$$u_j^{n+1} + \theta\left(\frac{CFL}{2}\right)(u_{j+1}^{n+1} - u_{j-1}^{n+1}) = u_j^n - (1-\theta)\frac{CFL}{2}(u_{j+1}^n - u_{j-1}^n)$$

$\theta = 0.5$

$$u_j^{n+1} + \frac{CFL}{4}\left(u_{j+1}^{n+1} - u_{j-1}^{n+1}\right) = u_j^n - \frac{CFL}{4}\left(u_{j+1}^{n} - u_{j-1}^{n}\right)$$

Stability: $e^{a\Delta t} + \frac{CFL}{4}\left(e^{a\Delta t}e^{ik\Delta x} - e^{a\Delta t}e^{-ik\Delta x}\right) = 1 - \frac{CFL}{4}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)$

$G = |e^{a\Delta t}| = 1-a \,/\, 1+a$ with $a = \frac{CFL}{4}\left(2i\sin(k\Delta x)\right)$

$$= (1-a)^2 / 1-a^2 = \frac{1 - 4i\sin(k\Delta x)\frac{CFL}{4} - 4\sin^2(k\Delta x)\frac{CFL^2}{16}}{1 + \frac{CFL^2}{2^2}\sin^2(k\Delta x)}$$

By numerical analysis the scheme is unconditionally unstable ($G = 1$ for all CFL).

Accuracy: $\cancel{u} + \Delta t\frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + \dots + \frac{CFL}{4}\left[\left(\cancel{u} + \cancel{\Delta t\frac{\partial u}{\partial t}} + \Delta x\frac{\partial u}{\partial x} + \cancel{\frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2}} + \cancel{\frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}} + \dots\right) - \left(\cancel{u} + \cancel{\Delta t\frac{\partial u}{\partial t}} - \Delta x\frac{\partial u}{\partial x} + \cancel{\frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2}} + \cancel{\frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}} \dots\right)\right]$

$$= \cancel{u} - \frac{CFL}{4}\left[\left(\cancel{u} + \Delta x\frac{\partial u}{\partial x} + \cancel{\frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}} + \dots\right) - \left(\cancel{u} - \Delta x\frac{\partial u}{\partial x} + \cancel{\frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}} \dots\right)\right]$$

$$\Rightarrow \cancel{\Delta t}\frac{\partial u}{\partial t} + \frac{\Delta t^{\cancel{2}}}{2}\frac{\partial^2 u}{\partial t^2} + \dots + \frac{c\cancel{\Delta t}}{4\cancel{\Delta x}}\left(2\cancel{\Delta x}\frac{\partial u}{\partial x} + 2\frac{\Delta x^{\cancel{3}2}}{6}\frac{\partial^3 u}{\partial x^3} + \dots\right) = -\frac{c\cancel{\Delta t}}{4\cancel{\Delta x}}\left(2\cancel{\Delta x}\frac{\partial u}{\partial x} + 2\frac{\Delta x^{\cancel{3}2}}{6}\frac{\partial^3 u}{\partial x^3} + \dots\right)$$

$$\Rightarrow \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = -\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} - \frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3} - \dots$$

Space: 2^{nd}, Time: 1^{st}

Consistent