1. b) Forward Time-Centered Space

$$u_j^{n+1} = u_j^n - \frac{c\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right)$$

Stability: $\varepsilon_j^{n+1} = \varepsilon_j^n - \dfrac{c\Delta t}{2\Delta x}\left(\varepsilon_{j+1}^n - \varepsilon_{j-1}^n\right)$

so $G = \left|\dfrac{\varepsilon_j^{n+1}}{\varepsilon_j^n}\right| = \left|1 - \dfrac{c\Delta t}{2\Delta x}\,\dfrac{\varepsilon_{j+1}^n - \varepsilon_{j-1}^n}{\varepsilon_j^n}\right|$

$$= \left|1 - \frac{c\Delta t}{2\Delta x}\,\frac{e^{at}e^{ik_m(x+\Delta x)} - e^{at}e^{ik_m(x-\Delta x)}}{e^{at}e^{ik_m x}}\right|$$

$$= \left|1 - \frac{c\Delta t}{2\Delta x}\left(e^{ik_m\Delta x} - e^{-ik_m\Delta x}\right)\right|$$

$$= \left|1 - \frac{c\Delta t}{2\Delta x}\left(2i\sin(k_m\Delta x)\right)\right|$$

$$= \sqrt{1^2 + \left(\frac{\Delta t}{\Delta x}c\right)^2 \sin^2(k_m\Delta x)}$$

So $G < 1$ if $1 + \left(\dfrac{c\Delta t}{\Delta x}\right)^2 < 1 \iff \dfrac{c\Delta t}{\Delta x} < 0$

This scheme is unconditionally unstable.

Accuracy: $\cancel{u} + \Delta t\dfrac{\partial u}{\partial t} + \dfrac{\Delta t^2}{2}\dfrac{\partial^2 u}{\partial t^2} + \dots = \cancel{u} - \dfrac{c\Delta t}{2\cancel{\Delta x}}\left[\left(\cancel{u} + \cancel{\Delta x}\dfrac{\partial u}{\partial x} + \cancel{\dfrac{\Delta x^2}{2}\dfrac{\partial^2 u}{\partial x^2}} + \dots\right)\right.$

$\left.- \left(\cancel{u} - \cancel{\Delta x}\dfrac{\partial u}{\partial x} + \cancel{\dfrac{\Delta x^2}{2}\dfrac{\partial^2 u}{\partial x^2}} \dots\right)\right]$

$$\Rightarrow \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = -\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} - c\frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3} + \dots$$

The scheme is $\begin{cases} 1^{st} \text{ order in time} \\ 2^{nd} \text{ order in space} \end{cases}$ and is consistent.

d) Lax-Wendroff

$$u_j^{n+1} = u_j^n - \frac{c\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{1}{2}\left(\frac{c\Delta t}{\Delta x}\right)^2\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

Stability: $\varepsilon_j^{n+1} = \varepsilon_j^n - \frac{c\Delta t}{2\Delta x}\left(\varepsilon_{j+1}^n - \varepsilon_{j-1}^n\right) + \frac{1}{2}\left(\frac{c\Delta t}{\Delta x}\right)^2\left(\varepsilon_{j+1}^n - 2\varepsilon_j^n + \varepsilon_{j-1}^n\right)$

so $G = \left| 1 - \frac{c\Delta t}{2\Delta x}\frac{\varepsilon_{j+1}^n - \varepsilon_{j-1}^n}{\varepsilon_j^n} + \frac{1}{2}\left(\frac{c\Delta t}{\Delta x}\right)^2 \frac{\varepsilon_{j+1}^n - 2\varepsilon_j^n + \varepsilon_{j-1}^n}{\varepsilon_j^n} \right|$

$$= \left| 1 - \frac{c\Delta t}{\cancel{2}\Delta x}\left[\cancel{2} i \sin(k_m \Delta x)\right] + \frac{1}{\cancel{2}}\left(\frac{c\Delta t}{\Delta x}\right)^2\left[-\cancel{2} + \cancel{2}\cos(k_m \Delta x)\right] \right|$$

$$= \left| \left[1 + \left(\frac{c\Delta t}{\Delta x}\right)^2(\cos(k_m\Delta x) - 1)\right] + \left[-\frac{c\Delta t}{\Delta x} i \sin(k_m \Delta x)\right] \right|$$

$$= \left| \left[1 - \left(\frac{c\Delta t}{\Delta x}\right)^2 2\sin^2\left(\frac{k_m\Delta x}{2}\right)\right] + \left[-\frac{c\Delta t}{\Delta x} i \sin(k_m \Delta x)\right] \right|$$

$$= \left[1 - 4\left(\frac{c\Delta t}{\Delta x}\right)^2 \sin^2\left(\frac{k_m\Delta x}{2}\right) + 4\left(\frac{c\Delta t}{\Delta x}\right)^4 \sin^4\left(\frac{k_m\Delta x}{2}\right) + \left(\frac{c\Delta t}{\Delta x}\right)^2 \sin^2(k_m\Delta x)\right]^{1/2}$$

By numerical analysis (see figure), the scheme is stable when CFL ≤ 1 ...

Accuracy: $\cancel{u} + \Delta t\frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} = \cancel{u} - \frac{c\Delta t}{2\cancel{\Delta x}}\left(2\cancel{\Delta x}\frac{\partial u}{\partial x} + \frac{\Delta x^{\cancel{2}}}{3}\frac{\partial^3 u}{\partial x^3} + \right) + \frac{1}{2}\left(\frac{c^2\Delta t}{\Delta x^{\cancel{2}}}\right)^{\cancel{2}}$

$\left(\cancel{\Delta x\frac{\partial u}{\partial x}} + \frac{\Delta x^{\cancel{2}}}{2}\frac{\partial^2 u}{\partial x^2} + \dots - \cancel{\Delta x\frac{\partial u}{\partial x}} + \frac{\Delta x^{\cancel{2}}}{2}\frac{\partial^2 u}{\partial x^2} - \dots\right)$

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = -\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} - c\frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3} + c^2\frac{\Delta t}{2}\frac{\partial^2 u}{\partial x^2} + \dots$$

The scheme is $\begin{cases} 1^{st} \text{ order in time} \\ 2^{nd} \text{ order in space} \end{cases}$ and is consistent