QUESTION 1 c)

STABILITY : $U_j^{n+1} = U_j^{n-1} - \frac{c\Delta t}{\Delta x}\left(U_{j+1}^n - U_{j-1}^n\right)$

$U_m^k = \bar{U}_m^k + \epsilon_m^k$

$$\Rightarrow \cancel{\bar{U}_j^{n+1}} + \epsilon_j^{n+1} = \cancel{\bar{U}_j^{n-1}} + \epsilon_j^{n-1} - \frac{c\Delta t}{\Delta x}\left(\cancel{\bar{U}_{j+1}^n} + \epsilon_{j+1}^n - \cancel{\bar{U}_{j-1}^n} + \epsilon_{j-1}^n\right)$$

$$\Rightarrow \epsilon_j^{n+1} = \epsilon_j^{n-1} - \frac{c\Delta t}{\Delta x}\left(\epsilon_{j+1}^n - \epsilon_{j-1}^n\right)$$

$$\Rightarrow G = \frac{\epsilon_j^{n+1}}{\epsilon_j^n} = \frac{\epsilon_j^{n-1}}{\epsilon_j^n} - \frac{c\Delta t}{\Delta x}\left(\frac{\epsilon_{j+1}^n}{\epsilon_j^n} - \frac{\epsilon_{j-1}^n}{\epsilon_j^n}\right)$$

$\epsilon_j^n = e^{at}e^{ik_n\Delta t}$; $\frac{c\Delta t}{\Delta x}$

$$\Rightarrow G = \frac{e^{a(t-\Delta t)}e^{ik_n x}}{e^{at}e^{ik_n x}} - \sigma\left(\frac{e^{at}e^{ik_n(x+\Delta x)}}{e^{at}e^{ik_n x}} - \frac{e^{at}e^{ik_n(x-\Delta x)}}{e^{at}e^{ik_n x}}\right)$$

$$e^{-a\Delta t} - \sigma\left(e^{ik_n\Delta x} - e^{-ik_n\Delta x}\right)$$

$$\Rightarrow G = 1/G - \sigma\left(2i\sin(k_n\Delta x)\right)$$

We Multiply by G on each side

$$\Rightarrow G^2 = 1 - \sigma G\left(2i\sin(k_n\Delta x)\right)$$

$$\Rightarrow G^2 + \sigma G\left(2i\sin(k_n\Delta x)\right) - 1$$

$$G_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$b = 2i\,\sigma \sin(k_m \Delta x) \Rightarrow b^2 = -4\sigma^2 \sin^2(k_m \Delta x)$$

$$\Rightarrow b^2 - 4ac = -4\sigma^2 \sin^2(k_m \Delta x) + 4$$

$$\Rightarrow G = \frac{-i\cancel{2}\sigma \sin(k_m \Delta x) \pm \sqrt{\cancel{4}\left(1 - \sigma^2 \sin^2(k_m \Delta x)\right)}}{\cancel{2}}$$

$$\Rightarrow |G| = \left| -i\sigma \sin(k_m \Delta x) \pm \sqrt{1 - \sigma^2 \sin^2(k_m \Delta x)} \right| \leq 1$$

$$\Rightarrow |G|^2 = \sigma^2 \sin^2(k_m \Delta x) + 1 - \sigma^2 \sin^2(k_m \Delta x) = 1$$

$\Rightarrow |G|^2 = 1 = |G|$ THE CONDITION $|G| \leq 1$ IS RESPECTED

HOWEVER

WE NEED TO VERIFY THE $\Delta = 1 - \sigma^2 \sin^2(k_m \Delta x) \geq 0$

$$\Rightarrow \sigma^2 \sin^2(k_m \Delta x) \leq 1$$

WORST CASE $\sin^2(k_m \Delta x) = 1$

$$\Rightarrow \sigma^2 \leq 1$$

$$\Rightarrow \sigma = \frac{C\Delta t}{\Delta x} \leq 1$$

if $\Delta = b^2 - 4ac \leq 0$ THERE WOULD BE

An imaginary part that would add to $-i\sigma \sin(k_m \Delta x)$

ACCURACY

LETS DEVELLOP IN TAYLOR

$$U_j^{n+1} = U_j^{n-1} - \frac{c\Delta t}{\Delta x}\left(U_{j+1}^n - U_{j-1}^n\right) \Rightarrow U_j^{n+1} - U_j^{n-1} = \frac{c\Delta t}{\Delta x}\left(U_{j+1}^n - U_{j-1}^n\right) \quad (1)$$

$$U_j^{n+1} = U_j^n + \Delta t\frac{\partial u}{\partial t} + \Delta t^2\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t} + \dots$$

$$U_j^{n-1} = U_j^n - \Delta t\frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} - \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} + \dots$$

0 2x 0 2x 0 2x ...

$$U_j^{n+1} - U_j^{n-1} = 2\Delta t\frac{\partial u}{\partial t} + \frac{2\Delta t^3}{6}\frac{\partial^3 u}{\partial t} + \frac{2\Delta t^5}{120}\frac{\partial^5 u}{\partial t^5} + \dots \quad (2)$$

$$U_{j+1}^n - U_{j-1}^n = 2\Delta x\frac{\partial u}{\partial x} + \frac{2\Delta x^3}{6}\frac{\partial^3 u}{\partial x} + \frac{2\Delta x^5}{120}\frac{\partial^5 u}{\partial x^5} + \dots \quad (3)$$

(2) (3) into (1)

$$\Rightarrow 2\Delta t\frac{\partial u}{\partial t} + \frac{2\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} + \frac{2\Delta t^5}{120}\frac{\partial^5 u}{\partial t^5} + \dots = -\frac{c\Delta t}{\Delta x}\left(2\Delta x\frac{\partial u}{\partial x} + \frac{2\Delta x^3}{6}\frac{\partial^3 u}{\partial x} + \frac{2\Delta x^5}{120}\frac{\partial^5 u}{\partial x^5} + \dots\right)$$

$$\Rightarrow \cancel{\Delta t}\frac{\partial u}{\partial t} + \frac{\Delta t^{3\,2}}{6}\frac{\partial^3 u}{\partial t^3} + \frac{\Delta t^{5\,4}}{120}\frac{\partial^5 u}{\partial t^5} + \dots = -\frac{c\cancel{\Delta t}}{\cancel{\Delta x}}\left(\cancel{\Delta x}\frac{\partial u}{\partial x} + \frac{\Delta x^{3\,2}}{6}\frac{\partial^3 u}{\partial x} + \frac{\Delta x^{5\,4}}{120}\frac{\partial^5 u}{\partial x^5} + \dots\right)$$

$$\Rightarrow \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3} + \frac{\Delta x^4}{120}\frac{\partial^5 u}{\partial x^5} - \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^3} - \frac{\Delta t^4}{120}\frac{\partial^5 u}{\partial t^5}$$

$$\Rightarrow \quad = \mathcal{O}(\Delta x^2) + \mathcal{O}(\Delta t^2)$$

CONSISTENCY

$$\lim_{\substack{\Delta t \to 0 \\ \Delta x \to 0}} \left(\frac{\overset{\to 0}{\Delta x^2}}{6} \frac{\partial^3 u}{\partial x^3} + \frac{\overset{\to 0}{\Delta x^4}}{120} \frac{\partial^5 u}{\partial x^5} - \frac{\overset{\to 0}{\Delta t^2}}{6} \frac{\partial^3 u}{\partial t^3} - \frac{\overset{\to 0}{\Delta t^4}}{120} \frac{\partial^5 u}{\partial t^5} \right) = 0$$

<u>THIS ALGORITHM IS CONSISTENT</u>