

QUESTION 7)

$$a) v_j^{n+1} = v_j^n - c \frac{\Delta t}{\Delta x} (v_j^n - v_{j-1}^n)$$

STABILITY

$$T_k = \bar{T}_k + \epsilon_k$$

$$\Rightarrow v_j^{n+1} = v_j^n - c \frac{\Delta t}{\Delta x} (v_j^n - v_{j-1}^n)$$

$$\Rightarrow \bar{v}_j^{n+1} + \epsilon_j^{n+1} = \bar{v}_j^n + \epsilon_j^n - c \frac{\Delta t}{\Delta x} (\bar{v}_j^n + \epsilon_j^n - \bar{v}_{j-1}^n - \epsilon_{j-1}^n)$$

$$\Rightarrow \epsilon_j^{n+1} = \epsilon_j^n - c \frac{\Delta t}{\Delta x} (\epsilon_j^n - \epsilon_{j-1}^n)$$

$$\Rightarrow \epsilon_j^{n+1} = \epsilon_j^n - c \frac{\Delta t}{\Delta x} \epsilon_j^n + \frac{c \Delta t}{\Delta x} \epsilon_{j-1}^n$$

$$\Rightarrow \frac{\epsilon_j^{n+1}}{\epsilon_j^n} = 1 - \frac{c \Delta t}{\Delta x} + \frac{c \Delta t}{\Delta x} \frac{\epsilon_{j-1}^n}{\epsilon_j^n}$$

$$\epsilon = e^{at} e^{ik_n x} \Rightarrow \epsilon_j^n = e^{at} e^{ik_n x}; \epsilon_{j-1}^n = e^{at} e^{ik_n (x-\Delta x)}$$

$$\Rightarrow \frac{\epsilon_j^{n+1}}{\epsilon_j^n} = 1 - \frac{c \Delta t}{\Delta x} + \frac{c \Delta t}{\Delta x} \left(\frac{e^{at} e^{ik_n (x-\Delta x)}}{e^{at} e^{ik_n x}} \right)$$

$$\Rightarrow \frac{\epsilon_j^{n+1}}{\epsilon_j^n} = 1 - \frac{c \Delta t}{\Delta x} + \frac{c \Delta t}{\Delta x} \left(e^{-ik_n \Delta x} \right) = 1 - \frac{c \Delta t}{\Delta x} \left(1 - e^{-ik_n \Delta x} \right) = G$$

$$e^{-ik_n \Delta x} = \cos(k_n \Delta x) - i \sin(k_n \Delta x); \frac{c \Delta t}{\Delta x} = \sigma$$

$$\Rightarrow |G| = \left| 1 - \sigma \left(1 - \cos(k_n \Delta x) + i \sin(k_n \Delta x) \right) \right|$$

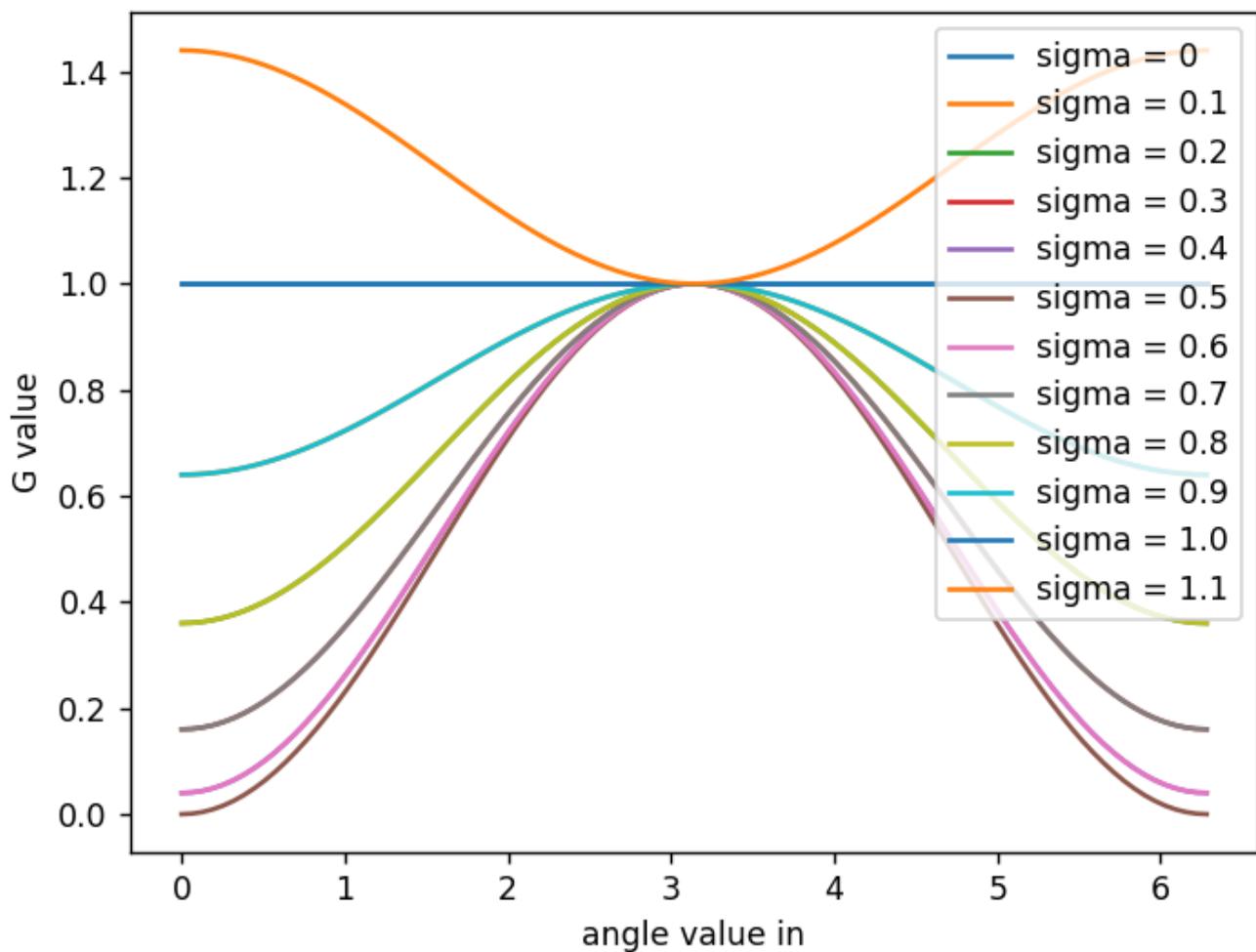
$$\Rightarrow |G|^2 = \left(1 - \sigma - \sigma \cos(k_n \Delta x) \right)^2 + \left(i \sigma \sin(k_n \Delta x) \right)^2$$

$$= 1 - \underline{\sigma} - \underline{\sigma \cos(k_n \Delta x)} - \underline{\sigma} + \underline{\sigma^2} + \underline{\sigma^2 \cos^2(k_n \Delta x)} - \underline{\sigma \cos(k_n \Delta x)} + \underline{\sigma^2 \cos^2(k_n \Delta x)} + \underline{\sigma^2 \sin^2(k_n \Delta x)} + \underline{\sigma^2 \sin^2(k_n \Delta x)}$$

$$\Rightarrow 1 - 2\sigma - 2\sigma \cos(k_n \Delta x) + \sigma^2 + 2\sigma^2 \cos^2(k_n \Delta x) + \sigma^2 \cos^2(k_n \Delta x) + \sigma^2 \sin^2(k_n \Delta x)$$

$$= (-26 - 26 \cos(k_m \Delta x) + 26^2 + 26^2 \cos(k_m \Delta x)) = |G|^2 < 1$$

We will show with a plot that $\sigma \leq 1$ to be stable when $K_{\text{PD}}x \in [0, \ell_\infty]$:



ACCURACY

We have to develop into Taylor

$$U_j^{n+1} = U_j^n - \frac{c \Delta t}{\Delta x} (U_j^n - U_{j-1}^n) \quad (1)$$

$$U_j^n = U_j^M \quad (2)$$

$$U_j^{n+1} = U_j^n + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2} \frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6} \frac{\partial^3 u}{\partial t^3} + \dots = \sum_{k=0}^{\infty} \frac{\Delta t^k}{k!} \frac{\partial^k u}{\partial t^k} \quad (3)$$

$$U_{j-1}^n = U_j^n - \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2} \frac{\partial^2 u}{\partial x^2} - \frac{\Delta x^3}{6} \frac{\partial^3 u}{\partial x^3} + \dots = \sum_{k=0}^{\infty} \frac{(-\Delta x)^k}{k!} \frac{\partial^k u}{\partial x^k} \quad (4)$$

We put (2), (3) and (4) into (1)

$$\Rightarrow U_j^n + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2} \frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6} \frac{\partial^3 u}{\partial t^3} = U_j^M - \frac{c \Delta t}{\Delta x} \left(U_j^M - U_j^n + \Delta x \frac{\partial u}{\partial x} - \frac{\Delta x^2}{2} \frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^3}{6} \frac{\partial^3 u}{\partial x^3} \right)$$

We put (2), (3) and (4) into (1)

$$\Rightarrow \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2} \frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6} \frac{\partial^3 u}{\partial t^3} = -c \Delta t \left(\Delta x \frac{\partial u}{\partial x} - \frac{\Delta x^2}{2} \frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^3}{6} \frac{\partial^3 u}{\partial x^3} \right)$$

$$\Rightarrow \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2} \frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6} \frac{\partial^3 u}{\partial t^3} = -c \Delta t \left(\frac{\partial u}{\partial x} - \frac{\Delta x}{2} \frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^2}{6} \frac{\partial^3 u}{\partial x^3} \right)$$

$$\Rightarrow \frac{\partial u}{\partial t} = -c \Delta t \left(\frac{\partial u}{\partial x} - \frac{\Delta x}{2} \frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^2}{6} \frac{\partial^3 u}{\partial x^3} \right) - \frac{\Delta t^2}{2} \frac{\partial^2 u}{\partial t^2} - \frac{\Delta t^3}{6} \frac{\partial^3 u}{\partial t^3}$$

Δt

$$\Rightarrow \frac{\partial u}{\partial t} - c \frac{\partial u}{\partial x} = \frac{c \Delta x}{2} \frac{\partial^2 u}{\partial x^2} + \frac{c \Delta x^2}{6} \frac{\partial^3 u}{\partial x^3} - \frac{\Delta t}{2} \frac{\partial^2 u}{\partial t^2} - \frac{\Delta t^2}{6} \frac{\partial^3 u}{\partial t^3}$$

The order of convergence for both time and space are 1

$\mathcal{O}(\Delta t) + \mathcal{O}(\Delta x)$

CONSISTENCY

The error term is $\frac{c \Delta x}{2} \frac{\partial^2 v}{\partial x^2} + \frac{c \Delta x^2}{6} \frac{\partial^3 v}{\partial x^3} - \frac{\Delta t}{2} \frac{\partial^2 v}{\partial x^2} - \frac{\Delta t^2}{6} \frac{\partial^3 v}{\partial x^3}$

$$\lim_{\substack{\Delta x \rightarrow 0 \\ \Delta t \rightarrow 0}} \frac{c \Delta x}{2} \frac{\partial^2 v}{\partial x^2} + \frac{c \Delta x^2}{6} \frac{\partial^3 v}{\partial x^3} - \frac{\Delta t}{2} \frac{\partial^2 v}{\partial x^2} - \frac{\Delta t^2}{6} \frac{\partial^3 v}{\partial x^3} = 0$$

Thus, the backward scheme is consistent

FORWARD

$$v_j^{n+1} = v_j^n - \frac{c \Delta t}{\Delta x} (v_{j+1}^n - v_j^n) \quad (1)$$

STABILITY

$$v_j^k = \bar{v}_j^k + \epsilon_j^k \quad (2)$$

(2) into (1)

$$\Rightarrow \bar{v}_j^{n+1} + \epsilon_j^{n+1} = \bar{v}_j^n + \epsilon_j^n - \frac{c \Delta t}{\Delta x} (\bar{v}_{j+1}^n + \epsilon_{j+1}^n - \bar{v}_j^n - \epsilon_j^n)$$

$$\Rightarrow \epsilon_j^{n+1} = \epsilon_j^n - \frac{c \Delta t}{\Delta x} (\epsilon_{j+1}^n - \epsilon_j^n)$$

lets form $|G| = \frac{\epsilon_j^{n+1}}{\epsilon_j^n}$

$$\Rightarrow \frac{\epsilon_j^{n+1}}{\epsilon_j^n} = 1 - \frac{c \Delta t}{\Delta x} \left(\frac{\epsilon_{j+1}^n}{\epsilon_j^n} - 1 \right) = G$$

$$\epsilon_j^k = e^{ak} e^{ikn\Delta x}$$

$$\Rightarrow G = 1 - \frac{c \Delta t}{\Delta x} \left(\frac{e^{at} e^{ikn(x+\Delta x)}}{e^{at} e^{ikn x}} - 1 \right)$$

$$= 1 - \frac{c \Delta t}{\Delta x} \left(e^{ikn \Delta x} - 1 \right)$$

$$\Rightarrow G = 1 - \frac{c \Delta t}{\Delta x} \left(\cos(kn \Delta x) + i \sin(kn \Delta x) - 1 \right) \leq 1$$

$$\sigma = \frac{c \Delta t}{\Delta x}$$

$$\Rightarrow |G|^2 = (\text{real part})^2 + (\text{im part})^2$$

$$= (1 - \sigma \cos(kn \Delta x) + \sigma)^2 + (-\sigma \sin(kn \Delta x))^2$$

$$= 1 - \sigma \cos(kn \Delta x) + \sigma - \sigma \cos(kn \Delta x) + \sigma^2 \cos^2(kn \Delta x) - \sigma^2 \cos(kn \Delta x) + \sigma - \sigma^2 \cos(kn \Delta x) + \sigma^2 + \sigma^2 \sin^2(kn \Delta x)$$

$$= \underbrace{(-\sigma \cos(k_n \Delta x) + \sigma)}_{-\sigma} - \underbrace{\sigma \cos(k_n \Delta x)}_{-\sigma^2} - \underbrace{\sigma^2 \cos(k_n \Delta x)}_{-\sigma^3} + \underbrace{\sigma}_{+\sigma^2} + \underbrace{\sigma^2}_{+\sigma^2}$$

$$|G|^2 = 1 - 2\sigma \cos(k_n \Delta x) + 2\sigma - 2\sigma^2 \cos(k_n \Delta x) + 2\sigma^2 \leq 1$$

For all case, we are just going to verify $|G|^2 \leq 1$
because $|G|^2 \geq 0$

Worst case scenario is when $\cos(k_n \Delta x) = -1$ $(|G_{\max}|)^2$

$$\Rightarrow |G_{\max}|^2 = 1 + 2\sigma + 2\sigma + 2\sigma^2 + 2\sigma^2$$

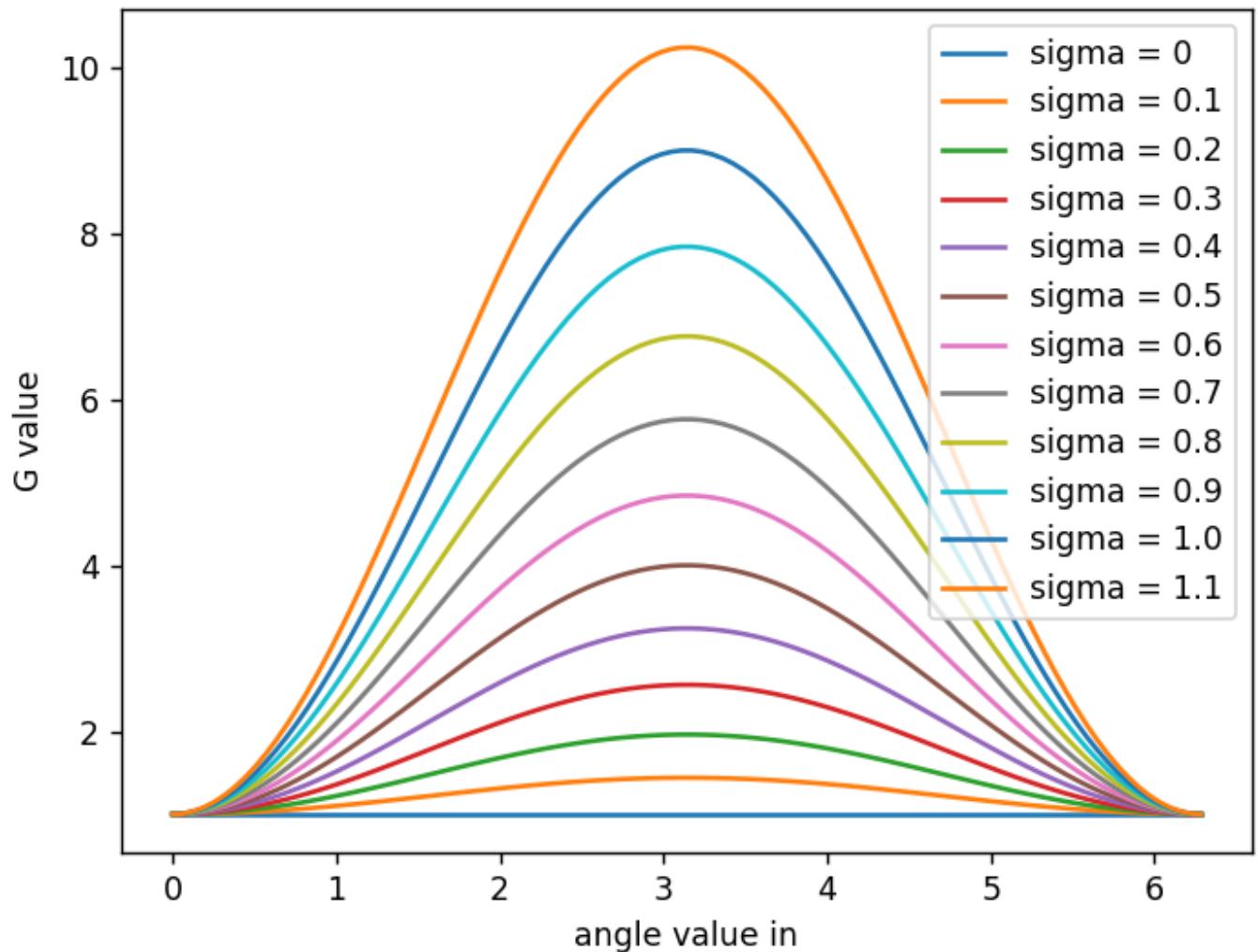
$$= 1 + 4\sigma + 4\sigma^2 \leq 1$$

$$\Rightarrow 4\sigma^2 + 4\sigma \leq 0$$

THIS IS NEVER TRUE WHEN $\sigma = \frac{cst}{\Delta x} \in \mathbb{R}$

THEFORE FORWARD SCHEME IS ALWAYS UNSTABLE, YOU CAN ALSO SEE IT WITH A PLOT:

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ACCURACY

$$v_j^{n+1} = v_j^n - \frac{c \Delta t}{\Delta x} (v_{j+1}^n - v_j^n) \quad (1)$$

$$v_j^{n+1} = v_j^n + \Delta t \frac{\partial v}{\partial x} + \frac{\Delta t^2}{2} \frac{\partial^2 v}{\partial x^2} + \frac{\Delta t^3}{6} \frac{\partial^3 v}{\partial x^3} + \dots \quad (2)$$

$$v_j^n = v_j^m \quad (3)$$

$$v_{j+1}^m = v_j^m + \Delta x \frac{\partial v}{\partial x} + \frac{\Delta x^2}{2} \frac{\partial^2 v}{\partial x^2} + \frac{\Delta x^3}{6} \frac{\partial^3 v}{\partial x^3} \quad (4)$$

We put (2), (3) and (4) into (1)

$$\Rightarrow v_j^m + \Delta t \frac{\partial v}{\partial x} + \frac{\Delta t^2}{2} \frac{\partial^2 v}{\partial x^2} + \frac{\Delta t^3}{6} \frac{\partial^3 v}{\partial x^3} = v_j^m - \frac{c \Delta t}{\Delta x} \left(v_j^m + \frac{\Delta x \partial v}{\partial x} + \frac{\Delta x^2}{2} \frac{\partial^2 v}{\partial x^2} + \frac{\Delta x^3}{6} \frac{\partial^3 v}{\partial x^3} - v_j^m \right)$$

$$= \Delta t \frac{\partial v}{\partial x} + \frac{\Delta t^2}{2} \frac{\partial^2 v}{\partial x^2} + \frac{\Delta t^3}{6} \frac{\partial^3 v}{\partial x^3} = - \frac{c \Delta t}{\Delta x} \left(\frac{\Delta x \partial v}{\partial x} + \frac{\Delta x^2}{2} \frac{\partial^2 v}{\partial x^2} + \frac{\Delta x^3}{6} \frac{\partial^3 v}{\partial x^3} \right)$$

$$\Rightarrow \frac{\partial v}{\partial t} = - \frac{c \Delta t}{\Delta x} \left(\frac{\Delta x \partial v}{\partial x} + \frac{\Delta x^2}{2} \frac{\partial^2 v}{\partial x^2} + \frac{\Delta x^3}{6} \frac{\partial^3 v}{\partial x^3} \right) - \frac{\Delta t^2}{2} \frac{\partial^2 v}{\partial x^2} - \frac{\Delta t^3}{6} \frac{\partial^3 v}{\partial x^3}$$

$\cancel{\Delta t}$

$$= -c \left(\frac{\partial v}{\partial x} + \frac{\Delta x}{2} \frac{\partial^2 v}{\partial x^2} + \frac{\Delta x^2}{6} \frac{\partial^3 v}{\partial x^3} \right) - \frac{\Delta t}{2} \frac{\partial^2 v}{\partial x^2} - \frac{\Delta t^2}{6} \frac{\partial^3 v}{\partial x^3}$$

$$\Rightarrow \frac{\partial v}{\partial t} - c \frac{\partial v}{\partial x} = -c \left(\frac{\Delta x \partial^2 v}{\partial x} + \frac{\Delta x^2}{6} \frac{\partial^3 v}{\partial x^3} \right) - \frac{\Delta t}{2} \frac{\partial^2 v}{\partial x^2} - \frac{\Delta t^2}{6} \frac{\partial^3 v}{\partial x^3}$$

$\downarrow O(\Delta x)$ $\downarrow O(\Delta t)$

CONSISTENCY

$$\lim_{\Delta x \rightarrow 0} -c \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^3} \right) - \frac{\Delta t}{2} \frac{\partial^2 u}{\partial t^2} - \frac{\Delta t^2}{6} \frac{\partial^3 u}{\partial t^3} = 0$$

THIS SCHEME IS CONSISTENT