Lab 4: Mathematical Modeling and Simulation

Objectives

The objective of this project is to model and simulate the oscillation of a pendulum using numerical methods and compare the results with those obtained through an analytical method and experimentation. In this laboratory, students will get familiar with the Linear Servo Base Unit device, referred to as the IP02 device, and use it to oscillate a pendulum. Students will use a SIMULINK interface to display the pendulum angles.

Description

The Linear Servo Base Unit (IP02) device by QUANSER shown in Figure 1 will be used to oscillate a pendulum. Table 1 provides a list of the principal elements of the IP02 device.

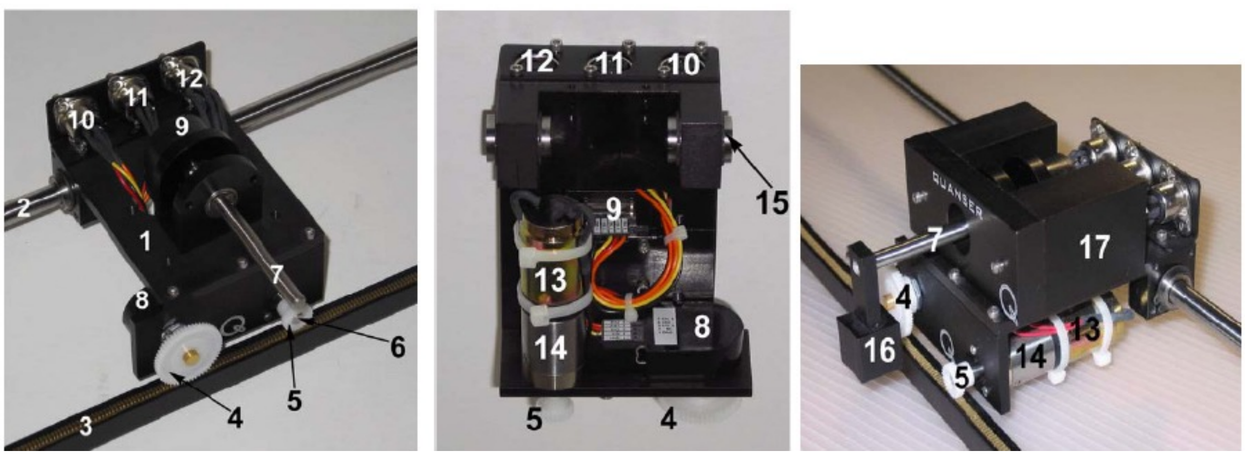

Figure 1. Principal components of the IP02 device

Table 1. List of principal components of the IP02 device

ID #	*Description*	*ID #*	*Description*
1	IP02 Cart	**2**	Stainless Steel Shaft
3	Rack	**4**	Cart Position Pinion
5	Cart Motor Pinion	**6**	Cart Motor Pinion Shaft
7	Pendulum Axis	**8**	IP02 Cart Encoder
9	IP02 Pendulum Encoder	**10**	IP02 Cart Encoder Connector
11	IP02 Pendulum Encoder Connector	**12**	Motor Connector
13	DC Motor	**14**	Planetary Gearbox
15	Linear Bearing	**16**	Pendulum Socket
17	IP02 Weight		

A picture of the pendulum is shown in Figure 2.

Figure 2. The IP02 pendulum

The experiment is conducted by connecting the pendulum to an axle on the IP02 cart. The pendulum is then positioned to a desired angle by hand before being released to begin the oscillations. An encoder is connected to the axle. The encoder sends a signal to the board. The board converts the signals into a readable signal for Simulink MATLAB. A Simulink code will output the pendulum angle and its linear velocity with respect to time.

In order to take the measurements, follow the steps below:

1. Run MATLAB and open Simulink file exp2.slx. Simulink diagram is shown in the Figure 3.
2. Set the mode to external (Figure 3).

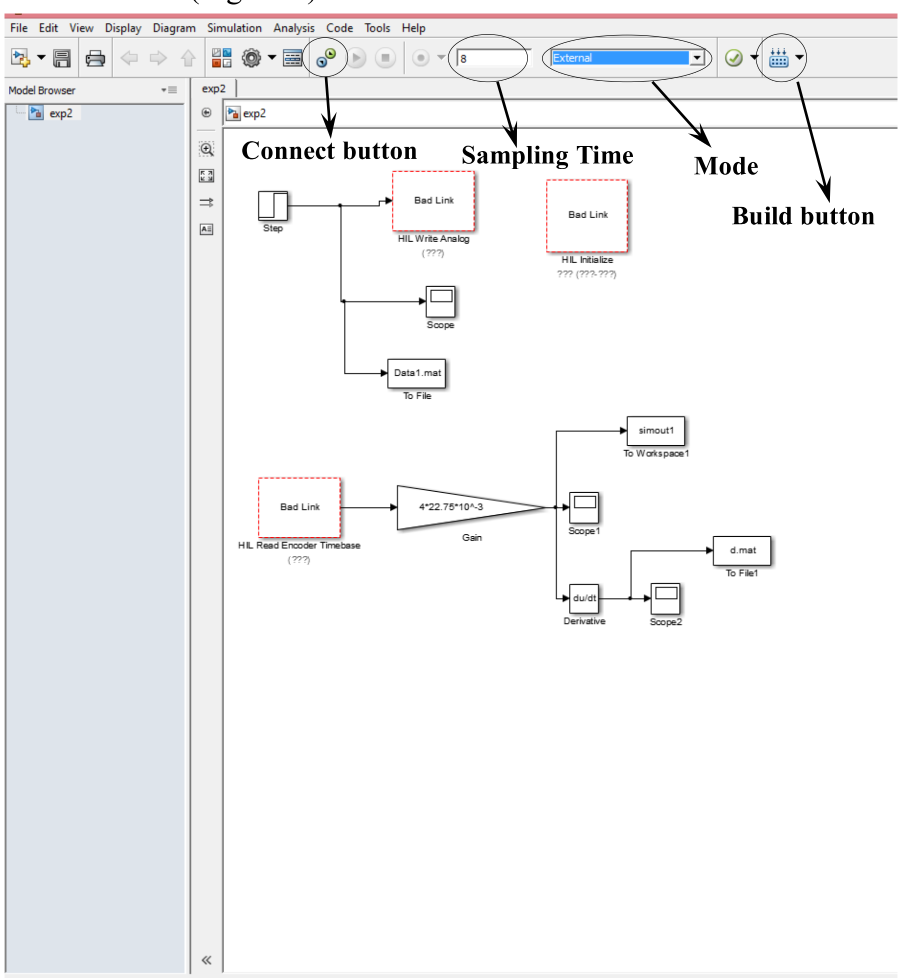

Figure 3. Simulink diagram

3. Set the sampling time to 8 (Figure 3).
4. Build the model first, and then click on the connect button (Figure 3).
5. Finally press the start button and let the pendulum go to begin the oscillations.
6. After 8 seconds (time of sampling) check the angle vs. time results by clicking on Scope 1, and the velocity vs. time results by clicking on Scope 2. The results of velocity vs. time are also written in a MATLAB file (d.mat) that will be saved in the same folder as exp2.slx.

1. Curve Fitting

In this section you will be developing an empirical model of the pendulum. Following the instructions given above:

1. Position the pendulum with 5 different initial amplitudes $(\theta = \theta_{max1}, \theta_{max2} \ldots \theta_{max5})$ (θ is the angle between the pendulum and vertical axis)

$$0 < \theta_{o1} < 15$$
$$15 < \theta_{o2} < 30$$
$$30 < \theta_{o3} < 45$$
$$45 < \theta_{o4} < 60$$
$$60 < \theta_{o5} < 75$$

   and monitor the maximum linear velocity of the pendulum. During an oscillation, the initial and final velocities are zero, the maximum velocity occurs when the pendulum is vertical ($\theta = 0$).

2. Use the first four maximum linear velocities, as determined by their initial amplitudes, $\theta_{max1}, \theta_{max2}, \theta_{max3}, \theta_{max4}$. Determine the linear regression by hand, this is find the slope and intercept. Also, use MATLAB's curve fitting tool (`polyfit`) and explain the relationship between the maximum linear velocity V_0 as a function of θ_{max}.
3. From the curve obtained in the previous step determine the maximum velocity for the last maximum angle you measured and compare the results.

2. Pendulum Oscillation Modelling

Consider the pendulum and with a length l and a mass m (Figure 1). Using equilibrium of forces in the polar system, the Ordinary Differential Equation (ODE) that describes the system can be written as shown in Eq. 1.

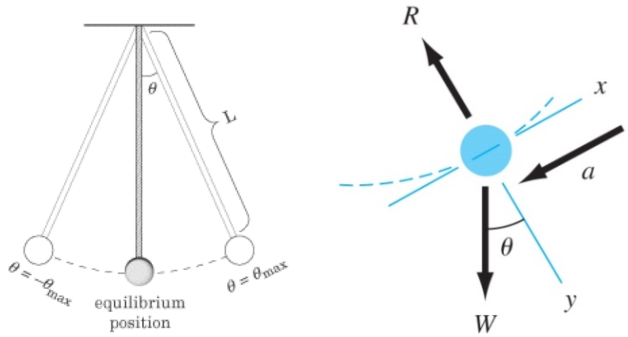

Figure 4. Free body diagram of a simple pendulum

$$\Sigma M = I_0\alpha \Rightarrow -mgr\sin\theta = I_0\alpha \Rightarrow -mgr\sin\theta = (I + mr^2)\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} = -\frac{mgr}{(I_G + mr^2)}\sin\theta \qquad (1)$$

where m is the mass of the pendulum, g is gravity, r is the distance from the pivot point to the centre of gravity of the pendulum (rod), and I_G is the moment of inertia about the centre of gravity.

Based on this mathematical model of the pendulum:

1. Find the equivalent system of two ordinary differential equations (ODE) based on the second order differential equation given in Eq. 1. Use the specifications for the real pendulum (Figure 2), which are given in the Table 2.

Table 2. Pendulum specifications

Mass (kg)	0.1270
Length (m)	0.3365
Distance from pivot point to centre of gravity (r) (m)	0.1778
Moment of inertia about centre of gravity (I_G) (kg.m^2)	1.2000E-3

2. Develop an algorithm for the Runge-Kutta 4th-order method in Matlab (See definition at the end of this document). Solve the nonlinear ODE numerically with the same five θ_o amplitudes. Simulate the pendulum for 8s with a time step of $h = 0.1s$. Include your code as an Appendix in your report.

3. Plot the velocities for all five θ_o amplitudes that resulted using the ODE (numerical method) and the experimental trials (d.mat). Comment the results.

Definition of Runge-Kutta 4th-order method for 2nd order ODE:

$$\frac{d^2x}{dt^2} = f(t, x, v), \quad \frac{dx}{dt} = v, \quad x(t_0) = x_0, \quad v(t_0) = v_0$$

$$t_{i+1} = t_i + h, \qquad h = stepsize$$

$$dx_1 = h \times v_i$$

$$dv_1 = h \times f(t_i, x_i, v_i)$$

$$dx_2 = h\left(v_i + \frac{1}{2} dv_1\right)$$

$$dv_2 = h \times f\left(t_i + \frac{1}{2}h, x_i + \frac{1}{2}dx_1, v_i + \frac{1}{2}dv_1\right)$$

$$dx_3 = h\left(v_i + \frac{1}{2} dv_2\right)$$

$$dv_3 = h \times f\left(t_i + \frac{1}{2}h, x_i + \frac{1}{2}dx_2, v_i + \frac{1}{2}dv_2\right)$$

$$dx_4 = h(v_i + dv_3)$$

$$dv_4 = h \times f(t_i + h, x_i + dx_3, v_i + dv_3)$$

$$x_{i+1} = x_i + \frac{1}{6}(dx_1 + 2dx_2 + 2dx_3 + dx_4)$$

$$v_{i+1} = v_i + \frac{1}{6}(dv_1 + 2dv_2 + 2dv_3 + dv_4)$$