LETS CREATE OUR OWN ALGORITHM

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

SECOND ORDER SPACE

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{u_{j+1}^{n} - u_{j-1}^{n}}{2\Delta x}$$

FOURTH ORDER IN TIME

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{-u_j^{n+2} + 8u_j^{n+1} - 8u_j^{n-1} + u_j^{n-2}}{12\Delta t}$$

$$\Rightarrow \frac{-u_j^{n+2} + 8u_j^{n+1} - 8u_j^{n-1} + u_j^{n-2}}{12\Delta t} + c\left(\frac{u_{j+1}^{n} - u_{j-1}^{n}}{2\Delta x}\right) = 0$$

$$\Rightarrow \Rightarrow \frac{-\frac{1}{12}u_j^{n+2} + \frac{2}{3}u_j^{n+1} - \frac{2}{3}u_j^{n-1} + \frac{1}{12}u_j^{n-2}}{\Delta t} + c\left(\frac{u_{j+1}^{n} - u_{j-1}^{n}}{2\Delta x}\right) = 0$$

$$\Rightarrow u_j^{n+1} = \frac{1}{8}u_j^{n+2} + u_j^{n-1} - \frac{1}{8}u_j^{n-2} - \frac{3c\Delta t}{4\Delta x}\left(u_{j+1}^{n} - u_{j-1}^{n}\right)$$

$$\Rightarrow \frac{1}{8}u_j^{n+2} - u_j^{n+1} + u_j^{n-1} - \frac{1}{8}u_j^{n-2} - 3\frac{c\Delta t}{4\Delta x}\left(u_{j+1}^{n} - u_{j-1}^{n}\right) = 0$$

$$u_j^{n\pm 2} = u_j^n \pm 2\Delta t\frac{\partial u}{\partial t} + \frac{4\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} \pm \frac{8\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} + \frac{16}{24}\Delta t^4\frac{\partial^4 u}{\partial t} + \ldots$$

$$u_j^{n\pm 1} = u_j^n \pm \Delta t\frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} \pm \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} + \frac{\Delta t^4}{24}\frac{\partial^4 u}{\partial t} + \ldots$$

$$u_{j\pm 1}^n = u_j^n \pm \Delta x\frac{\partial u}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} \pm \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3} - \frac{\Delta x^4}{24}\frac{\partial^4 u}{\partial x} + \ldots$$

We SAW BEFORE....

$$u_{j+1} - u_{j-1} = 2\Delta x \frac{\partial u}{\partial x} + \frac{2}{6}\Delta x^3 \frac{\partial^3 u}{\partial x^3} + \frac{2}{120}\Delta x^5 \frac{\partial^5 u}{\partial x^5} + \dots$$

$$\Rightarrow \frac{1}{8}u_j^{n+2} - u_j^{n+1} + u_j^{n-1} - \frac{1}{8}u_j^{n-2} = \left(u_j^{n-1} - u_j^{n+1}\right) + \frac{1}{8}\left(u_j^{n+2} - u_j^{n-2}\right)$$

$$\Rightarrow u_j^{n-1} - u_j^{n+1} = -2\left(\Delta t \frac{\partial u}{\partial t} + \frac{1}{6}\Delta t^3 \frac{\partial^3 u}{\partial t^3} + \frac{1}{120}\Delta t^5 \frac{\partial^5 u}{\partial t^5} + \dots\right)$$

$$\Rightarrow u_j^{n+2} - u_j^{n-2} = 2\left(2\Delta t \frac{\partial u}{\partial t} + \frac{8\Delta t^3}{6}\frac{\partial u}{\partial t} + \frac{32}{120}\Delta t^5 \frac{\partial u}{\partial t^5}\right)$$

$$\Rightarrow 2\cdot\frac{1}{8}\left(\left(2\Delta t \frac{\partial u}{\partial t} + \frac{8\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} + \frac{32}{120}\Delta t^5 \frac{\partial u}{\partial t^5}\right) - \left(\Delta t \frac{\partial u}{\partial t} + \frac{1}{6}\Delta t^3 \frac{\partial^3 u}{\partial t^3} + \frac{1}{120}\Delta t^5 \frac{\partial u}{\partial t^5} + \dots\right)\right) - \frac{3}{4}\frac{c\Delta t}{\Delta x}\left(2\left(\Delta x \frac{\partial u}{\partial x} + \frac{1}{6}\Delta x^3 \frac{\partial^3 u}{\partial x^3} + \frac{1}{120}\Delta x^5 \frac{\partial^5 u}{\partial x^5}\right)\right) = 0$$

$$\Rightarrow 2\cdot\left(\left(2\Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^3}{\cancel{6}}\frac{\partial^3 u}{\partial t^3} + \frac{4}{120}\Delta t^5 \frac{\partial u}{\partial t^5}\right) - \left(\Delta t \frac{\partial u}{\partial t} + \cancel{\frac{1}{6}\Delta t^3 \frac{\partial^3 u}{\partial t^3}} + \frac{1}{120}\Delta t^5 \frac{\partial u}{\partial t^5} + \dots\right)\right) - \frac{3}{4}\frac{c\Delta t}{\Delta x}\left(\cancel{2}\left(\Delta x \frac{\partial u}{\partial x} + \frac{1}{6}\Delta x^3 \frac{\partial^3 u}{\partial x^3} + \frac{1}{120}\Delta x^5 \frac{\partial^5 u}{\partial x^5}\right)\right)$$

$$\Rightarrow \Delta t \frac{\partial u}{\partial t} + \frac{1}{40}\Delta t^5 \frac{\partial^5 u}{\partial t^5} + \dots - \frac{c\Delta t}{\cancel{\Delta x}}\left(\cancel{\Delta x}\frac{\partial u}{\partial x} + \frac{1}{6}\Delta x^{\overset{2}{\cancel{3}}}\frac{\partial^3 u}{\partial x^3} + \frac{1}{120}\Delta x^{\overset{4}{\cancel{5}}}\frac{\partial^5 u}{\partial x^5}\right) = 0$$

DIVIDE PER Δt; TAKE NOTE THAT THE TERM Δt^3 IS GONE

$$\Rightarrow \frac{\partial u}{\partial t} + \frac{1}{40}\Delta t^4 \frac{\partial^5 u}{\partial t^5} - \frac{3}{4}c\left(\frac{\partial u}{\partial x} + \frac{1}{6}\Delta x^2 \frac{\partial^3 u}{\partial x^3} + \frac{1}{120}\Delta x^4 \frac{\partial^5 u}{\partial x}\right) = 0$$

$$\Rightarrow \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = \underbrace{-\frac{1}{40}\Delta t^4 \frac{\partial^5 u}{\partial t} + \frac{3}{4}c\left(\frac{1}{6}\Delta x^2 \frac{\partial^3 u}{\partial x^3} + \frac{1}{120}\Delta x^4 \frac{\partial^5 u}{\partial x}\right)}_{\text{ERROR}}$$

We see that it's $O(\Delta t^4) + O(\Delta x^2)$

CONSISTENCY

$$\text{ERROR} = -\frac{1}{40}\Delta t^4 \frac{\partial^5 u}{\partial t} + \frac{3c}{4}\left(\frac{1}{6}\Delta x^2 \frac{\partial^3 u}{\partial x^3} + \frac{1}{120}\Delta x^4 \frac{\partial^5 u}{\partial x}\right)$$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \text{ERROR} = 0$$

THUS, THIS ALGORITHM IS <u>CONSISTENT</u>

STABILITY

$$\frac{1}{8}U_j^{n+2} - U_j^{n+1} + U_j^{n-1} - \frac{1}{8}U_j^{n-2} - \frac{3}{4}\frac{c\Delta t}{\Delta x}\left(U_{j+1}^{n} - U_{j-1}^{n}\right) = 0$$

LINEAR EQUATION SO:

$$\Rightarrow \frac{1}{8}\epsilon_j^{n+2} - \epsilon_j^{n+1} + \epsilon_j^{n-1} - \frac{1}{8}\epsilon_j^{n-2} - \frac{3}{4}\frac{c\Delta t}{\Delta x}\left(\epsilon_{j+1}^{n} - \epsilon_{j-1}^{n}\right) = 0$$

AFTER DIVISION BY $\epsilon_j^n = e^{at}e^{ik_m\Delta x}$; $\sigma = 3/4\,\frac{c\Delta t}{\Delta x}$

$$\Rightarrow \frac{1}{8}e^{2a\Delta t} - e^{a\Delta t} + e^{-a\Delta t} - \frac{1}{8}e^{2a\Delta t} - \sigma\left(e^{ik_m\Delta x} - e^{-ik_m\Delta x}\right) = 0$$

$$\Rightarrow \frac{1}{8}e^{2a\Delta t} - e^{a\Delta t} + e^{-a\Delta t} - \frac{1}{8}e^{2a\Delta t} - \sigma\left(2i\sin(k_m\Delta x)\right) = 0$$

$$\Rightarrow \frac{1}{8}(e^{a\Delta t})^2 - (e^{a\Delta t}) + (e^{a\Delta t})^{-1} - \frac{1}{8}(e^{a\Delta t})^{-2} - \sigma(2i\sin k_m\Delta x) = 0$$

We HAVE TO FIND $e^{a\Delta t} = G$

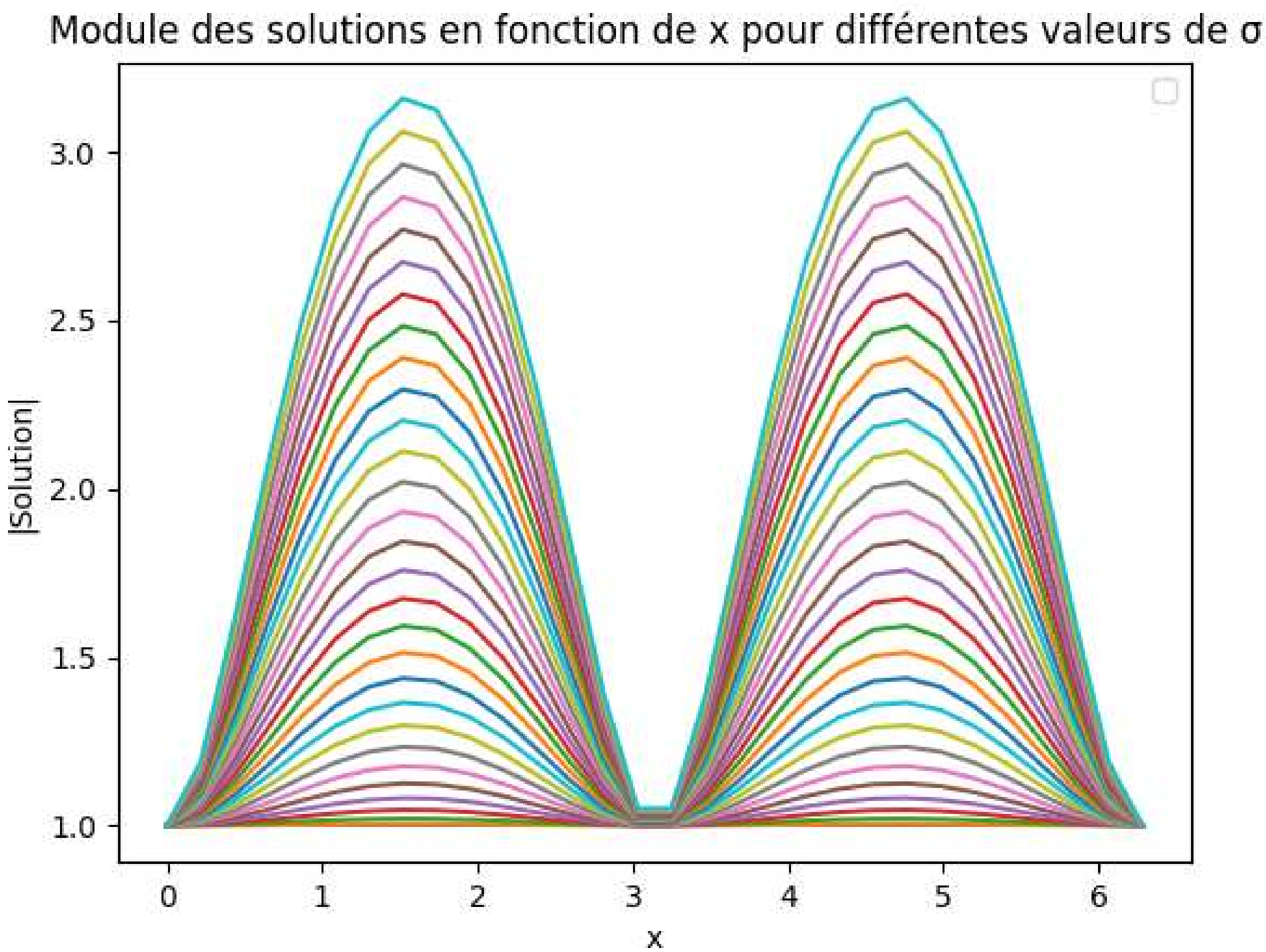
$$\Rightarrow \frac{1}{8}G^2 - G + 1/G - \frac{1}{8}\,1/G^2 - \sigma\left(2i\sin(k_m\Delta x)\right) = 0$$

HARD EQATION BUT... WE SEE THAT WHEN $\sigma = 0 \Rightarrow G = 1$. THUS IF $\sigma \neq 0$ THE SINUS PART WILL JUST MAKES $|G|$ OSCILATTE AROUND 1 ANS MOST IMPORTANTY AT THE VALUES >1 WHICH WILL MAKE IT UNSTABLE...

WE DID NEWTON-RALPHSON ANALYSIS TO PROVE OUR POINT

WE SEE IN THE PLOT THAT $\sigma \in [0, 2]$ WILL ALWAYS MAKE $G > 1$ WHICH MAKES IT UNSTABLE

```
def f(x,C):
    if x == 0:
        return np.inf
    else:
        return 1/8 * x**2 - x + x**-1 - 1/8 * x **-2 - C

def fp(x):
    if x == 0:
        return np.inf
    else: return 1/4*x - 1 + -x**-2 - 1/4 * x**-3

def newton_raphson(C,x0, tol = 1e-6, max_iter = 400):
    iteration = 0
    while iteration < max_iter:
        fx = f(x0,C)
        fpx = fp(x0)
        if fpx == 0:
            print("you are cooked bruv")
            return None

        x1 = x0 - fx/fpx
        if all(abs (x1 - x0)) < tol:
            return x1
    x0 = x1
    iteration += 1

    print ("ur cooked")



x = np.linspace(0,2*np.pi,30)
sigma_mat = np.linspace(0,3,30)
for sigma in sigma_mat:
    C = j*sigma*(2*np.sin(x))

    x0 = 1 + 0j
    sol = newton_raphson(C,x0, tol = 1e-6, max_iter = 100)
    print(abs(sol))
    plt.plot(x,abs(sol))

plt.xlabel('x')
plt.ylabel('|Solution|')
plt.title('Module des solutions en fonction de x pour différentes valeurs de σ')
plt.legend()
plt.show()
```


Module des solutions en fonction de x pour différentes valeurs de σ
|Solution|
x
3.0
2.5
2.0
1.5
1.0
0
1
2
3
4
5
6