Computational Physics

Ordinary Differential Equations II

- Numerically solving a orbit program
- 2nd order Runge-Kutta methods
- 4th order Runge-Kutta methods

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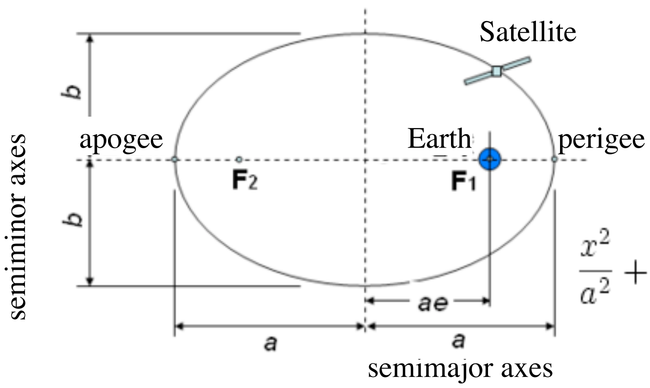
No class on Oct. 27 (Fri), Nov. 3 (Fri) The Culture Day

P02: Orbit of Satellite or Comet

Elliptical orbit about Sun



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

gravitational force

$$\mathbf{F} = -\frac{GmM}{|\mathbf{r}|^3}\mathbf{r}$$

total energy

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

- Basic Equations

$$m\frac{d\mathbf{v}}{dt} = -\frac{GmM}{|\mathbf{r}|^3}\mathbf{r}$$

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}$$

the mass of Sun

M=1.99E30 kg

the gravitational constant

G=6.67E-11 m^3/kg•s^2

Normalization of Equations

Unit of distance (the mean Earth-Sun distance) AU = 1.496E11 [m]

Unit of time: Year
(a period of a circular orbit of radius 1 AU)

yr = 31556925.9747 [s]
= 365.242198781 [days]

v_0 = AU/yr = 4743 [m/s]

- Normalized Basic Equations

$$\frac{d\bar{\mathbf{v}}}{d\bar{t}} = -\frac{4\pi^2}{|\bar{\mathbf{r}}|^2}\hat{\mathbf{r}}$$

$$\frac{d\bar{\mathbf{r}}}{d\bar{t}} = \bar{\mathbf{v}}$$

the gravitational constant

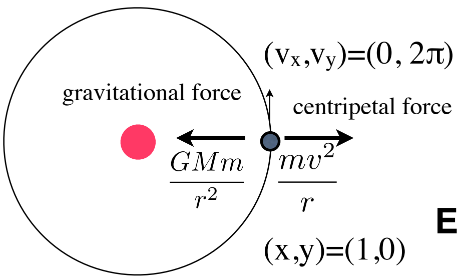
G = 6.67E-11 [$m^3/kg \cdot s^2$]

GM = 39.427 AU^3/yr^2 = $4\pi^2$ AU^3/yr^2

the mass of Sun

M = 1.99E30 [kg]

Orbit problem of satellite



$$\boxed{\frac{d\bar{\mathbf{r}}}{d\bar{t}} = \bar{\mathbf{v}} \qquad \frac{d\bar{\mathbf{v}}}{d\bar{t}} = -\frac{4\pi^2}{|\bar{\mathbf{r}}|^2}\hat{\mathbf{r}}}$$

Euler scheme (1st order)

$$\mathbf{r}^{n+1} = \mathbf{r}^n + \tau\mathbf{v}^n + O(\tau^2)$$

$$\mathbf{v}^{n+1} = \mathbf{v}^n + \tau\mathbf{a}^n + O(\tau^2)$$

τ=0.02, 200 time step
calculation time 4 yr

kinetic energy

$$E_k = \frac{v_x^2 + v_y^2}{2}$$

potential energy

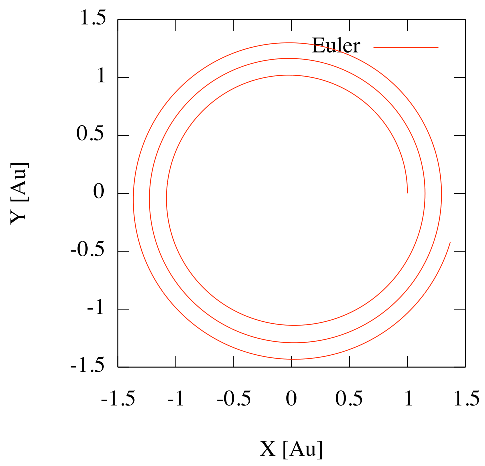
$$E_p = -\frac{GM}{r}$$

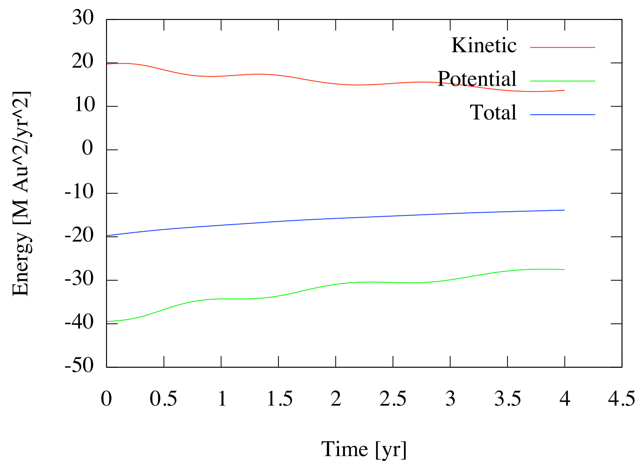
See the energy conservation to check the accuracy of the numerical result.

Forward scheme (Euler scheme)

Euler
τ=0.002, 2000 steps

Initial values: (x,y)=(1,0)
$(v_x, v_y) = (0, 2\pi)$





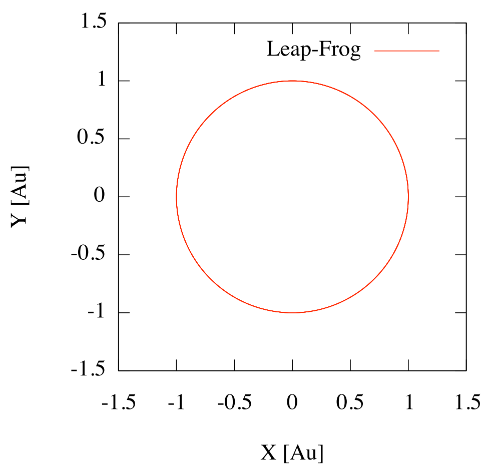
Euler scheme (1st order)

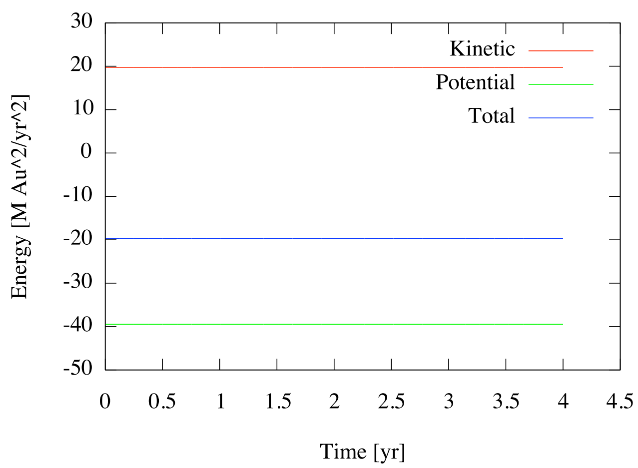
$$\mathbf{r}^{n+1} = \mathbf{r}^n + \tau \mathbf{v}^n + O(\tau^2)$$
$$\mathbf{v}^{n+1} = \mathbf{v}^n + \tau \mathbf{a}^n + O(\tau^2)$$

Centered scheme (Leap-Frog scheme)

Leap-Frog
τ=0.002, 2000 steps

Initial values: (x,y)=(1,0)
$(v_x, v_y) = (0, 2\pi)$





Leap-Frog scheme (2nd order)

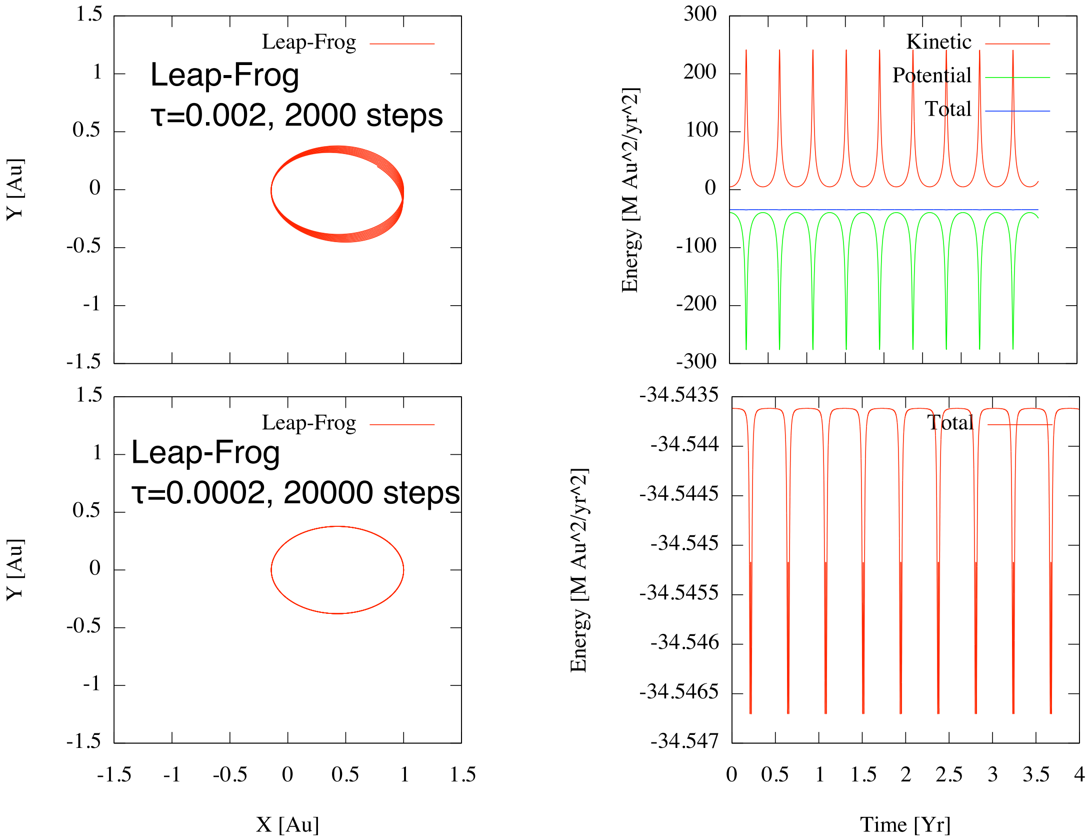
$$\mathbf{r}^{n+1/2} = \mathbf{r}^{n-1/2} + \tau \mathbf{v}^n + O(\tau^3)$$
$$\mathbf{v}^{n+1} = \mathbf{v}^n + \tau \mathbf{a}^{n+1/2} + O(\tau^3)$$

Centered scheme (Leap-Frog scheme)

- test of elliptical orbit -

Initial values: (x,y)=(1,0)

$(v_x, v_y) = (0, \pi)$



Runge-Kutta Methods

- second order RK -

General ODE

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t)$$

where

$$\mathbf{x}(t) = [x(t), y(t), v_x(t), v_y(t)]$$

$$\mathbf{f}(\mathbf{x}, t) = \left[\frac{dx}{dt}, \frac{dy}{dt}, \frac{dv_x}{dt}, \frac{dv_y}{dt}\right]$$

$[v_x, v_y, a_x, a_y]$

Euler method

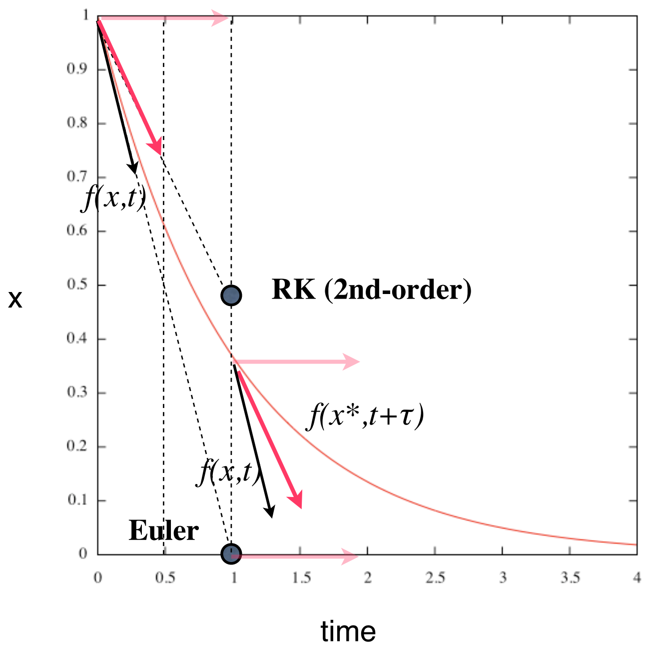
$$\mathbf{x}(t+\tau) = x(t) + \tau\mathbf{f}(\mathbf{x}, t)$$

Using a half time shifted values is better guess,

$$x(t+\tau) = x(t) + \frac{1}{2}\tau\left\{f(x,t) + f(x^*, t+\tau)\right\}$$

$$x^*(t+\tau) = x(t) + \tau f(x,t)$$

Graphical image of RK2



$$\frac{dx}{dt} = -x$$

$$x(t=0) = 1$$

$$x(t+\tau) = x(t) + \frac{1}{2}\tau\left\{f(x,t) + f(x^*, t+\tau)\right\}$$
$$x^*(t+\tau) = x(t) + \tau f(x,t)$$

Runge-Kutta Methods

- Fourth order RK -

General ODE $$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t)$$

where

$$\mathbf{x}(t) = [x(t), y(t), v_x(t), v_y(t)]$$

$$\mathbf{f}(\mathbf{x}, t) = \left[\frac{dx}{dt}, \frac{dy}{dt}, \frac{dv_x}{dt}, \frac{dv_y}{dt}\right]$$

$[v_x, v_y, a_x, a_y]$

$$x(t+\tau) = x(t) + \frac{1}{6}\tau\left[\mathbf{F}_1 + 2\mathbf{F}_2 + 2\mathbf{F}_3 + \mathbf{F}_4\right]$$

where

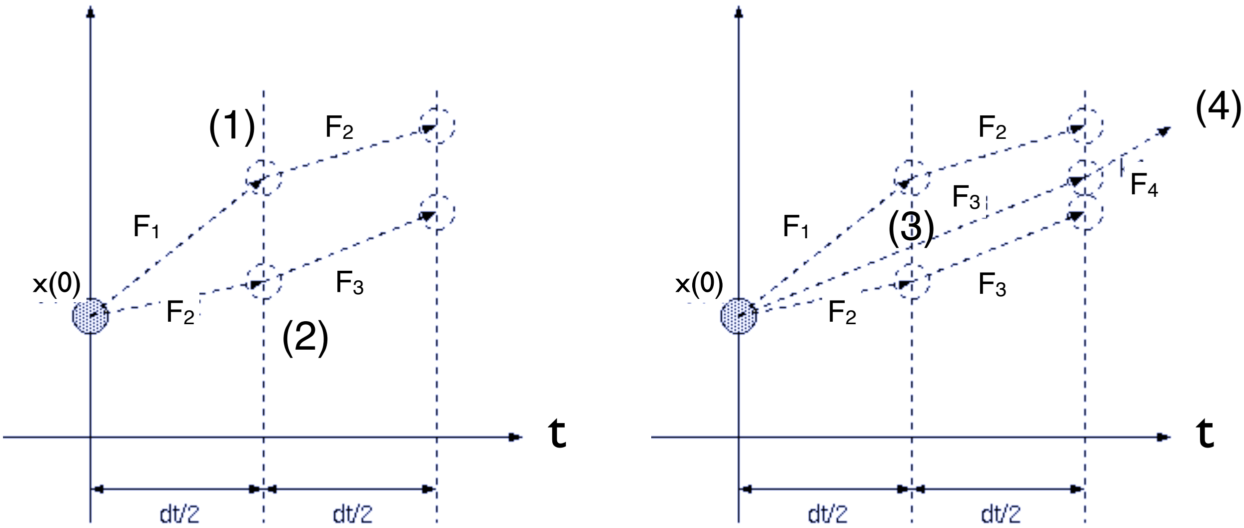
$$\mathbf{F}_1 = f(x,t)$$

$$\mathbf{F}_2 = f(x + \frac{1}{2}\tau\mathbf{F}_1, t + \frac{1}{2}\tau)$$

$$\mathbf{F}_3 = f(x + \frac{1}{2}\tau\mathbf{F}_2, t + \frac{1}{2}\tau)$$

$$\mathbf{F}_4 = f(x + \tau\mathbf{F}_3, t + \tau)$$

Graphical image of RK4



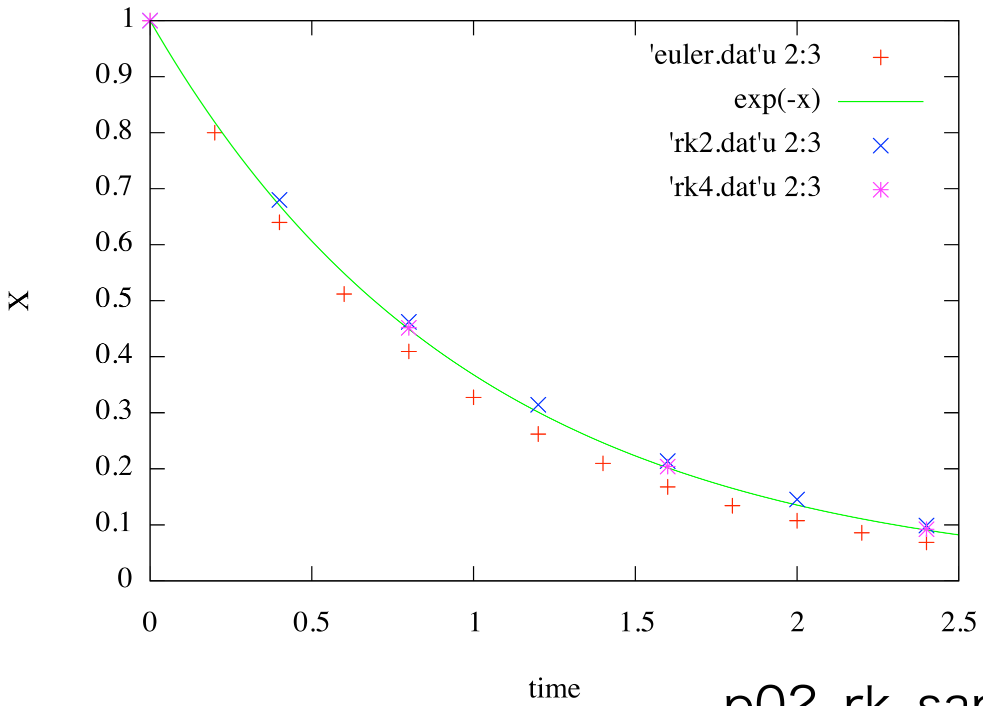
(1) calculate x(dt/2) with F1 at x(0) and calculate F2 at x(dt/2)

(2) using F2 get x(dt/2) and calculate F3 at x(dt/2)

(3) using F3 get x(dt) and calculate F4 at x(dt)

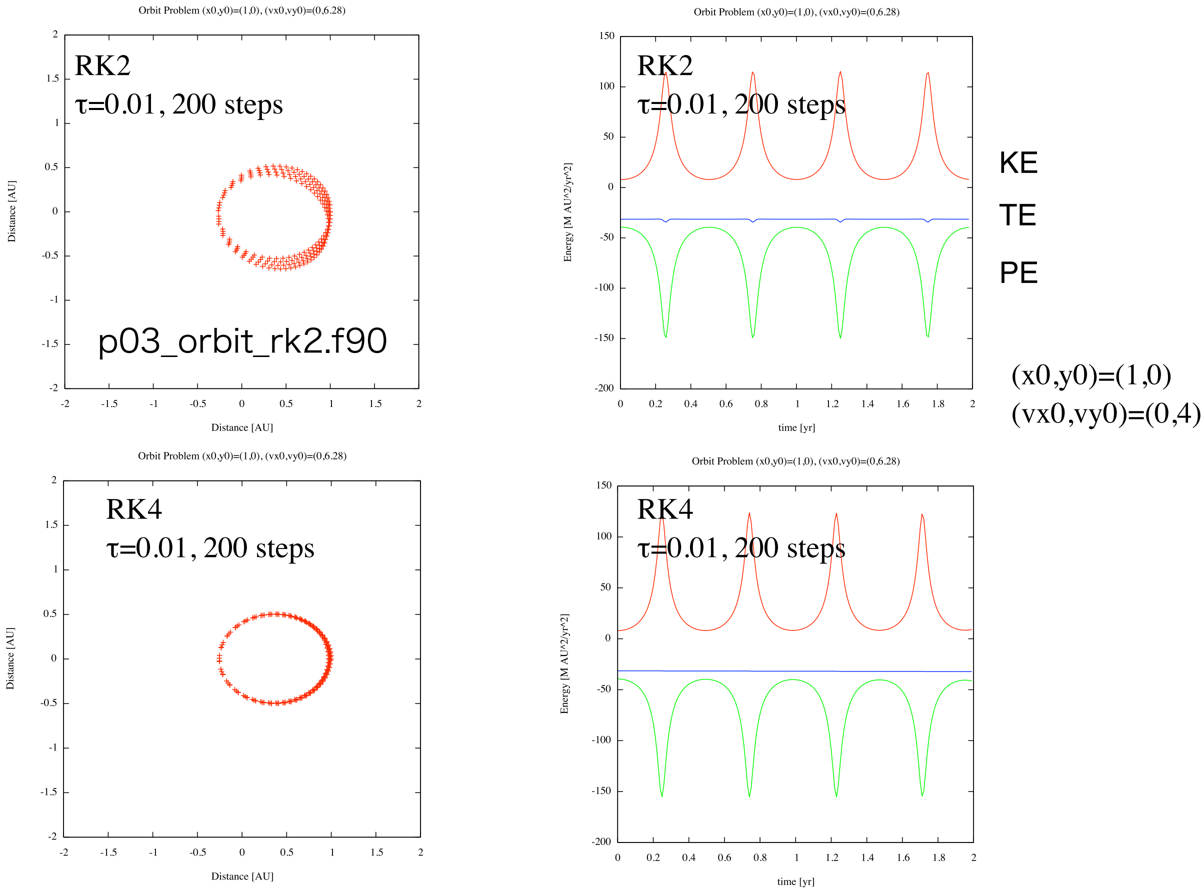
(4) calculate average value by x(dt) = dt $(F_1+2F_2+2F_3+F_4)/6$

Simple test for RK schemes
- dx/dt = -x, x(t=0)=1 -> x(t)=exp(-t) -



p02_rk_sample.f90

Test of elliptical orbit



Homework 02

- Change the Euler program to the 'Leap-Frog' scheme and calculate the orbit with τ=0.002, 2000 steps starting from (x,y)=(1,0) and (vx, vy)=(0,2π). Plot the orbit and the energy evolution in time.

- Write a subroutine of 4th-order Runge-Kutta rk4(), apply to the orbit program. Compute the orbit with the following initial parameters. (the initial radial distance 1AU, the initial tangential velocity is π AU/yr, the time step τ = 0.002, the total calculation time 4 yr). Make graphs of trajectory and energy evolution in time. Compare with the 2nd-order program.

- Calculate Halley's comet orbit in 200 yrs. (aphelion 35 AU, calculate the velocity at aphelion using Kepler's third law and the orbital speed. Use the 4-th order RK (τ = 0.001, number of time step=200000, 200yr).

- Kepler's third law

$$T^2 = \frac{4\pi^2}{GM} a^3$$

- orbital speed

$$v = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)}$$

name	T (yrs)	e	q (AU)	First Pass
Halley	76.03	0.967	0.587	239 B.C.

due date: 11/09 (Thr) 6:00pm