f) Hybrid explicit-implicit

$$u_j^{n+1} + \theta\left(\frac{CFL}{2}\right)\left(u_{j+1}^{n+1} - u_{j-1}^{n+1}\right) = u_j^n - (1-\theta)\frac{CFL}{2}\left(u_{j+1}^n - u_{j-1}^n\right)$$

* with $\theta = 0$, this scheme is equivalent of the one studied in 1. b) so unstable, 1st order in time, 2nd order in space, and consistent.

* with $\theta = 1$: $u_j^{n+1} + \left(\frac{CFL}{2}\right)\left(u_{j+1}^{n+1} - u_{j-1}^{n+1}\right) = u_j^n$

Stability:

$$e^{a\Delta t} + \frac{CFL}{2}\left(e^{a\Delta t}e^{ik\Delta x} - e^{a\Delta t}e^{-ik\Delta x}\right) = 1$$

$$e^{a\Delta t}\left[1 + \frac{CFL}{2}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)\right] = 1$$

$$e^{a\Delta t} = 1 \Big/ 1 + \frac{CFL}{2}\left(2i\sin(k\Delta x)\right)$$

see figure (always stable)

$$= 1 - \frac{CFL}{\cancel{2}}\left(\cancel{2}i\sin(k\Delta x)\right) \Big/ 1 + CFL^2 \sin^2(k\Delta x)$$

Accuracy

$CFL = \frac{c\,\Delta t}{\Delta x}$

$$\cancel{u} + \Delta t\frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + \dots + \left(\frac{CFL}{2}\right)\left[\left(\cancel{u} + \Delta t\frac{\partial u}{\partial t} + \cancel{\Delta x\frac{\partial u}{\partial x}} + \dots\right) - \left(\cancel{u} + \Delta t\frac{\partial u}{\partial t} - \cancel{\Delta x\frac{\partial u}{\partial x}} + \dots\right)\right] = \cancel{u}$$

$$\Rightarrow \cancel{\Delta t}\frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + \dots + \left(\frac{CFL}{2}\right)\left[2\Delta t\frac{\partial u}{\partial t} + \dots + 2\frac{\Delta x^2}{\cancel{2}}\frac{\partial^2 u}{\partial x^2} \dots\right] = \frac{1}{\Delta t}$$

$$\Rightarrow \cancel{\frac{\partial u}{\partial t}} + \Delta t\frac{\partial u}{\partial t} + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} + \dots + \frac{CFL}{2}\left[2\cancel{\Delta x}\frac{\partial u}{\partial x} + \right] = 1\,\Delta t$$

1st time   2nd space

$$\Rightarrow \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = -\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} + (\Delta t) - \frac{CFL}{2}\left[2\frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3}\right] \rightarrow -\frac{c}{12}\Delta x^2\frac{\partial^3 u}{\partial x^3}$$