Calculus II

Kenneth M Monks
Jenna M Allen

FRONT RANGE COMMUNITY COLLEGE
BOULDER COUNTY CAMPUS

Contents

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We hope you have fun with this resource and find it helpful! It is a beautiful subject, and we tried to honor that with a beautiful text. The text is still in its infancy, and we welcome any and all feedback you could give us. Thank you in advance for any comments, complaints, suggestions, and questions.

Your book makers,
Kenneth M Monks, Front Range Community College - Boulder County Campus

`kenneth.monks@frontrange.edu`

Jenna M Allen, University of Colorado Boulder

`jenna.m.allen@colorado.edu`

Chapter 1

Overview

A Three-Part Course

The topics of Calculus II fall into three parts that each have an appropriate place in the story of the calculus sequence.

- **Part I: Integration.** The first part of the course ties loose ends from Calculus I. The ending of Calculus I showed that antiderivatives can be used to evaluate integrals via the Fundamental Theorem of Calculus. However, by the end of Calculus I, only the very simplest antiderivatives can actually be computed. Part one expands the student's knowledge of **techniques of antidifferentiation**. These techniques are subsequently put to use computing **length, area, volume, and center of mass**.
- **Part II: Sequences and Series.** This is the topic that makes up the body of Calculus II. **Sequences and series** embody the beauty of mathematics; from simple beginnings (a sequence is just a list... a series is just adding up a list of numbers...) it quickly leads to incredible structure, surprises, complexity, and open problems. **Power series** redefine commonly used transcendental functions (functions that are not computed using algebra, e.g. cosine). If you've ever wondered **what your calculator does when you press the cosine button**, this is where you find out! (**Hint:** It does not have a circle of radius one spinning around with a team of elves that measure x coordinates.)
- **Part III: Coming Attractions.** By the end of Calculus II, the student is ready for a *lot* of other classes. The end of Calculus II thus ends with a sampler platter of topics that show the vast knowledge base built upon the foundation laid in Calculus II. Here the text takes a bite out of **Differential Equations**, serves some polar and parametric coordinates as a palate cleanser before **Calculus III**, and tastes some **Complex Analysis** to aid in digestion of Differential Equations. For dessert, it serves a scoop of **Probability** with both discrete and continuous colored sprinkles.

How to Use This Book

This book is meant to facilitate *Active Learning* for students, instructors, and learning assistants. Active Learning is the process by which the student participates directly in the learning process by reading, writing, and interacting with peers. This contrasts the traditional model where the student passively listens to lecture while taking notes. This book is designed as a self-guided step-by-step exploration of the concepts. The text incorporates theory and examples together in order to lead the student to discovering new results while still being able to relate back to familiar topics of mathematics. Much of this text can be done independently by the student for class preparation. During class sessions, the instructor and/or learning assistants may find it advantageous to encourage group work while being available to assist students, and work with students on a one-on-one or small group basis. This is highly desirable as extensive research has shown that active learning improves student success and retention. (For example, see `www.pnas.org/content/111/23/8410` for Scott Freeman's metaanalysis of 225 studies supporting this claim.)

What is Different about this Book

If you leaf through the text, you'll quickly notice two major structural differences from many traditional calculus books:

1. The exercises are very intermingled with the readings. Gone is the traditional separation into "section" versus "exercises".

2. Whitespace was included for the student to write and work through exercises. Parts of pages have indeed been intentionally left blank.

A consequence of this structure is that the readings and exercise are closely linked. It is intended for the student to do the readings and exercises concurrently.

Ok... Why?

The goal of this structure is to help the student simulate the process by which a mathematician reads mathematics. When a mathematician reads a paper or book, he/she always has a pen in hand and is constantly working out little examples alongside and scribbling incomprehensible notes in bad handwriting. It takes a *long* time and a lot of experience to know how to come up with good questions to ask oneself or to know what examples to work out in order to to help oneself absorb the subject. Hence, the exercises sprinkled throughout the readings are meant to mimic the margin scribbles or side work a mathematician engages in during the act of reading mathematics.

The Legend of Coffee

A potential hazard of this self-guided approach is that while most examples are meant to be simple exercises to help with absorption of the topics, there are some examples that students may find quite difficult. To prevent students from spinning their wheels in frustration, we have labeled the difficulty of all exercises using coffee cups as follows:

Coffee Cup Legend

Symbol	Number of Cups	Description of Difficulty
☕	A One-Cup Problem	Easy warm-up suitable for class prep.
☕☕	A Two-Cup Problem	Slightly harder, solid groupwork exercise.
☕☕☕	A Three-Cup Problem	Substantial problem requiring significant effort.
☕☕☕☕	A Four-Cup Problem	Difficult problem requiring effort and creativity!

Glossary of Symbols

In Precalculus and Calculus I, there is a wide range of how much notation from Set Theory gets used. To get everyone on the same page, here is a short list of some notation we will use in this text.

Sets and Elements

Often in mathematics, we construct collections of objects called *sets*.

- If an object x is in a set A, we say x is an *element* of A and write $x \in A$.
- If an object x is not in a set A, we say x is not an element of A and write $x \notin A$.

Any particular object is either an element of a set or it is not. We do not allow for an object to be partially contained in a set, nor do we allow for an object to appear multiple times in a set. Often we use curly braces around a comma-separated list to indicate what the elements are.

Example 1.0.0.1. A Prime Example

Suppose P is the set of all prime numbers. We write

$$P = \{2, 3, 5, 7, 11, 13, 17, \ldots\}$$

For example, $2 \in P$ and $65,537 \in P$, but $4 \notin P$.

Some Famous Sets of Numbers

The following are fundamental sets of numbers used in Calculus 2.

- **Natural Numbers:** The set $\mathbb{N}$ of natural numbers is the set of all positive whole numbers, along with zero. That is,

  $$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \ldots\}$$

  Note that in many other sources, zero is not included in the natural numbers. Both are widely used; be aware the choice on this convention will change throughout your mathematical travels!

- **Integers:** The set of integers $\mathbb{Z}$ is the set of all whole numbers, whether they are positive, negative, or zero. That is,

  $$\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

- **Rational Numbers:** The set of rational numbers $\mathbb{Q}$ is the set of all numbers expressible as a fraction whose numerator and denominator are both integers.

- **Real Numbers:** The set of real numbers $\mathbb{R}$ is the set of all numbers expressible as a decimal.

- **Complex Numbers:** The set $\mathbb{C}$ of complex numbers is the set of all numbers formed as a real number (called the real part) plus a real number times i (called the imaginary part), where i is a symbol such that $i^2 = -1$.

Set-Builder Notation

The most common notation used to construct sets is *set-builder notation*, in which one specifies a name for the elements being considered and then some property $P(x)$ that is the membership test for an object x to be an element of the set. Specifically,

$$A = \{x \in B : S(x)\}$$

means that an object x chosen from B is an element of the set A if and only if the claim $S(x)$ is true about x. Sometimes the "$\in B''$ gets dropped if it is clear from context what set the elements are being chosen from. The set-builder notation above gets read as "the set of all x in B such that $S(x)$". One can think of this as running through all elements of B and throwing away any that do not meet the condition described by S.

Example 1.0.0.2. Interval Notation

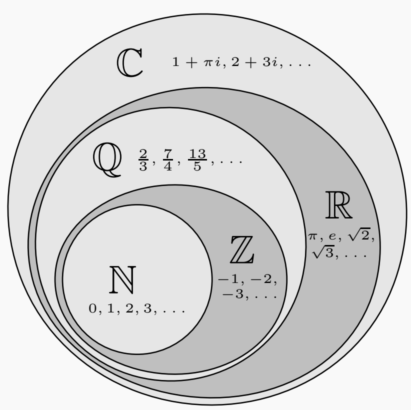
Interval notation can be expressed in set-builder notation as follows:

- $(a, b) = \{x \in \mathbb{R} : a < x < b\}$
- $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$
- $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$
- $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$

Example 1.0.0.3. Rational, Real, and Complex in Set-Builder Notation

Set-builder notation is often used to express the sets of rational, real, and complex numbers as follows:

- $\mathbb{Q} = \left\{\frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0\right\}$
- $\mathbb{R} = \{0.a_0a_1a_2a_3a_4 \ldots \times 10^n : n \in \mathbb{N}, a_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ where $i \in \mathbb{N}\}$ Note this is essentially scientific notation; the concatenation of the a_i's represents the digits in a base-ten decimal expansion.
- $\mathbb{C} = \{a + bi : a \in \mathbb{R}, b \in \mathbb{R}\}$



Part I

Integration

1.1 Integrating Products of Powers of Sine and Cosine

In this section, we give an algorithm to find an antiderivative of the form

$$\int \sin^n(x)\cos^m(x)\,\mathtt{d}x$$

for $n, m \in \mathbb{N}$.

Exercise 1.1.0.1. Knowledge is Power ☕

There are two exponents in the integrand above.

- What symbol above is the exponent of sine?

  A Solution: n

- What symbol above is the exponent of cosine?

  A Solution: m

Note that some sine-cosine integrals can be done by techniques you have already learned. For example, n or m is equal to 1, ordinary u-substitution will work just fine!

Exercise 1.1.0.2. u-sub with Sines and Cosines ☕☕

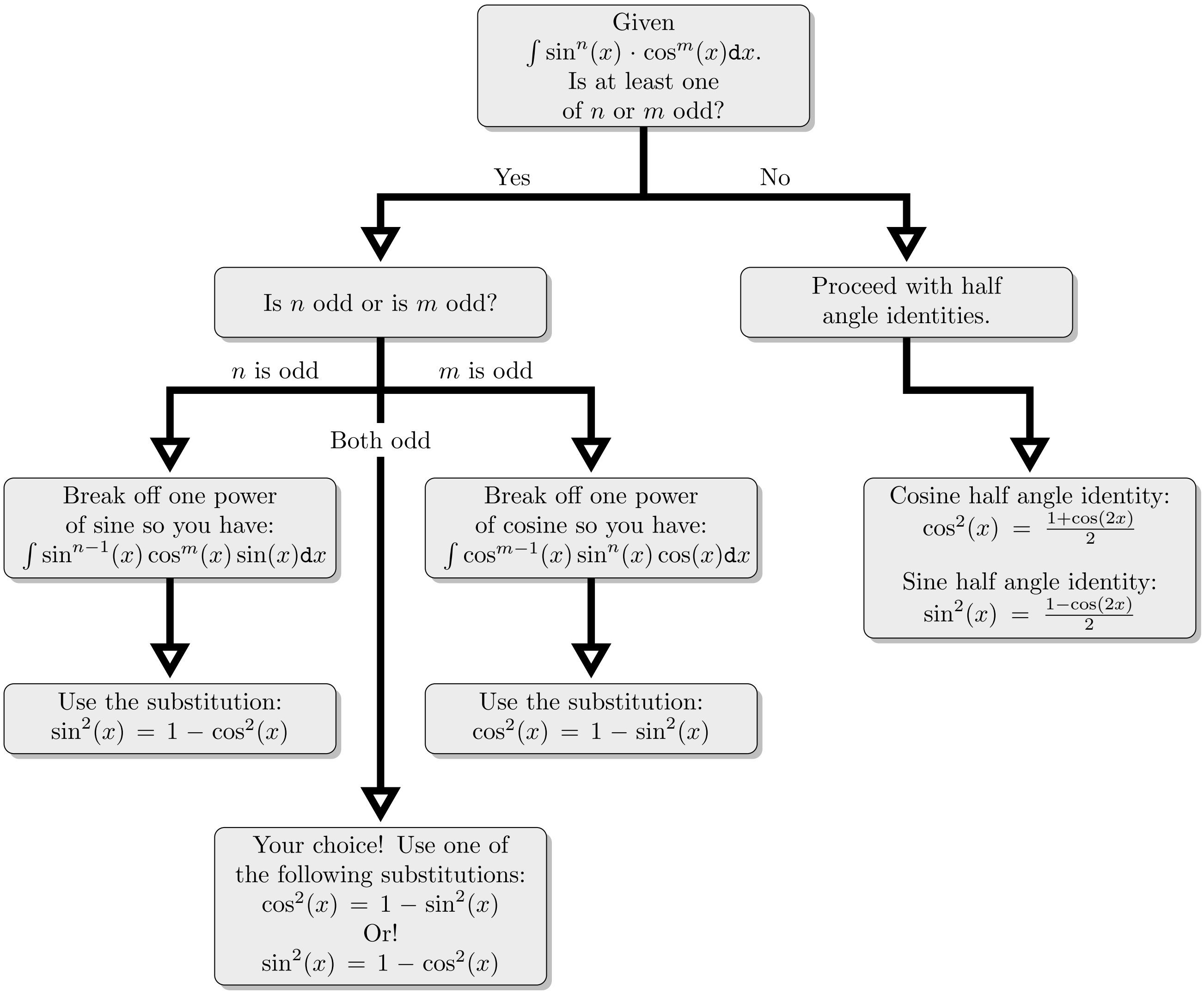
Evaluate the following integral using the substitution $u = \sin(x)$:

$$\int \sin^2(x)\cos(x)\,\mathtt{d}x$$

A Solution: Let $u = \sin(x)$, $\mathtt{d}u = \cos(x)$. Then:

$$\begin{aligned}\int \sin^2(x)\cos(x)\,\mathtt{d}x &= \int u^2\,\mathtt{d}u\\ &= \frac{1}{3}u^3 + C\\ &= \frac{1}{3}\sin^3(x) + C\end{aligned}$$

There are two types of integrals containing powers of sine and cosine. The first type is the case where we have at least one odd exponent; the second type is where both exponents are even. We show an overview of how to handle each case in the following awesome flow chart:



1.1.1 At Least One Odd Power

Recall the Pythagorean identity for sine and cosine (written in two useful forms here):

Pythagorean Theorem Slightly Rewritten	
$\cos^2(x) = 1 - \sin^2(x)$	$\sin^2(x) = 1 - \cos^2(x)$

If at least one exponent is odd, we pull one of those functions out for the "$\mathtt{d}u$" and perform u-sub. We then use the Pythagorean trig identity to rewrite sine and cosine in terms of each other as needed.

Example 1.1.1.1. Odd Power Case

Here we compute the integral

$$\int \sin^7(x)\cos^2(x)\,\mathtt{d}x$$

. In this case, we proceed using the substitution $u = \cos(x)$, so $\mathtt{d}x = \frac{1}{-\sin(x)}\,\mathtt{d}u$.

$$\begin{aligned}
\int \sin^7(x)\cos^2(x)\,\mathtt{d}x &= \int \sin^6(x)\cos^2(x)\sin(x)\,\mathtt{d}x \\
&= \int \left(\sin^2(x)\right)^3 \cos^2(x)\sin(x)\frac{1}{-\sin(x)}\,\mathtt{d}u \\
&= \int \left(1-\cos^2(x)\right)^3 \cos^2(x)(-1)\,\mathtt{d}u \\
&= -\int \left(1-u^2\right)^3 u^2\,\mathtt{d}u \\
&= -\int \left(1-3u^2+3u^4-u^6\right) u^2\,\mathtt{d}u \\
&= -\int \left(u^2-3u^4+3u^6-u^8\right)\mathtt{d}u \\
&= -\left(\frac{1}{3}u^3-\frac{3}{5}u^5+\frac{3}{7}u^7-\frac{1}{9}u^9\right)+C \\
&= -\frac{1}{3}\cos^3(x)+\frac{3}{5}\cos^5(x)-\frac{3}{7}\cos^7(x)+\frac{1}{9}\cos^9(x)+C
\end{aligned}$$

Exercise 1.1.1.2. Why Odd Mattered ☕

In Example 1.1.1.**??**, the exponent of sine (in this case, the number 7) being odd really mattered. If that 7 were replaced by an even number instead, why would this approach have failed? Answer in a few short sentences below.

A Solution: If the exponent of sine had been even, then we couldn't have used the Pythagorean identity to express it in terms of cosine, and still had an extra sine to use with the u-sub, which would have prevented us from expressing all the parts in terms of a single expression.

Exercise 1.1.1.3. Try a Few with Odd Exponents ☕☕

- Find an antiderivative for the function $\sin^5(x)\cos^2(x)$.

A Solution:

$$\begin{aligned}\int \sin^5(x)\cos^2(x)\,\mathtt{d}x &= \int \sin^4(x)\cos^2(x)\sin(x)\,\mathtt{d}x\\ &= \int \left(\sin^2(x)\right)^2\cos^2(x)\sin(x)\,\mathtt{d}x\\ &= \int \left(1-\cos^2(x)\right)^2\cos^2(x)\sin(x)\,\mathtt{d}x\end{aligned}$$

Let $u = \cos(x)$ so $\mathtt{d}u = -\sin(x)$ and $\mathtt{d}x = \frac{1}{-\sin(x)}\,\mathtt{d}u$

$$\begin{aligned}\int \left(1-\cos^2(x)\right)^2\cos^2(x)\sin(x)\,\mathtt{d}x &= \int \left(1-u^2\right)^2 u^2\sin(x)\frac{1}{-\sin(x)}\,\mathtt{d}u\\ &= \int \left(1-u^2\right)^2 u^2(-1)\,\mathtt{d}u\\ &= -\int \left(1-2u^2+u^4\right)u^2\,\mathtt{d}u\\ &= -\int u^2-2u^4+u^6\,\mathtt{d}u\\ &= -\left(\frac{1}{3}u^3-\frac{2}{5}u^5+\frac{1}{7}u^7+C\right)\\ &= -\frac{1}{3}\cos^3(x)+\frac{2}{5}\cos^5(x)-\frac{1}{7}\cos^7(x)+C\end{aligned}$$

- Evaluate $\int \cos^9(x)\,\mathtt{d}x$. (**Hint:** Pascal's Triangle will be extremely helpful!)

A Solution:

$$\begin{aligned}\int \cos^9(x)\,\mathtt{d}x &= \int \cos^8(x)\cos(x)\,\mathtt{d}x\\ &= \int (1-\sin^2(x))^4\cos(x)\,\mathtt{d}x\\ &\text{Let } u=\sin(x)\\ &= \int (1-u^2)^4\,\mathtt{d}u\\ &= \int 1-4u^2+6u^2-4u^6+u^8\,\mathtt{d}u\\ &= u-\frac{4}{3}u^3+\frac{6}{3}u^3-\frac{4}{7}u^7+\frac{1}{9}u^9+C\\ &= \sin(x)-\frac{4}{3}\sin^3(x)+\frac{6}{3}\sin^3(x)-\frac{4}{7}\sin^7(x)+\frac{1}{9}\sin^9(x)+C\end{aligned}$$

Exercise 1.1.1.4. Two Different Options

- Consider $\int \cos(x)\sin^3(x)\,\mathtt{d}x$.
  - Compute this integral using $u = \cos(x)$.

A Solution:

$$\begin{aligned}
\int \cos(x)\sin^3(x)\,\mathtt{d}x &= \int \cos(x)\left(1-\cos^2(x)\right)\sin(x)\,\mathtt{d}x \\
&= -\int u\left(1-u^2\right)\mathtt{d}u \\
&= -\int u - u^3\,\mathtt{d}u \\
&= -\left(\frac{1}{2}u^2 - \frac{1}{4}u^4\right) + C \\
&= -\frac{1}{2}\cos^2(x) + \frac{1}{4}\cos^4(x) + C
\end{aligned}$$

  - Compute this integral using $u = \sin(x)$.

A Solution:

$$\begin{aligned}
\int \cos(x)\sin^3(x)\,\mathtt{d}x &= \int u^3\,\mathtt{d}u \\
&= \frac{1}{4}\sin^4(x) + C
\end{aligned}$$

  - Your two answers will appear very different! Show that they are in fact compatible.

A Solution:

$$\begin{aligned}
-\frac{1}{2}\cos^2(x) + \frac{1}{4}\cos^4(x) + C &= -\frac{1}{2}(1-\sin^2(x)) + \frac{1}{4}(1-\sin^2(x))^2 + C \\
&= -\frac{1}{2} + \frac{1}{2}\sin^2(x) + \frac{1}{4}(1 - 2\sin^2(x) + \sin^4(x)) + C \\
&= -\frac{1}{2} + \frac{1}{2}\sin^2(x) + \frac{1}{4} - \frac{1}{2}\sin^2(x) + \frac{1}{4}\sin^4(x) + C \\
&= -\frac{1}{4} + \frac{1}{4}\sin^4(x) + C \\
&-\frac{1}{4} \text{ is absorbed into } C \text{ leaving} \\
&= \frac{1}{4}\sin^4(x) + C
\end{aligned}$$

- Consider $\int \cos^3(x)\sin^{11}(x)\,\mathtt{d}x$.
  - Can you compute this integral using $u = \cos(x)$? Explain.

    A Solution: Taking $u = \cos(x)$, we can express the integral as $\int \cos^3(x)\left(1-\cos^2(x)\right)^5\cos(x)\,\mathtt{d}x = \int u^3\left(1-u^2\right)^5\,\mathtt{d}u$

  - Can you compute this integral using $u = \sin(x)$? Explain.

    A Solution: We can express the integral as $\left(1-sin^2(x)\right)\sin^1 1(x)\cos(x)\,\mathtt{d}x = \int\left(1-u^2\right)\left(u^{11}\right)\mathtt{d}x$

  - Which of the two above substitutions will be easier to use? Carry out the integration, using the easier of the two.

    A Solution: The second option, $u = \sin(x)$ will be easier because it avoids expading a binomial to the 5th power. Let $u = \sin(x)$

$$\begin{aligned}\int \cos^3(x)\sin^{11}(x)\,\mathtt{d}x &= \left(1-u^2\right)u^{11}\,\mathtt{d}u\\ &= \int u^{11}-u^{13}\,\mathtt{d}u\\ &= \frac{1}{12}u^{12}-\frac{1}{14}u^{14}+C\\ &= \frac{1}{12}\sin^{12}(x)-\frac{1}{14}\sin^{14}(x)+C\end{aligned}$$

1.1.2 Both Even Powers

Recall the Half-Angle Identities!

Half-Angle Identities	
$\cos^2(x) = \frac{1+\cos(2x)}{2}$	$\sin^2(x) = \frac{1-\cos(2x)}{2}$

If the powers of sine and cosine are both even, we use the half-angle identities for both sine and cosine. This can get quite messy, but it works!

Exercise 1.1.2.1. Just Cosines without Sine ☕

Consider the following integral:

$$\int \cos^6(x)\,\mathtt{d}x$$

Here the exponent on cosine is the even number 6. What is the exponent of sine in that integrand? Is that an even number?

A Solution: The exponent on sine is zero, which is indeed even. Thus both exponents are even in this case.

Example 1.1.2.2. Carrying Out Antidifferentiation with the Half-Angle Identities

We now show how the half-angle identities help antidifferentiate the sixth power of cosine.

$$\begin{aligned}\int \cos^6(x)\,\mathtt{d}x &= \int \left(\cos^2(x)\right)^3 \mathtt{d}x\\ &= \int \left(\frac{1+\cos(2x)}{2}\right)^3 \mathtt{d}x\\ &= \frac{1}{8}\int 1 + 3\cos(2x) + 3\cos^2(2x) + \cos^3(2x)\,\mathtt{d}x\\ &= \frac{1}{8}\left(\int 1\,\mathtt{d}x + \int 3\cos(2x)\,\mathtt{d}x + \int 3\cos^2(2x)\,\mathtt{d}x + \int \cos^3(2x)\,\mathtt{d}x\right)\end{aligned}$$

Notice that we now have four integrals. The first is easy, the second is a u-substitution, and the third is another even power of cosine (where we again use the half-angle identity). Finally, the fourth is an odd power of cosine, so we can use the technique from Section 1.1.1.

Exercise 1.1.2.3. Finishing the Example ☕☕

Carry out each of these processes to compute the four integrals:

- $\int 1\,\mathtt{d}x$

  A Solution:

  $$\int 1\,\mathtt{d}x = x + C$$

- $\int 3\cos(2x)\,\mathtt{d}x$

A Solution: Let $u = 2x$. Then,

$$\int 3\cos 2x \, \mathtt{d}x = \frac{3}{2}\int \cos u \, \mathtt{d}u = \frac{3}{2}\sin 2x + C$$

- $\int 3\cos^2(2x) \, \mathtt{d}x$

A Solution:

$$\int 3\cos^2(2x) \, \mathtt{d}x = 3\int \frac{1+\cos 4x}{2} \, \mathtt{d}x = 3\left(\frac{1}{2}x + \frac{1}{8}\sin 4x + C\right) = \frac{3}{2}x + \frac{3}{8}\sin 4x + C$$

- $\int \cos^3(2x) \, \mathtt{d}x$

A Solution: Let $u = \sin 2x$ and $\mathtt{d}u = 2\cos 2x \, \mathtt{d}x$. Then,

$$\begin{aligned}
\int \cos^3(2x) \, \mathtt{d}x &= \int \cos 2x \left(1 - \sin^2 2x\right) \mathtt{d}x \\
&= \frac{1}{2}\int 1 - u^2 \, \mathtt{d}u \\
&= \frac{1}{2}u - \frac{1}{6}u^3 + C \\
&= \frac{1}{2}\sin 2x - \frac{1}{6}\sin^3 2x + C
\end{aligned}$$

Add your antiderivatives together and combine like terms to produce your final answer for the integral! Oh and remember that one-eighth.

$$\int \cos^6(x) \, \mathtt{d}x =$$

A Solution:

$$\begin{aligned}
&\frac{1}{8}\left(x + \frac{3}{2}\sin 2x + \frac{3}{2}x + \frac{3}{8}\sin 4x + \frac{1}{2}\sin 2x - \frac{1}{6}\sin^3 2x + C\right) \\
&= \frac{1}{8}\left(\frac{5}{2}x + 2\sin 2x + \frac{3}{8}\sin 4x - \frac{1}{6}\sin^3 2x + C\right) \\
&= \frac{5}{16}x + \frac{1}{4}\sin(2x) - \frac{1}{48}\sin^3(2x) + \frac{3}{64}\sin(4x) + C
\end{aligned}$$

Exercise 1.1.2.4. Checking the Previous Example ☕☕☕

Differentiate your answer and verify you get the original integrand back.

A Solution:

$$\frac{\mathtt{d}}{\mathtt{d}x}\left(\frac{5}{16}x+\frac{1}{4}\sin(2x)-\frac{1}{48}\sin^3(2x)+\frac{3}{64}\sin(4x)+C\right)$$

Before we differentiate, first bash everything back down to an "x" in the argument using double angle identities. This produces

$$\frac{5}{16}x+\frac{1}{2}\sin(x)\cos(x)-\frac{1}{6}\sin^3(x)\cos^3(x)+\frac{3}{16}\sin(x)\cos^3(x)-\frac{3}{16}\sin^3(x)\cos(x)+C$$

Factor out a sine and use the Pythagorean Identity to get everything else in terms of cosine. This produces

$$\frac{5}{16}x+\sin(x)\left(\frac{5}{16}\cos(x)+\frac{5}{24}\cos^3(x)+\frac{1}{6}\cos^5(x)\right)+C$$

Then we differentiate and obtain

$$\frac{5}{16}+\cos(x)\left(\frac{5}{16}\cos(x)+\frac{5}{24}\cos^3(x)+\frac{1}{6}\cos^5(x)\right)-\sin^2(x)\left(\frac{5}{16}+\frac{5}{8}\cos^2(x)+\frac{5}{6}\cos^4(x)\right)$$

to which we apply the Pythagorean Identity $\sin^2(x)=1-\cos^2(x)$ to produce

$$\frac{5}{16}+\cos(x)\left(\frac{5}{16}\cos(x)+\frac{5}{24}\cos^3(x)+\frac{1}{6}\cos^5(x)\right)-\left(1-\cos^2(x)\right)\left(\frac{5}{16}+\frac{5}{8}\cos^2(x)+\frac{5}{6}\cos^4(x)\right)$$

This will simplify to $\cos^6(x)$ once you expand and combine like terms.

Exercise 1.1.2.5. Practice with the Even Case ☕☕

- Find an antiderivative for the function $\sin^2(3x)$.

A Solution:

$$\begin{aligned}\int\sin^2(3x)&=\int\frac{1-\cos(6x)}{2}\,\mathtt{d}x\\&=\int\frac{1}{2}-\frac{1}{2}\cos(6x)\,\mathtt{d}x\\\text{Let } u=6x, \mathtt{d}u=6\,\mathtt{d}x\\&=\frac{1}{2}x-\frac{1}{2}\cdot\frac{1}{6}\int\cos u\,\mathtt{d}u\\&=\frac{1}{2}x-\frac{1}{12}\sin u+C\\&=\frac{1}{2}x-\frac{1}{12}\sin 6x+C\end{aligned}$$

- Find an antiderivative for the function $\sin^4(x)$.

A Solution:

$$\begin{aligned}
\int \sin^4(x)\,\mathtt{d}x &= \int \left(\sin^2(x)\right)^2 \mathtt{d}x \\
&= \int \left(\frac{1-\cos(2x)}{2}\right)^2 \mathtt{d}x \\
&= \int \frac{1-2\cos(2x)+cos^2(2x)}{4}\,\mathtt{d}x \\
&= \int \frac{1}{4}-\frac{1}{2}\cos(2x)+\frac{1}{4}\cdot\frac{1+\cos(4x)}{2}\,\mathtt{d}x \\
&= \int \frac{1}{4}-\frac{1}{2}\cos(2x)+\frac{1}{8}+\frac{1}{8}\cos(4x)\,\mathtt{d}x \\
&= \int \frac{3}{8}-\frac{1}{2}\cos(2x)+\frac{1}{8}\cos(4x)\,\mathtt{d}x \\
&= \frac{3}{8}x-\frac{1}{4}\sin(2x)+\frac{1}{32}\sin(4x)+C
\end{aligned}$$

- Find an antiderivative for the function $\sin^2(x)\cos^2(x)$.

A Solution:

$$\begin{aligned}
\int \sin^2(x)\cos^2(x)\,\mathtt{d}x &= \int \sin^2(x)\left(1-\sin^2(x)\right)\mathtt{d}x \\
&= \int \sin^2(x)-\sin^4(x)\,\mathtt{d}x
\end{aligned}$$

From previous examples:

$$\begin{aligned}
\int \sin^2(x)\,\mathtt{d}x &= \frac{1}{2}x-\frac{1}{2}\sin x+C \\
\int \sin^4(x)\,\mathtt{d}x &= \frac{3}{8}x-\frac{1}{4}\sin(2x)+\frac{1}{32}\sin(4x)+C \\
\int \sin^2(x)-\sin^4(x)\,\mathtt{d}x &= \frac{1}{2}x-\frac{1}{2}\sin x-\left(\frac{3}{8}x-\frac{1}{4}\sin(2x)+\frac{1}{32}\sin(4x)\right)+C \\
&= \frac{1}{2}x-\frac{1}{2}\sin x-\frac{3}{8}x+\frac{1}{4}\sin(2x)-\frac{1}{32}\sin(4x)+C \\
&= \frac{1}{8}x-\frac{1}{2}\sin x+\frac{1}{4}\sin(2x)-\frac{1}{32}\sin(4x)+C
\end{aligned}$$

Part II

Coming Attractions

Chapter 2

Introduction to Calculus III: Parametric and Polar

We begin by briefly thinking about the word *dimension*.

Exercise 2.0.0.1. Dimension ☕

One intuitive notion of dimension comes from the idea of how you would assign units to measure it. If an object has length, you would call it one-dimensional. If an object has area, it is called two-dimensional. If it has volume, it is called three-dimensional. State the dimension of each of the following objects:

- $\{x \in \mathbb{R} : x < 2\}$
- $\{(x, y) \in \mathbb{R}^2 : x < 2\}$
- The closed interval $[2, 3]$
- The circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$
- The disc $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$

In Calculus III, you will redo all of the key concepts of Calculus I and II but in three (or more) dimensions. Often the difficultly of higher-dimensional calculus is notational more than anything! In three or more dimensions, it becomes messier to write down the same concepts. To make this cleaner, we develop better languages for points and curves beyond our standard coordinate system.

2.1 Parametric Curves

Many of the objects we study, like circles or graphs of functions, are one-dimensional objects even though we usually view them as embedded in a two-dimensional plane. Thus, we can represent both x and y

(the two dimensions) in terms of the same parameter t.

Definition 2.1.0.1. Parametric Curve

Let $x(t)$ and $y(t)$ be functions of t and let $D \subset \mathbb{R}$. The corresponding *parametric curve* is the set of points

$$\{(x(t), y(t)) : t \in D\}.$$

Typically, D is an interval or union of intervals. We can graph most curves by just selecting t values from the domain D and plotting the corresponding points.

Exercise 2.1.0.2. A Warm-up Parametric Curve

Consider the parametric curve

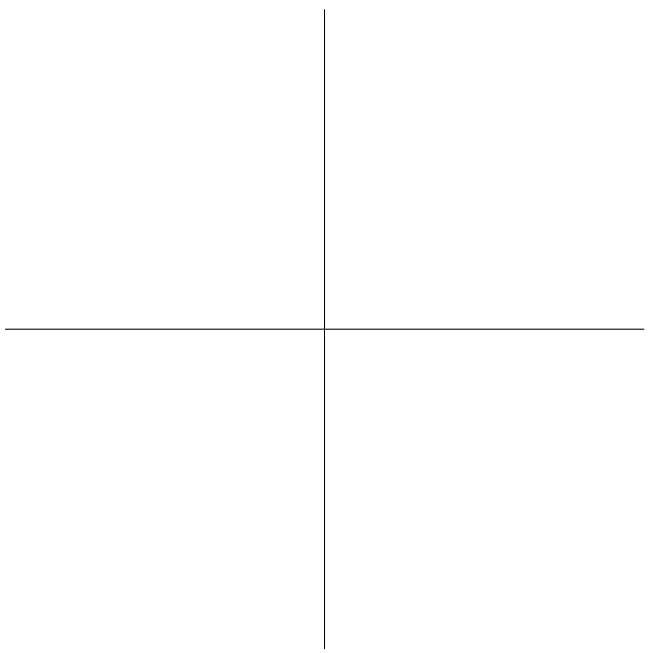
$$\{(2t, 3t + 1) : t \in [-1, 3]\}.$$

That is, $x(t) = 2t$, $y(t) = 3t + 1$, and $-1 \leq t \leq 3$.

- Use the above formulas for $x(t)$ and $y(t)$ and the following t-values selected from $D = [-1, 3]$ to fill out the following table:

t	-1	0	1	2	3
$x(t)$					
$y(t)$					

- Plot those five points on the axes below. What type of shape does it appear to be?



- Solve the equation $x = 2t$ for t. Substitute this expression for t into the equation $y = 3t + 1$.

What does this new equation tell you about the parametric curve?

Here is an example of a parametric curve used in Trigonometry (though not called so at the time).

Exercise 2.1.0.3. The Unit Circle ☕☕

- Explain why the parametric curve

$$C_1 = \{(\cos(t), \sin(t)) : t \in [0, 2\pi]\}.$$

is the familiar unit circle from trigonometry.

- Consider the curve

$$C_2 = \{(\sin(t), \cos(t)) : t \in [0, 2\pi]\}.$$

How are the curves C_1 and C_2 similar? How are they different?

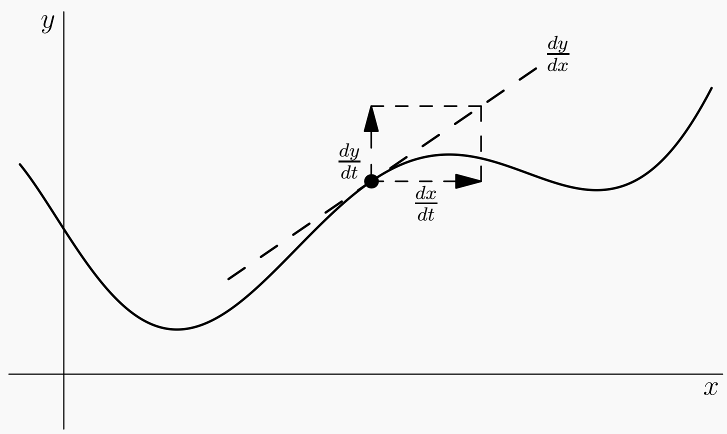
2.2 Derivatives of Parametric Curves: Slopes of Tangent Lines

To compute the derivative of a parametric curve, we recall that the slope of a line is the change in y-coordinate divided by the change in x-coordinate. In the context of parametric curves, these can be computed as rates of change with respect to the parameter t.

Definition 2.2.0.1. Parametric Derivatives

Let $(x(t), y(t))$ be a parametric curve. Then the slope of the tangent line can be computed as

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$



Notice that the above formula is just a slightly rearranged version of the chain rule. In particular, if we consider a portion of the graph of $(x(t), y(t))$ that passes the Vertical Line Test, then we can consider y as a function of x, which x in turn is a function of t. So if we wanted to ask how y changes with respect to t, we would have to take the rate of change of y with respect to x and multiply it by the rate of change of x with respect to t (by the chain rule). Expressing this Chain Rule in symbols instead of words, we have

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

Exercise 2.2.0.2. Understanding the Definition ☕

How would you get from the chain rule application shown above to our definition of parametric derivatives?

Example 2.2.0.3. Parametric Derivatives on a Parabola

Consider the parametric curve given by

$$\left\{\left(t^2, t\right) : t \in [0, \infty)\right\}.$$

To find the slope of a tangent line to this parabola, we can use the parametric derivative formula as follows:

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \\ &= \frac{\frac{d}{dt}(t)}{\frac{d}{dt}(t^2)} \\ &= \frac{1}{2t}.\end{aligned}$$

Alternately, we could convert the curve to a cartesian equation and differentiate with respect to y. Proceeding, we notice this curve is contained in the graph of $y = \sqrt{x}$, since the formulas $x = t^2$ and $y = t$ satisfy that relationship. Thus, we can differentiate y with the power rule.

$$\begin{aligned}\frac{dy}{dx} &= \left(\sqrt{x}\right)' \\ &= \frac{1}{2\sqrt{x}}\end{aligned}$$

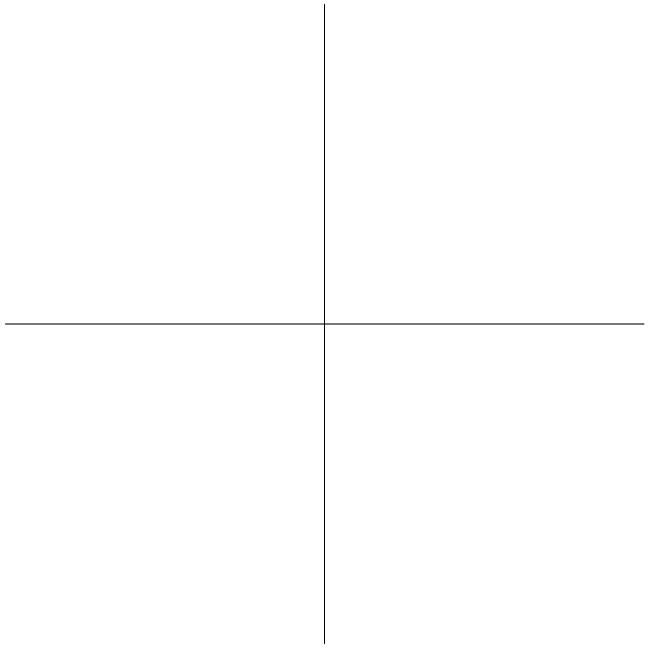
Exercise 2.2.0.4. Equivalence of the Results ☕☕

In the above example, we have two distinct expressions for $\frac{dy}{dx}$. Explain why they are in fact equivalent.

Exercise 2.2.0.5. The Tangent Line to an Ellipse ☕☕

- Plot the parametric curve given by

$$\{(2\cos(t), \sin(t)) : t \in [0, 2\pi]\}.$$



- Find the point on the graph located at $t = \pi/4$, and find the slope of the tangent line at that point using the parametric derivative formula. Sketch the tangent line on your graph above.

- Verify the above curve is in fact the ellipse given by $\frac{x^2}{4} + y^2 = 1$.

- Use implicit differentiation on the equation $\frac{x^2}{4} + y^2 = 1$ to find $\frac{dy}{dx}$ at that same point and verify your answers match!

Exercise 2.2.0.6. Finding a Parameterization ☕☕☕

Find a parameterization of the path that consists of two full clockwise laps around the ellipse given by

$$\frac{(x-3)^2}{4}+(y-3)^2=1$$

starting from the point (3,2).

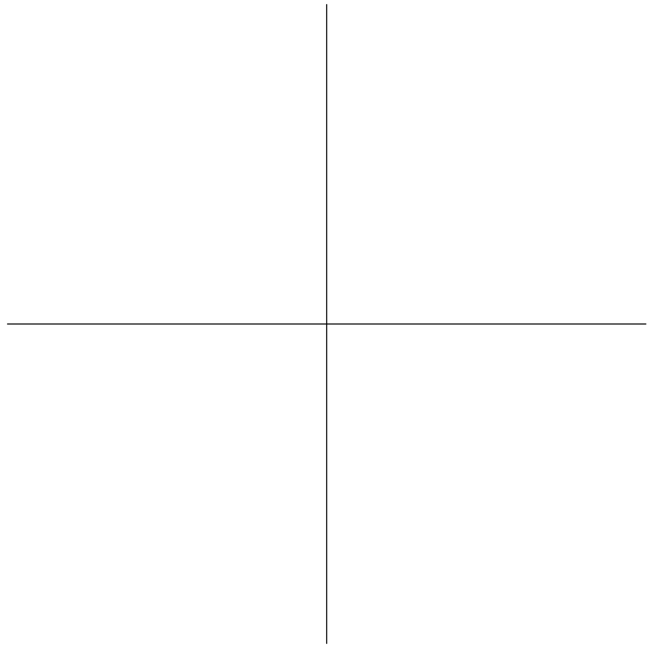
Exercise 2.2.0.7. A Hyperbola ☕☕

Consider the parametric curve given by the following:

$$\begin{aligned} x(t) &= e^t - e^{-t}, \\ y(t) &= e^t + e^{-t}, \\ t &\in [0,\infty). \end{aligned}$$

- Show that the above curve is contained in the hyperbola $y^2-x^2=4$.

- Graph the parametric curve.



- Find dy/dx using the parametric formula for derivatives. Take the limit as t approaches infinity and interpret on your graph.

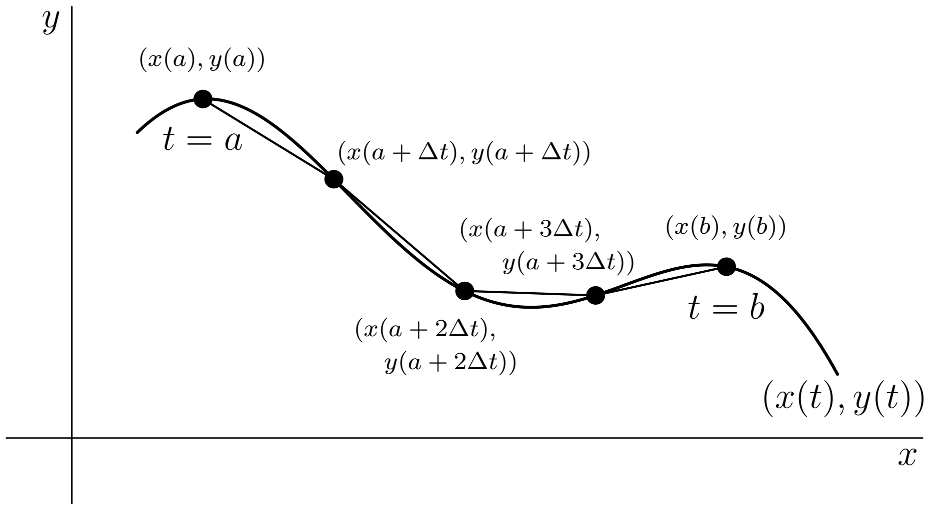
2.3 Integrals of Parametric Curves: Arc Length

The length of a parametric curve is given by the following formula.

Theorem 2.3.0.1. Parametric Arc Length

Let a parametric curve C be given by $(x(t), y(t))$ for $a \leq t \leq b$. Then the arc length is computed via

$$\int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, \mathtt{d}t.$$



The construction here is nearly identical to the construction of the arc length of the graph of a function in Section **??**. We select points corresponding to t-values along the curve, compute the sum of the lengths of the line segments connecting them, and take the limit as the number of line segments goes to infinity.

Exercise 2.3.0.2. Fill in the Blanks! Derivation of the Arc Length Formula ☕☕

Let $t_0, t_1, t_2, \ldots, t_n$ be equally spaced points in the interval $[a, b]$. That is, $t_0 = a$, $t_n = b$, and for each $i \in \{0, 1, 2, \ldots, n-1\}$, $\Delta t =$ ______________________.
With this setup, if we want the length of a line segment connecting points $(x(t_{i+1}), y(t_{i+1}))$ and $(x(t_i), y(t_i))$, we would use the Pythagorean Theorem to obtain

$$\sqrt{\left(\qquad\qquad\qquad\qquad\right)}$$

as the length.

$$\begin{aligned} L &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \sqrt{\qquad\qquad\qquad\qquad} \\ &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \sqrt{\qquad\qquad\qquad\qquad} \sqrt{(\Delta t)^2} \\ &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \sqrt{\qquad\qquad\qquad\qquad} \Delta t \\ &= \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, \mathrm{d}t \end{aligned}$$

As usual, when first trying out a new tool, it is best to use it in a case where you already know the answer.

Exercise 2.3.0.3. Checking the Circumference of a Circle ☕☕

Consider the parametrization

$$x(t) = r\cos(t)$$
$$y(t) = r\sin(t)$$

for $t \in [0, 2\pi]$.

- Explain why this is a parameterization of a circle of radius r.

- Use the parametric arc length formula to compute the length of the curve. Compare it to your known formula for the circumference of a circle. Does the answer make sense?

Exercise 2.3.0.4. A Familiar Conic in Disguise ☕☕☕

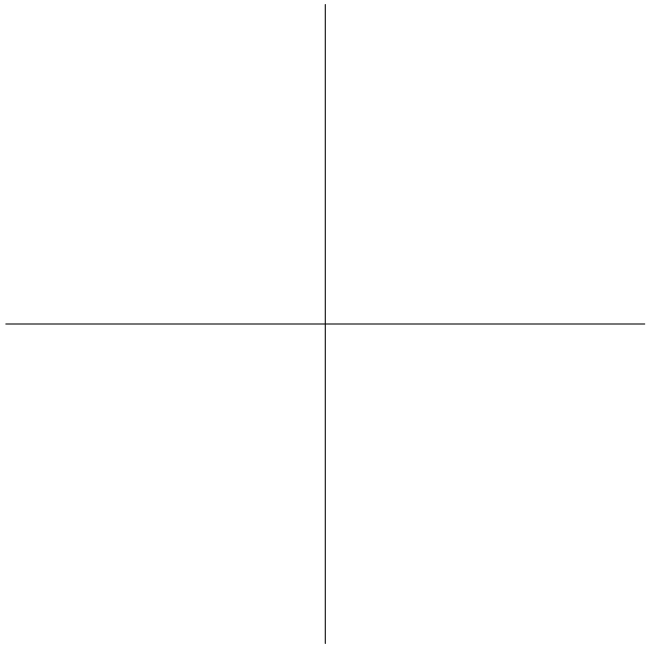
Consider the parametrization

$$x(t) = \sqrt{|t|}$$
$$y(t) = 3t - 1$$

for $t \in [-2, 2]$.

- Convert this to a cartesian equation. What kind of shape is it?

- Sketch the curve. Indicate any vertical or horizontal tangent lines and where they occur.



- Use the parametric arc length formula to compute the length of the curve. Does the answer make sense?

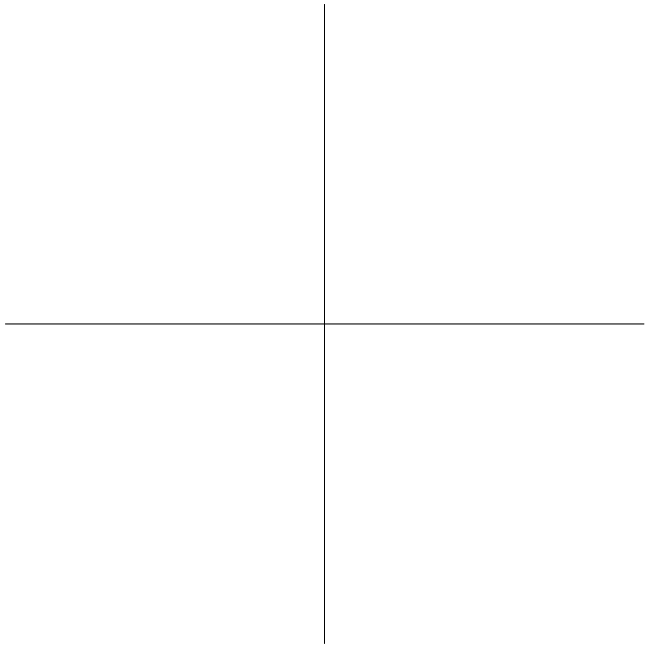
Ok, time to finally play with a curve that is not just a conic.

Exercise 2.3.0.5. Analyzing a Stranger Curve

- Sketch the graph of the following parametric curve C:

$$C = \left\{ \left(e^t \cos(t), e^t \sin(t) \right) : 0 \leq t \leq 2\pi \right\}.$$

Include labels of points on the graph at $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.



- Where does the above graph have vertical tangent lines? Where does the above graph have horizontal tangent lines? Mark them on your graph.

- What is the length of C?

2.4 Hyperbolic Sine and Cosine

You may have seen in a previous course (or if not, then here they are!) the definitions of the *hyperbolic sine* and *hyperbolic cosine* functions. They are typically defined as follows:

- $\cosh(t) = \frac{e^t + e^{-t}}{2}$
- $\sinh(t) = \frac{e^t - e^{-t}}{2}$

This of course prompts the question: “why do these e things get called sine or cosine?” We answer this question below.

Exercise 2.4.0.1. Power Series for Hyperbolic Sine and Cosine ☕☕

- Find a power series for cosh by using what we know about the series for the exponential function. How does the resulting series relate to cosine?

- Find a power series for sinh by using what we know about the series for the exponential function. How does the resulting series relate to sine?

And of course there are more questions prompted here: “Why do these e things get called hyperbolic? What do these hyperbolic functions have to do with hyperbolas?”

Exercise 2.4.0.2. Parametric Curve Generated by Hyperbolic Sine and Cosine ☕☕☕

Consider the parametric curve

$$\{(\cosh(t), \sinh(t)) : t \in \mathbb{R}\}.$$

- Verify this parametric curve satisfies the cartesian equation for a hyperbola given by

$$x^2 - y^2 = 1$$

by plugging the exponential definitions for our hyperbolic trig functions in for x and y.

- Again, verify this parametric curve satisfies the cartesian equation for the same hyperbola given by
$$x^2 - y^2 = 1$$
by plugging the power series formulas for our hyperbolic trig functions in for x and y.

2.5 Polar Coordinates

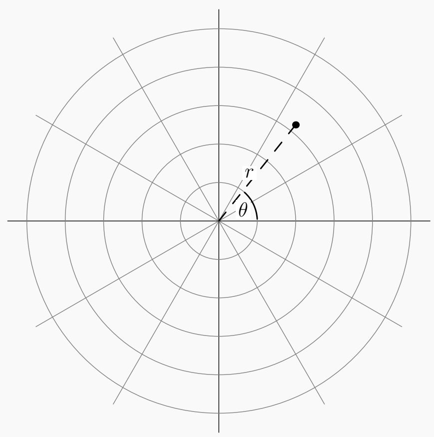
We use the *(horizontal,vertical)* coordinate system so much that it is easy to think that it is somehow inherent to a plane. However, a plane is just a geometric object; a coordinate system is an arbitrary system of labels that we slap on after the fact. Here we explore a different commonly used coordinate system, *polar coordinates*.

2.5.1 Points in Polar Coordinates

Assume we chose an origin in the plane, a direction that we call the positive x-axis, and some point along that ray that marks off unit distance. We now define coordinates for the rest of the plane based on these choices.

Definition 2.5.1.1. Plotting Points in Polar Coordinates

The point (θ, r) is the point located at an angle θ radians counterclockwise from the positive x-axis, a distance of r units from the origin.



Notice the angles are measured in the same manner as on the unit circle in trigonometry. The difference here is we allow any real number r as radius, rather than only radius one. We do allow r to be a negative number, in which case we travel "backwards" along the ray given by θ.

Example 2.5.1.2. Polar Coordinates are not Unique!

Be warned that any given point will have many different representations in polar coordinates. For example, consider the cartesian point (1,-1). In polar coordinates, we have many ways to represent this point. We can think of the angle as $\theta = -\pi/4$ and the radius as $r = \sqrt{2}$. We can also think of the angle as $\theta = 7\pi/4$ and the radius as $r = \sqrt{2}$. Yet another valid way to reach that same point is to use angle $\theta = 3\pi/4$ and the radius $r = -\sqrt{2}$. Thus, in polar coordinates we have that

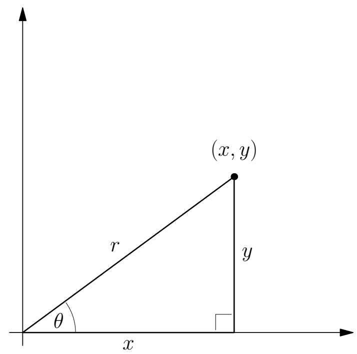
$$\left(-\pi/4, \sqrt{2}\right) = \left(7\pi/4, \sqrt{2}\right) = \left(3\pi/4, -\sqrt{2}\right)$$

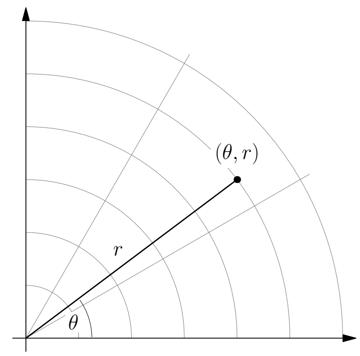
all represent the same point.

We now see how right-triangle trigonometry allows us to convert between polar coordinates and cartesian coordinates.

Exercise 2.5.1.3. Converting Between Polar and Cartesian ☕☕☕

- See the diagram below, with a point in QI labeled with both cartesian and polar measurements. For each of the conversion formulas listed below, write a short sentence afterwards explaining how it comes from the diagram.





  - $x^2 + y^2 = r^2$
  - $x = r\cos(\theta)$
  - $y = r\sin(\theta)$
  - $\tan(\theta) = \frac{y}{x}$
  - $\theta = \arctan\left(\frac{y}{x}\right)$

- If the point of interest were in a different quadrant, do the above formulas still hold? Do any of them require adjustment? Explain.

Exercise 2.5.1.4. Plotting in Polar ☕☕

- Plot the polar point $(5\pi/4{,}4)$. What are its cartesian coordinates?

- Consider the cartesian point (2,0). What are *all* possible ways of writing that point in polar coordinates?

Exercise 2.5.1.5. Do Any Points Have the Same Name? ☕☕☕

Do any points happen to have the same label in both polar and cartesian coordinates? Find all points that do, and explain why there are no more!

It is worth noting why "polar coordinates" are called what they are called. Cartesian coordinates look like a grid of horizontal and vertical lines. This is a great approximation of what latitude and longitude lines look like if you are standing at a random point on earth and think of your surroundings as approximated by a plane. But, if you are standing at the north or south pole, the latitude and longitude lines do not in any way look like a grid!

Exercise 2.5.1.6. Justifying the Name ☕☕

What do the lattitude and longitutde lines look like if you are standing at the north or south pole? Draw a small graph below.

Exercise 2.5.1.7. The Idea of Coordinate Systems ☕☕☕☕

Create another coordinate system for the plane that is not cartesian and is not polar! Describe your system of labeling all the points!

2.5.2 Graphs of Equations and Functions in Polar Coordinates

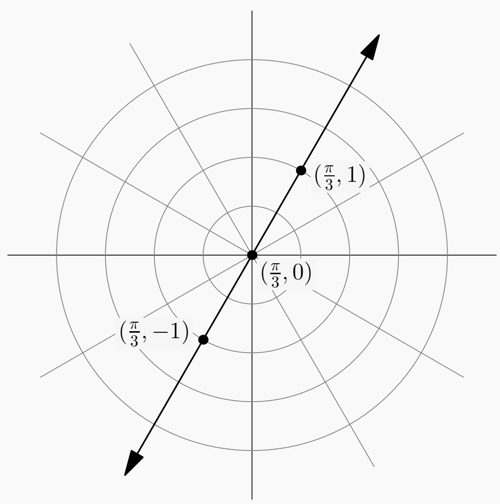
An equation in polar coordinates is an equality between expressions involving r and θ. We often wish to view the solutions visually by plotting all points (θ, r) in the plane that make the equations true (just as one would in cartesian).

Example 2.5.2.1. Graphing a Polar Equation

Suppose we wish to graph the equation

$$\theta = \pi/3$$

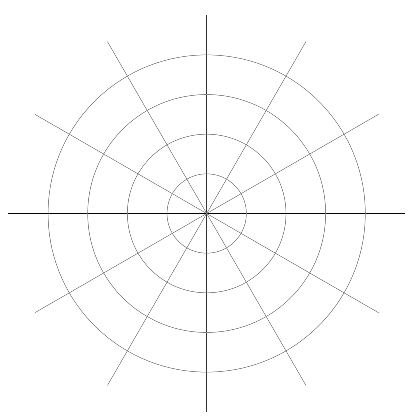
A point satisfies that equation if and only if the angle is $\pi/3$. The radius r is free to be any real number -positive, negative, or zero. For example, points that satisfy the equation include $(\pi/3, 1), (\pi/3, 0)$, and $(\pi/3, -1)$. Thus the graph is a line through the origin at 60° to the positive x-axis.



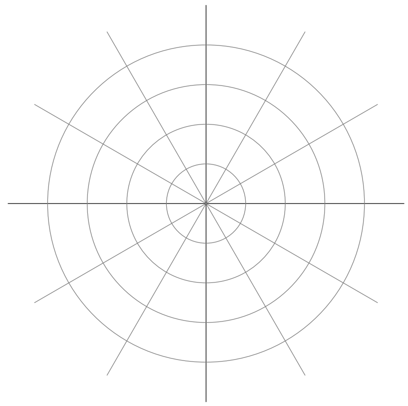
Exercise 2.5.2.2. Graphing Equations ☕☕

Graph the following equations.

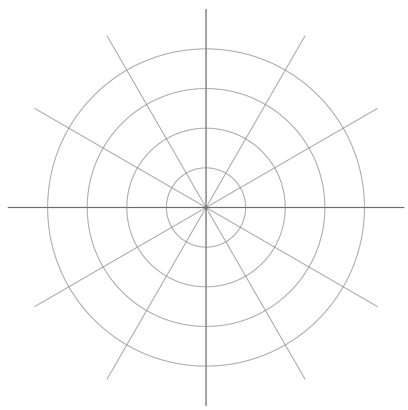
- $r = 2$



- $r = -2$



- $\theta^2 = \pi^2/4$



If the equation can be solved for r, we can consider r as a function of the independent variable θ. To graph a function, we simply make an input-output table of θ values and corresponding $r(\theta)$ values and plot the corresponding points $(\theta, r(\theta))$.

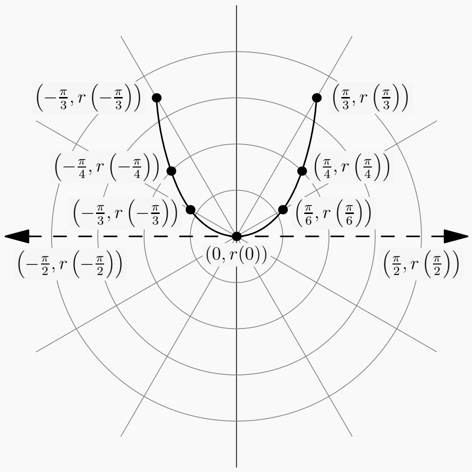
Example 2.5.2.3. Graphing a Polar Function

Plot the polar function

$$r(\theta) = \tan(\theta)$$

over the domain $-\pi/2 < \theta < \pi/2$. We select input values for θ that are clean unit circle values to plot.

θ	$-\pi/2$	$-\pi/3$	$-\pi/4$	$-\pi/6$	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$r(\theta)$	DNE	$-\sqrt{3}$	-1	$-\sqrt{3}/3$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	DNE



Exercise 2.5.2.4. Analyzing the Graph ☕☕

Does the graph appear to have any asymptotes? If so, where?

Looking at the graph prompts the question "can we find a Cartesian equation that describes the same set of points"? Here we use the formulas from Exercise 2.5.3 to rewrite all instances of r and θ in terms of x and y. Also, perhaps the cartesian equation can confirm our asymptote suspicions above!

Example 2.5.2.5. Converting to Cartesian

Let's find a cartesian equation for the graph of $r(\theta) = \tan(\theta)$ from the previous example. Since we do not have a particularly clean conversion formula for r itself but rather for r^2, it can be helpful to either multiply both sides by r or square both sides. In this case, squaring both sides will be cleaner so we take that path. Proceeding:

$$\begin{aligned}
r &= \tan(\theta) \\
r^2 &= (\tan(\theta))^2 \\
x^2 + y^2 &= \left(\frac{y}{x}\right)^2 \\
x^2 + y^2 &= \left(\frac{y}{x}\right)^2 \\
x^4 + x^2 y^2 &= y^2 \\
x^4 + \left(x^2 - 1\right) y^2 &= 0 \\
y^2 &= \frac{x^4}{1 - x^2} \\
y &= \pm \frac{x^2}{\sqrt{1 - x^2}}
\end{aligned}$$

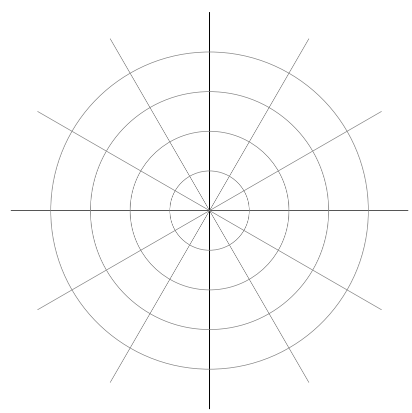
Exercise 2.5.2.6. Analyzing the Graph, Round II ☕☕

Does the cartesian formula tell you anything further about the apparent asymptotes on the graph?

The next exercise shows why converting a polar graph to cartesian coordinates can help analyze the geometry of the graph.

Exercise 2.5.2.7. Graphing a Function and Converting ☕☕☕

- Graph the function $r(\theta) = \sin(\theta)$. Does it look like a circle?



- Is it a circle? If so, what is the center and radius? Convert the equation to cartesian coordinates to confirm!

2.6 Derivatives in Polar Coordinates

Suppose we have the graph of a polar function $r(\theta)$, and we would like to find the slope of the tangent line at a point. We can consider this graph to be a parameterized curve by treating $t = \theta$ as the parameter. Specifically, the parameterization is given by

$$x(t) = r(t)\cos(t)$$
$$y(t) = r(t)\sin(t).$$

Exercise 2.6.0.1. Deriving the Derivative ☕☕

Use the formula for the derivative of a parametric curve to find the formula for the derivative of a polar graph.

Exercise 2.6.0.2. Using the Formula ☕☕

Use the polar derivative formula above to find the slope of the graph of $r(\theta) = \sec(\theta)$. What does this let you conclude about that graph?

2.7 Area in Polar Coordinates

To compute area in polar coordinates, we essentially repeat the process of taking a Riemann sum. Rather than using rectangles however, we use sectors of circles.

Exercise 2.7.0.1. Area of a Single Sector ☕☕

- What is the area of an entire circle with radius r? Draw the circle.

- Within your circle, draw a sector of that circle with angle θ. What proportion of the area of the entire circle does that sector occupy?

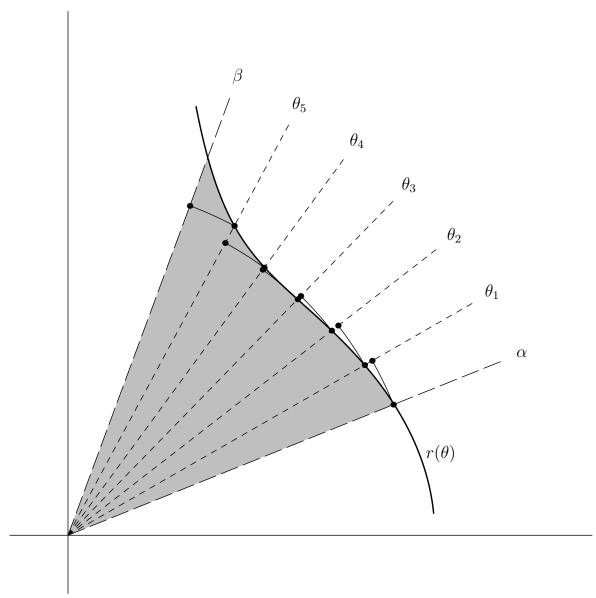
- Explain why the area of that sector is $A = \frac{1}{2}r^2\theta$.

We now repeat the process of taking a Riemann sum using sectors of circles. In particular, say we wish to find the area under the graph of $r(\theta)$ between two rays specified by angles $\theta = \alpha$ and $\theta = \beta$.

Let $\theta_0, \theta_1, \theta_2, \ldots, \theta_n$ be equally spaced angles from α to β. That is, $\theta_0 = \alpha$, $\theta_n = b$, and for each $i \in \{0, 1, 2, \ldots, n-1\}$, $\Delta\theta = \theta_{i+1} - \theta_i = \frac{\beta-\alpha}{n}$.

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{2} r^2(\theta_i)\,\Delta\theta \\ &= \frac{1}{2} \lim_{n\to\infty} \sum_{i=0}^{n-1} r^2(\theta_i)\,\Delta\theta \\ &= \frac{1}{2} \int_{\theta=\alpha}^{\theta=\beta} r^2(\theta)\,\mathrm{d}\theta \end{aligned}$$

Thus, we have the formula for polar area!



Theorem 2.7.0.2. Polar Area

The area under a polar curve $r(\theta)$ between $\theta = \alpha$ and $\theta = \beta$ is

$$A = \frac{1}{2} \int_{\theta=\alpha}^{\theta=\beta} r^2(\theta)\,\mathrm{d}\theta$$

Sometimes a helpful way to remember the above formula is to write it as

$$A = \int_{\theta=\alpha}^{\theta=\beta} \pi r^2(\theta) \frac{\mathrm{d}\theta}{2\pi}.$$

This way you can think of the integrand as the area of a circle being multiplied by what ratio of 2π radians the change in θ is occupying. Canceling the two π's and pulling the $\frac{1}{2}$ outside of the integral lands you back at the Polar Area formula.

Exercise 2.7.0.3. Looking for Patterns ☕☕☕

Fill out the table! Carry out the instructions listed below for each given value of n.

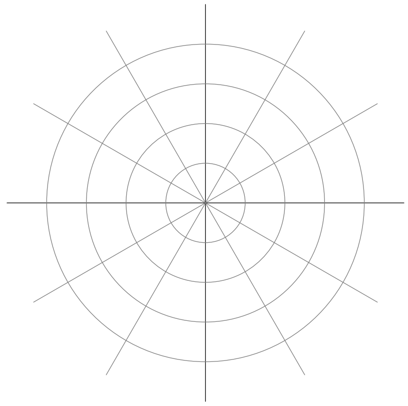
- Plot the graph of $r(\theta)$.
- Find the area inside just one "petal".
- What patterns in n do you see? What can you say about the percent of the unit circle that lies inside rather than outside the graph?

n	Graph of $r(\theta) = \sin(n\theta)$	Area of One Petal
2		
3		
4		
5		
n		

n	Graph of $r(\theta) = \cos(n\theta)$	Area of One Petal
2		
3		
4		
5		
n		

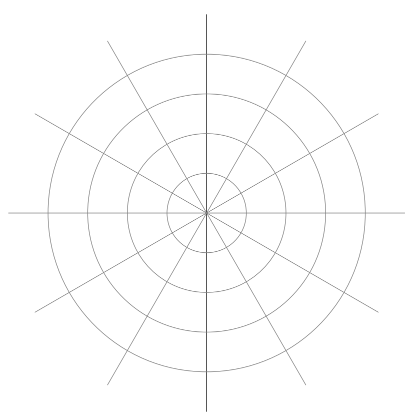
Exercise 2.7.0.4. Area Bounded by Two Polar Curves ☕☕☕

Plot both $r_1(\theta) = \frac{1}{2}\sec(\theta)$ and $r_2(\theta) = \cos(\theta)$ on the same set of axes. Find the area of the region that is to the right of r_1 but inside r_2.



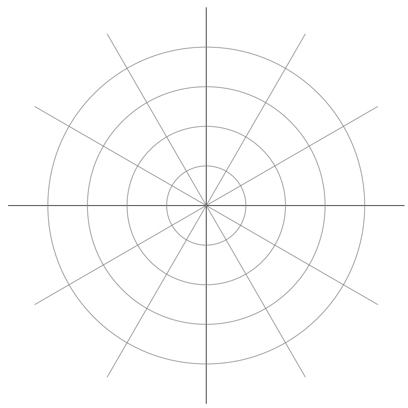
Exercise 2.7.0.5. Mixed Practice with Polar Curves ☕☕☕

- – Sketch the graph of $r(\theta) = 2\cos(2\theta)$.



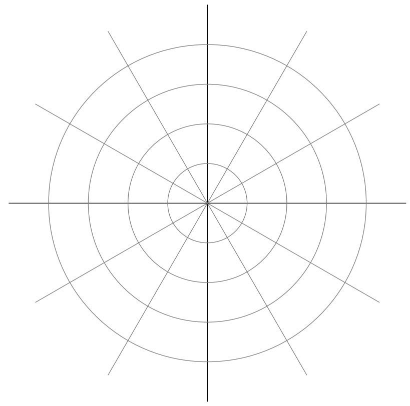
- Convert the above curve to cartesian. That is, find a polynomial equation in x and y whose solution set describes the same set of points. (**Hint:** Begin by applying the cosine double-angle identity!)

- - Sketch the graph of $r(\theta) = \frac{1}{2} + \cos(\theta)$.



  - Find the area enclosed by the inner loop of the graph.

- - Plot the graphs of both $r_1(\theta) = 1 + \cos(\theta)$ and $r_2(\theta) = 1 - \cos(\theta)$ on the same axes.



- Shade the region contained inside both curves. Find its area.

2.8 Microphone Design

A microphone is a device that picks up sound (variations in air pressure) and produces an electrical signal. For any microphone, sound engineers want what is called the *polar pattern*, a graph indicating all locations from which sound is picked up with equal intensity. Microphones that are physically designed differently will have different polar patterns.

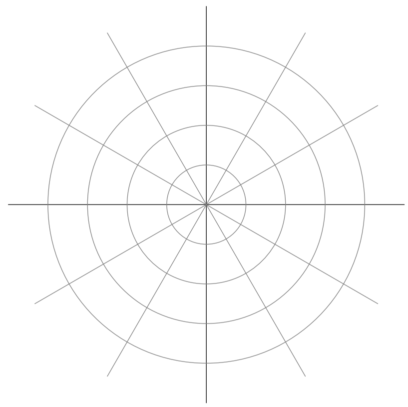
The key element to a microphone is some mechanical device that the waves of air pressure can compress. There are two basic types of devices:

2.8.1 Diaphragm

A spherical diaphragm responds equally to changes in air pressure from any side. Thus given a sound of a particular volume, the response in the microphone sensitivity is proportional to the distance from the diaphragm. A microphone with such a diaphragm is called an *omnidirectional microphone* and is represented by the polar pattern $r(\theta) = 1$, since the microphone has equal sensitivity to all points on a circle.

Exercise 2.8.1.1. Diaphragm Microphones

Plot the polar pattern for a diaphragm microphone function below.

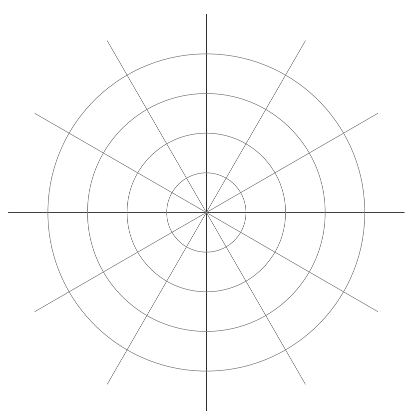


2.8.2 Ribbon

The other main type of device is a ribbon that floats in a magnetic field. Since it is a horizontal ribbon, it picks up changes in air pressure proportion to the sine of the angle to the source. (Imagine for example in physics a force pushing on a wall at an angle... the force that goes into the wall is not equal to the magnitude of the whole force but rather the magnitude times sine of the angle.) A microphone equipped with such a device is called a *ribbon microphone* or a *figure eight microphone* and has polar pattern given by $r(\theta) = |\sin(\theta)|$. Here we are taking absolute values because we are just denoting sensitivity, not the wave itself.

Exercise 2.8.2.1. Ribbon Microphones

Plot the polar pattern for a ribbon microphone function below.



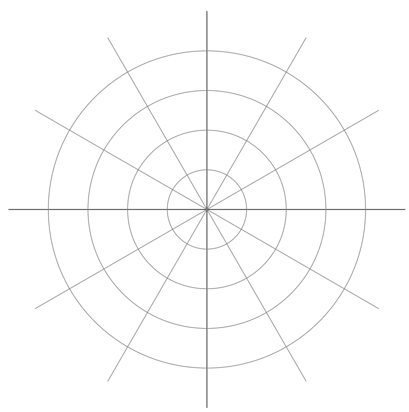
2.8.3 Cardioid

There are many situations where one of the above microphones is perfect for the purpose at hand. However, when a band is playing live music on a stage, the above two microphones do not work. The basic setup is the following: if a singer sings into the microphone, the main speakers are pointed towards the audience and not towards the singer. Thus, it is necessary to have monitors (smaller speakers pointing the opposite direction) so that the singer can hear herself. However, if the microphone picks up the sound coming out of the monitor, it's going to be again reproducing the same sound it just heard. The waves combine amplitude again and again, and this leads to that horrible high-pitched screeching noise known as feedback.

The solution to this is to design a microphone that picks up sound from one side but not from the other. The ingenious way engineers figured out how to do this was to simply make a microphone with *both* a ribbon and a diaphragm inside! The waves produced add to each other to make a single signal. Thus the sine of the ribbon will combine with the diaphragm's signal on one side, but cancel it out on the other! The polar pattern is given by the function $r(\theta) = 1 + \sin(\theta)$ (adding the waves together). Such a mic is called a *cardioid microphone* and is the standard mic for onstage live sound. The Shure 57 and Shure 58 are cardioid microphones and have been the standard mic used onstage for about 40 years now!

Exercise 2.8.3.1. Cardiod Microphones ☕☕

- Plot the polar pattern corresponding to the above described cardioid microphone.



- Find the area of the region where sounds are at least as sensitive as they are on the boundary of that cardioid. (That is, find the area enclosed by the above polar curve.)

2.9 Chapter Summary

Here we introduced two new languages for describing curves in the plane, parametric and polar.

1. **Parametric:** A **parametric curve** is the set of points $(x(t), y(t))$ for some specified domain of t values.

   (a) **Graphing:** Pick a helpful spread of t values and plot the resulting points $(x(t), y(t))$ to get some idea of the shape. Often **converting to cartesian** by eliminating t and finding a direct relationship between x and y can be helpful.

   (b) **Derivatives:** The **slope of the tangent line to a parametric curve** can be found by $\frac{\mathtt{d}y}{\mathtt{d}x} = \frac{\mathtt{d}y/\,\mathtt{d}t}{\mathtt{d}x/\,\mathtt{d}t}$.

   (c) **Integrals:** The **length of a parametric curve** can be found by integrating the distance formula. If the parameter domain is the closed interval $D = [a, b]$, then the length is

$$\int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, \mathtt{d}t.$$

2. **Polar:** The system of **polar coordinates** describes the plane as (θ, r) where θ is the counterclockwise angle from the positive x axis and r is the signed distance from the origin.

   (a) **Graphing:** Given a polar function $r(\theta)$, pick a helpful spread of θ values and plot the resulting points $(\theta, r(\theta))$ to get some idea of the shape. Often **converting to cartesian** can be helpful. To convert, use the relationships

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \\ r^2 &= x^2 + y^2 \end{aligned}$$

   or any other helpful relationship that follows from the triangle with angle θ, adjacent side x, opposite side y, and hypothenuse r.

   (b) **Derivatives:** To find the **slope of the tangent line to a polar graph**, convert to parametric by letting $t = \theta$. Specifically, set

$$\begin{aligned} x(t) &= r(t)\cos(t) \\ y(t) &= r(t)\sin(t) \end{aligned}$$

   and then use the formula for a parametric derivative.

   (c) **Integrals:** To find **area under a polar graph**, perform a Riemann sum with sectors of circles (rather than rectangles as we did initially). The π cancels to give us our polar area formula as seen below.

$$\begin{aligned} A &= \int_{\theta=\alpha}^{\theta=\beta} \underbrace{\pi r^2(\theta)}_{\text{Area of a circle}} \underbrace{\frac{\mathtt{d}\theta}{2\pi}}_{\text{proportion of full circle's radians}} \\ &= \frac{1}{2}\int_{\theta=\alpha}^{\theta=\beta} r^2(\theta)\,\mathtt{d}\theta \end{aligned}$$

2.10 Mixed Practice

2.10.1 Warm Ups

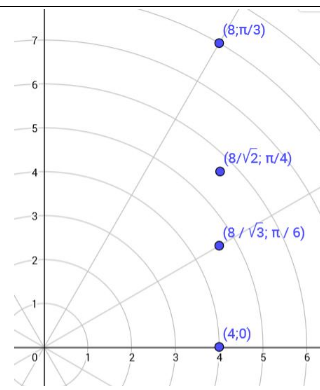
These are good problems for reinforcing the vocabulary and foundational concepts of this chapter.

Exercise 2.10.1.1.

a.) Graph the polar function $r(\theta) = 4\sec(\theta)$ by picking a spread of θ values and making an input-output table.

A Solution:

θ	0	$\pi/6$	$\pi/4$	$\pi/3$
$r(\theta)$	4	$8/\sqrt{3}$	$8/\sqrt{2}$	8



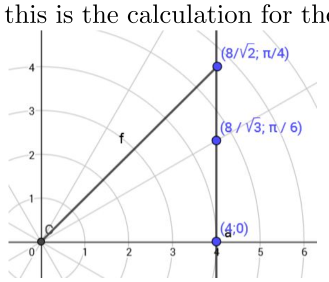
b.) Convert to cartesian coordinates to show that the graph is in fact just a line!

A Solution: $r = 4\sec\theta \Longrightarrow r\cos\theta = 4 \Longrightarrow x = 4$ is a vertical line.

c.) Sketch the polar region whose area corresponds to the following integral:

$$A = \int_{\theta=0}^{\theta=\pi/4} \frac{1}{2}\left(4\sec(\theta)\right)^2 \,\mathrm{d}\theta$$

A Solution: Since this is the calculation for the area under the curve $4\sec\theta$ in polar form, we have the region:



d.) What shape is the region sketched in c)? Find the area by basic geometry.

A Solution: It is an isosceles triangle with hypotenuse $4\sqrt{2}$ and sides 4 so $A = \frac{1}{2}b \cdot h = \frac{1}{2}4 \cdot 4 = 8$ so we get

e.) Find the area by computing the integral. Verify your answers match.

A Solution: $A = \int_0^{\pi/4} \frac{1}{2}(4\sec\theta)^2\ d\theta = \int_0^{\pi/4} \frac{1}{2}(16\sec^2\theta)\ d\theta = 8\tan\theta\Big|_0^{\pi/4} = 8(\tan\pi/4 - \tan 0) = 8 \cdot 1 - 8 \cdot 0 = 8$ They are the same.

Exercise 2.10.1.2. ☕☕

Consider the parameterization

$$\{(4t-1, 6t) : t \in [0,2]\}$$

That is, $x(t) = 4t - 1$ and $y(t) = 6t$ as t roams from 0 to 2.

a.) Describe the shape of this parametric curve.

A Solution: It is a line segment that lies on the line $\frac{x-1}{4} = t = \frac{y}{6} \leftrightarrow y = \frac{3}{2}x - \frac{3}{2}$ between $(-1, 0)$ and $(7, 12)$

b.) What is the slope of the tangent line to the parametric curve at $t = 1$?

A Solution: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6}{4} = \frac{3}{2}$ which is the slope of the line.

c.) Use the parametric arc length formula to compute the length of the parameterized curve from part a).

A Solution: $L = \int_{t=0}^{t=2} \sqrt{(x'(t))^2 + (y'(t))^2}\ dt = \int_0^2 \sqrt{4^2 + 6^2}\ dt = \int_0^2 \sqrt{52}\ dt = 2\sqrt{13}t\Big|_0^2 = 4\sqrt{13}$

2.10.2 Sample Test Problems

Exercise 2.10.2.1.

State the power series definitions for hyperbolic sine and cosine:

- $\cosh(t) =$
- $\sinh(t) =$

A Solution: $\cosh t = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \cdots$
$\sinh t = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots$

---

a.) Use the power series for hyperbolic sine to compute its derivative.

A Solution: $\frac{d\ \sinh(t)}{dt} = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots = \cosh(t)$

b.) Use the power series for hyperbolic cosine to compute its derivative.

A Solution: $\frac{d\ \cosh(t)}{dt} = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots = \sinh(t)$

c.) Consider the parametric curve

$$\{(\cosh(t), \sinh(t)) : t \in \mathbb{R}\}$$

Verify out to degree six that this parametric curve also satisfies the cartesian equation for the same hyperbola given by:

$$x^2 - y^2 = 1$$

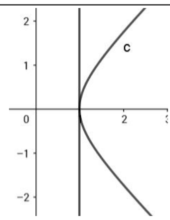
by plugging the power series formulas for our hyperbolic trig functions in for x and y.

A Solution: $\cosh^2 t - \sinh^2 t = \left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \cdots\right)^2 - \left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right)^2 = \left(1 + \frac{t^2}{2!} + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^4}{4!} + \frac{t^4}{2!2!} + \frac{t^6}{6!} + \frac{t^6}{6!} + \frac{t^6}{4!2!} + \frac{t^6}{4!2!} + \cdots\right) - \left(t^2 + \frac{t^4}{3!} + \frac{t^4}{3!} + \frac{t^6}{3!3!} + \frac{t^6}{5!} + \frac{t^6}{5!} + \cdots\right) = 1 - \left(\frac{2}{2!} - 1\right)t^2 + \left(\frac{2}{4!} + \frac{1}{2!2!} - \frac{2}{3!}\right)t^4 + \left(\frac{2}{6!} + \frac{2}{4!2!} - \frac{1}{3!3!} - \frac{2}{5!}\right)t^6 + \cdots = 1 + (1-1)t^2 + \left(\frac{1}{12} + \frac{1}{4} - \frac{1}{3}\right)t^4 + \left(\frac{1}{6\cdot5\cdot4\cdot3} + \frac{1}{4\cdot3\cdot2} - \frac{1}{36} - \frac{1}{5\cdot4\cdot3}\right)^6 + \cdots = 1 + 0t^2 + \left(\frac{1+3-4}{12}\right)t^4 + \left(\frac{1}{6\cdot5\cdot4\cdot3} + \frac{15}{6\cdot5\cdot4\cdot3} - \frac{10}{6\cdot5\cdot4\cdot3} - \frac{6}{6\cdot5\cdot4\cdot3}\right)t^6 \cdots = 1 + 0t^2 + 0t^4 + 0t^6 + \cdots = 1$

d.) Find the slope of the tangent line at $t = 0$ to the parametric curve using your derivatives computed above.

A Solution: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cosh(t)}{\sinh(t)}$ Evaluate at $t = 0$ to get $\frac{\cosh(0)}{\sinh(0)} = \frac{1 + \frac{0^2}{2!} + \frac{0^4}{4!} + \frac{0^6}{6!} + \cdots}{0 + \frac{0^3}{3!} + \frac{0^5}{5!} + \cdots} = \frac{1}{0}$ which is a vertical line e.) Graph the parametric curve along with the tangent line you found in the previous part.

A Solution:

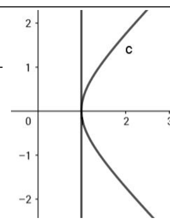


Exercise 2.10.2.2. ☕☕☕

a.) Graph the polar function $r(\theta) = \sin(2\theta)$.

A Solution:

θ	$r(\theta)$
0	1
$\pi/4$	0
$\pi/2$	-1
$3\pi/4$	0
π	1
$5\pi/4$	0
$3\pi/2$	-1
$7\pi/4$	0
2π	1



b.) Find the area enclosed by one loop of that function.

A Solution: $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}(\cos 2\theta)^2 \, d\theta = \frac{1}{2}\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \cos^2(2\theta) \, d\theta = \frac{1}{2}\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{1+\cos(4\theta)}{2} \, d\theta = \frac{1}{4}\left(\theta + \frac{\sin(4\theta)}{4}\right)\Big|_{\frac{-\pi}{4}}^{\frac{\pi}{4}} = \frac{1}{4}\left(\frac{\pi}{4} + \frac{\sin(\pi)}{4}\right) - \frac{1}{4}\left(\frac{-\pi}{4} + \frac{\sin(-\pi)}{4}\right) = \frac{\pi}{16} + \frac{\pi}{16} = \frac{\pi}{8}$

Chapter 3

Introduction to Differential Equations

In this course, we got really good at two things: finding antiderivatives and using power series. It is no accident that the study of differential equations relies primarily on those two techniques! Here we show just two methods for solving differential equations: separation of variables, based on antidifferentiation, and power series solutions, based on power series (really!).

3.1 What is a Differential Equation?

Definition 3.1.0.1. Differential Equation

A *differential equation* (DE) is an equation involving a variable (say y) that stands for some unknown function, and also involving one or more derivatives of y. The *solution* to a differential equation is the set of all functions y that make the equation true.

We begin with a nice bridge troll riddle. We ask "What functions are equal to their own derivative?".

Example 3.1.0.2. Functions Equal to Their Own Derivative

To state this question in the language of differential equations, we say that we wish to solve the DE

$$y' = y.$$

Exercise 3.1.0.3. Guess and Check ☕

Can you think of any functions that are equal to their own derivative? Do you think you have all of them, or are some likely still out there?

As you can see, guess and check is not a good method for solving even the simplest of differential equations. We now take a more structured approach.

3.2 Separable Equations

Definition 3.2.0.1. Separable

Let x be the independent variable and let y represent an unknown function of x. A differential equation is *separable* if and only if it can be written in the form

$$\frac{dy}{dx} = F(x)G(y)$$

for some functions F and G.

Our method for solving a separable differential equation is as follows:

1. Write right-hand side of the differential equation in factored form, one function of x times one function of y.
2. Separate variables by multiplying both sides by $\frac{1}{G(y)}\,\mathtt{d}x$.
3. Antidifferentiate both sides.
4. Solve for y, if possible. (If not, we at least have an implicit solution.)

We try out this method on the previous example.

Example 3.2.0.2. Separation of Variables

Notice the differential equation

$$y' = y$$

is separable because it can be rewritten as

$$\frac{\mathtt{d}y}{\mathtt{d}x} = (1)(y).$$

That is, our factored form uses the functions $F(x) = 1$ and $G(y) = y$. We now perform separation of variables and antidifferentiate both sides.

$$\frac{1}{y}\,\mathtt{d}y = 1\,\mathtt{d}x$$
$$\int \frac{1}{y}\,\mathtt{d}y = \int 1\,\mathtt{d}x$$
$$\ln|y| = x + C$$
$$|y| = e^{x+C}$$
$$y = \pm e^{C} e^{x}$$
$$y = Ce^{x}$$

Notice that on the last line for simplicity, we clean up the constant $\pm e^C$ by just calling it C.

Exercise 3.2.0.3. Analyzing the Example ☕

- Why were we able to just put a $+C$ on one side when we integrated? What would have happened if we put it on both sides?
- When we renamed $\pm e^C$ as C, we technically introduced a new solution. The expression $\pm e^C$ is incapable of being equal to zero, but C can be. Verify that the $C = 0$ solution is valid to include as a solution to the differential equation.

Exercise 3.2.0.4. More Complicated DEs ☕☕☕

- Solve the following differential equation via separation of variables:

$$\frac{dy}{dx} = xy + x$$

- Solve the following Initial Value Problem via separation of variables:

$$\frac{dy}{dx} = e^{y-x}\sec(y)(1+x^2)$$

Note that you will not be able to obtain an explicit formula for y in terms of x but rather an implicit solution. Use the initial condition $y(0) = 0$ to solve for C.

3.3 Power Series Solutions

Power series provide a very effective method for solving differential equations. The steps are simple:

- Set the unknown function y equal to an unknown power series.
- Plug the power series in for all occurrences of y. Expand and combine like terms.
- Equate coefficients one degree at a time (much like we do when solving for unknowns in a PFD).
- Solve for the coefficients a_0, a_1, $a_2, \ldots$ one at a time in terms of a_0.
- Plug those coefficients back into the power series expansion for y to obtain a power series solution.
- Try to recognize that power series as a known function or variant thereof.

The process is thus very mechanical, but sometimes working through the details becomes a bit messy. We repeat the previous example with this new method.

Example 3.3.0.1. Revisiting Our First DE

Here we solve

$$y' = y$$

using power series. First, let y be written as a generic unknown power series as follows:

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

Plug this expression into the differential equation and expand.

$$\begin{aligned}
\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots\right)' &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \\
a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots
\end{aligned}$$

We now equate coefficients one degree at a time, and solve for every coefficient in terms of a_0.

$$\begin{array}{lrcl}
\text{Degree 0:} & a_1 = a_0 & \Longrightarrow & a_1 = a_0 \\
\text{Degree 1:} & 2a_2 = a_1 & \Longrightarrow & a_2 = \frac{1}{2} a_0 \\
\text{Degree 2:} & 3a_3 = a_2 & \Longrightarrow & a_3 = \frac{1}{3!} a_0 \\
\text{Degree 3:} & 4a_4 = a_3 & \Longrightarrow & a_4 = \frac{1}{4!} a_0 \\
\vdots & \vdots & & \vdots \\
\text{Degree } n-1: & n a_n = a_{n-1} & \Longrightarrow & a_n = \frac{1}{n!} a_0
\end{array}$$

We can now plug all coefficients back into our expression for y and simplify until we obtain a closed form for y.

$$\begin{aligned}
y(x) &= a_0 + a_0 x + \frac{1}{2!} a_0 x^2 + \frac{1}{3!} a_0 x^3 + \frac{1}{4!} a_0 x^4 + \cdots \\
&= a_0 \left(1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \cdots\right) \\
&= a_0 e^x
\end{aligned}$$

Notice we have obtained the same solution as via separation of variables! Clearly, the power series solution was way more work. The reason it is so valuable though is that there are many DEs which are not separable but for which the power series method works just fine.

Exercise 3.3.0.2. Comparing the Methods

- – Show the differential equation $\frac{dy}{dx} = yx$ is separable and use this to separate variables and solve the differential equation.

  – Solve the same differential equation via power series. Confirm you get the same answer.

- – Explain why the differential equation $\frac{dy}{dx} = yx + x + 1$ is not separable.

- Solve the same differential equation via power series.

- Check your answer is correct by plugging it back into the original DE.

- Consider the DE given by

$$y(0) = 1$$
$$y'(0) = 0$$
$$y'' = -y.$$

Solve this DE via power series (use the initial conditions to solve for a_0 and a_1).

3.4 Modeling with Differential Equations

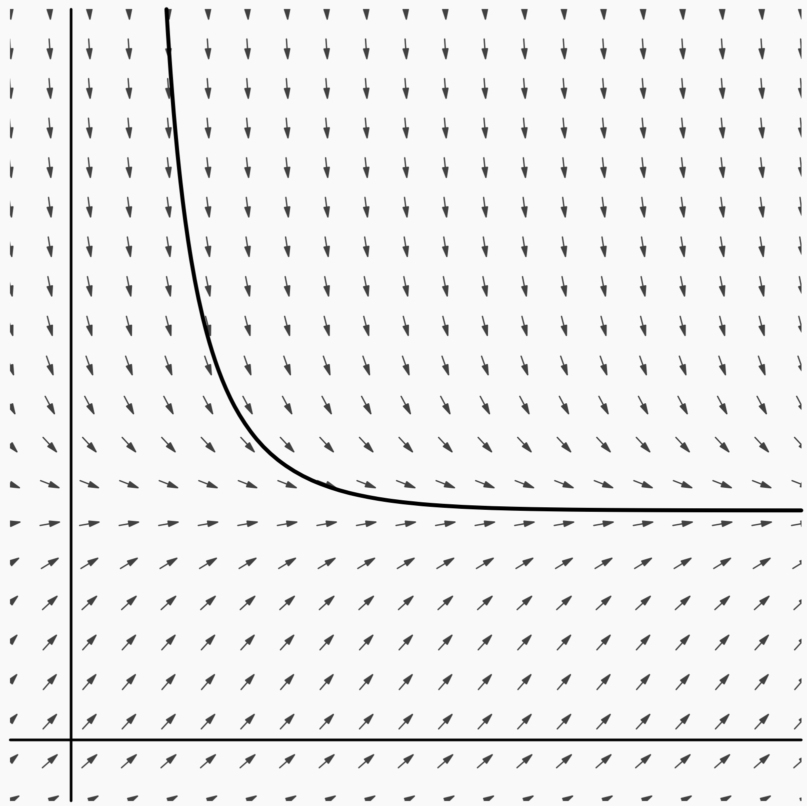
Differential equations are used extensively in applied mathematics and the sciences to describe models, which are then solved using mathematics to find explicit formulas for the quantities of interest.

Example 3.4.0.1. Newton's Law of Cooling

Newton's Law of Cooling states that the rate of change of the temperature of a small object in a room is proportional to the difference between room temperature and the temperature of the object. If A is the constant that represents the ambient temperature (room temperature), $T(t)$ represents the temperature of the room at time t, and k is the constant of proportionality, then this situation can be modeled by

$$\frac{dT}{dt} = k(T - A).$$

Here we introduce the idea of a *slope field*, a grid of small dashes that indicate the slope $\frac{\mathrm{d}T}{\mathrm{d}t}$ at every point (t, T) in the plane. Here we draw a slope field that governs solution curves to this model and show one sample solution curve.



Exercise 3.4.0.2. Newton's Law of Cooling ☕☕

- Label the above diagram. What variables do the axes correspond to? Can you find where the horizontal line $T = A$ is located?

- In this model, would it make sense that the proportionality constant k is positive or negative? Why?

- Solve the differential equation by separation of variables.

- Solve the differential equation by power series.

Note that those solutions give explicit formulas for the solution curves above, which is significantly more useful than just thinking of it intuitively as "following the arrows".

Exercise 3.4.0.3. Malthusian Population Model

A simple intuitive population model can be stated as follows:

If there are more individuals in a population, there will be more babies produced.

Here is a slightly more technical restatement of the same idea:

The rate of growth of a population is proportional to the size of the population.

- Let $P(t)$ be the size of the population at time t. Rewrite the above growth principal in the language of differential equations.

- Solve your differential equation using power series.

- Solve your differential equation using separation of variables and confirm that your answers match.

- Use your formula to find the limit of $P(t)$ as t approaches infinity.

- Under what real life conditions might this model be realistic? Under what conditions might this model be unrealistic?

As you probably noticed, the above model is slightly ridiculous for large time values since it would claim that eventually any species would fill up the entire visible universe with bodies. So let's adjust it to fix that unrealistic assumption. Here's an upgrade:

Exercise 3.4.0.4. Logistic Population Model ☕☕☕

If there are more individuals in a population, there will be more babies produced, but then it slows down as it approaches some sort of maximum possible population (a limit perhaps based on food supply, available habitat, etc).

Here is a slightly more technical restatement of the same idea:

The rate of growth of a population is jointly proportional to both the size of the population and the distance from some maximum possible population.

- Let $P(t)$ be the size of the population at time t and let M for maximum be a constant that the population cannot exceed. Rewrite the above growth principal in the language of differential equations.

- Solve your differential equation using power series out to a degree two approximation (this will be much too difficult to solve the whole thing using power series!).

- Solve your differential equation using separation of variables and confirm that your answers match out to the degree two approximation.

- Use your formula to find the limit of $P(t)$ as t approaches infinity.

- Suppose you started with population $P(0) = 2M$. What would your model predict would happen to the population?

3.5 Chapter Summary

A **differential equation** is an equation involving an unknown function $y(x)$ and one or more of its derivatives. The goal is to solve for an infinite family of functions $y(x)$ that satisfy the equation, or to find just a single function that solves the equation if a suitable **initial condition** is provided. There are many methods for solving DEs, but we focused on two in particular.

1. **Separation of variables:** To solve via separation of variables, we follow the following steps:

   - Write right-hand side of the differential equation in factored form, producing a DE in the form $\frac{dy}{dx} = F(x)G(y)$ (if possible).
   - Separate variables by multiplying both sides by $\frac{1}{G(y)}\,\mathtt{d}x$.
   - Antidifferentiate both sides.
   - Solve for y, if possible. (If not, we at least have an implicit solution.)

   This method is usually quite straightforward to carry out. The drawback is that most Differential Equations are not separable, meaning that it is impossible to write it as $\frac{dy}{dx} = F(x)G(y)$. Thus, the method usually fails as soon as it gets started.

2. **Power series solutions:** This method is far more robust as it does not depend on the DE being separable. It applies to more DEs, but be warned it is typically far messier! The steps are as follows:

   - Set the unknown function y equal to an unknown power series:
   $$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots .$$
   - Plug the power series into the DE for all occurrences of y. Expand and combine like terms.
   - Equate coefficients one degree at a time. This will create an infinite system of equations of the following form:
   $$\begin{aligned} \text{Left-hand side degree zero coefficient} &= \text{Right-hand side degree zero coefficient} \\ \text{Left-hand side degree one coefficient} &= \text{Right-hand side degree one coefficient} \\ \text{Left-hand side degree two coefficient} &= \text{Right-hand side degree two coefficient} \\ \text{Left-hand side degree three coefficient} &= \text{Right-hand side degree three coefficient} \\ &\vdots \end{aligned}$$
   - Solve for the coefficients a_0, a_1, $a_2, \ldots$ one at a time in terms of a_0. (If you have an initial condition, a_0 will just be a number. Otherwise leave everything in terms of a_0.)
   - Plug those coefficients back into the power series expansion for y to obtain a power series solution.
   - Try to recognize that power series as a known function or variant thereof.

   One can visualize solutions to DEs via a **slope field**, a grid of arrows that shows the value of $\frac{\mathtt{d}y}{\mathtt{d}x}$ at each point as the slope of the arrow. The solutions to the differential equation are functions that essentially follow the directions given by the arrows, starting at some initial condition.

3.6 Mixed Practice

3.6.1 Warm Ups

These are good problems for reinforcing the vocabulary and foundational concepts of this chapter.

Exercise 3.6.1.1. ☕

Find all functions that equal their own second derivative. That is to say, use power series to solve the following differential equation:

$$y'' = y$$

A Solution: Let $y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ then $y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots$ and $y'' = 2a_2 + 3\cdot 2a_3x + 4\cdot 3a_4x^2 + 5\cdot 4a_5x^3 \cdots$ Equate coefficients one degree at a time:
$a_0 = 2a_2 \to a_2 = \frac{a_0}{2}$
$a_1 = 3\cdot 2a_3 \to a_3 = \frac{a_1}{3!}$
$a_2 = 4\cdot 3a_4 \to a_4 = \frac{a_0}{4!}$
$a_3 = 5\cdot 4a_5 \to a_5 = \frac{a_1}{5!}$
So we have if n is odd then $a_n = \frac{a_1}{n!}$ and if n is even, we have $a_n = \frac{a_0}{n!}$
Put it all together to get $y = a_0 + a_1x + \frac{a_0}{2}x^2 + \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \cdots$ notice if you group the even and odd degrees you get $y = (a_0 + \frac{a_0}{2}x^2 + \frac{a_0}{4!}x^4 + \cdots) + (a_1x + \frac{a_1}{3!}x^3 + \frac{a_1}{5!} + \cdots) = a_0\cosh(x) + a_1\sinh(x)$
Thus, linear combinations of hyperbolic sine and hyperbolic cosine functions are the only functions that equal their own second derivatives.

3.6.2 Sample Test Problems

Exercise 3.6.2.1. ☕☕☕

a.) Find the set of all solutions to the following differential equation using power series. Do not leave your answer as a power series but rather turn it back into a closed explicit formula using familiar functions.

$$\frac{dy}{dx} = y - x - 2$$

A Solution: Let $y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ a power series. Then $y' = \frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + \cdots$ So we have $a_1 + 2a_2x + 3a_3x^2 + \cdots = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots - x - 2 = (a_0 - 2) + (a_1 - 1)x + a_2x^2 + a_3x^3 + \cdots$ Which means $a_1 = a_0 - 2$
$2a_2 = (a_1 - 1) = (a_0 - 2 - 1) = a_0 - 3$ so $a_2 = \frac{a_0 - 3}{2}$
$3a_3 = a_2 = a_2 = \frac{a_0-3}{2}$ So $a_3 = \frac{a_0-3}{3!}$
$4a_4 = a_3 = \frac{a_0-3}{3!}$ so $a_4 = \frac{a_0-3}{4!}$ etc
so we have $y = a_0 + (a_0 - 2)x + \frac{a_0-3}{2}x^2 + \frac{a_0-3}{3!}x^3 + \frac{a_0-3}{4!}x^4 + \cdots = a_0 + (a_0 - 2)x + (a_0 -$

$3)\frac{x^2}{2}+\frac{x^3}{3!}+\frac{x^4}{4!}x^4+\cdots = a_0+(a_0-2)x+(a_0-3)\sum_{x=0}^{\infty}\frac{x^n}{n!}-(a_0-3)-(a_0-3)x$ The part $-(a_0-3)-(a_0-3)x$ is necessary to exclude the first 2 terms of $(a_0-3)\sum_{x=0}^{\infty}\frac{x^n}{n!}=(a_0-3)e^x$ So we actually have $y=a_0-(a_0-3)+(a_0-2)x-(a_0-3)x+(a_0-3)e^x=3+x+(a_0-3)e^x$

---

b.) Plug your answer back into the DE to verify it is correct.

A Solution: If $y=3+x+(a_0-3)e^x$ then $\frac{dy}{dx}=1+(a_0-3)e^x$ but $y-x-2=3+x+(a_0-3)e^x-x-2=1+(a_0-3)e^x$ So they match.

Exercise 3.6.2.2. ☕☕☕

a.) Explain why one cannot use separation of variables to solve the differential equation

$$\frac{dy}{dx} = 2y + x$$

A Solution: The right-hand side $2y+x$ does not factor into a function of y times a function of x, so the equation is not separable.

---

b.) Solve the above differential equation using power series. Recognize your answer as a known function!

A Solution: Let $y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ then $\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots$ so we have $a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots = 2(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots) + x = 2a_0 + (2a_1 + 1)x + 2a_2x^2 + 2a_3x^3 + 2a_4x^4 + \cdots$ set corresponding coefficients equal to each other to get:
$a_1 = 2a_0$
$2a_2 = 2a_1 + 1 = 4a_0 + 1 \to a_2 = \frac{4a_0+1}{2}$
$3a_3 = 2a_2 = 8a_0 + 2 \to a_3 = \frac{8a_0+2}{3!}$
$4a_4 = 2a_3 = \frac{16a_0+4}{3!} \to a_4 = \frac{16a_0+4}{4}$
$a_n = \frac{2^n a_0 + 2^{n-2}}{n!}$ So $y = a_0 + 2a_0x + \frac{2^2 a_0 + 2^{2-2}}{2!}x^2 + \frac{2^3 a_0 + 2^{3-2}}{3!}x^3 + \cdots = a_0 + 2a_0x + \sum\limits_{n=2}^{\infty} \frac{2^n a_0 + 2^{n-2}}{n!}$
Note that $\sum\limits_{n=2}^{\infty} \frac{2^n a_0 + 2^{n-2}}{n!} = (a_0 + 2^{-2}) \sum\limits_{n=2}^{\infty} \frac{(2x)^n}{n!} = (a_0 + 2^{-2}) \sum\limits_{n=0}^{\infty} \frac{(2x)^n}{n!} - (a_0 + 2^{-2}) - (a_0 + 2^{-2})(2x) = (a_0 + 2^{-2})e^{2x} - (a_0 + 2^{-2}) - (a_0 + 2^{-2})(2x)$ So that we now have $y = a_0 + 2a_0x + (a_0 + 2^{-2})e^{2x} - (a_0 + 2^{-2}) - (a_0 + 2^{-2})(2x) = -\frac{1}{4} - \frac{1}{2}x + (a_0 + 2^{-2})e^{2x}$ Finally we have $y = -\frac{1}{4} - \frac{1}{2} + Ce^{2x}$

Chapter 4

Introduction to Complex Numbers

The extension from the real numbers to the complex numbers has far-reaching affects. In this chapter, we give a brief introduction to complex numbers and then show how they interact with almost every topic in the course!

4.1 Complex Numbers

The complex numbers arise out of the fact that the simple little equation $x^2 + 1 = 0$ has no solution over the reals. Thus, we create the number i to represent a root of that polynomial. That is, $i^2 + 1 = 0$.

Definition 4.1.0.1. Complex Numbers

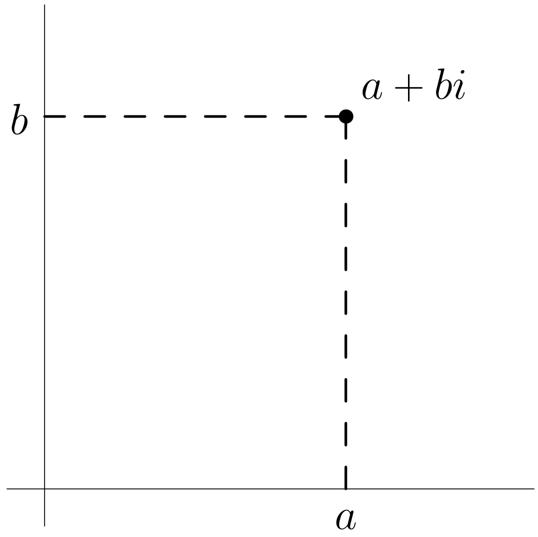
The set of *complex numbers* is the set of all numbers that can be written in the form $a + bi$ for real numbers a and b.

We perform arithmetic in the complex numbers using the usual rules of arithmetic and algebra along with the extra identity $i^2 = -1$.

Exercise 4.1.0.2. Containment of the Reals ☕

- Is 3 a complex number? Can you write 3 in the form $a + bi$ for real numbers a and b?
- Does the set of complex numbers contain all real numbers?

We can visualize complex numbers in the complex plane, where a (the *real part*) is the horizontal component and b (the *imaginary part*) is the vertical.



4.2 Euler's Identity and Consequences

Look again at the power series for the exponential function, sine, and cosine:

$$
\begin{aligned}
e^x &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 + \cdots \\
\cos(x) &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots \\
\sin(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots
\end{aligned}
$$

You have to wonder if there is some way to add together sine and cosine to get the exponential function! Sure the signs are off, but otherwise things seem so right. Sine has all the odd factorial denominators, cosine has all the even factorial denominators, and the exponential function has all of them! It turns out that i is exactly the constant we need to fix those minus signs!

Exercise 4.2.0.1. Proof of Euler's Identity ☕☕

- Write out a power series for $e^{i\theta}$.

- Write out a power series for $\cos(\theta) + i\sin(\theta)$.

- Verify the two are equal!

The fact that there is any relationship whatsoever between sine, cosine, and e is very surprising when you think of how differently those quantities are defined! We again state this incredible theorem, Euler's Identity!

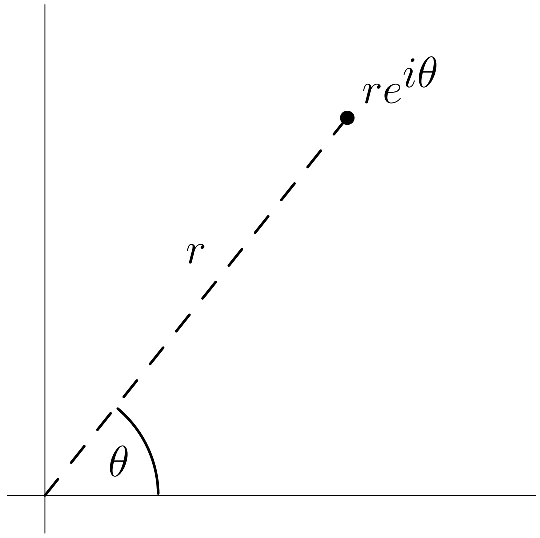
Theorem 4.2.0.2. Euler's Identity

For any real number θ,
$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

If we multiply both sides by a real number r, we then obtain

$$re^{i\theta} = r\cos(\theta) + ir\sin(\theta).$$

We notice that the horizontal component, $r\cos(\theta)$, is in fact the conversion for x into polar coordinates. Likewise, $r\sin(\theta)$ is the conversion for y into polar coordinates. This means that the complex number $re^{i\theta}$ is in fact the point located at angle θ and radius r in the complex plane.



4.2.1 Complex Roots

One interesting fact about the complex numbers is that the number of n^{th} roots of *every* real number is exactly n. So every number has two square roots, three cubed roots, and so on. We use $re^{i\theta}$ form to find these complex roots.

Example 4.2.1.1. The Cubed Roots of Two

To find all cubed roots of two, we solve the equation

$$z^3 = 2.$$

We begin by putting both z and 2 in complex polar form. We write $z = re^{i\theta}$ and $2 = 2e^{i0}$. We plug these into the equation, expand the powers.

$$\begin{aligned} z^3 &= 2 \\ \left(re^{i\theta}\right)^3 &= 2e^{i0} \\ r^3 e^{i3\theta} &= 2e^{i0} \end{aligned}$$

We now equate the radius and the angles as two separate equations.

- **Radius:** Since r is a real number, we obtain $r^3 = 2$, which implies $r = \sqrt[3]{2}$.
- **Angle:** The angles need to be equivalent but not necessarily equal. If they differ by a multiple of 2π, that is fine! Thus, we have $3\theta = 0 + 2\pi k$ for any integer k. Dividing both sides by 3, we have

$$\theta = \frac{0 + 2\pi k}{3} = \dots, \frac{-4\pi}{3}, \frac{-2\pi}{3}, 0, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{6\pi}{3}, \dots.$$

  However, if we use more values of θ beyond just $0, \frac{2\pi}{3}, \frac{4\pi}{3}$, the solutions will repeat since cosine and sine have period 2π. Thus, we use just those three angles.

Putting together our r and θ values, we have the following three roots:

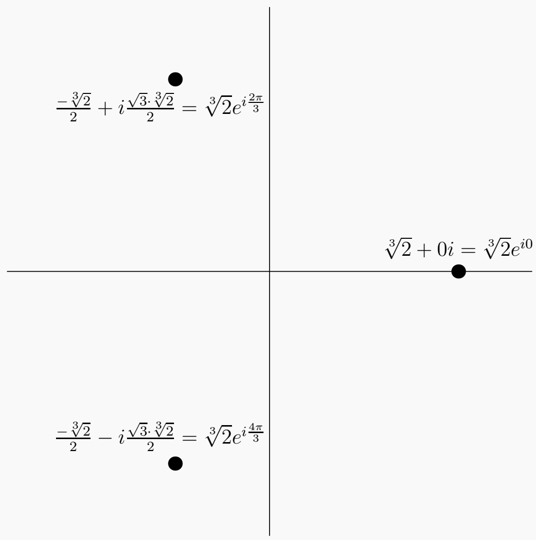
$$z = \sqrt[3]{2}e^{i0}, \sqrt[3]{2}e^{i\frac{2\pi}{3}}, \sqrt[3]{2}e^{i\frac{4\pi}{3}}.$$

Thus, we have our roots in complex polar form. We use Euler's Identity to turn these back into complex cartesian form as follows:

$$z = \sqrt[3]{2}\cos(0) + i\sqrt[3]{2}\sin(0), \sqrt[3]{2}\cos(\frac{2\pi}{3}) + i\sqrt[3]{2}\sin(\frac{2\pi}{3}), \sqrt[3]{2}\cos(\frac{4\pi}{3}) + i\sqrt[3]{2}\sin(\frac{4\pi}{3}).$$

At last, we use the unit circle to evaluate these and plot in the complex plane.

$$z = \sqrt[3]{2}, -\frac{\sqrt[3]{2}}{2} + i\frac{\sqrt[3]{2}\sqrt{3}}{2}, -\frac{\sqrt[3]{2}}{2} - i\frac{\sqrt[3]{2}\sqrt{3}}{2}$$



Exercise 4.2.1.2. Checking Once Again ☕☕

Cube each of the answers from the previous problem. Verify in each case you get 2!

It turns out to be of particular importance to find roots of 1. Define the n^{th} *roots of unity* to be the solutions to the equation

$$z^n = 1.$$

Lets play around and see if we can find some neat properties!

Exercise 4.2.1.3. Roots of Unity ☕☕☕☕

- Find all square roots of unity. Write your answers in both cartesian and polar complex form, and plot them in the complex plane. (the case where $n = 2$)

- Find all third roots of unity.

- Find all fourth roots of unity.

- Find all fifth roots of unity.

- Find all sixth roots of unity.

- Fill out the following table:

n	Σ_n	Π_n
2		
3		
4		
5		
6		

where Σ_n represents the sum of all n^{th} roots of unity and Π_n represents the product of all n^{th} roots of unity. (**Hint:** It's easier to add in cartesian, and easier to multiply in polar.)

- Based on your above data gathered, conjecture a formula for both Σ_n and Π_n. Prove your conjecture is correct. (**Hint:** Consider the roots of the polynomial $z^n - 1$ and how that polynomial would factor based on those roots. Then consider the degree zero and degree $n-1$ coefficients.)

Using the same techniques we can answer the following question, "what is the square root of i?" Keep in mind there are technically two square roots of i, the two solutions to the equation $z^2 = i$.

Exercise 4.2.1.4. Square Roots of i ☕☕

- Find the square roots of i. Write your answers in complex cartesian form.

- Square your answers back out (in complex cartesian form) and verify that you do in fact get

i when you square them.

Exercise 4.2.1.5. Cubed Roots of i ☕☕☕

- Find all cubed roots of i. That is, find all complex numbers z such that $z^3 = i$. Write your answers in $a + bi$ form.

- Take the cube of each of your roots to verify that you do in fact get i as the third power.

4.2.2 Proving Trig Identities

Remember how there are 47,000 useful but impossible to remember trigonometric identities? No? Well, that shows how hard they are to remember. Believe it or not, most of them can be constructed very quickly and easily from Euler's Identity!

Example 4.2.2.1. The Sine and Cosine Double-Angle Identities

To construct the sine and cosine double-angle formulas, we can manipulate the expression $e^{2\theta}$. We proceed with the following chain of equality:

$$\begin{aligned}\cos(2\theta) + i\sin(2\theta) &= e^{i\cdot 2\theta}\\ &= \left(e^{i\theta}\right)^2\\ &= (\cos(\theta) + i\sin(\theta))^2\\ &= \cos^2(\theta) + 2\cos(\theta) i\sin(\theta) + i^2\sin^2(\theta)\\ &= \left(\cos^2(\theta) - \sin^2(\theta)\right) + i\left(2\sin(\theta)\cos(\theta)\right).\end{aligned}$$

We now equate real parts to obtain the cosine double-angle identity

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta).$$

Similarly, we equate imaginary parts to obtain the sine double-angle identity

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta).$$

Exercise 4.2.2.2. Annotate! ☕

Write a short justification alongside each line of computation above.

Exercise 4.2.2.3. Angle-Sum Identities ☕☕

- Expand the expression $e^{i(A+B)}$ into real and imaginary parts using Euler's Identity.

- Expand the expression $e^{iA}e^{iB}$ into real and imaginary parts using Euler's Identity twice, once per factor. Multiply out the resulting terms into an expression of the format

$$f(A, B) + ig(A, B)$$

where f is the function corresponding the real part of that expression, and g corresponds to the imaginary part.

- Equate real and imaginary parts to produce the angle sum identities for cos and sin, respectively! (**Hint:** We're using the fact that two complex numbers $a + bi$ and $c + di$ are equal if and only if $a = c$ and $b = d$.)

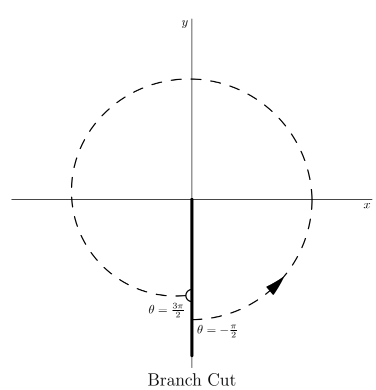
4.2.3 Natural Logarithm of a Complex Number

We now show how to compute the natural logarithm of a complex number. As usual, polar form will be critical.

- Given a complex number z, we first write z in polar form $z = re^{i\theta}$, where r is a positive real number and $\theta \in [-\pi/2, 3\pi/2)$. This choice of interval for θ is often called a *branch cut* and is essentially a domain restriction for the exponential function (since it fails to be one-to-one over the complex numbers).

- Split apart using the property of logarithms and cancel the log with the exponential as follows:

$$\ln(z) = \ln\left(re^{i\theta}\right) = \ln(r) + \ln\left(e^{i\theta}\right) = \ln(r) + i\theta.$$



Example 4.2.3.1. Natural Log of i

Here we compute the mysterious quantity $\ln(i)$. We begin by rewriting i as

$$i = 1 \cdot e^{i\pi/2}.$$

Notice that we chose the angle $\theta = \pi/2$ to be within the branch cut specified above. From here, we split using log properties as follows:

$$\begin{aligned} \ln(i) &= \ln\left(1 \cdot e^{i\pi/2}\right) \\ &= \ln(1) + \ln\left(e^{i\pi/2}\right) \\ &= 0 + i\frac{\pi}{2}. \end{aligned}$$

Thus, $\ln(i) = \frac{\pi}{2}i$.

Note that in principle there is no reason we had to pick our angle θ in that particular interval. One can construct a perfectly well-defined logarithm from choosing a different domain for θ. This is similar to the construction of the inverse trig functions, where one must restrict the domain in some manner, so we tend to just choose a default interval to restrict to and stick with it.

Exercise 4.2.3.2. Complex Logarithms ☕☕

Try the above method to compute each of the following logarithms. Write each in the standard complex cartesian form $a + bi$.

- $\ln(2)$
- $\ln(-2)$
- $\ln(1+i)$
- $\ln(3-4i)$

4.2.4 Complex Exponentials

Recall our trick for dealing with strange bases:

$$w_1{}^{w_2} = e^{\mathtt{ln}(w_1{}^{w_2})} = e^{w_2 \mathtt{ln}(w_1)}.$$

This provides the advantage of moving us back to the familiar base e from the unfamiliar base w_1. This will make a complex exponential base manageable!

Example 4.2.4.1. Computation of Hammurabi

Here we perform two exponentials; we have an i to an i, and a 2 to the 2^{th}. Using the above trick and the value of $\ln(i)$ computed in Example 4.2.3.1, we find

$$\begin{aligned} i^i 2^2 &= 4i^i \\ &= 4e^{\ln\left(i^i\right)} \\ &= 4e^{i \ln(i)} \\ &= 4e^{i \frac{\pi}{2} i} \\ &= 4e^{-\frac{\pi}{2}}. \end{aligned}$$

Notice there was no need to decompose further using Euler's Identity here; the end result of i raised to the i power is in fact a real number!

Exercise 4.2.4.2. Complex Exponentials ☕☕

- Use the above trick to compute $(1+i)^i$.

- Use the above trick to compute i^{1+i}.

- Use the above trick to compute $(1+i)^{1+i}$.

4.3 Partial Fractions via Complex Numbers

If we are using complex numbers, there are no more irreducible quadratics! This gives us an interesting alternate way to perform PFD, since all polynomials will fully factor into linear factors.

Exercise 4.3.0.1. PFD over the Complex Numbers ☕☕☕

- Find an antiderivative of $\frac{4-2x^2}{x^3+4x}$ via a partial fraction decomposition over the real numbers.

- Find an antiderivative of $\frac{4-2x^2}{x^3+4x}$ via a partial fraction decomposition over the complex numbers.

- Verify your answers are compatible.

Since Euler's Identity relates the exponential function to trigonometric functions, it is plausible that there would be analagous relationships out there between logarithmic and inverse trigonometric functions over the complex numbers! At a very crude level, one can think of just taking an inverse function of both sides of Euler's Identity. It turns out that PFD over the complex numbers is the right tool to make this formal!

Exercise 4.3.0.2. Inverse Tangent and Natural Log

Recall that by trigonometric substitution, we have

$$\int \frac{1}{x^2+1}\,\mathtt{d}x = \arctan(x) + C.$$

Compute the same antiderivative but using a PFD over the complex numbers. In particular, carry out the following steps:

- Factor x^2+1 over the complex numbers.

- Find A and B in the decomposition

$$\frac{1}{x^2+1} = \frac{A}{x+i} + \frac{B}{x-i}.$$

- Find the antiderivative, simply integrating terms of the form $\frac{c_1}{x+c_2}$ as $c_1 \ln(x+c_2)$ for complex constants c_1 and c_2, just as you would for real constants.

- Conclude that

$$\arctan(x) = \frac{1}{2i}\ln\left(\frac{x-i}{x+i}\right) + C$$

for some constant C.

- Solve for C by letting $x \to \infty$ on both sides to get a relationship between inverse tangent and the natural logarithm! (It is valid for positive x.)

4.4 Chapter Summary

The set of **complex numbers** is the set of all numbers expressible as $a + bi$ for real numbers a and b. In the **complex plane**, we plot a as the horizontal and b as the vertical. Allowing complex numbers in our calculus adventures relates many seemingly unrelated objects!

1. **Euler's Identity and consequences:** By setting $x = i\theta$ in our power series for the exponential function, we obtain **Euler's Identity**. This is the relationship

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

   which is usually then multiplied by r to obtain

$$re^{i\theta} = \underbrace{r\cos(\theta)}_{x \text{ in polar coords}} + i \underbrace{r\sin(\theta)}_{y \text{ in polar coords}}.$$

   This means that in the complex plane we have $re^{i\theta}$ as the point at angle θ and radius r. This relationship has many consequences, including the following:

   (a) **Proving trig identities:** Properties of exponentials can be turned into trig identities using Euler's Identity.

   (b) **Calculating roots of complex numbers:** To find the n^{th} roots of a complex number $a+bi$, notice that this is the same as solving the equation $z^n = a + bi$. Rewrite everything in polar form, distribute the n power, and equate radius and angles to find the roots.

   (c) **Calculating logarithms of complex numbers:** To compute $\ln(a + bi)$, write $a+bi$ in polar form with an angle chosen in the branch cut $-\pi/2 \leq \theta < 3\pi/2$. From there, use properties of logs to simplify the expression.

   (d) **Calculating exponentials with a complex base:** Rewrite as "e to the ln" of the expression and then use the method for complex logarithms described above.

2. **Revisiting PFD with complex numbers:** With complex numbers, there is no need for the irreducible quadratic case in a PFD. Instead, we can completely factor the denominator of any rational function into a product of powers of linear factors.

4.5 Mixed Practice

4.5.1 Warm Ups

These are good problems for reinforcing the vocabulary and foundational concepts of this chapter.

Exercise 4.5.1.1.

a.) Find all square roots of $-i$. Write your answers in complex cartesian form.

A Solution: If $z^2 = e^{i(-\pi/2)}$ and $z = re^{i\theta}$ then $r^2e^{i2\theta} = e^{i(-\pi/2}$ so $r = 1$ and $2\theta = \frac{-\pi}{2} + 2\pi k$ Thus $\theta = \frac{-\pi}{4} + \pi k = \frac{-\pi}{4}, \frac{3\pi}{4}$ So $z = e^{-i(\pi/4)}, e^{i(3\pi/4)} = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

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b.) Take your answers and square them to verify their square is $-i$ as you claim above.

A Solution: $\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)^2 = \frac{2}{4} - 2i\frac{2}{4} + i^2\frac{2}{4} = \frac{1}{2} - \frac{1}{2} - i = -i$ and $\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)^2 = \frac{2}{4} - 2i\frac{2}{4} + i^2\frac{2}{4} = \frac{1}{2} - \frac{1}{2} - i = -i$

4.5.2 Sample Test Problems

Exercise 4.5.2.1.

Compute the following complex numbers in standard $a + bi$ form for $a, b \in \mathbb{R}$. List all values if there are multiple answers.

a.) i^{2015}

A Solution: $i^{2015} = i^{2012+3} = i^{2012}i^3 = 1 \cdot i^3 = -i$

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b.) $i^{(2i)}$

A Solution: $i^{(2i)} = e^{\ln i^{(2i)}} = e^{2i \ln i}$ Note that $i = \cos(\pi/2) + i\sin(\pi/2) = e^{i\pi/2}$ So $e^{2i\ln i} = e^{2i \ln e^{i\pi/2}} = e^{2i \cdot i\pi/2} = e^{-\pi}$

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c.) $1 + i\pi - \frac{\pi^2}{2!} - \frac{i\pi^3}{3!} + \frac{\pi^4}{4!} + \frac{i\pi^5}{5!} - \frac{\pi^6}{6!} - \frac{i\pi^7}{7!} + \frac{\pi^8}{8!} + \cdots$

A Solution: $1 + i\pi - \frac{\pi^2}{2!} - \frac{i\pi^3}{3!} + \frac{\pi^4}{4!} + \frac{i\pi^5}{5!} - \frac{\pi^6}{6!} - \frac{i\pi^7}{7!} + \frac{\pi^8}{8!} + \cdots = 1 + i\pi + \frac{(i\pi)^2}{2!} + \frac{(i\pi)^3}{3!} +$

$\frac{(i\pi)^4}{4!} + \frac{(i\pi)^5}{5!} - \frac{(i\pi)^6}{6!} - \frac{(i\pi)^7}{7!} + \frac{(i\pi)^8}{8!} + \cdots = e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$

d.) $\sqrt[3]{i}$

A Solution: $\left(r(e^{i\theta})\right)^3 = r^3 e^{i3\theta} = i = e^{i\pi/2}$
$\Rightarrow r = 1$ and $3\theta = \frac{\pi}{2} + 2\pi k$
$\Rightarrow \theta = \frac{\pi}{6} + \frac{2\pi}{3}k \Rightarrow \frac{\pi}{6}, \frac{\pi}{6} + \frac{2\pi}{3}, \frac{\pi}{6} + \frac{4\pi}{3}$
$\Rightarrow \sqrt[3]{i} = e^{i\pi/6}, e^{i5\pi/6}, e^{i3\pi/2} = \frac{\sqrt{3}}{2} + \frac{i}{2}, -\frac{\sqrt{3}}{2} + \frac{i}{2}, -i$

e.) $\ln\left(4\sqrt{3} + i\right)$

A Solution: First convert $4\sqrt{3} + i$ to polar using a triangle where $4\sqrt{3}$ is the horizontal side and 1 is the vertical side. Then $r = \sqrt{(4\sqrt{3})^2 + 1^2} = \sqrt{16 \cdot 3 + 1} = \sqrt{49} = 7$ also, $\theta = \arctan\left(\frac{1}{4\sqrt{3}}\right) = \arctan\left(\frac{\sqrt{3}}{12}\right)$ Thus $ln(4\sqrt{3}+i) = \ln\left(7e^{i\arctan\sqrt{3}/12}\right) = \ln 7 + i\arctan\sqrt{3}/12$

Exercise 4.5.2.2.

a.) State and prove Euler's Identity using power series.

A Solution: $\cos(\theta) + i\sin(\theta) = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!}\cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!}\cdots\right) = \left(1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!}\cdots\right) = \left(1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!}\cdots\right) = e^{i\theta}$

b.) Multiply both sides of Euler's Identity by r. Explain how this formula relates to our conversion between polar and cartesian coordinates.

A Solution: $r\cos(\theta) + ri\sin(\theta) = re^{i\theta}$ This shows that if $x + iy$ is the **horizontal + i vertical** representation of a complex number, then $x = r\cos(\theta)$ and $y = ri\sin(\theta)$ so that r and θ represent a radius and an angle.

c.) Prove the sine double-angle identity using Euler's Identity.

A Solution: Prove $\sin(2\theta) = 2\cos(\theta)\sin(\theta)$
Start with $\cos(2\theta) + i\sin(2\theta) = e^{i(2\theta)} = (e^{i\theta})^2 = (\cos(\theta) + i\sin(\theta))^2 = \cos^2(\theta) + 2i\sin(\theta)\cos(\theta) + i^2\sin^2(\theta) = (\cos^2(\theta) - \sin^2(\theta)) + i(2\sin(\theta)\cos(\theta))$ So $\cos(2\theta) = \cos^2(\theta) -$

$\sin^2(\theta)$ and $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

Selected Answers and Hints

Exercise 1.1.0.2. The antiderivative is $\frac{1}{3}\sin^3(x) + C$.

Exercise 1.1.1.2. Since seven is odd, when we pulled out one factor of sine, we ended up with the sixth power of sine remaining. Since six is even, we were able to express it as a power of a perfect square of sine, which in turn let us rewrite as cosines using the Pythagorean identity.

Exercise 1.1.1.3. The first antiderivative is $-\frac{1}{3}\cos^3(x) + \frac{2}{5}\cos^5(x) - \frac{1}{7}\cos^7(x) + C$. For the second, rewrite as $(1 - \sin^2(x))^4\cos(x)$ and proceed by letting $u = \sin(x)$.

Exercise 1.1.1.4. Often when trying to show that two antiderivatives are compatible, it is easiest to verify that their difference is a constant.

Exercise 1.1.1.4. The substitution $u = \sin(x)$ is much cleaner since the other will involve having to expand a binomial to the fifth power. The antiderivative is $\frac{1}{12}\sin^{12}(x) - \frac{1}{14}\sin^{14}(x) + C$.

Exercise 1.1.2.1. The exponent on sine is zero, which is indeed even. Thus both exponents are even in this case.

Exercise 1.1.2.3. When all like terms are combined and the one-eighth is distributed, the result is $\frac{5}{16}x + \frac{1}{4}\sin(2x) - \frac{1}{48}\sin^3(2x) + \frac{3}{64}\sin(4x) + C$.

Exercise 1.1.2.4. The antiderivative to $\cos^6(x)$ came out to

$$\frac{5}{16}x + \frac{1}{4}\sin(2x) - \frac{1}{48}\sin^3(2x) + \frac{3}{64}\sin(4x) + C$$

Before we differentiate, first bash everything back down to an "x" in the argument using double angle identities. This produces

$$\frac{5}{16}x + \frac{1}{2}\sin(x)\cos(x) - \frac{1}{6}\sin^3(x)\cos^3(x) + \frac{3}{16}\sin(x)\cos^3(x) - \frac{3}{16}\sin^3(x)\cos(x) + C$$

Factor out a sine and use the Pythagorean Identity to get everything else in terms of cosine. This produces

$$\frac{5}{16}x + \sin(x)\left(\frac{5}{16}\cos(x) + \frac{5}{24}\cos^3(x) + \frac{1}{6}\cos^5(x)\right) + C$$

Then we differentiate and obtain

$$\frac{5}{16} + \cos(x)\left(\frac{5}{16}\cos(x) + \frac{5}{24}\cos^3(x) + \frac{1}{6}\cos^5(x)\right) - \sin^2(x)\left(\frac{5}{16} + \frac{5}{8}\cos^2(x) + \frac{5}{6}\cos^4(x)\right)$$

to which we apply the Pythagorean Identity $\sin^2(x) = 1 - \cos^2(x)$ to produce

$$\frac{5}{16} + \cos(x)\left(\frac{5}{16}\cos(x) + \frac{5}{24}\cos^3(x) + \frac{1}{6}\cos^5(x)\right) - \left(1 - \cos^2(x)\right)\left(\frac{5}{16} + \frac{5}{8}\cos^2(x) + \frac{5}{6}\cos^4(x)\right)$$

This will simplify to $\cos^6(x)$ once you expand and combine like terms.

Exercise 1.1.2.5. For the first, apply the identity $\sin^2(3x) = \frac{1-\cos(6x)}{2}$ and proceed. For the second, notice that $\sin^4(x)$ can be rewritten as $\left(\sin^2(x)\right)^2$, after which the half-angle identity can be applied.

Exercise 2.1.0.2. This parametric curve is the line $y = \frac{3}{2}x + 1$.

Exercise 2.1.0.3. The two curves are the same points in the plane. Both start at the point (1,0) at time $t = 0$, but C_1 then proceeds counter-clockwise while C_2 proceeds clockwise.

Exercise 2.3.0.4. The arc length is

$$\frac{6\sqrt{146} + \ln\left(\sqrt{73} + 6\sqrt{2}\right)}{6} \approx 12.55.$$

Also, to handle the absolute value, just find the arc length on the interval [0,2] where you can ignore the absolute value and then apply symmetry.

Exercise 2.3.0.5. The arc length is $\sqrt{2}\left(e^{2\pi} - 1\right)$.

Exercise 2.5.2.7. Yes, it is in fact a circle with cartesian center $(0, 1/2)$ and radius $1/2$. This can be verified by demonstrating the polar equation converts to the cartesian equation

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2.$$

Exercise 2.6.0.2. The derivative is a constant; thus the graph is a straight line!

Exercise 2.7.0.4. The area between the curves is $\frac{\pi}{8}$.

Exercise 2.7.0.5. The area inside the inner loop of $r(\theta) = \frac{1}{2} + \cos(\theta)$ is $\frac{\pi}{4} - \frac{3\sqrt{3}}{8}$.

Exercise 2.10.1.1. a. $r(0) = 4$, $r(\pi/6) = 8/\sqrt{3}$, $r(\pi/4) = 8/\sqrt{2} = 4\sqrt{2}$, $r(\pi/3) = 8$ b. $r = 4\sec\theta \implies r\cos\theta = 4 \implies x = 4$ is a vertical line. c. d. It is an isosceles triangle with hypotenuse $4\sqrt{2}$ and sides 4 $A = 8$ e. $A = 8$ They are the same.

Exercise 2.10.1.2. a.) It is a line segment that lies on the line $\frac{x-1}{4} = t = \frac{y}{6} \leftrightarrow y = \frac{3}{2}x - \frac{3}{2}$ between $(-1, 0)$ and $(7, 12)$
b.) $\frac{3}{2}$
c.) $4\sqrt{13}$

Exercise 2.10.2.1. $\cosh t = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \cdots$
$\sinh t = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots$
a. $\frac{d\ \sinh(t)}{dt} = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots = \cosh(t)$
b. $\frac{d\ \cosh(t)}{dt} = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots = \sinh(t)$
c. $\cosh^2 t - \sinh^2 t = \left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \cdots\right)^2 - \left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right)^2 = 1$
d. $\frac{1}{0}$ which is a vertical line

Exercise 2.10.2.2. b.) $\frac{\pi}{8}$

Exercise 3.2.0.4. Any solution to $\frac{dy}{dx} = xy + x$ can be written as $y = Ce^{\frac{x^2}{2}} - 1$ for some real number C. The second DE with initial condition has the solution

$$\frac{1}{2}e^{-y}\left(\sin(y) - \cos(y)\right) = -e^{-x}\left(3 + 2x + x^2\right) + \frac{5}{2}.$$

Exercise 3.6.1.1. Linear combinations of hyperbolic sine and hyperbolic cosine functions are the only functions that equal their own second derivatives.

Exercise 3.6.2.1. a. $y = 3 + x + (a_0 - 3)e^x$ b. If $y = 3 + x + (a_0 - 3)e^x$ then $\frac{dy}{dx} = 1 + (a_0 - 3)e^x$ but $y - x - 2 = 3 + x + (a_0 - 3)e^x - x - 2 = 1 + (a_0 - 3)e^x$ So they match.

Exercise 3.6.2.2. a.) The right-hand side $2y + x$ does not factor into a function of y times a function of x so there can be no separation of variables.
b.) $y = -\frac{1}{4} - \frac{1}{2}x + Ce^{2x}$

Exercise 4.2.3.2. $\bullet \ln(2) = \ln(2) + 0i$ $\bullet \ln(-2) = \ln(2) + \pi i$ $\bullet \ln(1 + i) = \ln\left(\sqrt{2}\right) + i\frac{\pi}{4}$ $\bullet \ln(3 - 4i) = \ln(5) + i \arctan\left(-\frac{3}{4}\right)$

Exercise 4.2.4.2. The number $(1 + i)^{1+i}$ can be written in complex cartesian form as

$$\left(e^{\ln\left(\sqrt{2}\right) - \frac{\pi}{4}} \cos\left(\ln\left(\sqrt{2}\right) + \frac{\pi}{4}\right)\right) + i\left(e^{\ln\left(\sqrt{2}\right) - \frac{\pi}{4}} \sin\left(\ln\left(\sqrt{2}\right) + \frac{\pi}{4}\right)\right).$$

Exercise 4.3.0.1. The PFD over the complex numbers is

$$\frac{4 - 2x^2}{x^3 + 4x} = \frac{1}{x} - \frac{\frac{3}{2}}{x + 2i} - \frac{\frac{3}{2}}{x - 2i}.$$

Exercise 4.3.0.2. In taking the limit, the logarithmic term will approach zero. Thus $C = \pi/2$.

Exercise 4.5.1.1. $z = e^{i(-\pi/4)}, e^{i(3\pi/4)} = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

Exercise 4.5.2.1. a.) $-i$
b.) $e^{-\pi}$
c.) -1
d.) $\frac{\sqrt{3}}{2} + \frac{i}{2}, -\frac{\sqrt{3}}{2} + \frac{i}{2}, -i$
e.) $\ln 7 + i \arctan \sqrt{3}/12$

Exercise 4.5.2.2. b. $r\cos(\theta) + ri\sin(\theta) = re^{i\theta}$ This shows that r and θ represent a radius and an angle.

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