QUESTION 1)

a) $u_j^{n+1} = u_j^n - c\frac{\Delta t}{\Delta x}\left(u_j^n - u_{j-1}^n\right)$

STABILITY

$T_k = \bar{T}_k + \varepsilon_k$

$$\Rightarrow u_j^{n+1} = u_j^n - c\frac{\Delta t}{\Delta x}\left(u_j^n - u_{j-1}^n\right)$$

$$\Rightarrow \cancel{\bar{u}_j^{n+1}} + \epsilon_j^{n+1} = \cancel{\bar{u}_j^n} + \epsilon_j^n - c\frac{\Delta t}{\Delta x}\left(\cancel{\bar{u}_j^n} + \epsilon_j^n - \cancel{\bar{u}_{j-1}^n} - \epsilon_{j-1}^n\right)$$

$$\Rightarrow \epsilon_j^{n+1} = \epsilon_j^n - \frac{c\Delta t}{\Delta x}\left(\epsilon_j^n - \epsilon_{j-1}^n\right)$$

$$\Rightarrow \epsilon_j^{n+1} = \epsilon_j^n - c\frac{\Delta t}{\Delta x}\epsilon_j^n + \frac{c\Delta t}{\Delta x}\epsilon_{j-1}^n$$

$$\Rightarrow \frac{\epsilon_j^{n+1}}{\epsilon_j^n} = 1 - \frac{c\Delta t}{\Delta x} + \frac{c\Delta t}{\Delta x}\frac{\epsilon_{j-1}^n}{\epsilon_j^n}$$

$$\epsilon = e^{at}e^{ik_m x} \Rightarrow \epsilon_j^n = e^{at}e^{ik_m x};\ \epsilon_{j-1}^n = e^{at}e^{ik_m(x-\Delta x)}$$

$$\Rightarrow \frac{\epsilon_j^{n+1}}{\epsilon_j^n} = 1 - \frac{c\Delta t}{\Delta x} + \frac{c\Delta t}{\Delta x}\left(\frac{\cancel{e^{at}}e^{ik_m(\cancel{x}-\Delta x)}}{\cancel{e^{at}}e^{ik_m(\cancel{x})}}\right)$$

$$\Rightarrow \frac{\epsilon_j^{n+1}}{\epsilon_j^n} = 1 - \frac{c\Delta t}{\Delta x} + \frac{c\Delta t}{\Delta x}\left(e^{-ik_m(\Delta x)}\right) = 1 - \frac{c\Delta t}{\Delta x}\left(1 - e^{-ik_m\Delta x}\right) = |G|$$

$$e^{-ik_m\Delta x} = \cos(k_m\Delta x) - i\sin(k_m\Delta x);\quad \frac{c\Delta t}{\Delta x} = \sigma$$

$$\Rightarrow |G| = 1 - \sigma\left(1 - \cos(k_m\Delta x) + i\sin(k_m\Delta x)\right)$$

$$\Rightarrow |G|^2 = \left(1 - \sigma - \sigma\cos(k_m\Delta x)\right)^2 + \left(i\sigma\sin(k_m\Delta x)\right)^2$$

$$= 1 \underline{-\sigma} - \underline{\sigma\cos(k_m\Delta x)} \underline{-\sigma} + \sigma^2 + \underline{\sigma^2\cos(k_m\Delta x)} \underline{-\sigma\cos(k_m\Delta x)}$$
$$+ \underline{\sigma^2\cos(k_m\Delta x)} + \sigma^2\cos^2(k_m\Delta x) + \sigma^2\sin^2(k_m\Delta x)$$

$$\Rightarrow 1 - 2\sigma - 2\sigma\cos(k_m\Delta x) + \sigma^2 + 2\sigma^2\cos(k_m\Delta x) +$$
$$\sigma^2\cos(k_m(\Delta) + \sigma^2\sin(k_m\Delta x)$$

$$= 1 - 2\sigma - 2\sigma \cos(k_m \Delta x) + 2\sigma^2 + 2\sigma^2 \cos(k_m \Delta x) = |G|^2 \leq 1$$

We will show WITH a plot that $\sigma \leq 1$ to be stable whe $k_m \Delta x \in [0, 2\pi]$:

ACCURACY

We have to develop into taylor

$$u_j^{n+1} = u_j^n - \frac{c\Delta t}{\Delta x}\left(u_j^n - u_{j-1}^n\right) \quad (1)$$

$$u_j^n = u_j^n \quad (2)$$

$$u_j^{n+1} = u_j^n + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} + \dots = \sum_{k=0}^{\infty} \frac{\Delta t^k}{k!}\frac{\partial^{(k)} u}{\partial t^{(k)}} \quad (3)$$

$$u_{j-1}^n = u_j^n - \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} - \Delta x^3 \frac{\partial^3 u}{\partial x^3} + \dots = \sum_{k=0}^{\infty} \frac{(-\Delta x)^k}{k!}\frac{\partial^{(k)} u}{\partial x^{(k)}} \quad (4)$$

We put (2),(3) and (4) into (1)

$$\Rightarrow \cancel{u_j^n} + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} = \cancel{u_j^n} - \frac{c\Delta t}{\Delta x}\left(\cancel{u_j^n} - \cancel{u_j^n} + \Delta x \frac{\partial u}{\partial x} - \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3}\right)$$

We put (2),(3) and (4) into (1)

$$\Rightarrow \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} = -\frac{c\Delta t}{\cancel{\Delta x}}\left(\cancel{\Delta x}\frac{\partial u}{\partial x} - \frac{\Delta x^{\cancel{2}\,1}}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^{\cancel{3}\,2}}{6}\frac{\partial^3 u}{\partial x^3}\right)$$

$$\Rightarrow \Delta t \frac{\partial u}{\partial t} + \Delta t^2 \frac{\partial^2 u}{\partial t} + \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} = -c\Delta t\left(\frac{\partial u}{\partial x} - \frac{\Delta x}{2}\frac{\partial^2 u}{\partial x} + \frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3}\right)$$

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{-c\cancel{\Delta t}\left(\frac{\partial u}{\partial x} - \frac{\Delta x}{2}\frac{\partial^2 u}{\partial x} + \frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3}\right) - \frac{\Delta t^{\cancel{2}\,1}}{2}\frac{\partial^2 u}{\partial t} - \frac{\Delta t^{\cancel{3}\,2}}{6}\frac{\partial^3 u}{\partial t^3}}{\cancel{\Delta t}}$$

$$\Rightarrow \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = \frac{c\Delta x}{2}\frac{\partial^2 u}{\partial x} + \frac{c\Delta x^2}{6}\frac{\partial^3 u}{\partial x} - \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t} - \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t}$$

The order of convergency for both time and space are 1

$$\mathcal{O}(\Delta t) + \mathcal{O}(\Delta x)$$

CONSISTENCY

The error term is: $\frac{c\Delta x}{2}\frac{\partial^2 u}{\partial x} + \frac{c\Delta x^2}{6}\frac{\partial^3 u}{\partial x} - \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t} - \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t}$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \frac{c\overset{\to 0}{\Delta x}}{2}\frac{\partial^2 u}{\partial x} + \frac{c\overset{\to 0}{\Delta x^2}}{6}\frac{\partial^3 u}{\partial x} - \frac{\overset{\to 0}{\Delta t}}{2}\frac{\partial^2 u}{\partial t} - \frac{\overset{\to 0}{\Delta t^2}}{6}\frac{\partial^3 u}{\partial t} = 0$$

Thus, the backward scheme is consistent

FORWARD

$$U_j^{n+1} = U_j^n - \frac{c\Delta t}{\Delta x}\left(U_{j+1}^n - U_j^n\right) \quad (1)$$

STABILITY

$$U_m^k = \bar{U}_m^k + \epsilon_m^k \quad (2)$$

(2) into (1)

$$\Rightarrow \cancel{\bar{U}_j^{n+1}} + \epsilon_j^{n+1} = \cancel{\bar{U}_j^n} + \epsilon_j^n - \frac{c\Delta t}{\Delta x}\left(\bar{U}_{j+1}^n + \epsilon_{j+1}^n - \cancel{U_j^n} - \epsilon_j^n\right)$$

$$\Rightarrow \epsilon_j^{n+1} = \epsilon_j^n - \frac{c\Delta t}{\Delta x}\left(\epsilon_{j+1}^n - \epsilon_j^n\right)$$

lets form $|G| = \frac{\epsilon_j^{n+1}}{\epsilon_j^n}$

$$\Rightarrow \frac{\epsilon_j^{n+1}}{\epsilon_j^n} = 1 - \frac{c\Delta t}{\Delta x}\left(\frac{\epsilon_{j+1}^n}{\epsilon_j^n} - 1\right) = G$$

$$\epsilon_l^k = e^{ak} e^{iK_m l}$$

$$\Rightarrow G = 1 - \frac{c\Delta t}{\Delta x}\left(\frac{\cancel{e^{at}} e^{iK_m(x+\Delta x)}}{\cancel{e^{at}} e^{iK_m \cancel{x}}} - 1\right)$$

$$= 1 - \frac{c\Delta t}{\Delta x}\left(e^{iK_m\Delta x} - 1\right)$$

$$\Rightarrow G = 1 - \frac{c\Delta t}{\Delta x}\left(\cos(K_m\Delta x) + i\sin(K_m\Delta x) - 1\right) \leq 1$$

$$\sigma = \frac{c\Delta t}{\Delta x}$$

$$\Rightarrow |G|^2 = (\text{real part})^2 + (\text{im part})^2$$

$$= \left(1 - \sigma\cos(K_m\Delta x) + \sigma\right)^2 + \left(-\sigma\sin(K_m\Delta x)\right)^2$$

$$= 1 - \sigma\cos(K_m\Delta x) + \sigma - \sigma\cos(K_m\Delta x) + \sigma^2\cos^2(K_m\Delta x) - \sigma^2\cos(K_m\Delta x) + \sigma - \sigma^2\cos(K_m\Delta x)$$
$$+ \sigma^2 + \sigma^2\sin^2(K_m\Delta x)$$

$$= |-\sigma\cos(K_m\Delta x)+\sigma-\sigma\cos(K_m\Delta x)-\sigma^2\cos(K_m\Delta x)+\sigma-\sigma^2\cos(K_m\Delta x)+\sigma^2+\sigma^2$$

$$|G|^2 = 1-2\sigma\cos(K_m\Delta x)+2\sigma-2\sigma^2\cos(K_m\Delta x)+2\sigma^2 \leq 1$$

For all case, we are just going to verify $|G|^2 \leq 1$

because $|G|^2 \geq 0$

Worst case scénario is when $\underline{\cos(K_m\Delta x)=-1}$ $(|G_{max}|)^2$

$$\Rightarrow |G_{max}|^2 = 1+2\sigma+2\sigma+2\sigma^2+2\sigma^2$$

$$= 1+4\sigma+4\sigma^2 \leq 1$$

$$\Rightarrow 4\sigma^2+4\sigma \leq 0$$

THIS IS NEVER TRUE WHEN $\sigma = \frac{c\Delta t}{\Delta x} \in \mathbb{R}$

THEREFORE FORWARD SCHEME IS ALWAYS

UNSTABLE, YOU CAN ALSO SEE IT WITH A PLOT:

ACCURACY

$$u_j^{n+1} = u_j^n - \frac{c\,\Delta t}{\Delta x}\left(u_{j+1}^n - u_j^n\right) \qquad (1)$$

$$u_j^{n+1} = u_j^n + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + \Delta t \frac{\partial^3 u}{\partial t^3} + \dots \qquad (2)$$

$$u_j^n = u_j^n \qquad (3)$$

$$u_{j+1}^n = u_j^n + \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3} \qquad (4)$$

We put (2), (3) and (4) into (1)

$$\Rightarrow \cancel{u_j^n} + \Delta t \frac{\partial u}{\partial t} + \Delta t^2 \frac{\partial^2 u}{\partial t^2} + \Delta t^3 \frac{\partial^3 u}{\partial t^3} = \cancel{u_j^n} - \frac{c\Delta t}{\Delta x}\left(\cancel{u_j^n} + \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3} - \cancel{u_j^n}\right)$$

$$= \quad \Delta t \frac{\partial u}{\partial t} + \Delta t^2 \frac{\partial^2 u}{\partial t^2} + \Delta t^3 \frac{\partial^3 u}{\partial t^3} = -\frac{c\Delta t}{\Delta x}\left(\Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3}\right)$$

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{-\frac{c\cancel{\Delta t}}{\cancel{\Delta x}}\left(\Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^{\cancel{2}\,1}}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^{\cancel{3}\,2}}{6}\frac{\partial^3 u}{\partial x^3}\right) - \frac{\Delta t^{\cancel{2}\,1}}{2}\frac{\partial^2 u}{\partial t^2} - \frac{\Delta t^{\cancel{3}\,2}}{6}\frac{\partial^3 u}{\partial t^3}}{\cancel{\Delta t}}$$

$$= -c\left(\frac{\partial u}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 u}{\partial x} + \Delta x^2 \frac{\partial^3 u}{\partial x^3}\right) - \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} - \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^3}$$

$$\Rightarrow \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = -c\left(\Delta x \frac{\partial^2 u}{\partial x} + \Delta x^2 \frac{\partial^3 u}{\partial x^3}\right) - \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} - \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^3}$$

$O(\Delta x)$ $\qquad$ $O(\Delta t)$

CONSISTENCY

$$\lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} -c\left(\overset{\to 0}{\Delta x}\,\frac{\partial^2 u}{\partial x} + \overset{\to 0}{\Delta x^2}\,\frac{\partial^3 u}{\partial x^3}\right) - \frac{\overset{\to 0}{\Delta t}}{2}\frac{\partial^2 u}{\partial t^2} - \frac{\overset{\to 0}{\Delta t^2}}{6}\frac{\partial^3 u}{\partial t^3} = 0$$

THIS SCHEME IS CONSISTENT