Inference and Regression

Conditions Necessary for Valid Inference in the Least Squares Regression Model

Testing Individual Regression Parameters

Addressing Nonsignificant Independent Variables

Multicollinearity

Inference and Regression

- **Statistical inference:**
  - Process of making estimates and drawing conclusions about one or more characteristics of a population (parameter) through the analysis of sample data drawn from the population.
- In regression, inference is commonly used to estimate and draw conclusions about:

  The regression parameters

$$\beta_0, \beta_1, \beta_2, \ldots, \beta_q.$$

  The mean value and/or the predicted value of the dependent variable *y* for specific values of the independent variables $x_1^*, x_2^*, \ldots, x_q^*.$

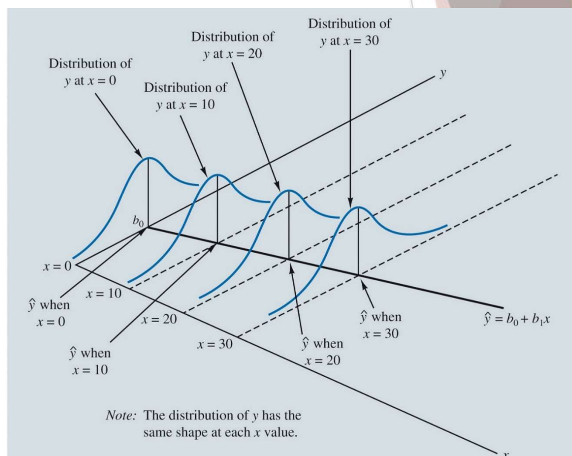
- Consider both **hypothesis testing** and **interval estimation**.

Inference and Regression

Conditions Necessary for Valid Inference in the Least Squares Regression Model:

- 1.For any given combination of values of the independent variables
  - $x_1, x_2, \ldots, x_q$, the population of potential error terms ε is normally distributed with a mean of 0 and a constant variance.
- 2. The values of ε are statistically independent

Illustration of the Conditions for Valid Inference in Regression



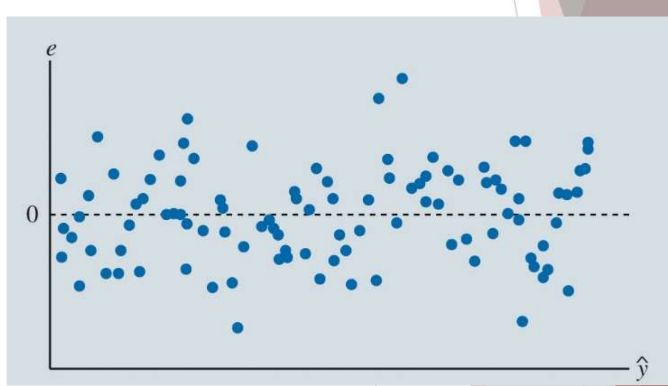
Inference and Regression

Are the conditions violated?

- 1.Center of the residuals should be approximately 0.
  - Mean 0
- 2. The spread in data should be about the same through out
  - Constant variance
- 3. Errors should be symmetrically distributed with values near 0 occurring more frequently
  - Normally Distributed
- 4. Independent
  - Current data points do not depend on previous points

These residuals look good! – No violations

Example of a Random Error Pattern in a Scatter Chart of Residuals and Predicted Values of the Dependent Variable



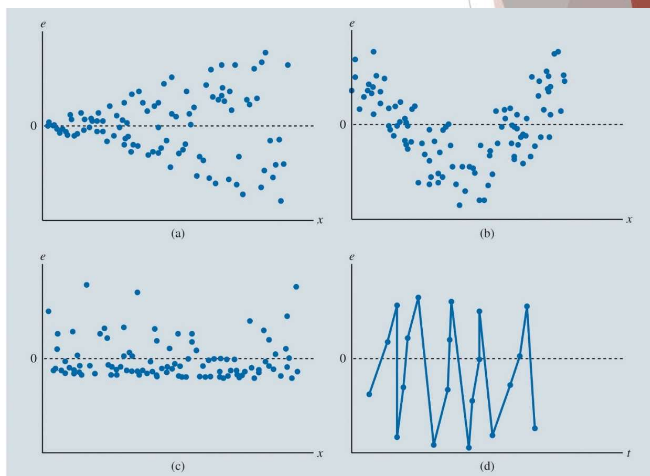
Inference and Regression

Are the conditions violated?

- 1.Center of the residuals should be approximately 0.
  - Mean 0
- 2. The spread in data should be about the same through out
  - Constant variance
- 3. Errors should be symmetrically distributed with values near 0 occurring more frequently
  - Normally Distributed
- 4. Independent
  - Current data points do not depend on previous points

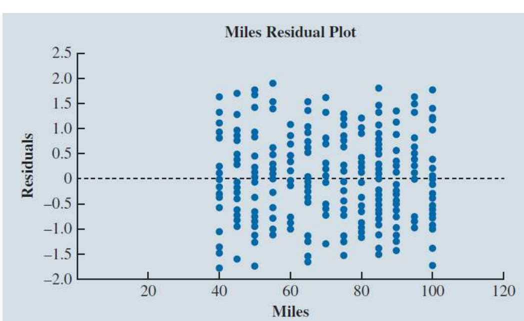
These residuals do **NOT** look good!

Examples of Diagnostic Scatter Charts of Residuals from Four Regressions



Inference and Regression

Figure 7.18: Excel Residual Plots for the Butler Trucking Company Multiple Regression



Inference and Regression

Table of the First Several Predicted Values $\hat{y}$ and Residuals e Generated by the Excel Regression Tool

23	RESIDUAL OUTPUT		
24			
25	*Observation*	*Predicted Time*	*Residuals*
26	1	9.605504464	−0.305504464
27	2	5.556419081	−0.756419081
28	3	9.605504464	−0.705504464
29	4	8.225507903	−1.725507903
30	5	4.8664208	−0.6664208
31	6	6.881873062	−0.681873062
32	7	7.235932632	0.164037368
33	8	7.254143492	−1.254143492
34	9	8.243688763	−0.643688763
35	10	7.553690482	−1.453690482
36	11	6.936415641	0.063584359
37	12	7.290505212	−0.290505212
38	13	9.287776613	0.312223387
39	14	5.874146931	0.625853069
40	15	6.954596501	0.245403499
41	16	5.556419081	0.443580919

Scatter chart of $\hat{y}$ vs Residuals e -

- used to assess whether the regression model satisfies the conditions needed for inference

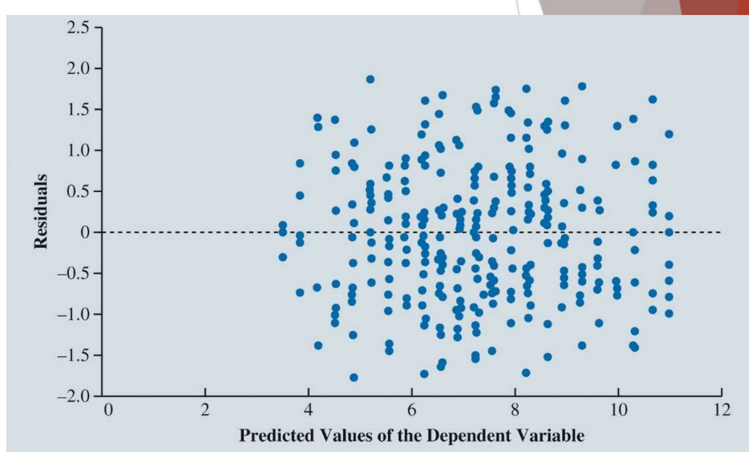
Inference and Regression

Scatter Chart of Predicted Values $\hat{y}$ and Residuals e

- Mean 0
- Similar Variance
- Concentrated around 0

No evidence for violation of the conditions

=> Trust the statistical inference!



Inference and Regression

Testing Individual Regression Parameters:

To determine whether statistically significant relationships exist between the dependent variable *y* and each of the independent variables $x_1, x_2, \ldots, x_q$, individually

If $\beta_j = 0$, there is no linear relationship between the dependent variable *y* and the independent variable x_j.

If $\beta_j \neq 0$, there is a linear relationship between *y* and x_j.

$$H_0: \beta_j = 0$$
$$H_a: \beta_j \neq 0$$

Inference and Regression

Testing Individual Regression Parameters (cont.):

- Use a t test to test the Null Hypothesis
- The test statistic for this t test is,

$$t = \frac{b_j}{s_{b_j}}$$

Where s_{b_j} is the estimated standard deviation of b_j

- As the magnitude of t increases (as t deviates from zero in either direction),
  - we are more likely to reject the hypothesis that the regression parameter β_j is 0.
  - Implies $\beta_j \neq 0$ and there is a relationship between y and x_j

Inference and Regression

Testing Individual Regression Parameters (cont.):

- Typically, most software will provide a p-value to determine if β_j is significant (not equal to 0)
- Confidence interval can be used to test whether each of the regression parameters

$\beta_0, \beta_1, \beta_2, \ldots, \beta_q$ is equal to zero as well.

- **Confidence interval:**
  - An estimated interval believed to contain the value of the parameter at some level of confidence.
    - Example 95% confidence interval

$$b_j \pm t_{\alpha/2} s_{b_j}$$

- **Confidence level:** α - Alpha
  - Indicates how frequently interval estimates will contain the true value of the parameter we are estimating.
    - Example = 0.05

Inference and Regression

Addressing Nonsignificant Independent Variables:

- If practical experience dictates that the nonsignificant independent variable has a relationship with the dependent variable
  - the independent variable should be left in the model.
- If the model sufficiently explains the dependent variable without the nonsignificant independent variable
  - then consider rerunning the regression without the nonsignificant independent variable.
- The appropriate treatment of the inclusion or exclusion of the y-intercept

when b_0 is not statistically significant may require special consideration.

Inference and Regression

Multicollinearity:

- the correlation among the independent variables in multiple regression analysis.
- In t tests for the significance of individual parameters, multicollinearity may lead to:
  - concluding that a parameter associated with one of the multicollinear independent variables is not significantly different from zero when the independent variable actually has a strong relationship with the dependent variable.
- This problem is avoided when there is little correlation among the independent variables.

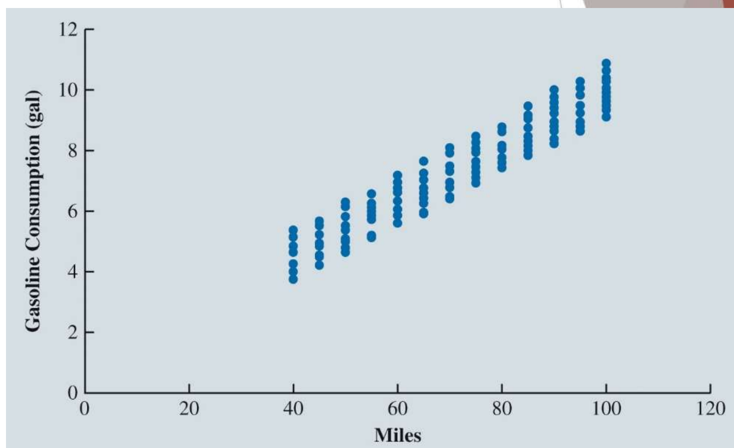
Inference and Regression

Figure 7.21: Excel Regression Output for the Butler Trucking Company with Miles and Gasoline Consumption as Independent Variables

	A	B	C	D	E	F	G	H	I
1	SUMMARY OUTPUT								
2									
3	*Regression Statistics*								
4	Multiple R	0.69406354							
5	R Square	0.481724198							
6	Adjusted R Square	0.478234125							
7	Standard Error	1.398077545							
8	Observations	300							
9									
10	ANOVA								
11		*df*	*SS*	*MS*	*F*	*Significance F*			
12	Regression	2	539.5808158	269.7904079	138.0269794	4.09542E-43			
13	Residual	297	580.5223842	1.954620822					
14	Total	299	1120.1032						
15									
16		*Coefficients*	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*	*Lower 99.0%*	*Upper 99.0%*
17	Intercept	2.493095385	0.33669895	7.404523781	1.36703E-12	1.830477398	3.155713373	1.620208758	3.365982013
18	Miles	0.074701825	0.014274552	5.233216928	3.15444E-07	0.046609743	0.102793908	0.037695279	0.111708371
19	Gasoline Consumption	–0.067506102	0.152707928	–0.442060235	0.658767336	–0.368032789	0.233020584	–0.463398955	0.328386751

Inference and Regression

Figure 7.22: Scatter Chart of Miles and Gasoline Consumed for Butler Trucking Company



Inference and Regression

Multicollinearity (cont.):

- Testing for an overall regression relationship:
  - Use an *F* test based on the *F* probability distribution.
  - If the *F* test leads us to reject the hypothesis that the values of

$$b_1, b_2, \ldots, b_q$$

are all zero:

- Conclude that there is an overall regression relationship.
- Otherwise, conclude that there is no overall regression relationship.

Inference and Regression

Multicollinearity (cont.):

- Testing for an overall regression relationship (cont.):
  - The test statistic generated by the sample data for this test is:

$$F = \frac{\text{SSR}/q}{\text{SSE}/(n-q-1)}$$

- SSR = Sum of squares due to regression.
- SSE = Sum of squares due to error.
- q = the number of independent variables in the regression model.
- n = the number of observations in the sample.

  - Larger values of F provide stronger evidence of an overall regression relationship.
  - For a small p-value => Reject null and conclude there is a regression relationship

Categorical Independent Variables

Butler Trucking Company and Rush Hour

Interpreting the Parameters

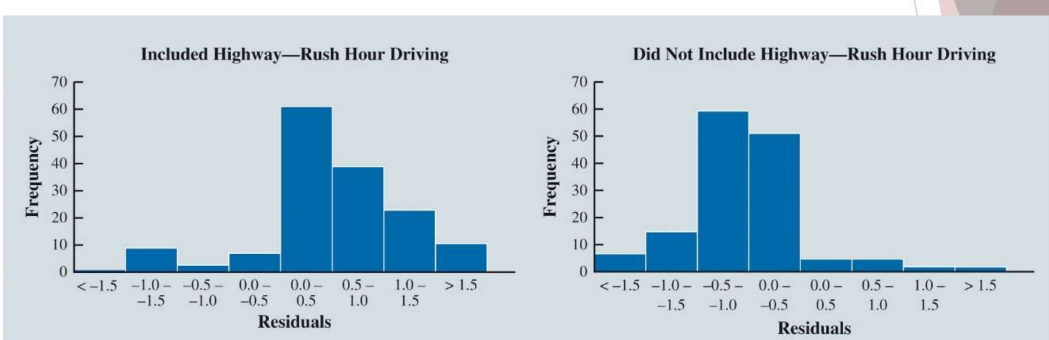
More Complex Categorical Variables

Categorical Independent Variables

Butler Trucking Company and Rush Hour:

- Dependent Variable, y : Travel Time
- Independent Variables
  - x_1 - Miles Traveled
  - x_2 - Number of Deliveries
  - x_3 - Rush Hour
    - Categorical Variable
    - $x_3 = 0$ if delivery trip took place during rush hour
    - $x_3 = 1$ if delivery trip did not take place during rush hour

Categorical Independent Variables



Categorical Independent Variables

Excel Data and Output for Butler Trucking with
Miles Traveled (x_1),
Number of Deliveries (x_2), and the
Highway Rush Hour Dummy
Variable (x_3), as the Independent
Variables

	A	B	C	D	E	F	G	H	I
1	SUMMARY OUTPUT								
2									
3	*Regression Statistics*								
4	Multiple R	0.940107228							
5	R Square	0.8838016							
6	Adjusted R Square	0.882623914							
7	Standard Error	0.663106426							
8	Observations	300							
9									
10	ANOVA								
11		*df*	*SS*	*MS*	*F*	*Significance F*			
12	Regression	3	989.9490008	329.9830003	750.455757	5.7766E–138			
13	Residual	296	130.1541992	0.439710132					
14	Total	299	1120.1032						
15									
16		*Coefficients*	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*	*Lower 99.0%*	*Upper 99.0%*
17	Intercept	–0.330229304	0.167677925	–1.969426232	0.04983651	–0.66022126	–0.000237349	–0.764941128	0.104482519
18	Miles	0.067220302	0.00196142	34.27125147	4.7852E-105	0.063360208	0.071080397	0.062135243	0.072305362
19	Deliveries	0.67351584	0.023619993	28.51465081	6.74797E-87	0.627031441	0.720000239	0.612280051	0.734751629
20	Highway	0.9980033	0.076706582	13.0106605	6.49817E-31	0.847043924	1.148962677	0.799138374	1.196868226

Categorical Independent Variables

Interpreting the Parameters:

- The model estimates that travel time **increases** by:
  - **0.0672 hours (about 4 minutes)** for every increase of 1 mile traveled, holding all other variables constant
  - **0.6735 hours (about 40 minutes)** for every delivery, holding all other variables constant
  - **0.9980 hours (about 60 minutes)** if the driving route took place during the afternoon rush hour period, holding all other variables constant
  - $R^2 = 0.8838$
    - indicates that the regression model explains approximately 88.4% of the variability in travel time for the driving assignments in the sample

Categorical Independent Variables

Interpreting the Parameters (cont.):

Compare the regression model for the case when $x_3 = 0$ and when $x_3 = 1$.

When $x_3 = 0$:

$$\begin{aligned} \hat{y} &= -0.3302 + 0.0672x_1 + 0.6735x_2 + 0.9980(0) \\ &= -0.3302 + 0.0672x_1 + 0.6735x_2 \end{aligned} \quad \textbf{(7.16)}$$

When $x_3 = 1$:

$$\begin{aligned} \hat{y} &= -0.3302 + 0.0672x_1 + 0.6735x_2 + 0.9980(1) \\ &= 0.6678 + 0.0672x_1 + 0.6735x_2 \end{aligned} \quad \textbf{(7.17)}$$

Categorical Independent Variables

More Complex Categorical Variables:

If a categorical variable has *k* levels, *k* minus 1 dummy variables are required, with each dummy variable corresponding to one of the levels of the categorical variable and coded as 0 or 1.

- Example:
  - Suppose a manufacturer of vending machines organized the sales territories for a particular state into three regions: A, B, and C.
  - Sales Region – Categorical variable with 3 levels (A, B, C)
  - Number of Dummy Variables = 3-1 = 2

Region	x_1	x_2
A	0	0
B	1	0
C	0	1

Categorical Independent Variables

More Complex Categorical Variables:

- **Example Continued:**
  - The regression equation:

$$\hat{y} = b_0 + b_1 x_1 + b_2 x_2$$

  - Observations corresponding to Region A -> $x_1 = 0, x_2 = 0,$
    - Estimated mean number of units sold in Region A

$$\hat{y} = b_0 + b_1(0) + b_2(0) = b_0$$

Categorical Independent Variables

More Complex Categorical Variables:

- **Example Continued:**
  - Observations corresponding to Region B -> $x_1 = 1, x_2 = 0,$
  - Estimated number of units sold in Region B:

$$\hat{y} = b_0 + b_1(1) + b_2(0) = b_0 + b_1$$

  - Observations corresponding to Region C -> $x_1 = 0, x_2 = 1,$
  - Estimated number of units sold in Region C:

$$\hat{y} = b_0 + b_1(0) + b_2(1) = b_0 + b_2$$

Modeling Nonlinear Relationships

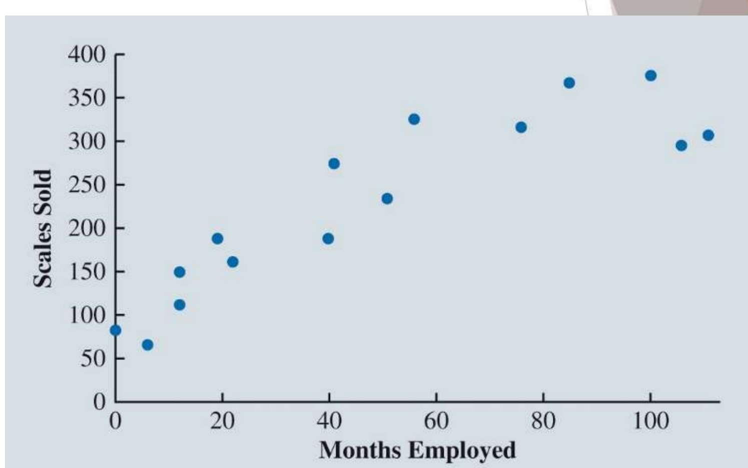
Quadratic Regression Models

Piecewise Linear Regression Models

Interaction Between Independent Variables

Modeling Nonlinear Relationships

Figure 7.25: Scatter Chart for the Reynolds Example



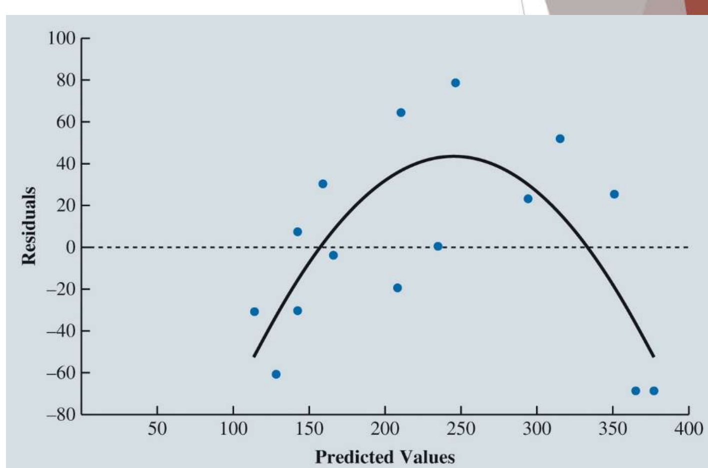
Modeling Nonlinear Relationships

Figure 7.26: Excel Regression Output for the Reynolds Example

	A	B	C	D	E	F	G	H	I
1	SUMMARY OUTPUT								
2									
3	*Regression Statistics*								
4	Multiple R	0.888897515							
5	R Square	0.790138792							
6	Adjusted R Square	0.773995622							
7	Standard Error	48.49087146							
8	Observations	15							
9									
10	ANOVA								
11		*df*	*SS*	*MS*	*F*	*Significance F*			
12	Regression	1	115089.1933	115089.1933	48.94570268	9.39543E–06			
13	Residual	13	30567.74	2351.364615					
14	Total	14	145656.9333						
15									
16		*Coefficients*	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*	*Lower 95.0%*	*Upper 95.0%*
17	Intercept	113.7452874	20.81345608	5.464987985	0.000108415	68.78054927	158.7100256	68.78054927	158.7100256
18	Months Employed	2.367463621	0.338396631	6.996120545	9.39543E-06	1.636402146	3.098525095	1.636402146	3.098525095

Modeling Nonlinear Relationships

Figure 7.27: Scatter Chart of the Residuals and Predicted Values of the Dependent Variable for the Reynolds Simple Linear Regression



Modeling Nonlinear Relationships

- Equation (7.18) corresponds to a **quadratic regression model.**

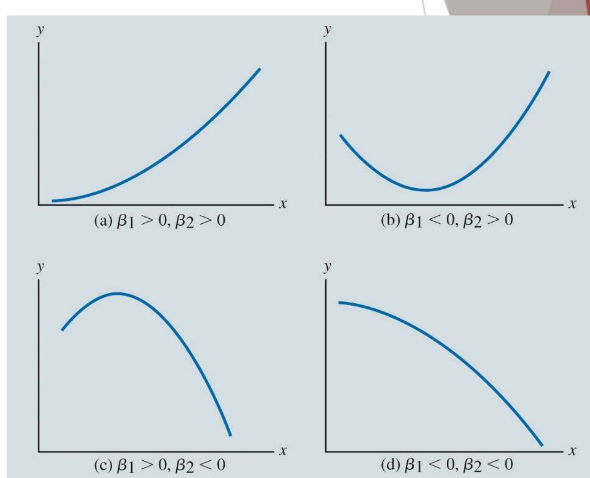
$$\hat{y} = b_0 + b_1x_1 + b_2x_1^2$$

Quadratic Regression Models:

- In the Reynolds example,
  - To account for the curvilinear relationship between months employed and scales sold,
  - include the square of the number of months the salesperson has been employed

Modeling Nonlinear Relationships

Figure 7.28: Relationships That Can Be Fit with a Quadratic Regression Model



Modeling Nonlinear Relationships

Figure 7.29: Excel Data for the Reynolds Quadratic Regression Model

	A	B	C
1	Months Employed	MonthsSq	Scales Sold
2	41	1,681	275
3	106	11,236	296
4	76	5,776	317
5	100	10,000	376
6	22	484	162
7	12	144	150
8	85	7,225	367
9	111	12,321	308
10	40	1,600	189
11	51	2,601	235
12	0	0	83
13	12	144	112
14	6	36	67
15	56	3,136	325
16	19	361	189

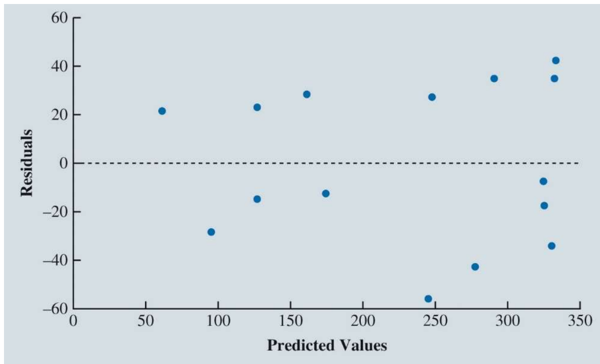
Modeling Nonlinear Relationships

Figure 7.30: Excel Output for the Reynolds Quadratic Regression Model

	A	B	C	D	E	F	G	H	I
1	SUMMARY OUTPUT								
2									
3	*Regression Statistics*								
4	Multiple R	0.949361402							
5	R Square	0.901287072							
6	Adjusted R Square	0.884834917							
7	Standard Error	34.61481184							
8	Observations	15							
9									
10	ANOVA								
11		*df*	*SS*	*MS*	*F*	*Significance F*			
12	Regression	2	131278.711	65639.35548	54.78231208	9.25218E-07			
13	Residual	12	14378.22238	1198.185199					
14	Total	14	145656.9333						
15									
16		*Coefficients*	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*	*Lower 99.0%*	*Upper 99.0%*
17	Intercept	61.42993467	20.57433536	2.985755485	0.011363561	16.60230882	106.2575605	−1.415187222	124.2750566
18	Months Employed	5.819796648	0.969766536	6.001234761	6.20497E-05	3.706856877	7.93273642	2.857606371	8.781986926
19	MonthsSq	−0.031009589	0.008436087	−3.675826286	0.003172962	−0.049390243	−0.012628935	−0.05677795	−0.005241228

Modeling Nonlinear Relationships

Figure 7.31: Scatter Chart of the Residuals and Predicted Values of the Dependent Variable for the Reynolds Quadratic Regression Model



Modeling Nonlinear Relationships

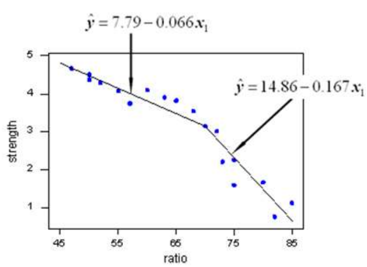
Piecewise Linear Regression Models:

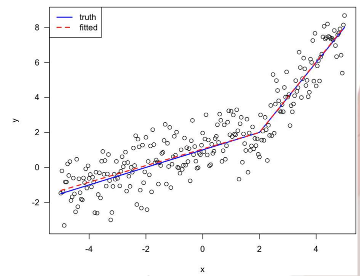
- For the Reynolds data, as an alternative to a quadratic regression model:
  - Recognize that up to a certain point of Months Employed
    - the relationship between Months Employed and Sales appears to be positive and linear.
  - After this point,
    - the relationship between Months Employed and Sales appears to be negative and linear
- **Piecewise linear regression model:**
  - This model will allow us to fit these relationships as two linear regressions
    - joined at the value of Months where the relationship between Months Employed and Sales changes.

Modeling Nonlinear Relationships

Piecewise Linear Regression Models (cont.):

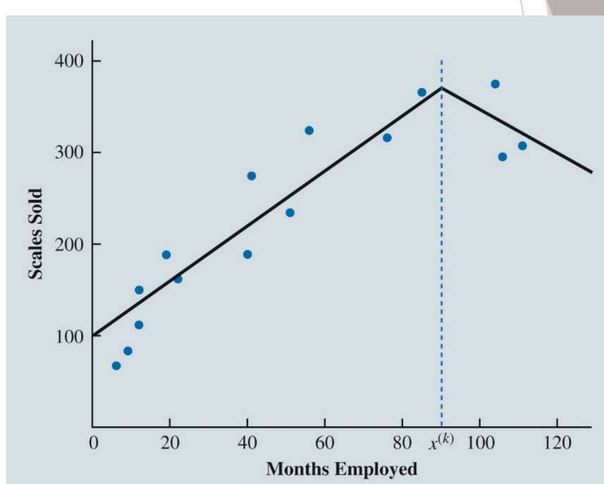
- **Knot:**
  - The value of the independent variable at which the relationship between dependent variable and independent variable changes;
  - also called *breakpoint*.





Modeling Nonlinear Relationships

Figure 7.32: Possible Position of Knot $x^{(k)}$



Modeling Nonlinear Relationships

Piecewise Linear Regression Models (cont.):

- Define a dummy variable:

$$x_k = \begin{cases} 0 \text{ if } x_1 \le x^{(k)} \\ 1 \text{ if } x_1 > x^{(k)} \end{cases}$$

$x_1 =$ Months.

$x^{(k)} =$ value of the knot (90 months for the Reynolds example).

$x_k =$ the knot dummy variable.

- Then fit the following estimated regression equation:

$$\hat{y} = b_0 + b_1 x_1 + b_2(x_1 - x^{(k)})x_k$$

Modeling Nonlinear Relationships

Figure 7.33: Data and Excel Output for the Reynolds Piecewise Linear Regression Model

	A	B	C	D	E	F	G	H	I
1	Knot Dummy	Months Employed	Knot Dummy* Months	Scales Sold					
2	0	41	0	275					
3	1	106	16	296					
4	0	76	0	317					
5	1	100	10	376					
6	0	22	0	162					
7	0	12	0	150					
8	0	85	0	367					
9	1	111	21	308					
10	0	40	0	189					
11	0	51	0	235					
12	0	0	0	83					
13	0	12	0	112					
14	0	6	0	67					
15	0	56	0	325					
16	0	19	0	189					
17									
18									
19	SUMMARY OUTPUT								
20									
21	*Regression Statistics*								
22	Multiple R	0.955796127							
23	R Square	0.913546237							
24	Adjusted R Square	0.899137276							
25	Standard Error	32.3941739							
26	Observations	15							
27									
28	ANOVA								
29		*df*	*SS*	*MS*	*F*	*Significance F*			
30	Regression	2	133064.3433	66532.17165	63.4012588	4.17545E-07			
31	Residual	12	12592.59003	1049.382502					
32	Total	14	145656.9333						
33									
34		*Coefficients*	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*	*Lower 99.0%*	*Upper 99.0%*
35	Intercept	87.21724231	15.31062519	5.696517369	9.9677E-05	53.85825572	120.5762289	40.45033153	133.9841531
36	Months Employed	3.409431979	0.338360666	10.07632484	3.2987E-07	2.67220742	4.146656538	2.375895931	4.442968028
37	Knot Dummy* Months	–7.872553259	1.902156543	–4.138751508	0.00137388	–12.01699634	–3.728110179	–13.68276572	–2.062340794

Modeling Nonlinear Relationships

Interaction Between Independent Variables:

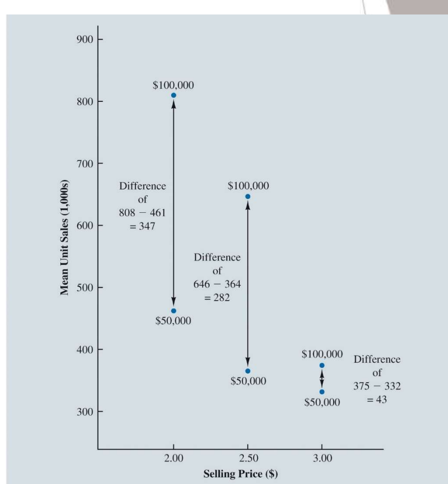
- **Interaction:**
  - This occurs when the relationship between the dependent variable and one independent variable is different at various values of a second independent variable.

- The estimated multiple linear regression equation is given as:

$$\hat{y} = b_0 + b_1x_1 + b_2x_2 + b_3x_1x_2$$

Modeling Nonlinear Relationships

Figure 7.34: Mean Unit Sales (1,000s) as a Function of Selling Price and Advertising Expenditures



Modeling Nonlinear Relationships

Figure 7.35: Excel Output for the Tyler Personal Care Linear Regression Model with Interaction

	A	B	C	D	E	F	G	H	I
1	SUMMARY OUTPUT								
2									
3	*Regression Statistics*								
4	Multiple R	0.988993815							
5	R Square	0.978108766							
6	Adjusted R Square	0.974825081							
7	Standard Error	28.17386496							
8	Observations	24							
9									
10	ANOVA								
11		*df*	*SS*	*MS*	*F*	*Significance F*			
12	Regression	3	709316	236438.6667	297.8692	9.25881E-17			
13	Residual	20	15875	793.7666667					
14	Total	23	5191.3333						
15									
16		*Coefficients*	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*	*Lower 99.0%*	*Upper 99.0%*
17	Intercept	–275.8333333	112.8421033	–2.444418575	0.023898351	–511.2178361	–40.44883053	–596.9074508	45.24078413
18	Price	175	44.54679188	3.928453489	0.0008316	82.07702045	267.9229796	48.24924412	301.7507559
19	Advertising Expenditure ($1,000s)	19.68	1.42735225	13.78776683	1.1263E-11	16.70259538	22.65740462	15.61869796	23.74130204
20	Price*Advertising	–6.08	0.563477299	–10.79014187	8.67721E-10	–7.255393049	–4.904606951	–7.683284335	–4.476715665