QUESTION 1a)

STABILITY

$$U_j^{n+1} = \left(\frac{U_{j+1}^n + U_{j-1}^n}{2}\right) - \frac{c\Delta t}{\Delta x}\left(U_{j+1}^n - U_{j-1}^n\right)$$

$U_n^k = \bar{U}_n^k + \epsilon_n^k$

$$\Rightarrow \cancel{\bar{U}_j^{n+1}} + \epsilon_j^{n+1} = \left(\frac{\cancel{\bar{U}_{j+1}^n} + \epsilon_{j+1}^n + \cancel{\bar{U}_{j-1}^n} + \epsilon_{j-1}^n}{2}\right) - \frac{c\Delta t}{2\Delta x}\left(\cancel{\bar{U}_{j+1}^n} + \epsilon_{j+1}^n - \cancel{\bar{U}_{j-1}^n} - \epsilon_{j-1}^n\right)$$

$$\Rightarrow \epsilon_j^{n+1} = \left(\frac{\epsilon_{j+1}^n + \epsilon_{j-1}^n}{2}\right) - \frac{c\Delta t}{2\Delta x}\left(\epsilon_{j+1}^n - \epsilon_{j-1}^n\right)$$

$\epsilon_n^k = e^{ak} e^{ik_m n}$; $\frac{c\Delta t}{2\Delta x} = \sigma$

$$\Rightarrow G = \frac{\epsilon_j^{n+1}}{\epsilon_j^n} = \frac{1}{2}\left(\frac{e^{at}e^{ik_m(x+\Delta x)}}{e^{at}e^{ik_m x}} + \frac{e^{at}e^{ik_m(x-\Delta x)}}{e^{at}e^{ik_m x}}\right) - \sigma\left(\frac{e^{at}e^{ik_m(x+\Delta x)}}{e^{at}e^{ik_m x}} - \frac{e^{at}e^{ik_m(x-\Delta x)}}{e^{at}e^{ik_m x}}\right)$$

$$\Rightarrow G = \frac{1}{2}\left(e^{ik_m\Delta x} + e^{-ik_m\Delta x}\right) - \sigma\left(e^{ik_m\Delta x} - e^{-ik_m\Delta x}\right)$$

$e^{ix} + e^{-ix} = 2\cos(x)$; $e^{ix} - e^{-ix} = -2i\sin(x)$;

$$\Rightarrow G = \cos(k_m\Delta x) + 2\sigma i\sin(k_m\Delta x)$$

$$\Rightarrow |G|^2 = \cos^2(k_m\Delta x) + 4\sigma^2\sin^2(k_m\Delta x) < 1$$

$$\Rightarrow \sigma^2 < \frac{-\cos^2(k_m\Delta x)}{4\sin^2(k_m\Delta x)} + 1$$

WE SEE ON THE GRAPH THAT SIGMA SHOULD B LOWER THAN 0,5.

$$\sigma = \frac{c\Delta t}{2\Delta X} \Rightarrow \frac{c\Delta t}{\Delta x} = CFL \leq 1$$

ACCURACY

$$u_j^{n+1} = \frac{\left(u_{j+1}^n + u_{j-1}^n\right)}{2} - \frac{c\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) \qquad (1)$$

$$u_j^{n+1} = u_j^n + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} + \dots \qquad (2)$$

$$u_{j+1}^n = u_j^n + \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3} + \dots$$

$$u_{j-1}^n = u_j^n - \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} - \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3} + \dots$$

$$\Rightarrow u_{j+1}^n + u_{j-1}^n = 2u_j^n + 2\frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} + 2\frac{\Delta x^4}{24}\frac{\partial^4 u}{\partial x^4} + \dots \qquad (3)$$

$$\Rightarrow u_{j+1}^n - u_{j-1} = 2\Delta x \frac{\partial u}{\partial x} + 2\frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x} + 2\frac{\Delta x^5}{120}\frac{\partial^5 u}{\partial x^5} + \dots \qquad (4)$$

(2)(3)(4) into (1)

$$\Rightarrow \cancel{u_j^n} + \Delta t\frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} + \dots = \cancel{\frac{1}{2}}\left(\cancel{2u_j^n} + \cancel{2}\frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} + \cancel{2}\frac{\Delta x^4}{24}\frac{\partial^4 u}{\partial x^4} + \dots\right) - \frac{c\Delta t}{\cancel{2\Delta x}}\left(\cancel{2\Delta x}\frac{\partial u}{\partial x} + \cancel{2}\frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x} + \cancel{2}\frac{\Delta x^5}{120}\frac{\partial^5 u}{\partial x^5} + \dots\right)$$

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{\Delta x^2}{2\Delta t}\frac{\partial^2 u}{\partial x} + \frac{\Delta x^4}{24\Delta t}\frac{\partial^3 u}{\partial x^4} + \dots - c\left(\frac{\partial u}{\partial x} + \frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x} + \frac{\Delta x^4}{120}\frac{\partial^5 u}{\partial x^5} + \dots\right) - \frac{\Delta t}{2}\frac{\partial^2 u}{\partial x} - \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \frac{\Delta x^2}{2\Delta t}\frac{\partial^2 u}{\partial x} + \frac{\Delta x^4}{24\Delta t}\frac{\partial^3 u}{\partial x^4} + \dots - c\left(\frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x} + \frac{\Delta x^4}{120}\frac{\partial^5 u}{\partial x^5} + \dots\right) - \frac{\Delta t}{2}\frac{\partial^2 u}{\partial x} - \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial x}$$

$$= O\left(\frac{\Delta x^2}{\Delta t}\right) + O(\Delta t) \qquad ?$$

CONSISTENCY

$$\lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \frac{\Delta x^2}{2\Delta t}\frac{\partial^2 u}{\partial x} + \frac{\Delta x^4}{24\Delta t}\frac{\partial^3 u}{\partial x^4} + \dots - c\left(\frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x} + \frac{\Delta x^4}{120}\frac{\partial^5 u}{\partial x^5} + \dots\right) - \frac{\Delta t}{2}\frac{\partial^2 u}{\partial x} - \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial x} \neq 0$$

THE TERMS $\frac{\Delta x^n}{\Delta t}$ MAKES THIS ALGORITHM INCONSISTENT..