QUESTION 1)

a) $u_j^{n+1} = u_j^n - c\dfrac{\Delta t}{\Delta x}\left(u_j^n - u_{j-1}^n\right)$

STABILITY

$T_k = \bar{T}_k + \varepsilon_k$

$\Rightarrow u_j^{n+1} = u_j^n - c\dfrac{\Delta t}{\Delta x}\left(u_j^n - u_{j-1}^n\right)$

$\Rightarrow \cancel{\bar{u}_j^{n+1}} + \epsilon_j^{n+1} = \cancel{\bar{u}_j^n} + \epsilon_j^n - c\dfrac{\Delta t}{\Delta x}\left(\cancel{\bar{u}_j^n} + \epsilon_j^n - \cancel{\bar{u}_{j-1}^n} - \epsilon_{j-1}^n\right)$

$\Rightarrow \epsilon_j^{n+1} = \epsilon_j^n - \dfrac{c\Delta t}{\Delta x}\left(\epsilon_j^n - \epsilon_{j-1}^n\right)$

$\Rightarrow \epsilon_j^{n+1} = \epsilon_j^n - c\dfrac{\Delta t}{\Delta x}\epsilon_j^n + \dfrac{c\Delta t}{\Delta x}\epsilon_{j-1}^n$

$\Rightarrow \dfrac{\epsilon_j^{n+1}}{\epsilon_j^n} = 1 - \dfrac{c\Delta t}{\Delta x} + \dfrac{c\Delta t}{\Delta x}\dfrac{\epsilon_{j-1}^n}{\epsilon_j^n}$

$\epsilon = e^{at}e^{ik_m x} \Rightarrow \epsilon_j^n = e^{at}e^{ik_m x}; \ \epsilon_{j-1}^n = e^{at}e^{ik_m(x-\Delta x)}$

$\Rightarrow \dfrac{\epsilon_j^{n+1}}{\epsilon_j^n} = 1 - \dfrac{c\Delta t}{\Delta x} + \dfrac{c\Delta t}{\Delta x}\left(\dfrac{\cancel{e^{at}}e^{ik_m(\cancel{x}-\Delta x)}}{\cancel{e^{at}}e^{ik_m(\cancel{x})}}\right)$

$\Rightarrow \dfrac{\epsilon_j^{n+1}}{\epsilon_j^n} = 1 - \dfrac{c\Delta t}{\Delta x} + \dfrac{c\Delta t}{\Delta x}\left(e^{-ik_m(\Delta x)}\right) = 1 - \dfrac{c\Delta t}{\Delta x}\left(1 - e^{-ik_m\Delta x}\right) = |G|$

$e^{-ik_m\Delta x} = \cos(k_m\Delta x) - i\sin(k_m\Delta x); \ \dfrac{c\Delta t}{\Delta x} = \sigma$

$\Rightarrow |G| = 1 - \sigma\left(1 - \cos(k_m\Delta x) + i\sin(k_m\Delta x)\right)$

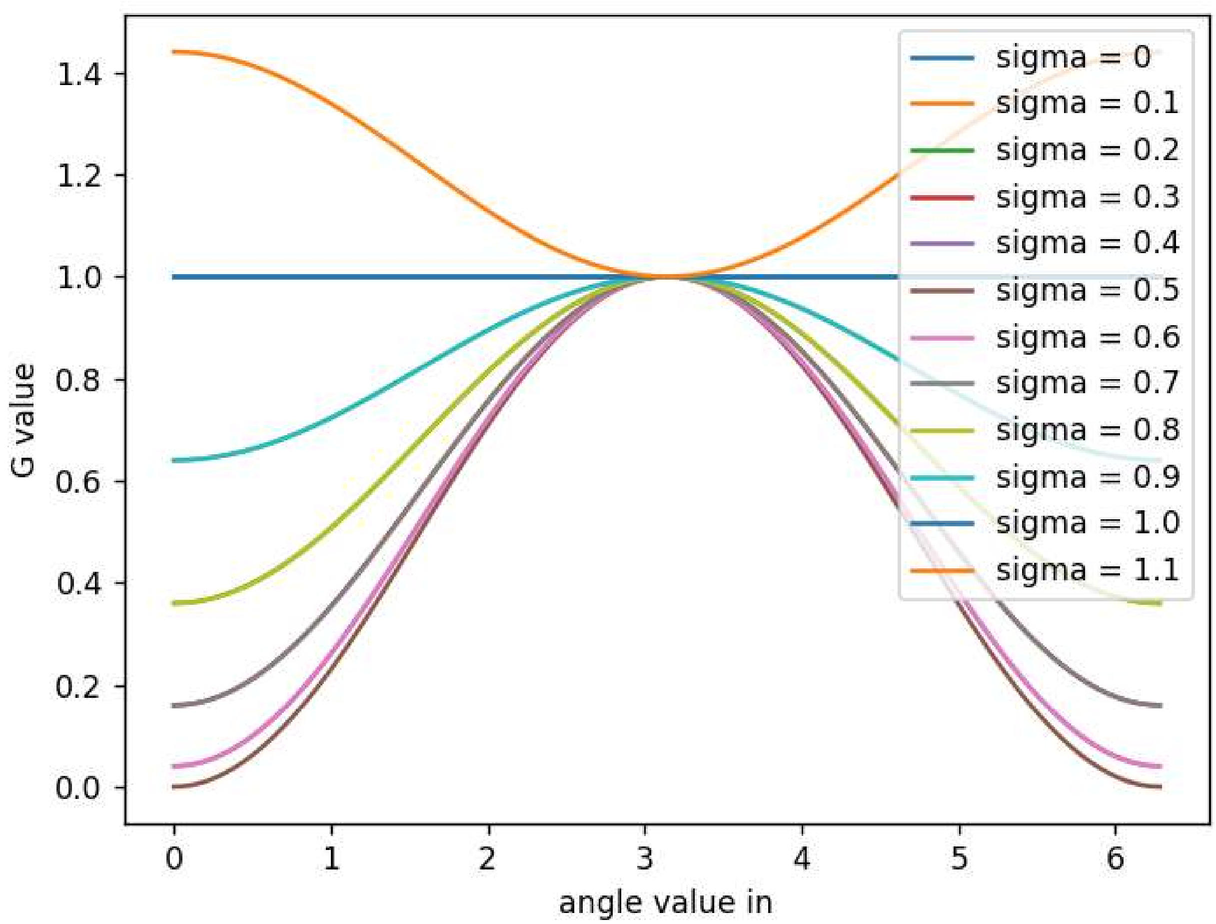
$\Rightarrow |G|^2 = \left(1 - \sigma - \sigma\cos(k_m\Delta x)\right)^2 + \left(i\sigma\sin(k_m\Delta x)\right)^2$

$= 1 - \sigma - \sigma\cos(k_m\Delta x) - \sigma + \sigma^2 + \sigma^2\cos(k_m\Delta x) - \sigma\cos(k_m\Delta x)$
$+ \sigma^2\cos(k_m\Delta x) + \sigma^2\cos^2(k_m\Delta x) + \sigma^2\sin^2(k_m\Delta x)$

$\Rightarrow 1 - 2\sigma - 2\sigma\cos(k_m\Delta x) + \sigma^2 + 2\sigma^2\cos(k_m\Delta x) +$
$\sigma^2\cos(k_m(\Delta) + \sigma^2\sin(k_m\Delta x)$

$$= 1 - 2\sigma - 2\sigma \cos(k_m \Delta x) + 2\sigma^2 + 2\sigma^2 \cos(k_m \Delta x) = |G|^2 \leq 1$$

We will show WITH a plot that $\sigma \leq 1$ to be stable whe $k_m \Delta x \in [0, 2\pi]$:



ACCURACY

We have to develop into taylor

$$u_j^{n+1} = u_j^n - \frac{c\Delta t}{\Delta x}\left(u_j^n - u_{j-1}^n\right) \quad (1)$$

$$u_j^n = u_j^n \quad (2)$$

$$u_j^{n+1} = u_j^n + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} + \dots = \sum_{k=0}^{\infty} \frac{\Delta t^k}{k!}\frac{\partial^{(k)} u}{\partial t^{(k)}} \quad (3)$$

$$u_{j-1}^n = u_j^n - \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} - \Delta x^3 \frac{\partial^3 u}{\partial x^3} + \dots = \sum_{k=0}^{\infty} \frac{(-\Delta x)^k}{k!}\frac{\partial^{(k)} u}{\partial x^{(k)}} \quad (4)$$

We put (2), (3) and (4) into (1)

$$\Rightarrow \cancel{u_j^n} + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} = \cancel{u_j^n} - \frac{c\Delta t}{\Delta x}\left(\cancel{u_j^n} - \cancel{u_j^n} + \Delta x \frac{\partial u}{\partial x} - \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3}\right)$$

We put (2), (3) and (4) into (1)

$$\Rightarrow \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} = \frac{-c\Delta t}{\cancel{\Delta x}}\left(\cancel{\Delta x}\frac{\partial u}{\partial x} - \frac{\Delta x^{2}}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^{3}}{6}\frac{\partial^3 u}{\partial x^3}\right)$$

$$\Rightarrow \Delta t \frac{\partial u}{\partial t} + \Delta t^2 \frac{\partial^2 u}{\partial t} + \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} = -c\Delta t\left(\frac{\partial u}{\partial x} - \frac{\Delta x}{2}\frac{\partial^2 u}{\partial x} + \frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3}\right)$$

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{-c\cancel{\Delta t}\left(\frac{\partial u}{\partial x} - \frac{\Delta x}{2}\frac{\partial^2 u}{\partial x} + \frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3}\right) - \frac{\Delta t^{2}}{2}\frac{\partial^2 u}{\partial t} - \frac{\Delta t^{3}}{6}\frac{\partial^3 u}{\partial t^3}}{\cancel{\Delta t}}$$

$$\Rightarrow \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = \frac{c\Delta x}{2}\frac{\partial^2 u}{\partial x} + \frac{c\Delta x^2}{6}\frac{\partial^3 u}{\partial x} - \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t} - \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t}$$

The order of convergence for both time and space are 1

$$O(\Delta t) + O(\Delta x)$$

CONSISTENCY

The error term is: $\dfrac{c\Delta x}{2}\dfrac{\partial^2 u}{\partial x} + \dfrac{c\Delta x^2}{6}\dfrac{\partial^3 u}{\partial x} - \dfrac{\Delta t}{2}\dfrac{\partial^2 u}{\partial t} - \dfrac{\Delta t^2}{6}\dfrac{\partial^3 u}{\partial t}$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \dfrac{c\overset{\to 0}{\Delta x}}{2}\dfrac{\partial^2 u}{\partial x} + \dfrac{c\overset{\to 0}{\Delta x^2}}{6}\dfrac{\partial^3 u}{\partial x} - \dfrac{\overset{\to 0}{\Delta t}}{2}\dfrac{\partial^2 u}{\partial t} - \dfrac{\overset{\to 0}{\Delta t^2}}{6}\dfrac{\partial^3 u}{\partial t} = 0$$

Thus, the backward scheme is consistent