

QUESTION 1 c)

$$\text{STABILITY: } v_j^{n+1} = v_j^{n-1} - c \frac{\Delta t}{\Delta x} (v_{j+1}^n - v_{j-1}^n)$$

$$v_m^K = \bar{v}_m^K + \epsilon_m^K$$

$$\Rightarrow \bar{v}_j^{n+1} + \epsilon_j^{n+1} = \bar{v}_j^{n-1} + \epsilon_j^{n-1} - \frac{c \Delta t}{\Delta x} \left(\bar{v}_{j+1}^n + \epsilon_{j+1}^n - \bar{v}_{j-1}^n + \epsilon_{j-1}^n \right)$$

$$\Rightarrow \epsilon_j^{n+1} = \epsilon_j^{n-1} - \frac{c \Delta t}{\Delta x} \left(\epsilon_{j+1}^n - \epsilon_{j-1}^n \right)$$

$$\Rightarrow G = \frac{\epsilon_j^{n+1}}{\epsilon_j^n} = \frac{\epsilon_j^{n-1}}{\epsilon_j^n} - \frac{c \Delta t}{\Delta x} \left(\frac{\epsilon_{j+1}^n}{\epsilon_j^n} - \frac{\epsilon_{j-1}^n}{\epsilon_j^n} \right)$$

$$\epsilon_j^n = e^{at} e^{ik_n \Delta x} \quad ; \quad \frac{c \Delta t}{\Delta x}$$

$$\Rightarrow G = \frac{e^{at - \Delta t} e^{ik_n \Delta x}}{e^{at} e^{ik_n \Delta x}} - \sigma \left(\frac{e^{at} e^{ik_n(x+\Delta x)}}{e^{at} e^{ik_n x}} - \frac{e^{at} e^{ik_n(x-\Delta x)}}{e^{at} e^{ik_n x}} \right)$$

$$e^{-at} - \sigma \left(e^{ik_n \Delta x} - e^{-ik_n \Delta x} \right)$$

$$\Rightarrow G = \frac{1}{G} - \sigma \left(2 i \sin(k_n \Delta x) \right)$$

We Multiply by G on each side

$$\Rightarrow G^2 = 1 - \sigma G \left(2 i \sin(k_n \Delta x) \right)$$

$$\Rightarrow G^2 + \sigma G (2 i \sin(k_n \Delta x)) - 1$$

$$G_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$b = 2i \sigma \sin(kn\Delta x) \Rightarrow b^2 = -4\sigma^2 \sin^2(kn\Delta x)$$

$$\Rightarrow b^2 - 4ac = -4\sigma^2 \sin^2(kn\Delta x) + 4$$

$$\Rightarrow G = \frac{-i2\sigma \sin(kn\Delta x) \pm \sqrt{4(1 - \sigma^2 \sin^2(kn\Delta x))}}{2}$$

$$\Rightarrow |G| = \left| \frac{-i\sigma \sin(kn\Delta x) \pm \sqrt{4(1 - \sigma^2 \sin^2(kn\Delta x))}}{2} \right| \leq 1$$

$$\Rightarrow |G|^2 = \sigma^2 \sin^2(kn\Delta x) + 1 - \sigma^2 \sin^2(kn\Delta x) = 1$$

$$\Rightarrow |G|^2 = 1 = |G| \quad \text{THE CONDITION } |G| \leq 1 \text{ IS RESPECTED
ITS "MGT STABLE"}$$

BUT, WE NEED TO VERIFY THE $\Delta = 1 - \sigma^2 \sin^2(kn\Delta x) > 0$
BECAUSE OTHERWISE $|G| \geq 1$ FOR SOME VALUES OF $kn\Delta x$

$$\Rightarrow \sigma^2 \sin^2(kn\Delta x) \leq 1$$

$$\Rightarrow \sigma^2 \leq 1$$

WORST CASE $\sin^2(kn\Delta x) = 1$

$$\Rightarrow G = \frac{c\Delta t}{\Delta x} \leq 1$$

ACCURACY

LET'S DEVELOP IN TAYLOR

$$v_j^{n+1} = v_j^n - \frac{c \Delta t}{\Delta x} (v_{j+1}^n - v_{j-1}^n) \Rightarrow v_j^{n+1} - v_j^n = \frac{c \Delta t}{\Delta x} (v_{j+1}^n - v_{j-1}^n) \quad (1)$$

$$v_j^{n+1} = v_j^n + \Delta t \frac{\partial v}{\partial t} + \Delta t^2 \frac{\partial^2 v}{\partial t^2} + \frac{\Delta t^3}{6} \frac{\partial^3 v}{\partial t^3} + \dots$$

$$v_j^n = v_j^n - \Delta t \frac{\partial v}{\partial t} + \frac{\Delta t^2}{2} \frac{\partial^2 v}{\partial t^2} - \frac{\Delta t^3}{6} \frac{\partial^3 v}{\partial t^3} + \dots$$

$$v_j^{n+1} - v_j^n = 2 \Delta t \frac{\partial v}{\partial t} + \frac{2 \Delta t^3}{6} \frac{\partial^3 v}{\partial t^3} + \frac{2 \Delta t^5}{120} \frac{\partial^5 v}{\partial t^5} + \dots \quad (2)$$

$$v_{j+1}^n - v_{j-1}^n = 2 \Delta x \frac{\partial v}{\partial x} + \frac{2 \Delta x^3}{6} \frac{\partial^3 v}{\partial x^3} + \frac{2 \Delta x^5}{120} \frac{\partial^5 v}{\partial x^5} + \dots \quad (3)$$

(2) (3) into (1)

$$\Rightarrow 2 \Delta t \frac{\partial v}{\partial t} + \frac{2 \Delta t^3}{6} \frac{\partial^3 v}{\partial t^3} + \frac{2 \Delta t^5}{120} \frac{\partial^5 v}{\partial t^5} + \dots = - \frac{c \Delta t}{\Delta x} \left(2 \Delta x \frac{\partial v}{\partial x} + \frac{2 \Delta x^3}{6} \frac{\partial^3 v}{\partial x^3} + \frac{2 \Delta x^5}{120} \frac{\partial^5 v}{\partial x^5} + \dots \right)$$

$$\Rightarrow \frac{\Delta t \partial v}{\partial t} + \frac{\Delta t^3}{6} \frac{\partial^3 v}{\partial t^3} + \frac{\Delta t^5}{120} \frac{\partial^5 v}{\partial t^5} + \dots = - \frac{c \Delta t}{\Delta x} \left(\frac{\Delta x \partial v}{\partial x} + \frac{\Delta x^3}{6} \frac{\partial^3 v}{\partial x^3} + \frac{\Delta x^5}{120} \frac{\partial^5 v}{\partial x^5} + \dots \right)$$

$$\Rightarrow \frac{\partial v}{\partial t} + c \frac{\partial v}{\partial x} = \frac{\Delta x^2}{6} \frac{\partial^3 v}{\partial x^3} + \frac{\Delta x^4}{120} \frac{\partial^5 v}{\partial x^5} - \frac{\Delta t^2}{6} \frac{\partial^3 v}{\partial t^3} - \frac{\Delta t^4}{120} \frac{\partial^5 v}{\partial t^5}$$

$$\Rightarrow = \mathcal{O}(\Delta x^2) + \mathcal{O}(\Delta t^2)$$

CONSISTENCY

$$\lim_{\Delta t \rightarrow 0} \left(\frac{\Delta x^2}{6} \frac{\partial^3 u}{\partial x^3} + \frac{\Delta x^4}{120} \frac{\partial^5 u}{\partial x^5} - \frac{\Delta t^2}{6} \frac{\partial^3 u}{\partial t^3} - \frac{\Delta t^4}{120} \frac{\partial^5 u}{\partial t^5} \right) = 0$$

THIS ALGORITHM IS CONSISTENT