Práctica individual : Transformada de Fourier

1. a.

$$F^{-1}(F(\tilde{k}))(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} F(k)\, e^{\frac{2\pi i n k}{N}}$$

$$= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \left[\frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} f(m)\, e^{\frac{-2\pi i m k}{N}} \right] e^{\frac{2\pi i n k}{N}}.$$

(definición de $\mathcal{F}$)

$$= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} f(m)\, e^{\frac{2\pi i k (n-m)}{N}}$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} f(m) \underbrace{\left(\sum_{k=0}^{N-1} e^{\frac{2\pi i k (n-m)}{N}} \right)}_{\circledast}$$

$$\circledast \quad \sum_{k=0}^{N-1} e^{\frac{2\pi i k (n-m)}{N}} = \begin{cases} \dfrac{1 - e^{\cancel{2\pi i (n-m)}}}{1 - e^{\frac{2\pi i (n-m)}{N}}} = 0 & \text{si } n \neq m \\ N & \text{si } n = m \end{cases} \Rightarrow \circledast = N \cdot \delta_{n,m} = \begin{cases} N & n=m \\ 0 & n \neq m \end{cases}$$

luego

$$F^{-1}(F(k))(n) = \frac{1}{N} \sum_{m=0}^{N-1} f(m) \cdot N \cdot \delta_{n,m} = \sum_{m=0}^{N-1} f(m) \cdot \delta_{n,m} = f(n).$$

b. $$F[f * g] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} (f * g)(n)\, e^{\frac{-2\pi i n k}{N}}$$

definición de convolución (discreta)

$$= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \left(\sum_{m=0}^{N-1} f(m)\, g(n-m) \right) e^{\frac{-2\pi i n k}{N}}$$

$$= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} f(m)\, e^{\frac{-2\pi i m k}{N}}\, g(n-m)\, e^{\frac{-2\pi i (n-m) k}{N}}$$

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b. $\mathrm{DFT}(f * g) = \mathrm{DFT}\left(\sum_{m=0}^{N-1} f(m) \cdot g(n-m)\right)$ definición de convolución.

$= \sum_{m=0}^{N-1} f(m)\, \mathrm{DFT}\left(g(n-m)\right)$ linealidad de DFT

Veamos que

$$\mathcal{F}(f(n-d)) = \mathrm{DFT}(f(n-d)) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(n-d)\, e^{\frac{-2\pi i n k}{N}}$$

$$= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(n-d)\, e^{\frac{-2\pi i}{N} k(n-d)} \cdot e^{\frac{-2\pi i k d}{N}} \qquad j = n-d$$

$$= \frac{1}{\sqrt{N}} \sum_{j=-d}^{N-1-d} f(j)\, e^{\frac{-2\pi i k j}{N}}\, e^{\frac{-2\pi i k d}{N}}$$

$$= e^{\frac{-2\pi i k d}{N}}\, \mathcal{F}(f(n)) = e^{\frac{-2\pi i k d}{N}}\, \mathrm{DFT}(f).$$

luego

$$= \sum_{m=0}^{N-1} f(m)\, e^{\frac{-2\pi i k m}{N}}\, \mathrm{DFT}(g).$$

$$= \mathrm{DFT}(f)\, \mathrm{DFT}(g)$$

$$= \mathcal{F}(f) \cdot \mathcal{F}(g)$$

La prueba está mal por el término $\frac{1}{\sqrt{N}}$, si definimos la transformada como

$$\sum_{n=0}^{N-1} f(n)\, e^{\frac{-2\pi i n k}{N}}$$

está bien, pero tal como está en el enunciado esta prueba estaría incorrecta.

c.

$$\mathcal{F}(f)(k+N) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(n)\, e^{\frac{-2\pi i n (k+N)}{N}}$$

$$= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(n)\, e^{\frac{-2\pi i n k}{N}}\, e^{-2\pi i n}$$

como $n \in \mathbb{Z}$, entonces $1 = e^{-2\pi i n} = \cancel{\ }$

$$= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(n)\, e^{\frac{-2\pi i n k}{N}} = \mathcal{F}(f)(n).$$

d.

$$\mathcal{F}(f)^*(-k) = \overline{\mathcal{F}(f)(-k)} = \overline{\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(n)\, e^{\frac{-2\pi i n (-k)}{N}}} = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(n)\, \overline{e^{\frac{-2\pi i n (-k)}{N}}}$$

$f \in \mathbb{R}$ $(\forall n)$ hipotesis.

$$= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(n)\, e^{\frac{-2\pi i n k}{N}} = \mathcal{F}(f)(k).$$

e.

$|F(k)| = \sqrt{F(k) \cdot F^*(k)}$ por definición. (Suponga como hip. que $f(n)$ es real).

$$|F(-k)| = \sqrt{F(-k) \cdot F^*(-k)} = \sqrt{F(-k)\, F(k)} = \sqrt{F(k) \cdot F(-k)} = \sqrt{F(k)\, F^*(k)} = |F(k)|$$

punto (d) — prop. conmutativa — (d).

f.

$$F^*(N-k) = \overline{\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(n)\, e^{\frac{-2\pi i n (N-k)}{N}}} = \overline{\frac{1}{\sqrt{N}} \sum_{n=0} f(n)\, e^{\frac{2\pi i n k}{N}}\, \underbrace{e^{-2\pi i n}}_{=1}}$$

xq $n \in \mathbb{Z}$.

$$= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(n)\, e^{\frac{-2\pi n i k}{N}} = F(k).$$

Suponiendo que $f(n)$ es real.

3. (1) recordemos que $e^{i\pi n} = e^{-i\pi n}$ $(\forall n \in \mathbb{Z})$

$$\cos(\pi n) + i\underbrace{\sin(n\pi)}_{=0} = \cos(n\pi) + i\underbrace{\sin(-n\pi)}_{=0}$$ identidad de euler.

$$\cos(\pi n) = \cos(-n\pi)$$ es una función par.

$$F^*\left(\frac{N}{2}+k\right) = \overline{\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} f(n)\, e^{\frac{-2\pi i n\left(\frac{N}{2}+k\right)}{N}}} = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} \overline{f(n)}\, e^{\frac{+2\pi n i}{N}\frac{N}{2} + \frac{2\pi i n k}{N}} =$$

$$= \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} \overline{f(n)}\, e^{\pi n i}\, e^{\frac{2\pi n i k}{N}} \underset{(1)}{=} \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} \overline{f(n)}\, e^{-\pi n i}\, e^{\frac{2\pi n i k}{N}} =$$

$$= \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} \overline{f(n)}\, e^{-\frac{\pi n i 2N}{2N} + \frac{2\pi n i k}{N}} = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} \overline{f(n)}\, e^{\frac{-2\pi i n\left(\frac{N}{2}-k\right)}{N}} = F\left(\frac{N}{2}-k\right).$$

2) a. i $\mathcal{F}(a f(n,m))(k,l) = \frac{1}{N}\sum_{n,m=0}^{N-1} a\cdot f(n,m)\, e^{\frac{-2\pi i (nk+ml)}{N}}$

$$= a\,\frac{1}{N}\sum_{n,m=0}^{N-1} f(n,m)\, e^{\frac{-2\pi i (nk+ml)}{N}}.$$

$$= a\,\mathcal{F}(f(n,m))(k,l).$$

ii. $\mathcal{F}\left(f(an,bm)\right) = \sum_{n,m=0}^{N-1} f(an,bm)\, e^{\frac{-2\pi i (nk+ml)}{N}}$

$$= \sum_{n,m=0}^{N-1} f(an,bm)\, e^{\frac{-2\pi i n k a}{Na} - \frac{2\pi i m l b}{Nb}}.$$

hacemos el cambio de variable $n' = an$ $m' = bm$.

$$= \sum_{n'=0}^{a(N-1)} \sum_{m'=0}^{b(N-1)} f(n',m')\, e^{\frac{-2\pi i n' k}{Na} - \frac{2\pi i m' l}{Nb}}$$

$$= \mathcal{F}\left(\frac{k}{a}, \frac{l}{b}\right)$$

b.

$$\mathcal{F}(f(n-n_0, m-m_0))(k,l) = \frac{1}{N}\sum_n \sum_m f(n-n_0, m-m_0)\, e^{\frac{-2\pi i\,((n-\cancel{n_0})k + (m-\cancel{m_0})l)}{N}}$$

hacemos el cambio de variables $n' = n - n_0$
$m' = m - m_0$

$$= \frac{1}{N}\sum_{n'}\sum_{m'} f(n', m')\, e^{\frac{-2\pi i\,((n'+n_0)k + (m'+m_0)l)}{N}}$$

$$= \frac{1}{N}\sum_{n'}\sum_{m'} f(n', m')\, e^{\frac{-2\pi i\,(n'k + m'l)}{N}}\, e^{\frac{-2\pi i\,(n_0 k + m_0 l)}{N}}.$$

$$= e^{\frac{-2\pi i\,(n_0 k + m_0 l)}{N}}\, \mathcal{F}(f(n,m))(k,l).$$

c.

$$\mathcal{F}^{-1}(\mathcal{F}(u\ldots))\cancel{(n,m)}(a,b) = \frac{1}{N}\sum_{k}\sum_{l} \mathcal{F}(k-u_0, l-v_0)\, e^{2\pi i\,\frac{(ak+bl)}{N}}$$

hacemos el cambio de variable $\begin{cases} k' = k - u_0 \\ l' = l - v_0 \end{cases}$

$$= \frac{1}{N}\sum_{k', l'} \mathcal{F}(k', l')\, e^{2\pi i\,\frac{(a(k'+u_0) + b(l'+v_0))}{N}}$$

$$= \frac{1}{N}\sum_{k'}\sum_{l'} \mathcal{F}(k', l')\, e^{2\pi i\,\frac{(ak' + bl')}{N}}\, e^{2\pi i\,\frac{(au_0 + bv_0)}{N}}.$$

$$= \mathcal{F}^{-1}(\mathcal{F}(k,l))(a,b) \cdot e^{2\pi i\,\frac{(au_0 + bv_0)}{N}}.$$

No puede hacer el a iii.

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