

QUESTION 1a)

## STABILITY

$$v_j^{n+1} = \left( \frac{v_{j+1}^n + v_{j-1}^n}{2} \right) - \frac{c \Delta t}{\Delta x} \left( v_{j+1}^n - v_{j-1}^n \right)$$

$$v_n^k = \bar{v}_n^k + \epsilon_n^k$$

$$\Rightarrow \bar{v}_j^{n+1} + \epsilon_j^{n+1} = \left( \frac{\bar{v}_{j+1}^n + \epsilon_{j+1}^n + \bar{v}_{j-1}^n + \epsilon_{j-1}^n}{2} \right) - \frac{c \Delta t}{\Delta x} \left( \bar{v}_{j+1}^n + \epsilon_{j+1}^n - \bar{v}_{j-1}^n - \epsilon_{j-1}^n \right)$$

$$\Rightarrow \epsilon_j^{n+1} = \left( \frac{\epsilon_{j+1}^n + \epsilon_{j-1}^n}{2} \right) - \frac{c \Delta t}{\Delta x} \left( \epsilon_{j+1}^n - \epsilon_{j-1}^n \right)$$

$$\epsilon_n^k = e^{\sigma k} e^{i k n \Delta x} ; \quad \frac{c \Delta t}{\Delta x} = \sigma$$

$$\Rightarrow G = \frac{\epsilon_j^{n+1}}{\epsilon_j^n} = \frac{1}{2} \left( \frac{e^{\sigma \Delta t} e^{i k n (\Delta x + \Delta x)} + e^{\sigma \Delta t} e^{i k n (\Delta x - \Delta x)}}{e^{\sigma \Delta t} e^{i k n \Delta x}} \right) - \sigma \left( \frac{e^{\sigma \Delta t} e^{i k n (\Delta x + \Delta x)} - e^{\sigma \Delta t} e^{i k n (\Delta x - \Delta x)}}{e^{\sigma \Delta t} e^{i k n \Delta x}} \right)$$

$$\Rightarrow G = \frac{1}{2} \left( e^{i k n \Delta x} + e^{-i k n \Delta x} \right) - \sigma \left( e^{i k n \Delta x} - e^{-i k n \Delta x} \right)$$

$$e^{i x} + e^{-i x} = 2 \cos(x) ; \quad e^{i x} - e^{-i x} = -2i \sin(x) ;$$

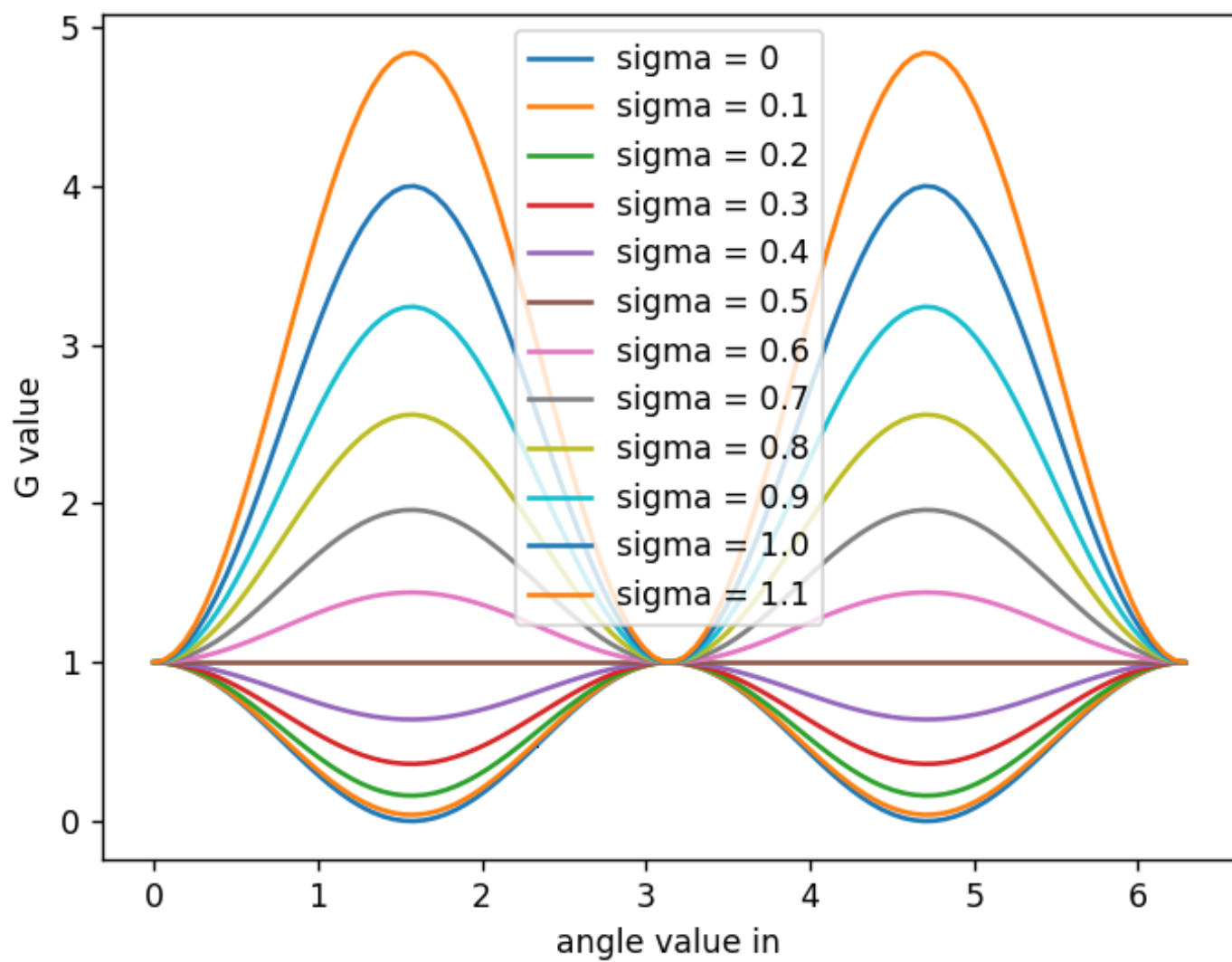
$$\Rightarrow G = \cos(k n \Delta x) + 2\sigma i \sin(k n \Delta x)$$

$$\Rightarrow |G|^2 = \cos^2(k n \Delta x) + 4\sigma^2 \sin^2(k n \Delta x) < 1$$

$$\Rightarrow \sigma^2 < \frac{-\cos^2(k n \Delta x)}{4 \sin^2(k n \Delta x)} + 1$$

WE SEE ON THE GRAPH THAT SIGMA SHOULD BE LOWER THAN 0.5.

$$\sigma = \frac{c \Delta t}{\Delta x} \Rightarrow \frac{c \Delta t}{\Delta x} = CFL \leq 1$$



# ACCURACY

$$u_j^{n+1} = \frac{(u_{j+1}^n + u_{j-1}^n)}{2} - \frac{c \Delta t}{2 \Delta x} (u_{j+1}^n - u_{j-1}^n) \quad (1)$$

$$u_j^{n+1} = u_j^n + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2} \frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6} \frac{\partial^3 u}{\partial t^3} + \dots \quad (2)$$

$$u_{j+1}^n = u_j^n + \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2} \frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^3}{6} \frac{\partial^3 u}{\partial x^3} + \dots$$

$$u_{j-1}^n = u_j^n - \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2} \frac{\partial^2 u}{\partial x^2} - \frac{\Delta x^3}{6} \frac{\partial^3 u}{\partial x^3} + \dots$$

$$\Rightarrow u_{j+1}^n + u_{j-1}^n = 2u_j^n + 2 \frac{\Delta x^2}{2} \frac{\partial^2 u}{\partial x^2} + 2 \frac{\Delta x^4}{24} \frac{\partial^4 u}{\partial x^4} + \dots \quad (3)$$

$$\Rightarrow u_{j+1}^n - u_{j-1}^n = 2 \Delta x \frac{\partial u}{\partial x} + 2 \frac{\Delta x^3}{6} \frac{\partial^3 u}{\partial x^3} + 2 \frac{\Delta x^5}{120} \frac{\partial^5 u}{\partial x^5} + \dots \quad (4)$$

(2)(3)(4) into (1)

$$\Rightarrow \cancel{u_j^n} + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2} \frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6} \frac{\partial^3 u}{\partial t^3} + \dots - \frac{1}{2} \left( \cancel{2u_j^n} + 2 \frac{\Delta x^2}{2} \frac{\partial^2 u}{\partial x^2} + 2 \frac{\Delta x^4}{24} \frac{\partial^4 u}{\partial x^4} + \dots \right) - \frac{c \Delta t}{2 \Delta x} \left( 2 \Delta x \frac{\partial u}{\partial x} + 2 \frac{\Delta x^3}{6} \frac{\partial^3 u}{\partial x^3} + 2 \frac{\Delta x^5}{120} \frac{\partial^5 u}{\partial x^5} + \dots \right)$$

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{\Delta x^2}{2 \Delta t} \frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^4}{24 \Delta t} \frac{\partial^4 u}{\partial x^4} + \dots - c \left( \frac{\partial u}{\partial x} + \frac{\Delta x^2}{6} \frac{\partial^3 u}{\partial x^3} + \frac{\Delta x^4}{120} \frac{\partial^5 u}{\partial x^5} + \dots \right) - \frac{\Delta t}{2} \frac{\partial^2 u}{\partial x^2} - \frac{\Delta t^3}{6} \frac{\partial^4 u}{\partial x^4}$$

$$\Rightarrow \frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = \frac{\Delta x^2}{2 \Delta t} \frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^4}{24 \Delta t} \frac{\partial^4 u}{\partial x^4} + \dots - c \left( \frac{\Delta x^2}{6} \frac{\partial^3 u}{\partial x^3} + \frac{\Delta x^4}{120} \frac{\partial^5 u}{\partial x^5} + \dots \right) - \frac{\Delta t}{2} \frac{\partial^2 u}{\partial x^2} - \frac{\Delta t^3}{6} \frac{\partial^4 u}{\partial x^4}$$

$$= O\left(\frac{\Delta x^2}{\Delta t}\right) + O(\Delta t) \quad ?$$

## CONSISTENCY

$$\lim_{\substack{\Delta x \rightarrow 0 \\ \Delta t \rightarrow 0}} \frac{\Delta x^2}{2\Delta t} \frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^4}{24\Delta t} \frac{\partial^4 u}{\partial x^4} + \dots - c \left( \frac{\Delta x^2}{6} \frac{\partial^3 u}{\partial x^3} + \frac{\Delta x^4}{120} \frac{\partial^5 u}{\partial x^5} + \dots \right) - \frac{\Delta t}{2} \frac{\partial^2 u}{\partial t^2} - \frac{\Delta t^3}{6} \frac{\partial^4 u}{\partial t^4} \neq 0$$

THE TERMS  $\frac{\Delta x^4}{\Delta t}$  MAKES THIS ALGORITHM

INCONSISTENT..