LETS CREATE OUR OWN ALGORITHM

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

SECOND ORDER SPACE

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{u_{j+1}^{n} - u_{j-1}^{n}}{2\,\Delta x}$$

FOURTH ORDER IN TIME

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{-u_j^{n+2} + 8u_j^{n+1} - 8u_j^{n-1} + u_j^{n-2}}{12\,\Delta t}$$

$$\Rightarrow \frac{-u_j^{n+2} + 8u_j^{n+1} - 8u_j^{n-1} + u_j^{n-2}}{12\,\Delta t} + c\left(\frac{u_{j+1}^{n} - u_{j-1}^{n}}{2\,\Delta x}\right) = 0$$

$$\Rightarrow \Rightarrow \frac{-\frac{1}{12}u_j^{n+2} + \frac{2}{3}u_j^{n+1} - \frac{2}{3}u_j^{n-1} + \frac{1}{12}u_j^{n-2}}{\Delta t} + c\left(\frac{u_{j+1}^{n} - u_{j-1}^{n}}{2\,\Delta x}\right) = 0$$

$$\Rightarrow u_j^{n+1} = \frac{1}{8}u_j^{n+2} + u_j^{n-1} - \frac{1}{8}u_j^{n-2} - 3\frac{c\,\Delta t}{4\Delta x}\left(u_{j+1}^{n} - u_{j-1}^{n}\right)$$

$$\Rightarrow \frac{1}{8}u_j^{n+2} - u_j^{n+1} + u_j^{n-1} - \frac{1}{8}u_j^{n-2} - 3\frac{c\Delta t}{4\Delta x}\left(u_{j+1}^{n} - u_{j-1}^{n}\right) = 0$$

$$u_j^{n\pm 2} = u_j^n \pm 2\Delta t\frac{\partial u}{\partial t} + \frac{4\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} \pm \frac{8\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} + \frac{16}{24}\Delta t^4\frac{\partial^4 u}{\partial t^4} + \dots$$

$$u_j^{n\pm 1} = u_j^n \pm \Delta t\frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} \pm \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} + \frac{\Delta t^4}{24}\frac{\partial^4 u}{\partial t^4} + \dots$$

$$u_{j\pm 1}^{n} = u_j^n \pm \Delta x\frac{\partial u}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} \pm \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3} - \frac{\Delta x^4}{24}\frac{\partial^4 u}{\partial x^4} + \dots$$

We SAW BEFORE....

$$u_{j+1} - u_{j-1} = 2\Delta x \frac{\partial u}{\partial x} + \frac{2}{6}\Delta x^3 \frac{\partial^3 u}{\partial x^3} + \frac{2}{120}\Delta x^5 \frac{\partial^5 u}{\partial x^5} + \dots$$

$$\Rightarrow \frac{1}{8} u_j^{n+2} - u_j^{n+1} + u_j^{n-1} - \frac{1}{8} u_j^{n-2} = \left(u_j^{n-1} - u_j^{n+1}\right) + \frac{1}{8}\left(u_j^{n+2} - u_j^{n-2}\right)$$

$$\Rightarrow u_j^{n-1} - u_j^{n+1} = -2\left(\Delta t \frac{\partial u}{\partial t} + \frac{1}{6}\Delta t^3 \frac{\partial^3 u}{\partial t^3} + \frac{1}{120}\Delta t^5 \frac{\partial^5 u}{\partial t^5} + \dots\right)$$

$$\Rightarrow u_j^{n+2} - u_j^{n-2} = 2\left(2\Delta t \frac{\partial u}{\partial t} + \frac{8\Delta t^3}{6} \frac{\partial u}{\partial t} + \frac{32}{120}\Delta t^5 \frac{\partial u}{\partial t^5}\right)$$

$$\Rightarrow 2\cdot\frac{1}{8}\left(\left(2\Delta t \frac{\partial u}{\partial t} + \frac{8\Delta t^3}{6} \frac{\partial^3 u}{\partial t^3} + \frac{32}{120}\Delta t^5 \frac{\partial u}{\partial t^5}\right) - \left(\Delta t \frac{\partial u}{\partial t} + \frac{1}{6}\Delta t^3 \frac{\partial^3 u}{\partial t^3} + \frac{1}{120}\Delta t^5 \frac{\partial^5 u}{\partial t^5} + \dots\right)\right) - \frac{3}{4}\frac{c\Delta t}{\Delta x}\left(2\left(\Delta x \frac{\partial u}{\partial x} + \frac{1}{6}\Delta x^3 \frac{\partial^3 u}{\partial x^3} + \frac{1}{120}\Delta x^5 \frac{\partial^5 u}{\partial x^5}\right)\right) = 0$$

$$\Rightarrow 2\cdot\left(\left(2\Delta t \frac{\partial u}{\partial t} + \cancel{\frac{\Delta t^3}{6} \frac{\partial^3 u}{\partial t^3}} + \frac{4}{120}\Delta t^5 \frac{\partial u}{\partial t^5}\right) - \left(\Delta t \frac{\partial u}{\partial t} + \cancel{\frac{1}{6}\Delta t^3 \frac{\partial^3 u}{\partial t^3}} + \frac{1}{120}\Delta t^5 \frac{\partial^5 u}{\partial t^5} + \dots\right)\right) - \frac{3}{4}\frac{c\Delta t}{\Delta x}\left(\cancel{2}\left(\Delta x \frac{\partial u}{\partial x} + \frac{1}{6}\Delta x^3 \frac{\partial^3 u}{\partial x^3} + \frac{1}{120}\Delta x^5 \frac{\partial^5 u}{\partial x^5}\right)\right)$$

$$\Rightarrow \Delta t \frac{\partial u}{\partial t} + \frac{1}{40}\Delta t^5 \frac{\partial^5 u}{\partial t^5} + \dots - \frac{c\Delta t}{\cancel{\Delta x}}\left(\cancel{\Delta x} \frac{\partial u}{\partial x} + \frac{1}{6}\Delta x^{\cancel{3}\,2} \frac{\partial^3 u}{\partial x^3} + \frac{1}{120}\Delta x^{\cancel{5}\,4} \frac{\partial^5 u}{\partial x^5}\right) = 0$$

DIVIDE PER Δt; TAKE NOTE THAT THE TERM Δt^3 IS GONE

$$\Rightarrow \frac{\partial u}{\partial t} + \frac{1}{40}\Delta t^4 \frac{\partial^5 u}{\partial t^5} - \frac{3}{4}c\left(\frac{\partial u}{\partial x} + \frac{1}{6}\Delta x^2 \frac{\partial^3 u}{\partial x^3} + \frac{1}{120}\Delta x^4 \frac{\partial^5 u}{\partial x}\right) = 0$$

$$\Rightarrow \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = \underbrace{-\frac{1}{40}\Delta t^4 \frac{\partial^5 u}{\partial t} + \frac{3}{4}c\left(\frac{1}{6}\Delta x^2 \frac{\partial^3 u}{\partial x^3} + \frac{1}{120}\Delta x^4 \frac{\partial^5 u}{\partial x}\right)}_{\text{ERROR}}$$

We see that it's $O(\Delta t^4) + O(\Delta x^2)$

CONSISTENCY

$$ERROR = -\frac{1}{40}\Delta t^4 \frac{\partial^5 u}{\partial t} + \frac{3c}{4}\left(\frac{1}{6}\Delta x^2 \frac{\partial^3 u}{\partial x^3} + \frac{1}{120}\Delta x^4 \frac{\partial^5 u}{\partial x}\right)$$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} ERROR = 0$$

THUS, THIS ALGORITHM IS CONSISTENT

STABILITY

$$\frac{1}{8} U_j^{n+2} - U_j^{n+1} + U_j^{n-1} - \frac{1}{8} U_j^{n-2} - \frac{3}{4}\frac{c\,\Delta t}{\Delta x}\left(U_{j+1}^{n} - U_{j-1}^{n}\right) = 0$$

LINEAR EQUATION SO:

$$\Rightarrow \frac{1}{8} \epsilon_j^{n+2} - \epsilon_j^{n+1} + \epsilon_j^{n-1} - \frac{1}{8} \epsilon_j^{n-2} - \frac{3}{4}\frac{c\,\Delta t}{\Delta x}\left(\epsilon_{j+1}^{n} - \epsilon_{j-1}^{n}\right) = 0$$

AFTER DIVISION BY $\epsilon_j^n = e^{at} e^{ik_m \Delta x}$; $\sigma = 3/4 \frac{c\Delta t}{\Delta x}$

$$\Rightarrow \frac{1}{8} e^{2a\Delta t} - e^{a\Delta t} + e^{-a\Delta t} - \frac{1}{8} e^{-2a\Delta t} - \sigma\left(e^{ik_m\Delta x} - e^{-ik_m\Delta x}\right) = 0$$

$$\Rightarrow \frac{1}{8} e^{2a\Delta t} - e^{a\Delta t} + e^{-a\Delta t} - \frac{1}{8} e^{-2a\Delta t} - \sigma\left(2i\sin(k_m\Delta x)\right) = 0$$

$$\Rightarrow \frac{1}{8}(e^{a\Delta t})^2 - (e^{a\Delta t}) + (e^{a\Delta t})^{-1} - \frac{1}{8}(e^{a\Delta t})^{-2} - \sigma(2i\sin k_m\Delta x) = 0$$

We HAVE TO FIND $e^{a\Delta t} = G$