Linearization of ODEs
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$$\vec{g}(\dot{\vec{\eta}}, \vec{\eta}, t, \vec{\beta}, \vec{v}) = \vec{0}_g \quad \text{(ODEs)}$$

$$\frac{dg}{d\beta} = \frac{\partial \vec{g}}{\partial \dot{\vec{\eta}}}\cdot\frac{\partial \dot{\vec{\eta}}}{\partial \vec{\beta}} + \frac{\partial \vec{g}}{\partial \vec{\eta}}\cdot\frac{\partial \vec{\eta}}{\partial \vec{\beta}} + \frac{\partial \vec{g}}{\partial t}\cdot\overset{\vec{0}}{\cancel{\frac{\partial t}{\partial \vec{\beta}}}} + \frac{\partial \vec{g}}{\partial \vec{\beta}}\cdot\overset{I_\beta}{\cancel{\frac{\partial \vec{\beta}}{\partial \vec{\beta}}}} + \frac{\partial \vec{g}}{\partial \vec{v}}\cdot\overset{\vec{0}}{\cancel{\frac{\partial \vec{v}}{\partial \vec{\beta}}}} = \vec{0}_{g\times\beta}$$

$$\Rightarrow \vec{0}_{g\times\beta} = \frac{\partial \vec{g}}{\partial \dot{\vec{\eta}}}\cdot\frac{\partial \dot{\vec{\eta}}}{\partial \vec{\beta}} + \frac{\partial \vec{g}}{\partial \vec{\eta}}\cdot\frac{\partial \vec{\eta}}{\partial \vec{\beta}} + \frac{\partial \vec{g}}{\partial \vec{\beta}}$$

$$\frac{\partial \vec{g}}{\partial \dot{\vec{\eta}}}\cdot\frac{\partial \dot{\vec{\eta}}}{\partial \vec{\beta}} + \frac{\partial \vec{g}}{\partial \vec{\eta}}\cdot\frac{\partial \vec{\eta}}{\partial \vec{\beta}} + \frac{\partial \vec{g}}{\partial \vec{\beta}} = \vec{0}_{g\times\beta}$$

$$(n_g\times n_\eta)\,(n_\eta\times n_p) + (n_g\times n_\eta)(n_\eta\times n_p) + (n_g\times n_p) = (n_g\times n_p)$$

- g differential equations / number of eqs.
- η state variable
- t time
- β model parameters
- v control variables
- $\vec{x}$ vector
- $\dot{x}$ time derivative
- x^T transpose

$$\vec{g} = \begin{bmatrix} \frac{dC_A}{dt} - F_{in}C_{A,in} - \nu_A^{(1)} r_1 - \nu_A^{(2)} r_2 \\ \frac{dC_B}{dt} - F_{in}C_{B,in} - \nu_B^{(1)} r_1 - \nu_B^{(2)} r_2 \\ \frac{dC_C}{dt} - F_{in}C_{C,in} - \nu_C^{(1)} r_1 - \nu_C^{(2)} r_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{\eta}^T = \left[C_A, C_B, C_C\right]; \quad \dot{\vec{\eta}}^T = \left[\frac{dC_A}{dt}, \frac{dC_B}{dt}, \frac{dC_C}{dt}\right] \qquad \nu = \begin{bmatrix} \nu_A^{(1)} & \nu_A^{(2)} \\ \nu_B^{(1)} & \nu_B^{(2)} \\ \nu_C^{(1)} & \nu_C^{(2)} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$r_1 = k_1 C_A, \quad r_2 = k_2 C_B;$$

$$\vec{g} = \begin{bmatrix} \frac{dC_A}{dt} - F_{in}C_{A,in} + k_1 C_A \\ \frac{dC_B}{dt} - F_{in}C_{B,in} - k_1 C_A + k_2 C_B \\ \frac{dC_C}{dt} - F_{in}C_{C,in} - k_2 C_B \end{bmatrix}$$

$$\frac{\partial \vec{g}}{\partial \dot{\vec{\eta}}} = \begin{bmatrix} \frac{\partial g_1}{\partial \dot{C}_A} & \frac{\partial g_1}{\partial \dot{C}_B} & \frac{\partial g_1}{\partial \dot{C}_C} \\ \frac{\partial g_2}{\partial \dot{C}_A} & \frac{\partial g_2}{\partial \dot{C}_B} & \frac{\partial g_2}{\partial \dot{C}_C} \\ \frac{\partial g_3}{\partial \dot{C}_A} & \frac{\partial g_3}{\partial \dot{C}_B} & \frac{\partial g_3}{\partial \dot{C}_C} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{\partial \vec{g}}{\partial \vec{\eta}} = \begin{bmatrix} \frac{\partial g_1}{\partial C_A} & \frac{\partial g_1}{\partial C_B} & \frac{\partial g_1}{\partial C_C} \\ \frac{\partial g_2}{\partial C_A} & \frac{\partial g_2}{\partial C_B} & \frac{\partial g_2}{\partial C_C} \\ \frac{\partial g_3}{\partial C_A} & \frac{\partial g_3}{\partial C_B} & \frac{\partial g_3}{\partial C_C} \end{bmatrix} = \begin{bmatrix} k_1 & 0 & 0 \\ -k_1 & k_2 & 0 \\ 0 & -k_2 & 0 \end{bmatrix}$$

$$\frac{\partial \vec{g}}{\partial \vec{\beta}} = \begin{bmatrix} \frac{\partial g_1}{\partial k_1} & \frac{\partial g_1}{\partial k_2} \\ \frac{\partial g_2}{\partial k_1} & \frac{\partial g_2}{\partial k_2} \\ \frac{\partial g_3}{\partial k_1} & \frac{\partial g_3}{\partial k_2} \end{bmatrix} = \begin{bmatrix} C_A & 0 \\ -C_A & C_B \\ 0 & -C_B \end{bmatrix}$$

Putting things together:

$$\cancel{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} \frac{d}{dt}\left(\frac{\partial C_A}{\partial k_1}\right) & \frac{d}{dt}\left(\frac{\partial C_A}{\partial k_2}\right) \\ \frac{d}{dt}\left(\frac{\partial C_B}{\partial k_1}\right) & \frac{d}{dt}\left(\frac{\partial C_B}{\partial k_2}\right) \\ \frac{d}{dt}\left(\frac{\partial C_C}{\partial k_1}\right) & \frac{d}{dt}\left(\frac{\partial C_C}{\partial k_2}\right) \end{bmatrix} + \begin{bmatrix} k_1 & 0 & 0 \\ -k_1 & k_2 & 0 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial C_A}{\partial k_1} & \frac{\partial C_A}{\partial k_1} \\ \frac{\partial C_B}{\partial k_1} & \frac{\partial C_B}{\partial k_2} \\ \frac{\partial C_C}{\partial k_1} & \frac{\partial C_C}{\partial k_3} \end{bmatrix} + \begin{bmatrix} C_A & 0 \\ -C_A & C_B \\ 0 & -C_B \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{d}{dt}\left(\frac{\partial C_A}{\partial k_1}\right) & \frac{d}{dt}\left(\frac{\partial C_A}{\partial k_2}\right) \\ \frac{d}{dt}\left(\frac{\partial C_B}{\partial k_1}\right) & \frac{d}{dt}\left(\frac{\partial C_B}{\partial k_2}\right) \\ \cdots & \cdots \end{bmatrix} + \begin{bmatrix} k_1 & 0 & 0 \\ -k_1 & k_2 & 0 \\ \cdots \end{bmatrix} \begin{bmatrix} \frac{\partial C_A}{\partial k_1} & \frac{\partial C_A}{\partial k_2} \\ \frac{\partial C_B}{\partial k_1} & \frac{\partial C_B}{\partial k_2} \\ \cdots & \cdots \end{bmatrix} + \begin{bmatrix} C_A & 0 \\ -C_A & C_B \\ \cdots \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \cdots \end{bmatrix}$$

$$\begin{bmatrix} \frac{d}{dt}\left(\frac{\partial C_A}{\partial k_1}\right) & \frac{d}{dt}\left(\frac{\partial C_A}{\partial k_2}\right) \\ \frac{d}{dt}\left(\frac{\partial C_B}{\partial k_1}\right) & \frac{d}{dt}\left(\frac{\partial C_B}{\partial k_2}\right) \\ \frac{d}{dt}\left(\frac{\partial C_C}{\partial k_1}\right) & \frac{d}{dt}\left(\frac{\partial C_C}{\partial k_2}\right) \end{bmatrix} + \begin{bmatrix} k_1 & 0 & 0 \\ -k_1 & k_2 & 0 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial C_A}{\partial k_1} & \frac{\partial C_A}{\partial k_2} \\ \frac{\partial C_B}{\partial k_1} & \frac{\partial C_B}{\partial k_2} \\ \frac{\partial C_C}{\partial k_1} & \frac{\partial C_C}{\partial k_2} \end{bmatrix} + \begin{bmatrix} C_A & 0 \\ -C_A & C_B \\ 0 & -C_B \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} k_1 \frac{\partial C_A}{\partial k_1} & k_1 \frac{\partial C_A}{\partial k_2} \\ -k_1 \frac{\partial C_A}{\partial k_1} + k_2 \frac{\partial C_B}{\partial k_1} & -k_1 \frac{\partial C_A}{\partial k_2} + k_2 \frac{\partial C_B}{\partial k_2} \\ -k_2 \frac{\partial C_B}{\partial k_1} & -k_2 \frac{\partial C_B}{\partial k_2} \end{bmatrix} + = \begin{bmatrix} 0 \end{bmatrix}$$

$$\left.\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial C_A}{\partial k_1}\right) + k_1 \frac{\partial C_A}{\partial k_1} + C_A = 0 \\
&\frac{d}{dt}\left(\frac{\partial C_B}{\partial k_1}\right) - k_1 \frac{\partial C_A}{\partial k_1} + k_2 \frac{\partial C_B}{\partial k_1} - C_A = 0 \\
&\frac{d}{dt}\left(\frac{\partial C_C}{\partial k_1}\right) - k_2 \frac{\partial C_B}{\partial k_1} = 0 \\
&\frac{d}{dt}\left(\frac{\partial C_A}{\partial k_2}\right) + k_1 \frac{\partial C_A}{\partial k_2} = 0 \\
&\frac{d}{dt}\left(\frac{\partial C_B}{\partial k_2}\right) - k_1 \frac{\partial C_A}{\partial k_2} + k_2 \frac{\partial C_B}{\partial k_2} + C_B = 0 \\
&\frac{d}{dt}\left(\frac{\partial C_C}{\partial k_2}\right) - k_2 \frac{\partial C_B}{\partial k_2} - C_B = 0
\end{aligned}\right\} \text{sensitivity equations.}$$