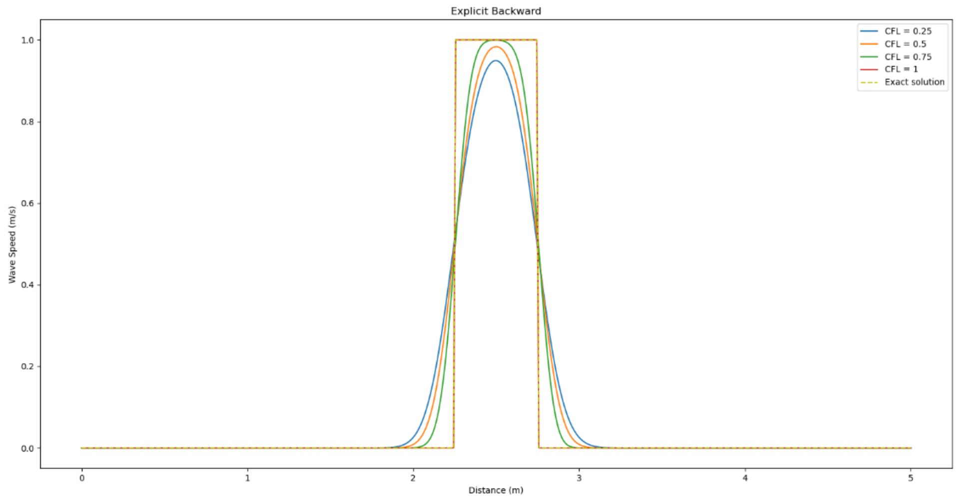
$$\frac{\partial u}{\partial t} \approx \frac{-u_j^{n+2} + 8u_j^{n+1} - 8u_j^{n-1} + u_j^{n-2}}{12\Delta t} \tag{9}$$

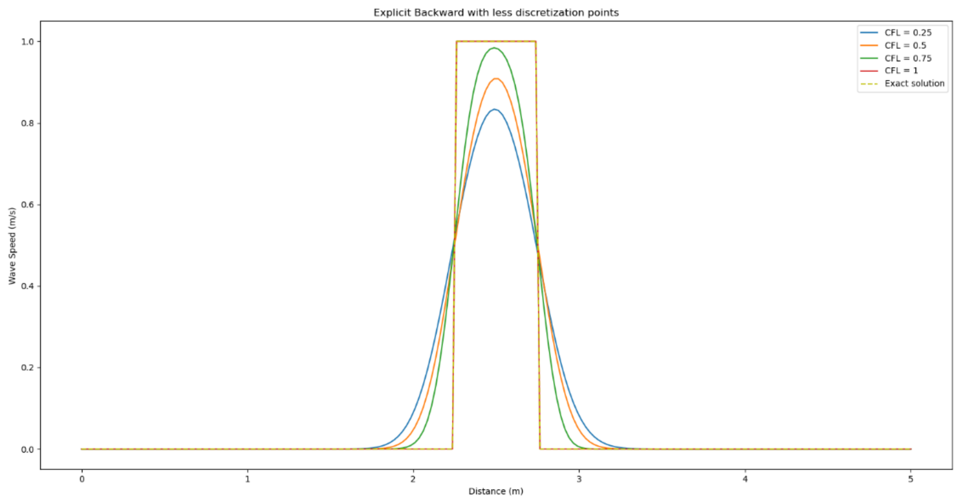
$$u_j^{n+1} = \frac{1}{8}u_j^{n+2} + u_j^{n-1} - \frac{1}{8}u_j^{n-2} - \frac{3c\Delta t}{4\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) \tag{10}$$

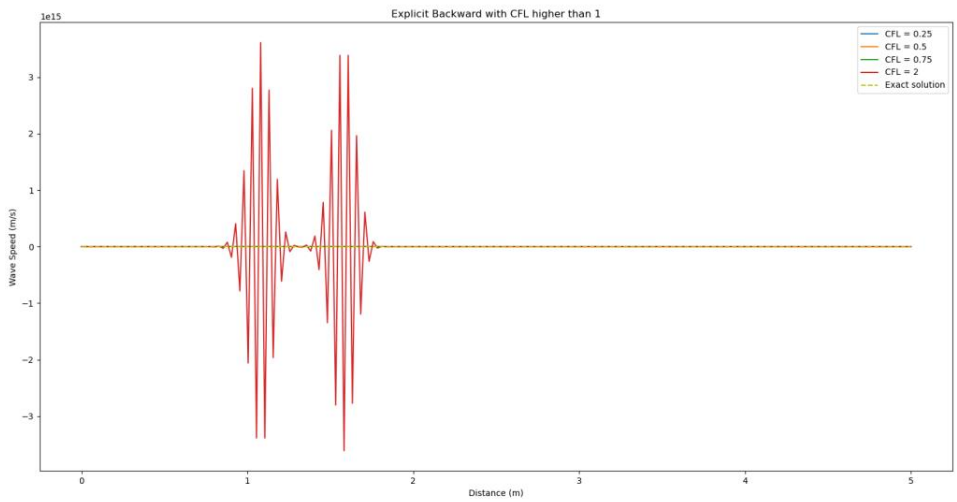
- Unstable
- 4nd order in time, 2nd order in space
- Consistent

Of course, every calculation will be found on the annexe 1

2 NUMERICAL ANALYSIS

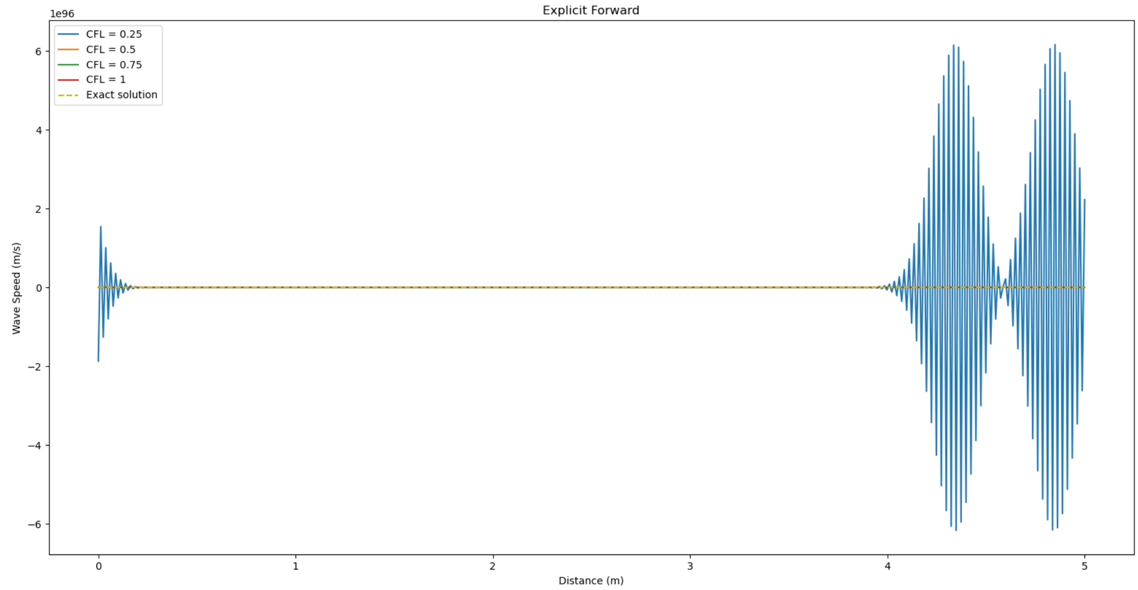

Explicit Backward
CFL = 0.25
CFL = 0.5
CFL = 0.75
CFL = 1
Exact solution
Wave Speed (m/s)
Distance (m)


Explicit Backward with less discretization points
CFL = 0.25
CFL = 0.5
CFL = 0.75
CFL = 1
Exact solution
Wave Speed (m/s)
Distance (m)

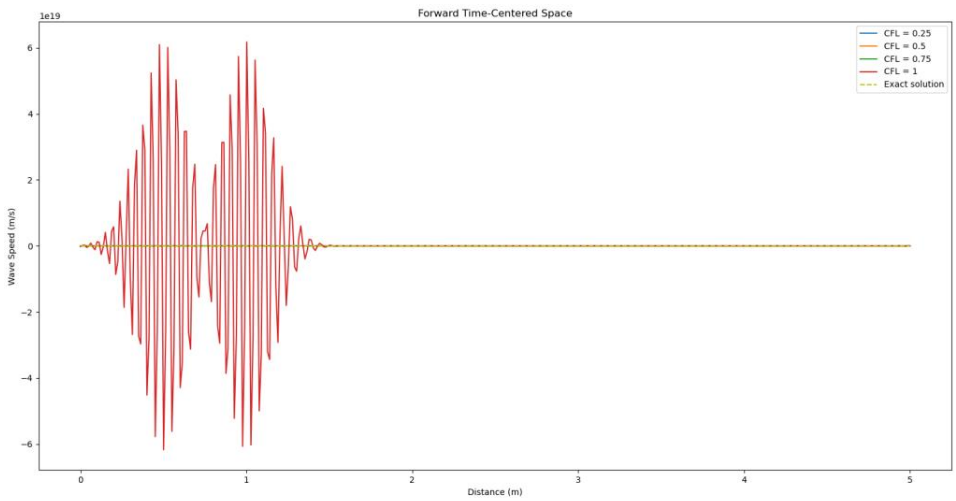

Explicit Backward with CFL higher than 1
1e15
CFL = 0.25
CFL = 0.5
CFL = 0.75
CFL = 2
Exact solution
Wave Speed (m/s)
Distance (m)

With the Explicit Backward scheme, we have some diffusion when the CFL number tends to 0. However, when the CFL number is equal to 1, the solution is near an exact solution, without any diffusion, dispersion, or loss. We will take this result as our "Exact solution" in the following figures.

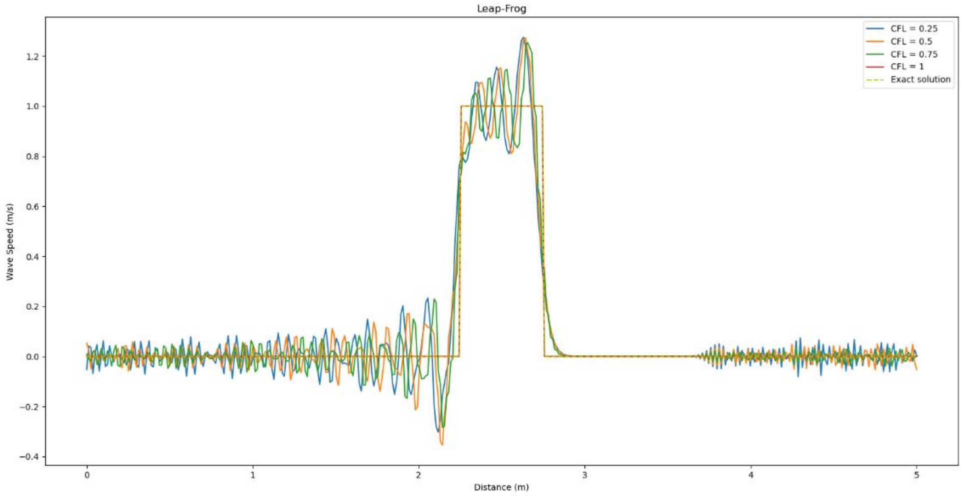
The accuracy increases with the number of discretization and the scheme is stable for CFL less than 1.



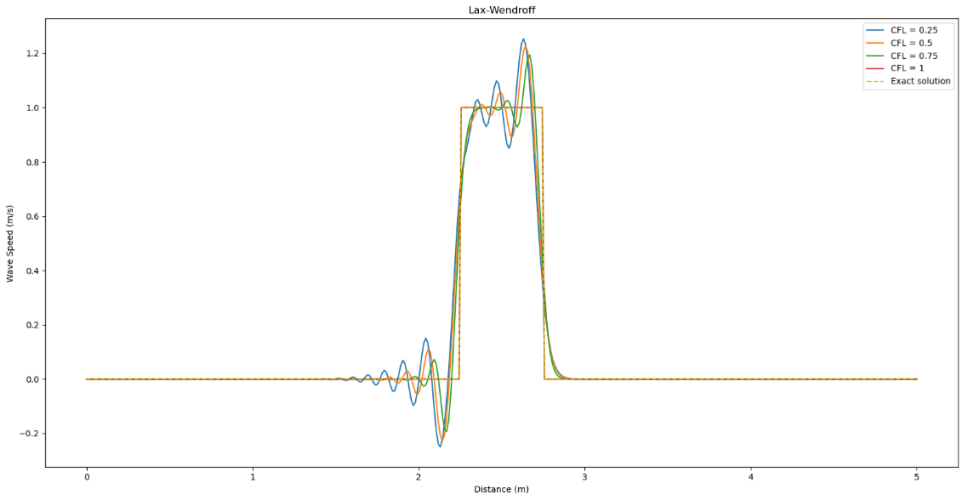
The Explicit Forward scheme is always unstable, so we cannot make any comment on the accuracy with a numerical analysis.



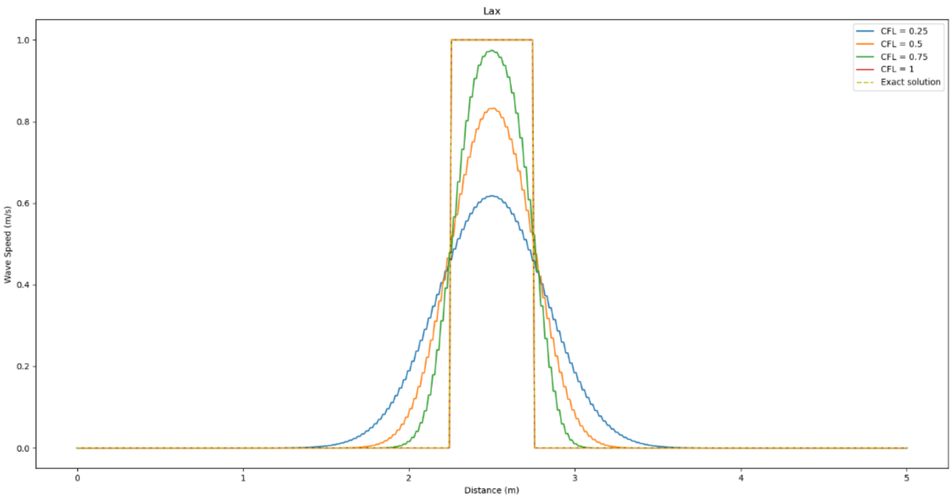
As for the Explicit Forward scheme, the Forward Time-Centered Space is always unstable and we cannot make any comment on the accuracy with a numerical analysis.



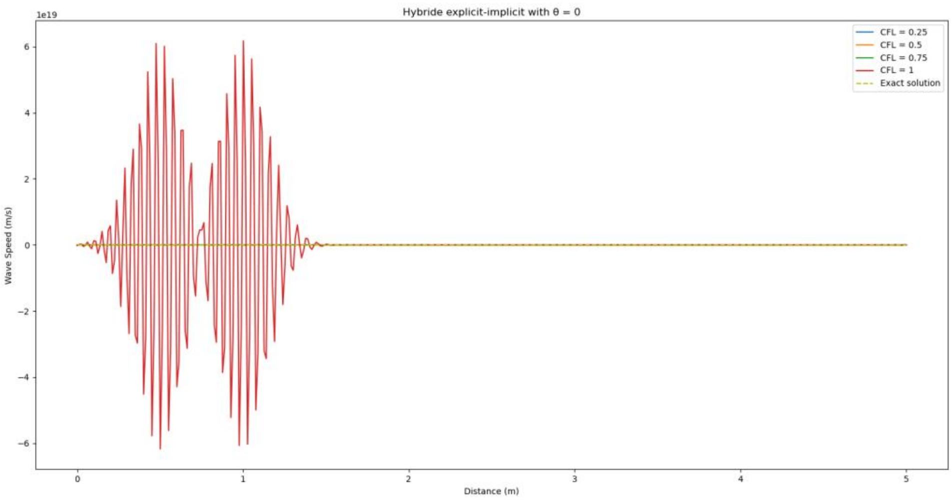
The Leap-Frog scheme has a lot of dispersion in its numerical solutions, resulting in a lot of oscillations. But with CFL equal one, we get the "Exact solution". This scheme is stable for CFL less than 1.



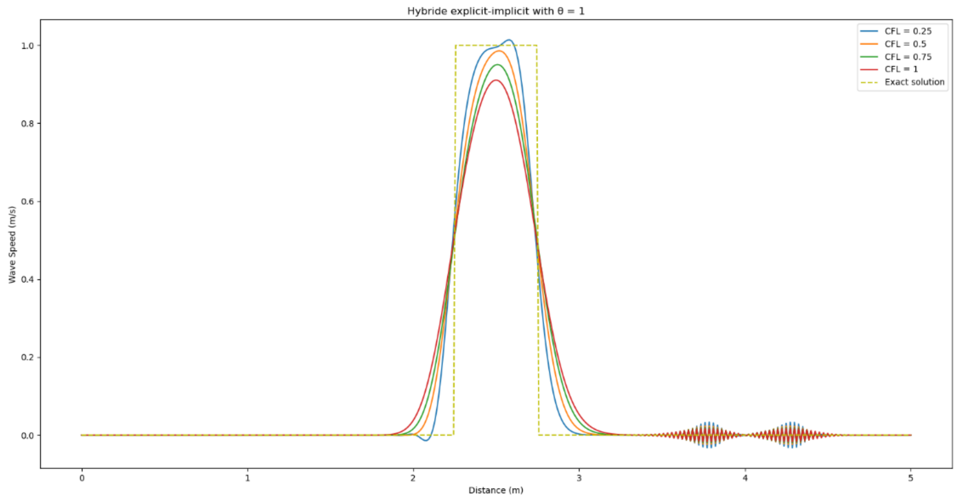
Like the previous scheme, Lax-Wendroff triggers some oscillations (dispersion), but they are more concentrated on the edges of the wave, resulting in a solution more accurate. It is also unstable for CFL higher than 1.

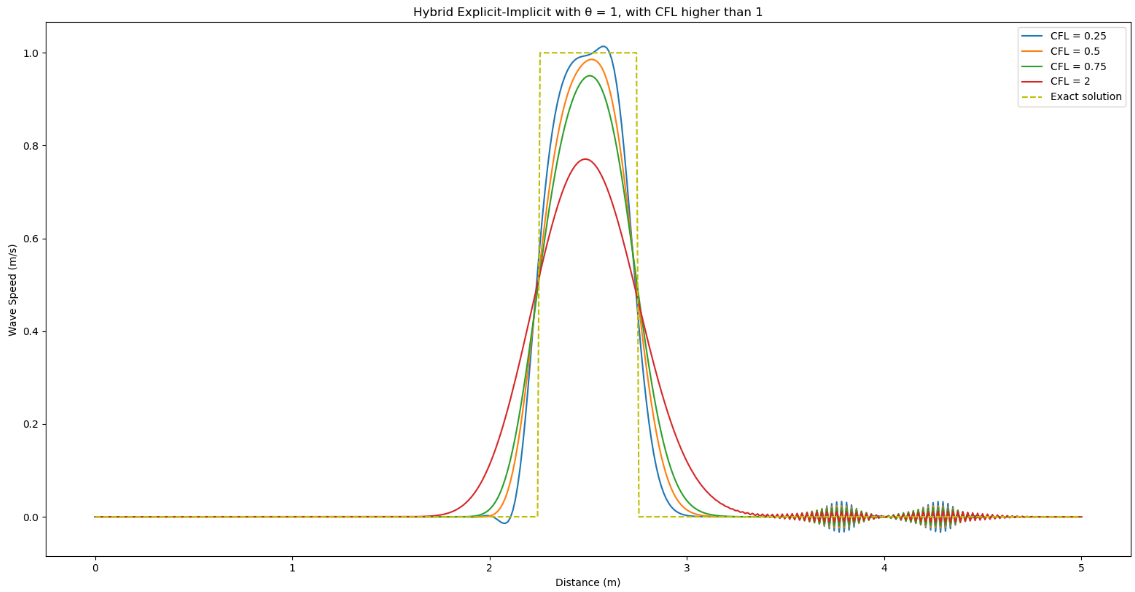


The Lax scheme is closer to the first scheme studied, the Explicit Backward. Indeed, there are some diffusions in the results, but they are larger than the Explicit Backward scheme. It is also stable for CFL less than 1.

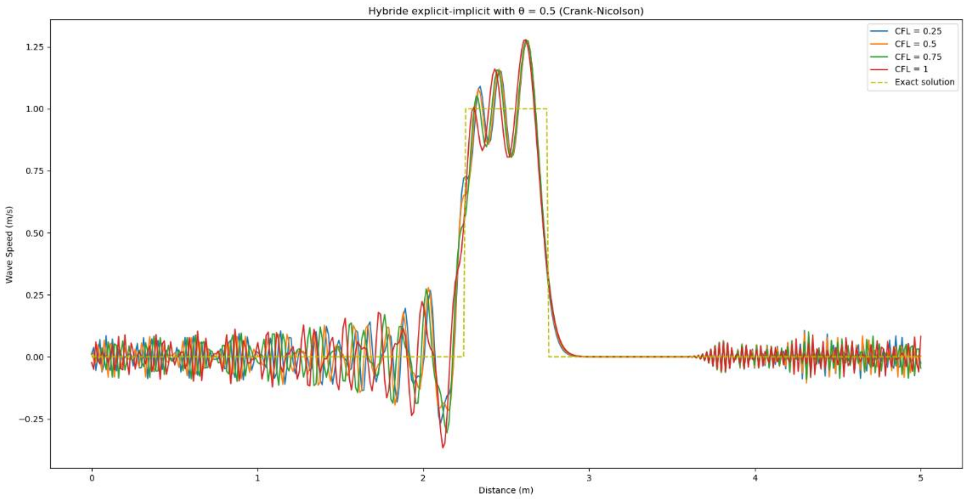


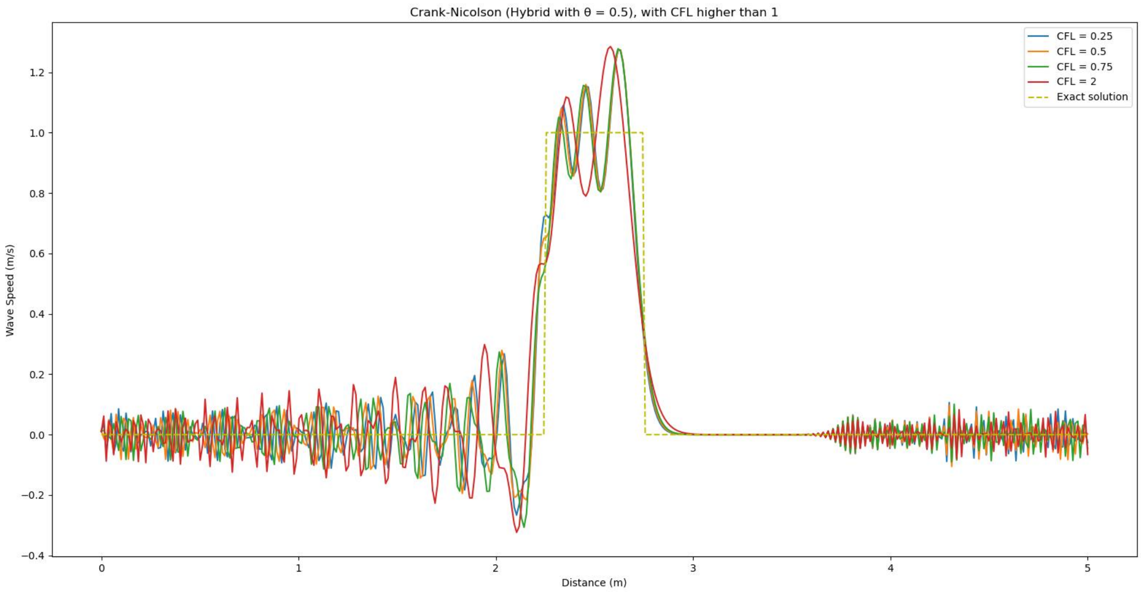
The Hybrid Explicit-Implicit with θ equal to 0 scheme is always unstable, as it is the same scheme as the Explicit Forward one.



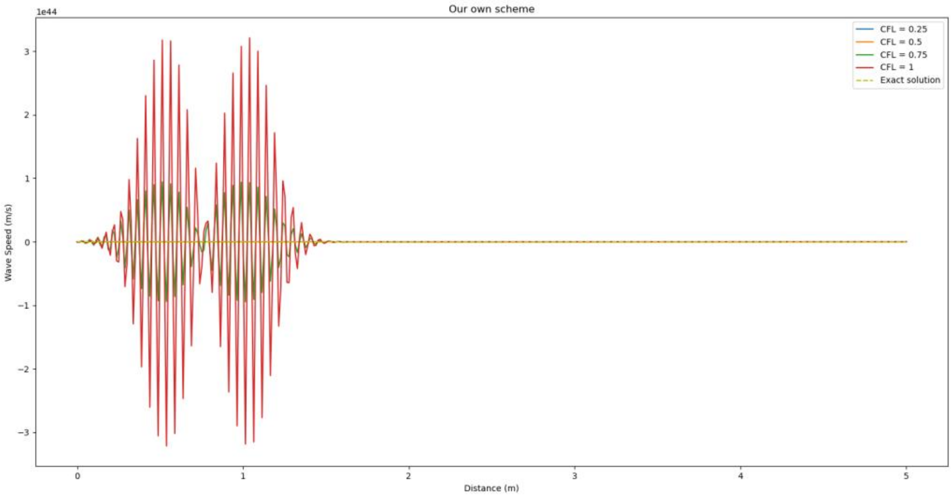


The Hybrid Explicit-Implicit with θ equal to 1 scheme is always stable. This scheme has some diffusion and some dispersion (after the wave we can see some oscillations). The accuracy is better near CFL equal 1.





Crank-Nicolson is stable for all CFL. Is it not what we have found in our derivation (we found a growth-rate always equal to 1). There are a lot of dispersion in the results and changing the CFL number seems to have no effect on the results.



The scheme we studied (4th order in time and 2nd order in space) is unstable for all CFL number. Therefore we cannot analyse the accuracy of the scheme with numerical analysis.

3 ANNEXE 1

QUESTION 1)

a) $u_j^{n+1} = u_j^n - c\frac{\Delta t}{\Delta x}\left(u_j^n - u_{j-1}^n\right)$

STABILITY

$T_k = \bar{T}_k + \varepsilon_k$

$\Rightarrow u_j^{n+1} = u_j^n - c\frac{\Delta t}{\Delta x}\left(u_j^n - u_{j-1}^n\right)$

$\Rightarrow \cancel{\bar{u}_j^{n+1}} + \epsilon_j^{n+1} = \cancel{\bar{u}_j^n} + \epsilon_j^n - c\frac{\Delta t}{\Delta x}\left(\cancel{u_j^n} + \epsilon_j^n - \cancel{\bar{u}_{j-1}^n} - \bar{\epsilon}_{j-1}^n\right)$

$\Rightarrow \epsilon_j^{n+1} = \epsilon_j^n - \frac{c\Delta t}{\Delta x}\left(\epsilon_j^n - \epsilon_{j-1}^n\right)$

$\Rightarrow \epsilon_j^{n+1} = \epsilon_j^n - c\frac{\Delta t}{\Delta x}\epsilon_j^n + \frac{c\Delta t}{\Delta x}\epsilon_{j-1}^n$

$\Rightarrow \frac{\epsilon_j^{n+1}}{\epsilon_j^n} = 1 - \frac{c\Delta t}{\Delta x} + \frac{c\Delta t}{\Delta x}\frac{\epsilon_{j-1}^n}{\epsilon_j^n}$

$\epsilon = e^{at}e^{ik_m x} \Rightarrow \epsilon_j^n = e^{at}e^{ik_m x}\,;\ \epsilon_{j-1}^n = \epsilon^{at}\epsilon^{ik_m(x-\Delta x)}$

$\Rightarrow \frac{\epsilon_j^{n+1}}{\epsilon_j^n} = 1 - \frac{c\Delta t}{\Delta x} + \frac{c\Delta t}{\Delta x}\left(\frac{\cancel{\epsilon^{at}}\epsilon^{ik_m(\cancel{x}-\Delta x)}}{\cancel{\epsilon^{at}}\epsilon^{ik_m(\cancel{x})}}\right)$

$\Rightarrow \frac{\epsilon_j^{n+1}}{\epsilon_j^n} = 1 - \frac{c\Delta t}{\Delta x} + \frac{c\Delta t}{\Delta x}\left(\epsilon^{-ik_m(\Delta x)}\right) = 1 - \frac{c\Delta t}{\Delta x}\left(1 - \epsilon^{-ik_m\Delta x}\right) = |G|$

$\epsilon^{ik_m\Delta x} = \cos(k_m\Delta x) - i\sin(k_m\Delta x)\,;\ \frac{c\Delta t}{\Delta x} = \sigma$

$\Rightarrow |G| = 1 - \sigma\left(1 - \cos(k_m\Delta x) + i\sin(k_m\Delta x)\right)$

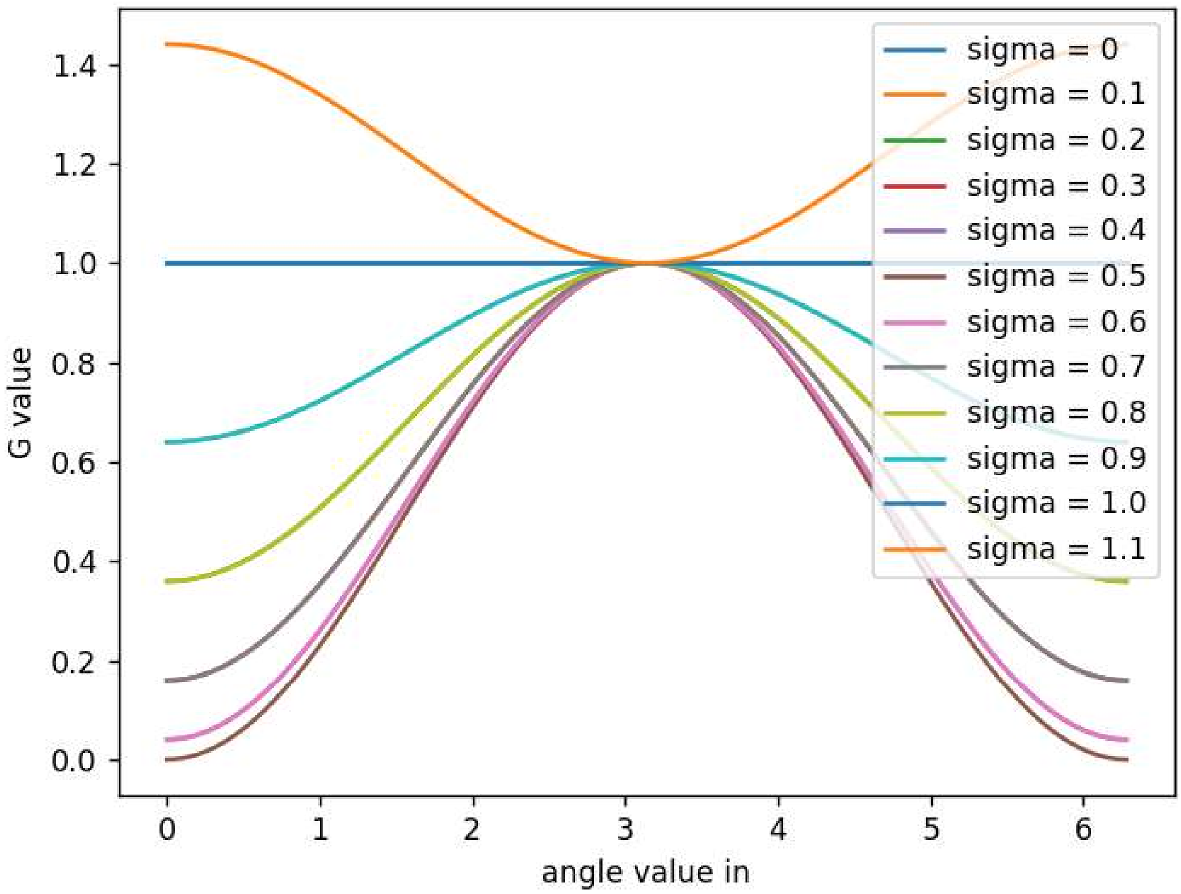
$\Rightarrow |G|^2 = \left(1 - \sigma - \sigma\cos(k_m\Delta x)\right)^2 + \left(i\sigma\sin(k_m\Delta x)\right)^2$

$= 1 - \sigma - \sigma\cos(k_m\Delta x) - \sigma + \sigma^2 + \sigma^2\cos(k_m\Delta x) - \sigma\cos(k_m\Delta x)$
$+ \sigma^2\cos(k_m\Delta x) + \sigma^2\cos^2(k_m\Delta x) + \sigma^2\sin^2(k_m\Delta x)$

$\Rightarrow 1 - 2\sigma - 2\sigma\cos(k_m\Delta x) + \sigma^2 + 2\sigma^2\cos(k_m\Delta x) +$
$\sigma^2\cos(k_m(\Delta) + \sigma^2\sin(k_m\Delta x)$

$$= 1 - 2\sigma - 2\sigma \cos(k_m \Delta x) + 2\sigma^2 + 2\sigma^2 \cos(k_m \Delta x) = |G|^2 \leq 1$$

We will show WITH a plot that $\sigma \leq 1$ to be stable when $k_m \Delta x \in [0, 2\pi]$:



ACCURACY

We have to develop into taylor

$$u_j^{n+1} = u_j^n - \frac{c\Delta t}{\Delta x}\left(u_j^n - u_{j-1}^n\right) \quad (1)$$

$$u_j^n = u_j^n \quad (2)$$

$$u_j^{n+1} = u_j^n + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} + \dots = \sum_{k=0}^{\infty} \frac{\Delta t^k}{k!}\frac{\partial^{(k)} u}{\partial t^{(k)}} \quad (3)$$

$$u_{j-1}^n = u_j^n - \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} - \Delta x^3 \frac{\partial^3 u}{\partial x^3} + \dots = \sum_{k=0}^{\infty} \frac{(-\Delta x)^k}{k!}\frac{\partial^{(k)} u}{\partial x^{(k)}} \quad (4)$$

We put (2), (3) and (4) into (1)

$$\Rightarrow \cancel{u_j^n} + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} = \cancel{u_j^n} - \frac{c\Delta t}{\Delta x}\left(\cancel{u_j^n} - \cancel{u_j^n} + \Delta x \frac{\partial u}{\partial x} - \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3}\right)$$

We put (2), (3) and (4) into (1)

$$\Rightarrow \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} = -\frac{c\Delta t}{\cancel{\Delta x}}\left(\cancel{\Delta x}\frac{\partial u}{\partial x} - \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3}\right)$$

$$\Rightarrow \Delta t \frac{\partial u}{\partial t} + \Delta t^2 \frac{\partial^2 u}{\partial t} + \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} = -c\Delta t\left(\frac{\partial u}{\partial x} - \frac{\Delta x}{2}\frac{\partial^2 u}{\partial x} + \frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3}\right)$$

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{-c\cancel{\Delta t}\left(\frac{\partial u}{\partial x} - \frac{\Delta x}{2}\frac{\partial^2 u}{\partial x} + \frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3}\right) - \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t} - \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3}}{\cancel{\Delta t}}$$

$$\Rightarrow \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = \frac{c\Delta x}{2}\frac{\partial^2 u}{\partial x} + \frac{c\Delta x^2}{6}\frac{\partial^3 u}{\partial x} - \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t} - \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t}$$

The order of convergeng for both time and space are 1

$$O(\Delta t) + O(\Delta x)$$

CONSISTENCY

The error term is: $\frac{c\,\Delta x}{2}\frac{\partial^2 u}{\partial x} + \frac{c\,\Delta x^2}{6}\frac{\partial^3 u}{\partial x} - \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t} - \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t}$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \overset{\to 0}{\frac{c\,\Delta x}{2}\frac{\partial^2 u}{\partial x}} + \overset{\to 0}{\frac{c\,\Delta x^2}{6}\frac{\partial^3 u}{\partial x}} - \overset{\to 0}{\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t}} - \overset{\to 0}{\frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t}} = 0$$

Thus, the backward scheme is consistent

QUESTION 1 c)

STABILITY : $U_j^{n+1} = U_j^{n-1} - \frac{c\Delta t}{\Delta x}\left(U_{j+1}^n - U_{j-1}^n\right)$

$U_m^k = \bar{U}_m^k + \epsilon_m^k$

$$\Rightarrow \cancel{\bar{U}_j^{n+1}} + \epsilon_j^{n+1} = \cancel{\bar{U}_j^{n-1}} + \epsilon_j^{n-1} - \frac{c\Delta t}{\Delta x}\left(\cancel{\bar{U}_{j+1}^n} + \epsilon_{j+1}^n - \cancel{\bar{U}_{j-1}^n} + \epsilon_{j-1}^n\right)$$

$$\Rightarrow \epsilon_j^{n+1} = \epsilon_j^{n-1} - \frac{c\Delta t}{\Delta x}\left(\epsilon_{j+1}^n - \epsilon_{j-1}^n\right)$$

$$\Rightarrow G = \frac{\epsilon_j^{n+1}}{\epsilon_j^n} = \frac{\epsilon_j^{n-1}}{\epsilon_j^n} - \frac{c\Delta t}{\Delta x}\left(\frac{\epsilon_{j+1}^n}{\epsilon_j^n} - \frac{\epsilon_{j-1}^n}{\epsilon_j^n}\right)$$

$\epsilon_j^n = e^{at} e^{ik_n \Delta t}$; $\frac{c\Delta t}{\Delta x}$

$$\Rightarrow G = \frac{e^{a(t-\Delta t)} e^{ik_n x}}{e^{at} e^{ik_n x}} - \sigma\left(\frac{e^{at} e^{ik_n(x+\Delta x)}}{e^{at} e^{ik_n x}} - \frac{e^{at} e^{ik_n(x-\Delta x)}}{e^{at} e^{ik_n x}}\right)$$

$$e^{-a\Delta t} - \sigma\left(e^{ik_n \Delta x} - e^{-ik_n \Delta x}\right)$$

$$\Rightarrow G = 1/G - \sigma\left(2i\sin(k_n \Delta x)\right)$$

We Multiply by G on each side

$$\Rightarrow G^2 = 1 - \sigma G\left(2i\sin(k_n\Delta x)\right)$$

$$\Rightarrow G^2 + \sigma G\left(2i\sin(k_n\Delta x)\right) - 1$$

$$G_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$b = 2i\,\sigma \sin(k_m \Delta x) \Rightarrow b^2 = -4\sigma^2 \sin^2(k_m \Delta x)$$

$$\Rightarrow b^2 - 4ac = -4\sigma^2 \sin^2(k_m \Delta x) + 4$$

$$\Rightarrow G = \frac{-i\cancel{2}\sigma \sin(k_m \Delta x) \pm \sqrt{\cancel{4}\left(1 - \sigma^2 \sin^2(k_m \Delta x)\right)}}{\cancel{2}}$$

$$\Rightarrow |G| = \left| -i\sigma \sin(k_m \Delta x) \pm \sqrt{1 - \sigma^2 \sin^2(k_m \Delta x)} \right| \leqslant 1$$

$$\Rightarrow |G|^2 = \sigma^2 \sin^2(k_m \Delta x) + 1 - \sigma^2 \sin^2(k_m \Delta x) = 1$$

$$\Rightarrow |G|^2 = 1 = |G|$$

THE CONDITION $|G| \leqslant 1$ IS RESPECTED

HOWEVER

WE NEED TO VERIFY THE $\Delta = 1 - \sigma^2 \sin^2(k_m \Delta x) \geqslant 0$

$$\Rightarrow \sigma^2 \sin^2(k_m \Delta x) \leqslant 1$$

WORST CASE $\sin^2(k_m \Delta x) = 1$

$$\Rightarrow \sigma^2 \leqslant 1$$

$$\Rightarrow \sigma = \frac{c\Delta t}{\Delta x} \leqslant 1$$

if $\Delta = b^2 - 4ac \leqslant 0$ THERE WOULD BE

An imaginary part that would add to $-i\sigma \sin(k_m \Delta x)$

ACCURACY

LETS DEVELLOP IN TAYLOR

$$u_j^{n+1} = u_j^{n-1} - \frac{c\Delta t}{\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) \Rightarrow u_j^{n+1} - u_j^{n-1} = -\frac{c\Delta t}{\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) \quad (1)$$

$$u_j^{n+1} = u_j^n + \Delta t\frac{\partial u}{\partial t} + \Delta t^2\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t} + \dots$$

$$u_j^{n-1} = u_j^n - \Delta t\frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} - \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} + \dots$$

$0 \quad 2\times \quad 0 \quad 2\times \quad 0 \quad 2\times \quad \dots$

$$u_j^{n+1} - u_j^{n-1} = 2\Delta t\frac{\partial u}{\partial t} + \frac{2\Delta t^3}{6}\frac{\partial^3 u}{\partial t} + \frac{2\Delta t^5}{120}\frac{\partial^5 u}{\partial t^5} + \dots \quad (2)$$

$$u_{j+1}^n - u_{j-1}^n = 2\Delta x\frac{\partial u}{\partial x} + \frac{2\Delta x^3}{6}\frac{\partial^3 u}{\partial x} + \frac{2\Delta x^5}{120}\frac{\partial^5 u}{\partial x^5} + \dots \quad (3)$$

(2) (3) into (1)

$$\Rightarrow 2\Delta t\frac{\partial u}{\partial t} + \frac{2\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} + \frac{2\Delta t^5}{120}\frac{\partial^5 u}{\partial t^5} + \dots = -\frac{c\Delta t}{\Delta x}\left(2\Delta x\frac{\partial u}{\partial x} + \frac{2\Delta x^3}{6}\frac{\partial^3 u}{\partial x} + \frac{2\Delta x^5}{120}\frac{\partial^5 u}{\partial x^5} + \dots\right)$$

$$\Rightarrow \cancel{\Delta t}\frac{\partial u}{\partial t} + \frac{\Delta t^{\cancel{3}\,2}}{6}\frac{\partial^3 u}{\partial t^3} + \frac{\Delta t^{\cancel{5}\,4}}{120}\frac{\partial^5 u}{\partial t^5} + \dots = -\frac{c\cancel{\Delta t}}{\cancel{\Delta x}}\left(\cancel{\Delta x}\frac{\partial u}{\partial x} + \frac{\Delta x^{\cancel{3}\,2}}{6}\frac{\partial^3 u}{\partial x} + \frac{\Delta x^{\cancel{5}\,4}}{120}\frac{\partial^5 u}{\partial x^5} + \dots\right)$$

$$\Rightarrow \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3} + \frac{\Delta x^4}{120}\frac{\partial^5 u}{\partial x^5} - \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^3} - \frac{\Delta t^4}{120}\frac{\partial^5 u}{\partial t^5}$$

$$\Rightarrow \qquad = \mathcal{O}(\Delta x^2) + \mathcal{O}(\Delta t^2)$$

CONSISTENCY

$$\lim_{\substack{\Delta t \to 0 \\ \Delta x \to 0}} \left(\frac{\overset{\to 0}{\Delta x^2}}{6} \frac{\partial^3 u}{\partial x^3} + \frac{\overset{\to 0}{\Delta x^4}}{120} \frac{\partial^5 u}{\partial x^5} - \frac{\overset{\to 0}{\Delta t^2}}{6} \frac{\partial^3 u}{\partial t^3} - \frac{\overset{\to 0}{\Delta t^4}}{120} \frac{\partial^5 u}{\partial t^5} \right) = 0$$

<u>THIS ALGORITHM IS CONSISTENT</u>

QUESTION 1a)

STABILITY

$$U_j^{n+1} = \left(\frac{U_{j+1}^n + U_{j-1}^n}{2}\right) - \frac{c\Delta t}{\Delta x}\left(U_{j+1}^n - U_{j-1}^n\right)$$

$U_n^k = \bar{U}_n^k + \epsilon_n^k$

$$\Rightarrow \cancel{\bar{U}_j^{n+1}} + \epsilon_j^{n+1} = \left(\frac{\cancel{\bar{U}_{j+1}^n} + \epsilon_{j+1}^n + \cancel{\bar{U}_{j-1}^n} + \epsilon_{j-1}^n}{2}\right) - \frac{c\Delta t}{2\Delta x}\left(\cancel{\bar{U}_{j+1}^n} + \epsilon_{j+1}^n - \cancel{\bar{U}_{j-1}^n} - \epsilon_{j-1}^n\right)$$

$$\Rightarrow \epsilon_j^{n+1} = \left(\frac{\epsilon_{j+1}^n + \epsilon_{j-1}^n}{2}\right) - \frac{c\Delta t}{2\Delta x}\left(\epsilon_{j+1}^n - \epsilon_{j-1}^n\right)$$

$\epsilon_n^k = e^{a k} e^{i k_m n}$; $\frac{c\Delta t}{2\Delta x} = \sigma$

$$\Rightarrow G = \frac{\epsilon_j^{n+1}}{\epsilon_j^n} = \frac{1}{2}\left(\frac{e^{at}e^{ik_n(x+\Delta x)}}{e^{at}e^{ik_n x}} + \frac{e^{at}e^{ik_n(x-\Delta x)}}{e^{at}e^{ik_n x}}\right) - \sigma\left(\frac{e^{at}e^{ik_n(x+\Delta x)}}{e^{at}e^{ik_n x}} - \frac{e^{at}e^{ik_n(x-\Delta x)}}{e^{at}e^{ik_n x}}\right)$$

$$\Rightarrow G = \frac{1}{2}\left(e^{ik_m\Delta x} + e^{-ik_n\Delta x}\right) - \sigma\left(e^{ik_n\Delta x} - e^{-ik_m\Delta x}\right)$$

$e^{ix} + e^{-ix} = 2\cos(x)$; $e^{ix} - e^{-ix} = -2i\sin(x)$;

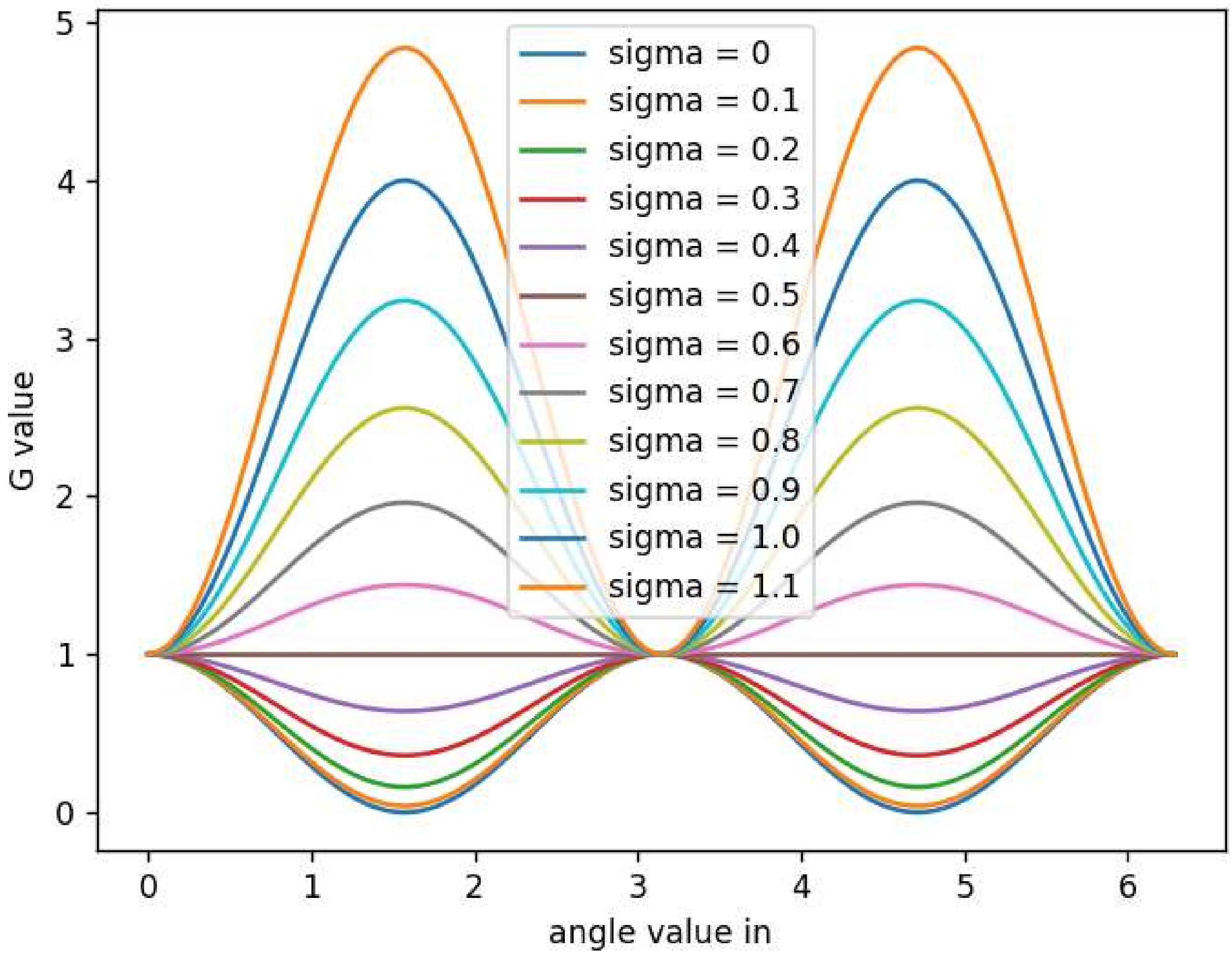
$$\Rightarrow G = \cos(k_n\Delta x) + 2\sigma\, i\sin(k_n\Delta x)$$

$$\Rightarrow |G|^2 = \cos^2(k_n\Delta x) + 4\sigma^2\sin^2(k_n\Delta x) < 1$$

$$\Rightarrow \sigma^2 < \frac{-\cos^2(k_n\Delta x)}{4\sin^2(k_n\Delta x)} + 1$$

WE SEE ON THE GRAPH THAT SIGMA SHOULD B LOWER THAN 0,5.

$$\sigma = \frac{c\Delta t}{2\Delta X} \Rightarrow \frac{c\Delta t}{\Delta x} = CFL \leq 1$$


sigma = 0
sigma = 0.1
sigma = 0.2
sigma = 0.3
sigma = 0.4
sigma = 0.5
sigma = 0.6
sigma = 0.7
sigma = 0.8
sigma = 0.9
sigma = 1.0
sigma = 1.1
G value
angle value in
0
1
2
3
4
5
6

ACCURACY

$$u_j^{n+1} = \frac{\left(u_{j+1}^n + u_{j-1}^n\right)}{2} - \frac{c\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) \quad (1)$$

$$u_j^{n+1} = u_j^n + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} + \dots \quad (2)$$

$$u_{j+1}^n = u_j^n + \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3} + \dots$$

$$u_{j-1}^n = u_j^n - \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} - \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3} + \dots$$

$$\Rightarrow u_{j+1}^n + u_{j-1}^n = 2u_j^n + 2\frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} + 2\frac{\Delta x^4}{24}\frac{\partial^4 u}{\partial x^4} + \dots \quad (3)$$

$$\Rightarrow u_{j+1}^n - u_{j-1} = 2\Delta x \frac{\partial u}{\partial x} + 2\frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x} + 2\frac{\Delta x^5}{120}\frac{\partial^5 u}{\partial x^5} + \dots \quad (4)$$

(2) (3) (4) into (1)

$$\Rightarrow \cancel{u_j^n} + \Delta t\frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} + \dots = \frac{1}{2}\left(2\cancel{u_j^n} + \cancel{2}\frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} + \cancel{2}\frac{\Delta x^4}{24}\frac{\partial^4 u}{\partial x^4} + \dots\right) - \frac{c\Delta t}{\cancel{2\Delta x}}\left(\cancel{2\Delta x}\frac{\partial u}{\partial x} + \cancel{2}\frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x} + \cancel{2}\frac{\Delta x^5}{120}\frac{\partial^5 u}{\partial x^5} + \dots\right)$$

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{\Delta x^2}{2\Delta t}\frac{\partial^2 u}{\partial x} + \frac{\Delta x^4}{24\Delta t}\frac{\partial^4 u}{\partial x^4} + \dots - c\left(\frac{\partial u}{\partial x} + \frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x} + \frac{\Delta x^4}{120}\frac{\partial^5 u}{\partial x^5} + \dots\right) - \frac{\Delta t}{2}\frac{\partial^2 u}{\partial x} - \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \frac{\Delta x^2}{2\Delta t}\frac{\partial^2 u}{\partial x} + \frac{\Delta x^4}{24\Delta t}\frac{\partial^4 u}{\partial x^4} + \dots - c\left(\frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x} + \frac{\Delta x^4}{120}\frac{\partial^5 u}{\partial x^5} + \dots\right) - \frac{\Delta t}{2}\frac{\partial^2 u}{\partial x} - \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial x}$$

$$= O\left(\frac{\Delta x^2}{\Delta t}\right) + O(\Delta t) \quad ?$$

CONSISTENCY

$$\lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \frac{\Delta x^2}{2\Delta t}\frac{\partial^2 u}{\partial x} + \frac{\Delta x^4}{24\Delta t}\frac{\partial^4 u}{\partial x^4} + \dots - c\left(\frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x} + \frac{\Delta x^4}{120}\frac{\partial^5 u}{\partial x^5} + \dots\right) - \frac{\Delta t}{2}\frac{\partial^2 u}{\partial x} - \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial x} \neq 0$$

THE TERMS $\frac{\Delta x^n}{\Delta t}$ MAKES THIS ALGORITHM INCONSISTENT...

LETS CREATE OUR OWN ALGORITHM

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

SECOND ORDER SPACE

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{U_{j+1}^{n} - U_{j-1}^{n}}{2\Delta x}$$

FOURTH ORDER IN TIME

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{-U_j^{n+2} + 8U_j^{n+1} - 8U_j^{n-1} + U_j^{n-2}}{12\Delta t}$$

$$\Rightarrow \frac{-U_j^{n+2} + 8U_j^{n+1} - 8U_j^{n-1} + U_j^{n-2}}{12\Delta t} + c\left(\frac{U_{j+1}^{n} - U_{j-1}^{n}}{2\Delta x}\right) = 0$$

$$\Rightarrow \Rightarrow \frac{-\frac{1}{12}U_j^{n+2} + \frac{2}{3}U_j^{n+1} - \frac{2}{3}U_j^{n-1} + \frac{1}{12}U_j^{n-2}}{\Delta t} + c\left(\frac{U_{j+1}^{n} - U_{j-1}^{n}}{2\Delta x}\right) = 0$$

$$\Rightarrow U_j^{n+1} = \frac{1}{8}U_j^{n+2} + U_j^{n-1} - \frac{1}{8}U_j^{n-2} - \frac{3c\Delta t}{4\Delta x}\left(U_{j+1}^{n} - U_{j-1}^{n}\right)$$

$$\Rightarrow \frac{1}{8}U_j^{n+2} - U_j^{n+1} + U_j^{n-1} - \frac{1}{8}U_j^{n-2} - 3\frac{c\Delta t}{4\Delta x}\left(U_{j+1}^{n} - U_{j-1}^{n}\right) = 0$$

$$U_j^{n\pm 2} = U_j^n \pm 2\Delta t\frac{\partial u}{\partial t} + \frac{4\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} \pm \frac{8\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} + \frac{16}{24}\Delta t^4\frac{\partial^4 u}{\partial t} + \dots$$

$$U_j^{n\pm 1} = U_j^n \pm \Delta t\frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} \pm \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} + \frac{\Delta t^4}{24}\frac{\partial^4 u}{\partial t} + \dots$$

$$U_{j\pm 1}^n = U_j^n \pm \Delta x\frac{\partial u}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} \pm \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3} - \frac{\Delta x^4}{24}\frac{\partial^4 u}{\partial x} + \dots$$

We SAW BEFORE....

$$u_{j+1} - u_{j-1} = 2\Delta x \frac{\partial u}{\partial x} + \frac{2}{6}\Delta x^3 \frac{\partial^3 u}{\partial x^3} + \frac{2}{120}\Delta x^5 \frac{\partial^5 u}{\partial x^5} + \dots$$

$$\Rightarrow \frac{1}{8} u_j^{n+2} - u_j^{n+1} + u_j^{n-1} - \frac{1}{8} u_j^{n-2} = \left(u_j^{n-1} - u_j^{n+1}\right) + \frac{1}{8}\left(u_j^{n+2} - u_j^{n-2}\right)$$

$$\Rightarrow u_j^{n-1} - u_j^{n+1} = -2\left(\Delta t \frac{\partial u}{\partial t} + \frac{1}{6}\Delta t^3 \frac{\partial^3 u}{\partial t^3} + \frac{1}{120}\Delta t^5 \frac{\partial^5 u}{\partial t^5} + \dots\right)$$

$$\Rightarrow u_j^{n+2} - u_j^{n-2} = 2\left(2\Delta t \frac{\partial u}{\partial t} + \frac{8\Delta t^3}{6}\frac{\partial u}{\partial t} + \frac{32}{120}\Delta t^5 \frac{\partial u}{\partial t^5}\right)$$

$$\Rightarrow 2\cdot\frac{1}{8}\left(\left(2\Delta t\frac{\partial u}{\partial t} + \frac{8\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} + \frac{32}{120}\Delta t^5\frac{\partial u}{\partial t^5}\right) - \left(\Delta t\frac{\partial u}{\partial t} + \frac{1}{6}\Delta t^3\frac{\partial^3 u}{\partial t^3} + \frac{1}{120}\Delta t^5\frac{\partial^5 u}{\partial t^5} + \dots\right)\right) - \frac{3}{4}\frac{c\Delta t}{\Delta x}\left(\cancel{2}\left(\Delta x\frac{\partial u}{\partial x} + \frac{1}{6}\Delta x^3\frac{\partial^3 u}{\partial x^3} + \frac{1}{120}\Delta x^5\frac{\partial^5 u}{\partial x^5}\right)\right)$$

$$= 0$$

$$\Rightarrow 2\cdot\left(\left(2\Delta t\frac{\partial u}{\partial t} + \cancel{\frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3}} + \frac{4}{120}\Delta t^5\frac{\partial u}{\partial t^5}\right) - \left(\Delta t\frac{\partial u}{\partial t} + \cancel{\frac{1}{6}\Delta t^3\frac{\partial^3 u}{\partial t^3}} + \frac{1}{120}\Delta t^5\frac{\partial^5 u}{\partial t^5} + \dots\right)\right) - \frac{3}{4}\frac{c\Delta t}{\Delta x}\left(\cancel{2}\left(\Delta x\frac{\partial u}{\partial x} + \frac{1}{6}\Delta x^3\frac{\partial^3 u}{\partial x^3} + \frac{1}{120}\Delta x^5\frac{\partial^5 u}{\partial x^5}\right)\right)$$

$$\Rightarrow \Delta t\frac{\partial u}{\partial t} + \frac{1}{40}\Delta t^5\frac{\partial^5 u}{\partial t^5} + \dots - \frac{c\Delta t}{\cancel{\Delta x}}\left(\cancel{\Delta x}\frac{\partial u}{\partial x} + \frac{1}{6}\Delta x^{\overset{2}{\cancel{3}}}\frac{\partial^3 u}{\partial x^3} + \frac{1}{120}\Delta x^{\overset{4}{\cancel{5}}}\frac{\partial^5 u}{\partial x^5}\right) = 0$$

DIVIDE PER Δt; TAKE NOTE THAT THE TERM Δt^3 IS GONE

$$\Rightarrow \frac{\partial u}{\partial t} + \frac{1}{40}\Delta t^4\frac{\partial^5 u}{\partial t^5} - \frac{3}{4}c\left(\frac{\partial u}{\partial x} + \frac{1}{6}\Delta x^2\frac{\partial^3 u}{\partial x^3} + \frac{1}{120}\Delta x^4\frac{\partial^5 u}{\partial x}\right) = 0$$

$$\Rightarrow \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = \underbrace{-\frac{1}{40}\Delta t^4\frac{\partial^5 u}{\partial t} + \frac{3}{4}c\left(\frac{1}{6}\Delta x^2\frac{\partial^3 u}{\partial x^3} + \frac{1}{120}\Delta x^4\frac{\partial^5 u}{\partial x}\right)}_{\text{ERROR}}$$

We see that it's $O(\Delta t^4) + O(\Delta x^2)$

CONSISTENCY

$$ERROR = -\frac{1}{40}\Delta t^4 \frac{\partial^5 u}{\partial t} + \frac{3c}{4}\left(\frac{1}{6}\Delta x^2 \frac{\partial^3 u}{\partial x^3} + \frac{1}{120}\Delta x^4 \frac{\partial^5 u}{\partial x}\right)$$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} ERROR = 0$$

THUS, THIS ALGORITHM IS <u>CONSISTENT</u>

STABILITY

$$\frac{1}{8} U_j^{n+2} - U_j^{n+1} + U_j^{n-1} - \frac{1}{8} U_j^{n-2} - \frac{3}{4} \frac{c\Delta t}{\Delta x} \left(U_{j+1}^n - U_{j-1}^n \right) = 0$$

LINEAR EQUATION SO:

$$\Rightarrow \frac{1}{8} \epsilon_j^{n+2} - \epsilon_j^{n+1} + \epsilon_j^{n-1} - \frac{1}{8} \epsilon_j^{n-2} - \frac{3}{4} \frac{c\Delta t}{\Delta x} \left(\epsilon_{j+1}^n - \epsilon_{j-1}^n \right) = 0$$

AFTER DIVISION BY $\epsilon_j^n = e^{at} e^{ik_m \Delta x}$; $\sigma = \frac{3}{4} \frac{c\Delta t}{\Delta x}$

$$\Rightarrow \frac{1}{8} e^{2a\Delta t} - e^{a\Delta t} + e^{-a\Delta t} - \frac{1}{8} e^{-2a\Delta t} - \sigma \left(e^{ik_m \Delta x} - e^{-ik_m \Delta x} \right) = 0$$

$$\Rightarrow \frac{1}{8} e^{2a\Delta t} - e^{a\Delta t} + e^{-a\Delta t} - \frac{1}{8} e^{-2a\Delta t} - \sigma \left(2i \sin(k_m \Delta x) \right) = 0$$

$$\Rightarrow \frac{1}{8} (e^{a\Delta t})^2 - (e^{a\Delta t}) + (e^{a\Delta t})^{-1} - \frac{1}{8} (e^{a\Delta t})^{-2} - \sigma \left(2i \sin k_m \Delta x \right) = 0$$

WE HAVE TO FIND $e^{a\Delta t} = G$

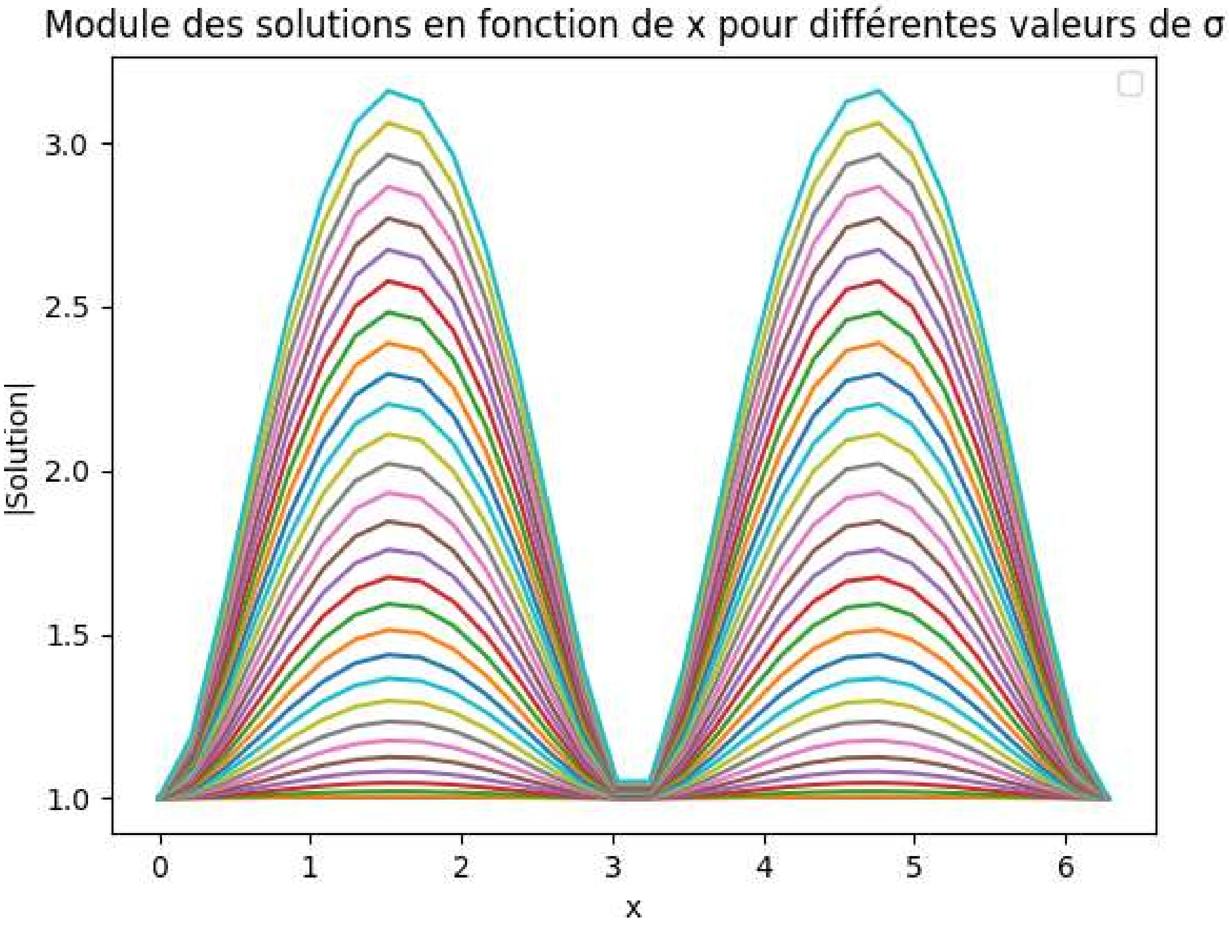
$$\Rightarrow \frac{1}{8} G^2 - G + \frac{1}{G} - \frac{1}{8} \frac{1}{G^2} - \sigma \left(2i \sin(k_m \Delta x) \right) = 0$$

HARD EQATION BUT... WE SEE THAT WHEN $\sigma = 0 \Rightarrow G = 1$. THUS IF $\sigma \neq 0$ THE SINUS PART WILL JUST MAKES $|G|$ OSCILATTE AROUND 1 ANS MOST IMPORTANTY AT THE VALUES > 1 WHICH WILL MAKE IT UNSTABLE...

WE DID NEWTON-RALPHSON ANALYSIS TO PROVE OUR POINT

WE SEE IN THE PLOT THAT $\sigma \in [0, 2]$ WILL ALWAYS MAKE $G > 1$ WHICH MAKES IT UNSTABLE

```
def f(x,C):
    if x == 0:
        return np.inf
    else:
        return 1/8 * x**2 - x + x**-1 - 1/8 * x **-2 - C

def fp(x):
    if x == 0:
        return np.inf
    else: return 1/4*x - 1 + -x**-2 - 1/4 * x**-3

def newton_raphson(C,x0, tol = 1e-6, max_iter = 400):
    iteration = 0
    while iteration < max_iter:
        fx = f(x0,C)
        fpx = fp(x0)
        if fpx == 0:
            print("you are cooked bruv")
            return None

        x1 = x0 - fx/fpx
        if all(abs (x1 - x0)) < tol:
            return x1
    x0 = x1
    iteration += 1

    print ("ur cooked")




x = np.linspace(0,2*np.pi,30)
sigma_mat = np.linspace(0,3,30)
for sigma in sigma_mat:
    C = j*sigma*(2*np.sin(x))

    x0 = 1 + 0j
    sol = newton_raphson(C,x0, tol = 1e-6, max_iter = 100)
    print(abs(sol))
    plt.plot(x,abs(sol))

plt.xlabel('x')
plt.ylabel('|Solution|')
plt.title('Module des solutions en fonction de x pour différentes valeurs de σ')
plt.legend()
plt.show()
```



Module des solutions en fonction de x pour différentes valeurs de σ
|Solution|
x
3.0
2.5
2.0
1.5
1.0
0
1
2
3
4
5
6

1. b) Forward Time-Centered Space

$$u_j^{n+1} = u_j^n - \frac{c\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right)$$

Stability: $\varepsilon_j^{n+1} = \varepsilon_j^n - \frac{c\Delta t}{2\Delta x}\left(\varepsilon_{j+1}^n - \varepsilon_{j-1}^n\right)$

so $G = \left|\frac{\varepsilon_j^{n+1}}{\varepsilon_j^n}\right| = \left| 1 - \frac{c\Delta t}{2\Delta x}\frac{\varepsilon_{j+1}^n - \varepsilon_{j-1}^n}{\varepsilon_j^n}\right|$

$= \left| 1 - \frac{c\Delta t}{2\Delta x}\frac{e^{at}e^{ik_m(x+\Delta x)} - e^{at}e^{ik_m(x-\Delta x)}}{e^{at}e^{ik_m x}}\right|$

$= \left| 1 - \frac{c\Delta t}{2\Delta x}\left(e^{ik_m\Delta x} - e^{-ik_m\Delta x}\right)\right|$

$= \left| 1 - \frac{c\Delta t}{2\Delta x}\left(2 i \sin(k_m\Delta x)\right)\right|$

$= \sqrt{1^2 + \left(\frac{\Delta t}{\Delta x}c\right)^2 \sin^2(k_m\Delta x)}$

So $G < 1$ if $1 + \left(\frac{c\Delta t}{\Delta x}\right)^2 < 1 \iff \frac{c\Delta t}{\Delta x} < 0$

This scheme is unconditionally unstable.

Accuracy: $\cancel{u} + \Delta t\frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + \dots = \cancel{u} - \frac{c\Delta t}{2\cancel{\Delta x}}\left[\left(\cancel{u} + \cancel{\Delta x}\frac{\partial u}{\partial x} + \cancel{\frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}} + \dots\right)\right.$

$\left. - \left(\cancel{u} - \cancel{\Delta x}\frac{\partial u}{\partial x} + \cancel{\frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}} \dots\right)\right]$

$\Rightarrow \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = -\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} - c\frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3} + \dots$

The scheme is $\begin{cases} 1^{st} \text{ order in time} \\ 2^{nd} \text{ order in space} \end{cases}$ and is consistent.

d) Lax - Wendroff

$$u_j^{n+1} = u_j^n - \frac{c\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{1}{2}\left(\frac{c\Delta t}{\Delta x}\right)^2\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

Stability: $\varepsilon_j^{n+1} = \varepsilon_j^n - \frac{c\Delta t}{2\Delta x}\left(\varepsilon_{j+1}^n - \varepsilon_{j-1}^n\right) + \frac{1}{2}\left(\frac{c\Delta t}{\Delta x}\right)^2\left(\varepsilon_{j+1}^n - 2\varepsilon_j^n + \varepsilon_{j-1}^n\right)$

so $G = \left| 1 - \frac{c\Delta t}{2\Delta x}\frac{\varepsilon_{j+1}^n - \varepsilon_{j-1}^n}{\varepsilon_j^n} + \frac{1}{2}\left(\frac{c\Delta t}{\Delta x}\right)^2 \frac{\varepsilon_{j+1}^n - 2\varepsilon_j^n + \varepsilon_{j-1}^n}{\varepsilon_j^n} \right|$

$= \left| 1 - \frac{c\Delta t}{\cancel{2}\Delta x}\left[\cancel{2} i \sin(k_m \Delta x)\right] + \frac{\cancel{1}}{\cancel{2}}\left(\frac{c\Delta t}{\Delta x}\right)^2\left[-\cancel{2} + \cancel{2}\cos(k_m\Delta x)\right] \right|$

$= \left| \left[1 + \left(\frac{c\Delta t}{\Delta x}\right)^2 (\cos(k_m\Delta x) - 1)\right] + \left[-\frac{c\Delta t}{\Delta x} i \sin(k_m \Delta x)\right] \right|$

$= \left| \left[1 - \left(\frac{c\Delta t}{\Delta x}\right)^2 2\sin^2\left(\frac{k_m\Delta x}{2}\right)\right] + \left[-\frac{c\Delta t}{\Delta x} i \sin(k_m \Delta x)\right] \right|$

$= \left[1 - 4\left(\frac{c\Delta t}{\Delta x}\right)^2 \sin^2\left(\frac{k_m\Delta x}{2}\right) + 4\left(\frac{c\Delta t}{\Delta x}\right)^4 \sin^4\left(\frac{k_m\Delta x}{2}\right) + \left(\frac{c\Delta t}{\Delta x}\right)^2 \sin^2(k_m\Delta x)\right]^{1/2}$

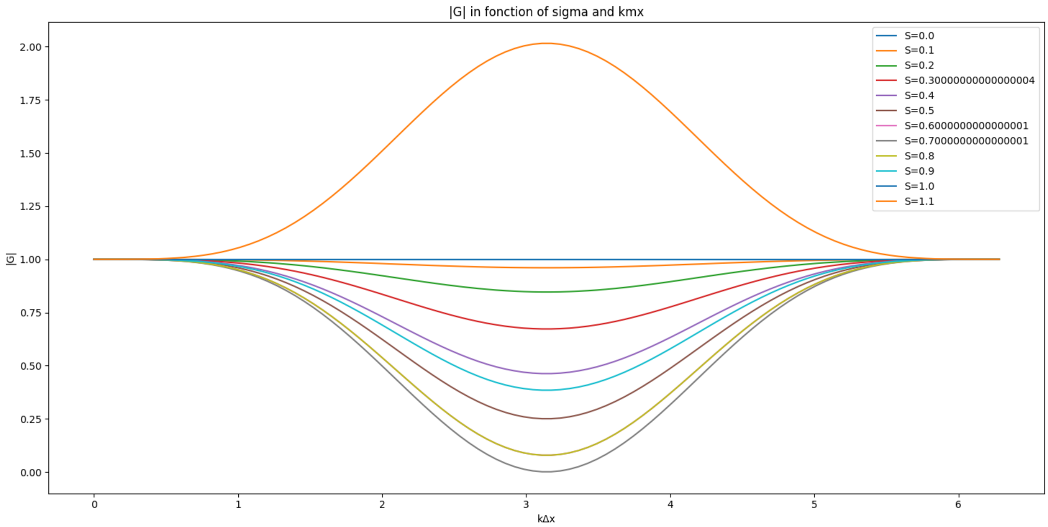
By numerical analysis (see figure), the scheme is stable when CFL < 1 ...

Accuracy: $\cancel{u} + \Delta t\frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} = \cancel{u} - \frac{c\Delta t}{\cancel{2\Delta x}}\left(\cancel{2\Delta x}\frac{\partial u}{\partial x} + \frac{\Delta x^2}{3}\frac{\partial^3 u}{\partial x^3} + \right) + \frac{1}{2}\left(\frac{c\Delta t}{\cancel{\Delta x^2}}\right)^2$

$\left(\cancel{\Delta x\frac{\partial u}{\partial x}} + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} + \dots - \cancel{\Delta x\frac{\partial u}{\partial x}} + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} - \dots\right)$

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = -\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} - c\frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3} + c^2\frac{\Delta t}{2}\frac{\partial^2 u}{\partial x^2} + \dots$$

The scheme is $\begin{cases} 1^{st} \text{ order in time} \\ 2^{nd} \text{ order in space} \end{cases}$ and is consistent



|G| in fonction of sigma and kmx
S=0.0
S=0.1
S=0.2
S=0.30000000000000004
S=0.4
S=0.5
S=0.6000000000000001
S=0.7000000000000001
S=0.8
S=0.9
S=1.0
S=1.1
|G|
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
0
1
2
3
4
5
6
kΔx

f) Hybrid explicit-implicit

$$u_j^{n+1} + \theta\left(\frac{CFL}{2}\right)\left(u_{j+1}^{n+1} - u_{j-1}^{n+1}\right) = u_j^n - (1-\theta)\frac{CFL}{2}\left(u_{j+1}^n - u_{j-1}^n\right)$$

* with $\theta = 0$, this scheme is equivalent of the one studied in 1.b) so unstable, 1st order in time, 2nd order in space, and consistent.

* with $\theta = 1$: $u_j^{n+1} + \left(\frac{CFL}{2}\right)\left(u_{j+1}^{n+1} - u_{j-1}^{n+1}\right) = u_j^n$

Stability: $e^{a\Delta t} + \frac{CFL}{2}\left(e^{a\Delta t}e^{ik\Delta x} - e^{a\Delta t}e^{-ik\Delta x}\right) = 1$

$$e^{a\Delta t}\left[1 + \frac{CFL}{2}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)\right] = 1$$

$$e^{a\Delta t} = 1 \Big/ 1 + \frac{CFL}{2}\left(2i\sin(k\Delta x)\right)$$

see figure (always stable)

$$= 1 - \frac{CFL}{\cancel{2}}\left(\cancel{2}i\sin(k\Delta x)\right) \Big/ 1 + CFL^2 \sin^2(k\Delta x)$$

Accuracy

$CFL = \frac{c\Delta t}{\Delta x}$

$$\cancel{u} + \Delta t\frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + \dots + \left(\frac{CFL}{2}\right)\left[\left(\cancel{u} + \Delta t\frac{\partial u}{\partial t} + \cancel{\Delta x\frac{\partial u}{\partial x}} + \dots\right) - \left(\cancel{u} + \Delta t\frac{\partial u}{\partial t} - \cancel{\Delta x\frac{\partial u}{\partial x}} + \dots\right)\right] = \cancel{u}$$

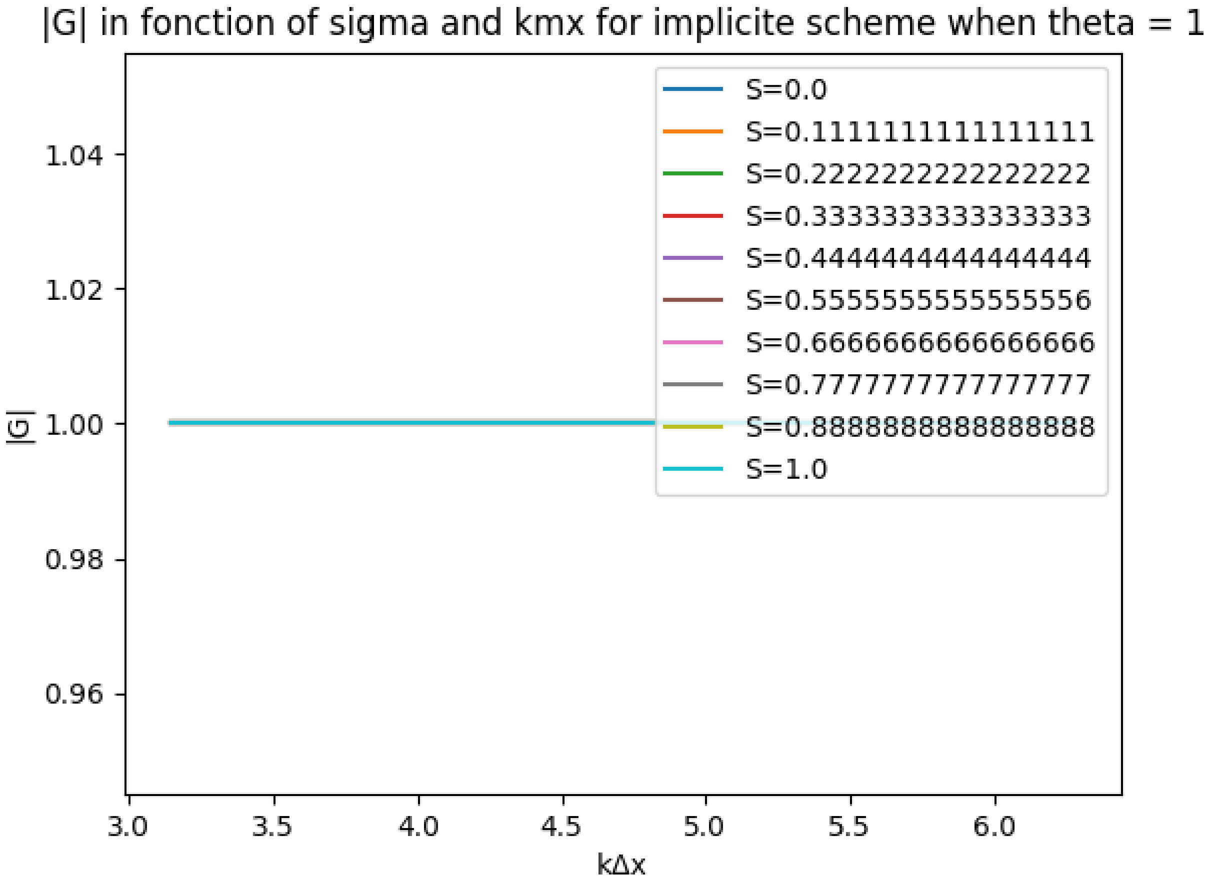
$$\Rightarrow \cancel{\Delta t}\frac{\partial u}{\partial t} + \frac{\Delta t^{\cancel{2}}}{2}\frac{\partial^2 u}{\partial t^2} + \dots + \left(\frac{CFL}{2}\right)\left[2\Delta t\frac{\partial u}{\partial t} + \dots + 2\frac{\Delta x^2}{\cancel{2}}\frac{\partial^2 u}{\partial x^2} + \dots\right] = \frac{1}{\Delta t}$$

$$\Rightarrow \cancel{\frac{\partial u}{\partial t}} + \cancel{\Delta t}\frac{\partial u}{\partial t} + \frac{\Delta t^{\cancel{1}}}{2}\frac{\partial^2 u}{\partial t^2} + \dots + \frac{CFL}{2}\left[2\cancel{\Delta x}\frac{\partial u}{\partial x} + \dots\right] = 1\,\Delta t$$

1st time

2nd space

$$\Rightarrow \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = -\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} + \Delta t - \frac{CFL}{2}\left[2\frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3}\right] \rightarrow -\frac{c}{12}\Delta x^2\frac{\partial^3 u}{\partial x^3}$$


|G| in fonction of sigma and kmx for implicite scheme when theta = 1
S=0.0
S=0.1111111111111111
S=0.2222222222222222
S=0.3333333333333333
S=0.4444444444444444
S=0.5555555555555556
S=0.6666666666666666
S=0.7777777777777777
S=0.8888888888888888
S=1.0
1.04
1.02
|G| 1.00
0.98
0.96
3.0
3.5
4.0
4.5
5.0
5.5
6.0
kΔx

$$u_j^{n+1} + \theta\left(\frac{CFL}{2}\right)(u_{j+1}^{n+1} - u_{j-1}^{n+1}) = u_j^n - (1-\theta)\frac{CFL}{2}(u_{j+1}^n - u_{j-1}^n)$$

$\theta = 0.5$

$$u_j^{n+1} + \frac{CFL}{4}(u_{j+1}^{n+1} - u_{j-1}^{n+1}) = u_j^n - \frac{CFL}{4}(u_{j+1}^n - u_{j-1}^n)$$

Stability: $e^{a\Delta t} + \frac{CFL}{4}(e^{a\Delta t}e^{ik\Delta x} - e^{a\Delta t}e^{-ik\Delta x}) = 1 - \frac{CFL}{4}(e^{ik\Delta x} - e^{-ik\Delta x})$

$G = |e^{a\Delta t}| = 1 - a / 1 + a$ with $a = \frac{CFL}{4}(2i\sin(k\Delta x))$

$= (1-a)^2 / 1 - a^2 = \dfrac{1 - 4i\sin(k\Delta x)\frac{CFL}{4} - 4\sin^2(k\Delta x)\frac{CFL^2}{16}}{1 + \frac{CFL^2}{2^2}\sin^2(k\Delta x)}$

By numerical analysis the scheme is unconditionally unstable ($G = 1$ for all CFL).

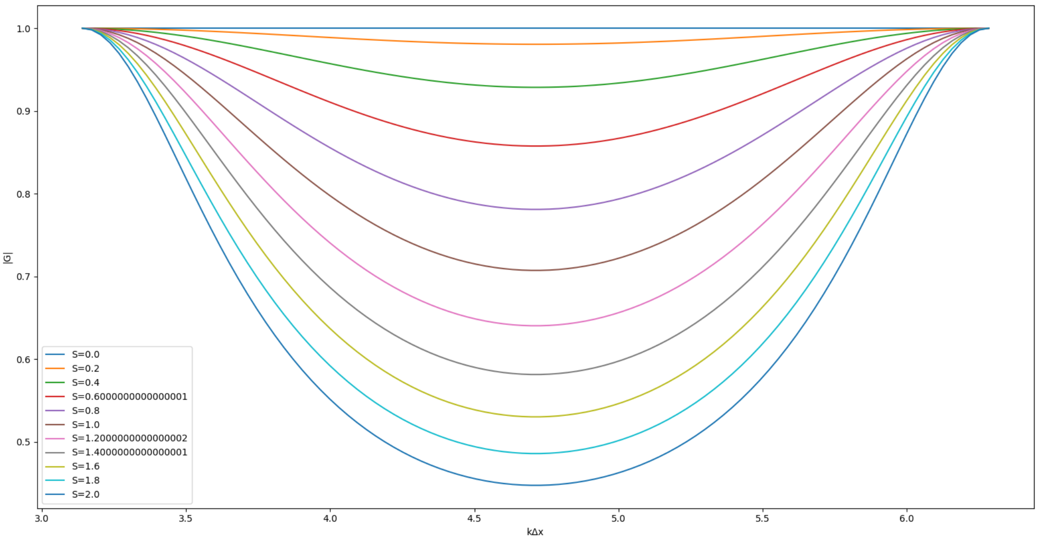
Accuracy $\cancel{u} + \Delta t\frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + \cdots + \frac{CFL}{4}\left[\left(\cancel{u} + \cancel{\Delta t\frac{\partial u}{\partial t}} + \Delta x\frac{\partial u}{\partial x} + \cancel{\frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2}} + \cancel{\frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}} + \cdots\right) - \left(\cancel{u} + \cancel{\Delta t\frac{\partial u}{\partial t}} - \Delta x\frac{\partial u}{\partial x} + \cancel{\frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2}} + \cancel{\frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}} \cdots\right)\right]$

$= \cancel{u} - \frac{CFL}{4}\left[\left(\cancel{u} + \Delta x\frac{\partial u}{\partial x} + \cancel{\frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}} + \cdots\right) - \left(\cancel{u} - \Delta x\frac{\partial u}{\partial x} + \cancel{\frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}} \cdots\right)\right]$

$\Rightarrow \cancel{\Delta t}\frac{\partial u}{\partial t} + \frac{\Delta t^{\cancel{2}}}{2}\frac{\partial^2 u}{\partial t^2} + \cdots + \frac{c\cancel{\Delta t}}{4\cancel{\Delta x}}\left(2\cancel{\Delta x}\frac{\partial u}{\partial x} + 2\frac{\Delta x^{\cancel{3}2}}{6}\frac{\partial^3 u}{\partial x^3} + \cdots\right) = -\frac{c\cancel{\Delta t}}{4\cancel{\Delta x}}\left(2\cancel{\Delta x}\frac{\partial u}{\partial x} + 2\frac{\Delta x^{\cancel{3}2}}{6}\frac{\partial^3 u}{\partial x^3} + \cdots\right)$

$\Rightarrow \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = -\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} - \frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3} - \cdots$ Space: 2nd, Time: 1st

Consistent



4 ANNEXE 2: GITHUB LINK

Click for Github Acces