Calculus II

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Contents

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We hope you have fun with this resource and find it helpful! It is a beautiful subject, and we tried to honor that with a beautiful text. The text is still in its infancy, and we welcome any and all feedback you could give us. Thank you in advance for any comments, complaints, suggestions, and questions.

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Chapter 1

Overview

A Three-Part Course

The topics of Calculus II fall into three parts that each have an appropriate place in the story of the calculus sequence.

- **Part I: Integration.** The first part of the course ties loose ends from Calculus I. The ending of Calculus I showed that antiderivatives can be used to evaluate integrals via the Fundamental Theorem of Calculus. However, by the end of Calculus I, only the very simplest antiderivatives can actually be computed. Part one expands the student's knowledge of **techniques of antidifferentiation**. These techniques are subsequently put to use computing **length, area, volume, and center of mass**.

- **Part II: Sequences and Series.** This is the topic that makes up the body of Calculus II. **Sequences and series** embody the beauty of mathematics; from simple beginnings (a sequence is just a list... a series is just adding up a list of numbers...) it quickly leads to incredible structure, surprises, complexity, and open problems. **Power series** redefine commonly used transcendental functions (functions that are not computed using algebra, e.g. cosine). If you've ever wondered **what your calculator does when you press the cosine button**, this is where you find out! (**Hint:** It does not have a circle of radius one spinning around with a team of elves that measure x coordinates.)

- **Part III: Coming Attractions.** By the end of Calculus II, the student is ready for a *lot* of other classes. The end of Calculus II thus ends with a sampler platter of topics that show the vast knowledge base built upon the foundation laid in Calculus II. Here the text takes a bite out of **Differential Equations**, serves some polar and parametric coordinates as a palate cleanser before **Calculus III**, and tastes some **Complex Analysis** to aid in digestion of Differential Equations. For dessert, it serves a scoop of **Probability** with both discrete and continuous colored sprinkles.

How to Use This Book

This book is meant to facilitate *Active Learning* for students, instructors, and learning assistants. Active Learning is the process by which the student participates directly in the learning process by reading, writing, and interacting with peers. This contrasts the traditional model where the student passively listens to lecture while taking notes. This book is designed as a self-guided step-by-step exploration of the concepts. The text incorporates theory and examples together in order to lead the student to discovering new results while still being able to relate back to familiar topics of mathematics. Much of this text can be done independently by the student for class preparation. During class sessions, the instructor and/or learning assistants may find it advantageous to encourage group work while being available to assist students, and work with students on a one-on-one or small group basis. This is highly desirable as extensive research has shown that active learning improves student success and retention. (For example, see `www.pnas.org/content/111/23/8410` for Scott Freeman's metaanalysis of 225 studies supporting this claim.)

What is Different about this Book

If you leaf through the text, you'll quickly notice two major structural differences from many traditional calculus books:

1. The exercises are very intermingled with the readings. Gone is the traditional separation into "section" versus "exercises".
2. Whitespace was included for the student to write and work through exercises. Parts of pages have indeed been intentionally left blank.

A consequence of this structure is that the readings and exercise are closely linked. It is intended for the student to do the readings and exercises concurrently.

Ok... Why?

The goal of this structure is to help the student simulate the process by which a mathematician reads mathematics. When a mathematician reads a paper or book, he/she always has a pen in hand and is constantly working out little examples alongside and scribbling incomprehensible notes in bad handwriting. It takes a *long* time and a lot of experience to know how to come up with good questions to ask oneself or to know what examples to work out in order to to help oneself absorb the subject. Hence, the exercises sprinkled throughout the readings are meant to mimic the margin scribbles or side work a mathematician engages in during the act of reading mathematics.

The Legend of Coffee

A potential hazard of this self-guided approach is that while most examples are meant to be simple exercises to help with absorption of the topics, there are some examples that students may find quite difficult. To prevent students from spinning their wheels in frustration, we have labeled the difficulty of all exercises using coffee cups as follows:

Coffee Cup Legend		
Symbol	Number of Cups	Description of Difficulty
☕	A One-Cup Problem	Easy warm-up suitable for class prep.
☕☕	A Two-Cup Problem	Slightly harder, solid groupwork exercise.
☕☕☕	A Three-Cup Problem	Substantial problem requiring significant effort.
☕☕☕☕	A Four-Cup Problem	Difficult problem requiring effort and creativity!

Glossary of Symbols

In Precalculus and Calculus I, there is a wide range of how much notation from Set Theory gets used. To get everyone on the same page, here is a short list of some notation we will use in this text.

Sets and Elements

Often in mathematics, we construct collections of objects called *sets*.

- If an object x is in a set A, we say x is an *element* of A and write $x \in A$.
- If an object x is not in a set A, we say x is not an element of A and write $x \notin A$.

Any particular object is either an element of a set or it is not. We do not allow for an object to be partially contained in a set, nor do we allow for an object to appear multiple times in a set. Often we use curly braces around a comma-separated list to indicate what the elements are.

Example 1.0.0.1. A Prime Example

Suppose P is the set of all prime numbers. We write

$$P = \{2, 3, 5, 7, 11, 13, 17, \ldots\}$$

For example, $2 \in P$ and $65,537 \in P$, but $4 \notin P$.

Some Famous Sets of Numbers

The following are fundamental sets of numbers used in Calculus 2.

- **Natural Numbers:** The set $\mathbb{N}$ of natural numbers is the set of all positive whole numbers, along with zero. That is,

  $$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \ldots\}$$

  Note that in many other sources, zero is not included in the natural numbers. Both are widely used; be aware the choice on this convention will change throughout your mathematical travels!

- **Integers:** The set of integers $\mathbb{Z}$ is the set of all whole numbers, whether they are positive, negative, or zero. That is,

  $$\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

- **Rational Numbers:** The set of rational numbers $\mathbb{Q}$ is the set of all numbers expressible as a fraction whose numerator and denominator are both integers.

- **Real Numbers:** The set of real numbers $\mathbb{R}$ is the set of all numbers expressible as a decimal.

- **Complex Numbers:** The set $\mathbb{C}$ of complex numbers is the set of all numbers formed as a real number (called the real part) plus a real number times i (called the imaginary part), where i is a symbol such that $i^2 = -1$.

Set-Builder Notation

The most common notation used to construct sets is *set-builder notation*, in which one specifies a name for the elements being considered and then some property $P(x)$ that is the membership test for an object x to be an element of the set. Specifically,

$$A = \{x \in B : S(x)\}$$

means that an object x chosen from B is an element of the set A if and only if the claim $S(x)$ is true about x. Sometimes the "$\in B''$ gets dropped if it is clear from context what set the elements are being chosen from. The set-builder notation above gets read as "the set of all x in B such that $S(x)$". One can think of this as running through all elements of B and throwing away any that do not meet the condition described by S.

Example 1.0.0.2. Interval Notation

Interval notation can be expressed in set-builder notation as follows:

- $(a, b) = \{x \in \mathbb{R} : a < x < b\}$
- $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$
- $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$
- $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$

Example 1.0.0.3. Rational, Real, and Complex in Set-Builder Notation ☕☕

Set-builder notation is often used to express the sets of rational, real, and complex numbers as follows:

- $\mathbb{Q} = \left\{\frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0\right\}$
- $\mathbb{R} = \{0.a_0a_1a_2a_3a_4 \ldots \times 10^n : n \in \mathbb{N}, a_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ where $i \in \mathbb{N}\}$ Note this is essentially scientific notation; the concatenation of the a_i's represents the digits in a base-ten decimal expansion.
- $\mathbb{C} = \{a + bi : a \in \mathbb{R}, b \in \mathbb{R}\}$

Part I

Integration

Chapter 2

How to Find an Antiderivative

The Fundamental Theorem of Calculus says that an integral (defined as the area under a curve) can be easily evaluated via antiderivative. However, it turns out to be very difficult and sometimes impossible to find an antiderivative! In this chapter, we give several commonly used methods for antidifferentiation.

Exercise 2.0.0.1. What is an Antiderivative Again? ☕

- Complete the definition of antiderivative. That is, if $f(x)$ is a function, then we say $F(x)$ is an antiderivative of $f(x)$ if and only if...
- How do you use antiderivatives to evaluate definite integrals? Describe in a short sentence below.
- Once you found an antiderivative, what could you do to check that it is correct? (Besides just computing it again!)

2.1 The Method of u-Substitution

2.1.1 Undoing the Chain Rule

The technique of u-substitution (affectionately known as "u-sub" from here on) can be seen as the reverse of the chain rule for antiderivatives.

Exercise 2.1.1.1. What Was the Chain Rule Again? ☕

- First, write down the chain rule.

$$(f(g(x)))' =$$

- Take the antiderivative of both sides of that equation.

$$\int \qquad\qquad\qquad \mathtt{d}x = f\left(g(x)\right) + C$$

In practice, we often make the substitution $u = g(x)$ to condense the notation. This will take a nastier integral with respect to x and replace it by a hopefully friendlier integral with respect to u. This process of transforming from x to u involves the following three steps:

1. **Choose u:** Pick u to be equal to some expression involving x. Frequently, it is helpful to pick u to be some "inner function" in a composition of functions that appears in the integrand. However, there is a *lot* of freedom regarding what substitution you make. Some choices of u will be helpful, and others will not be! It is important to be brave and just try some.
2. **Differentiate u:** Once you have a formula for u, differentiate with respect to x to get a formula for $\frac{\mathtt{d}u}{\mathtt{d}x}$. This will tell us what the conversion factor is between x units and u units.
3. **Solve for $\mathtt{d}x$:** Use your derivative to solve for $\mathtt{d}x$. Substitute that expression for the $\mathtt{d}x$ in the integral to replace it with $\mathtt{d}u$.

For the sake of having this process in a nice little formula box, here is the above paragraph rewritten concisely and precisely.

u-Substitution

$$\int f'\left(g(x)\right) \mathtt{d}x = \int f'(u)\,\mathtt{d}u = f(u) + C = f\left(g(x)\right) + C$$

Example 2.1.1.2. An Example of Integration via u-sub

To evaluate $\int x\cos\left(x^2\right)\mathtt{d}x$, we identify $u = x^2$ as a plausible choice based on our recollection of chain rule. This gives the following change of variables:

Three Steps of u-Substitution

Choice of u	Differentiate u	Solve for $\mathtt{d}x$
$u = x^2$	$\frac{\mathtt{d}u}{\mathtt{d}x} = 2x$	$\mathtt{d}x = \frac{1}{2x}\,\mathtt{d}u$

We now replace x^2 by u and replace $\mathtt{d}x$ by $\frac{1}{2x}\,\mathtt{d}u$ in our integral.

$$\int x\cos\left(x^2\right)\mathtt{d}x = \int x\cdot\cos(u)\frac{1}{2x}\,\mathtt{d}u = \frac{1}{2}\int \cos(u)\,\mathtt{d}u = \frac{1}{2}\sin(u) + C = \frac{1}{2}\sin\left(x^2\right) + C$$

Exercise 2.1.1.3. Checking Our Work ☕

As a follow up to the previous example, differentiate the answer to verify that you end up with the original integrand!

$$\frac{\mathtt{d}}{\mathtt{d}x}\left(\frac{1}{2}\sin\left(x^2\right)+C\right)=$$

Example 2.1.1.4. A Trickier u-sub

Suppose we wish to evaluate the following integral:

$$\int \frac{\sqrt[3]{x}}{\sqrt[3]{x}+1}\,\mathtt{d}x$$

One possible approach is to let u be the denominator. The denominator can be thought of as the "inner function" inside a reciprocal function and thus often makes a good choice for u.

Three Steps of u-Substitution		
Choice of u	Differentiate u	Solve for $\mathtt{d}x$
$u=\sqrt[3]{x}+1$	$\frac{\mathtt{d}u}{\mathtt{d}x}=\frac{1}{3}x^{-2/3}$	$\mathtt{d}x=3x^{2/3}\,\mathtt{d}u$

We now perform the substitutions on the denominator and the $\mathtt{d}x$.

$$\int \frac{\sqrt[3]{x}}{\sqrt[3]{x}+1}\,\mathtt{d}x=\int\frac{\sqrt[3]{x}}{u}3x^{2/3}\,\mathtt{d}u=3\int\frac{x}{u}\,\mathtt{d}u$$

At the moment, it seems like things are going very poorly! We hoped that x in the numerator would nicely cancel out, like it did back in the more civilized age of Exercise 2.1.1.2. To fix this, we solve for x in the equation $u=\sqrt[3]{x}+1$ to obtain $x=(u-1)^3$. We now substitute that expression for x in the integral.

$$\begin{aligned}
3\int\frac{x}{u}\,\mathtt{d}u &= 3\int\frac{(u-1)^3}{u}\,\mathtt{d}u\\
&= 3\int\frac{u^3-3u^2+3u-1}{u}\,\mathtt{d}u\\
&= 3\int u^2-3u+3-\frac{1}{u}\,\mathtt{d}u\\
&= u^3-\frac{9}{2}u^2+9u-3\ln|u|+C\\
&= \left(\sqrt[3]{x}+1\right)^3-\frac{9}{2}\left(\sqrt[3]{x}+1\right)^2+9\left(\sqrt[3]{x}+1\right)-3\ln\left|\sqrt[3]{x}+1\right|+C\\
&= x-\frac{3}{2}\sqrt[3]{x}^2+3\sqrt[3]{x}-3\ln\left|\sqrt[3]{x}+1\right|+C
\end{aligned}$$

Exercise 2.1.1.5. Missing Constants ☕

In the above example, all of the constant terms disappeared on the final step! Was that ok?

Exercise 2.1.1.6. Practice with u-sub ☕☕

- Evaluate $\int \frac{6x+3}{x^2+x+8}\,\mathtt{d}x$.

- Evaluate $\int \frac{(\ln(x))^2}{x}\,\mathtt{d}x$.

- Evaluate $\int xe^{-x^2}\,\mathtt{d}x$.

- Consider the integral

$$\int e^{\left(x^2\right)}\,\mathtt{d}x$$

Explain in words why the substitution $u = x^2$ will not work in this case. Where do you get stuck?

Exercise 2.1.1.7. Create an Integral! ☕☕☕

Come up with your own integral that can be evaluated by u-sub. Find a partner and trade! See if you can evaluate each other's integrals with u-sub, or explain to your partner why theirs cannot be evaluated using u-sub.

2.1.2 Change of Coordinates and u-Substitution

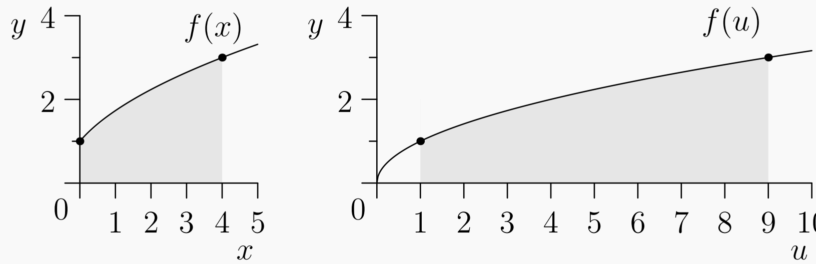
The method of u-substitution is actually a special case of a more general notion, *change of coordinates*. This will be studied more thoroughly and in more generality in Calculus 3.

Example 2.1.2.1. How u-sub Affects Area

Suppose we wish to compute the following integral:

$$\int_{x=0}^{x=4} \sqrt{2x+1}\,\mathtt{d}x$$

We interpret this as the area under the curve $f(x) = \sqrt{2x+1}$ from $x = 0$ to $x = 4$ as drawn below. We apply the substitution $u = 2x+1$. This transforms a region in the xy-plane to a region in the uy-plane.



Notice the heights are exactly the same, but the widths have changed by a factor of 2. Thus to equate the x integral to the u integral, we need to divide the u integral by 2 in order to fix the fact that we doubled all the widths! In particular,

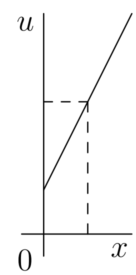
$$\int_{x=0}^{x=4} \sqrt{2x+1}\,\mathrm{d}x = \frac{1}{2}\int_{u=1}^{u=9} \sqrt{u}\,\mathrm{d}u.$$

Note that this geometric interpretation corresponds perfectly to what happens in the Riemann sum definition of an integral. For a fixed number of rectangles, the Δx in a Riemann sum of the area under $\sqrt{2x+1}$ would be exactly half of the Δu in a corresponding Riemann sum of the area under $\sqrt{u}$.

Exercise 2.1.2.2. Relationship between Δx and Δu

To illustrate the final claim of the example, draw an evenly-spaced four-rectangle Riemann Sum in each of the two shaded regions above.

- What is your value of Δx?
- What is your value of Δu?
- Explain where the u bounds of 1 and 9 come from.
- In the conversion $u = 2x+1$, the multiplication by 2 had all the effects described above. But, why did the "plus one" not affect area? (**Hint:** What would the u bounds have been, and what would the region have looked like, if the "plus one" was not there?)
- In the figure at right, what aspect of that graph corresponds to the scaling factor between x and u?



Graph of $u = 2x + 1$, constant derivative 2.

The moral to the story is that the scaling factor between Δx and Δu is in fact the derivative of u with respect to x. In order to translate from x to u in our integral, we must divide by $\frac{du}{dx}$. In this case, that was just the constant 2. What is fascinating is that this approach of dividing off by the scaling factor still works, even when the scaling factor is not just a constant. Hence if we use $u = x^2$, we must divide by $\frac{\mathtt{d}u}{\mathtt{d}x} = 2x$, as we do in many of the above examples. It is still a scaling factor, but one that changes depending on the input!

Exercise 2.1.2.3. Looking Graphically at a u-sub ☕☕

We now look at a case where the scaling factor depends on x.

- Plot a graph of the function $f(x) = x\sin(x^2)$ on the domain $[0, 3]$. (You may use a graphing utility to assist you.)

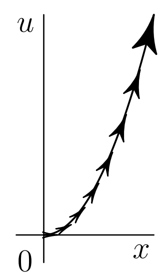
- Plot a graph of the function $h(u) = \sqrt{u}\sin(u)$ on the domain $[0, 9]$. (You may use a graphing utility to assist you.)

- Can you see the horizontal scaling factor change at different points of the graph? How stretched out does it seem to be at $x = 1$? How stretched out does it seem to be at $x = 3$?

- Show algebraically that $u = x^2$ is the substitution that turns that function $f(x)$ into the corresponding function $h(u)$.

- Evaluate the integral $\int_{x=0}^{x=3} x\sin(x^2)\,\mathtt{d}x$.

- In the figure at right, what aspect of that graph corresponds to the scaling factor between x and u?



Graph of $u = x^2$, changing derivative of $2x$.

2.1.3 Antiderivatives of the Six Trig Functions

In Calculus I, we found the derivatives of all six trig functions. List those below:

Exercise 2.1.3.1. Recalling the Derivatives of the Six Trig Functions ☕

Write the derivative of each trig function:

- $\frac{\mathtt{d}}{\mathtt{d}x}\left(\sin(x)\right) =$
- $\frac{\mathtt{d}}{\mathtt{d}x}\left(\cos(x)\right) =$
- $\frac{\mathtt{d}}{\mathtt{d}x}\left(\tan(x)\right) =$
- $\frac{\mathtt{d}}{\mathtt{d}x}\left(\cot(x)\right) =$
- $\frac{\mathtt{d}}{\mathtt{d}x}\left(\sec(x)\right) =$
- $\frac{\mathtt{d}}{\mathtt{d}x}\left(\csc(x)\right) =$

From these, we easily obtain the antiderivatives of sine and cosine.

Exercise 2.1.3.2. Integrals of Sine and Cosine ☕

Use the derivatives above to compute the following antiderivatives.

- $\int \sin(x)\,\mathtt{d}x =$
- $\int \cos(x)\,\mathtt{d}x =$

For tangent and cotangent, we need u-sub.

Example 2.1.3.3. Antiderivative of Tangent

We compute the antiderivative of tangent by rewriting as $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and then using the substitution $u = \cos(x)$. Differentiating both sides produces $\mathtt{d}x = \frac{\mathtt{d}u}{-\sin(x)}$. We now apply these substitutions:

$$\begin{aligned}
\int \tan(x)\,\mathtt{d}x &= \int \frac{\sin(x)}{\cos(x)}\,\mathtt{d}x \\
&= \int \frac{\sin(x)}{u}\frac{\mathtt{d}u}{-\sin(x)} \\
&= -\int \frac{1}{u}\,\mathtt{d}u \\
&= -\ln|u| + C \\
&= -\ln|\cos(x)| + C
\end{aligned}$$

The method used to antidifferentiate tangent can be adapted to also antidifferntiate cotangent.

Exercise 2.1.3.4. Integral of Cotangent ☕☕

Find the antiderivative of cotangent.

$$\int \cot(x)\,\mathtt{d}x =$$

The antiderivative of secant is much trickier! The process is not intuitive and requires a rabbit out of a hat.

Example 2.1.3.5. Integral of Secant

Since multiplication by 1 does not change the integrand, we are free to multiply by 1 whenever it is helpful. Here, it turns out to be helpful to multiply by $\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}$. This is the rabbit.

$$\begin{aligned}
\int \sec(x)\,\mathtt{d}x &= \int \sec(x)\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\,\mathtt{d}x \\
&= \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\,\mathtt{d}x \\
&= \int \frac{\sec^2(x)+\sec(x)\tan(x)}{u}\,\frac{1}{\sec(x)\tan(x)+\sec^2(x)}\,\mathtt{d}u \\
&= \int \frac{1}{u}\,\mathtt{d}u \\
&= \ln|u| + C \\
&= \ln|\sec(x)+\tan(x)| + C
\end{aligned}$$

The above method can be adapted to antidifferentiate cosecant.

Exercise 2.1.3.6. Integral of Cosecant ☕☕

Find the antiderivative of cosecant.

$$\int \csc(x)\,\mathtt{d}x =$$

2.2 Integration by Parts

Integration by parts (IBP) is the Product Rule spun around backwards to become a rule for antiderivatives rather than derivatives.

Exercise 2.2.0.1. Reversing the Product Rule ☕☕

Fill in the blanks in the following construction of integration by parts:

- Recall the Product Rule for derivatives.

$$(f(x)g(x))' =$$

- Take an antiderivative of both sides.

$$= \int (f'(x)g(x))\,\mathtt{d}x + \int (f(x)g'(x))\,\mathtt{d}x$$

- Rewrite the equation by subtracting the term $\int (f'(x)g(x))\,\mathtt{d}x$ from both sides.

$$=$$

- To condense the notation, it is customary to make the substitutions $u = f(x)$ and $v = g(x)$. Thus, we say $\frac{\mathtt{d}u}{\mathtt{d}x} = f'(x)$ and similarly $\frac{\mathtt{d}v}{\mathtt{d}x} = g'(x)$. Multiply the $\mathtt{d}x$ to the right-hand side in both of those equations, we obtain

$$\mathtt{d}u =$$

and

$$\mathtt{d}v =$$

- Use these substitutions to replace all instances of x, f, and g by u and v and conclude the IBP formula.

Just for sake of having it in its own box, here it is again!

Integration by Parts Formula

$$\int u\,\mathtt{d}v = uv - \int v\,\mathtt{d}u$$

We typically use this to integrate a product of functions in the case that u-substitution does not work. You can identify one factor of your integrand as u, the remaining factor as $\mathtt{d}v$, and plug into the IBP formula. There are three main types:

1. A product with one factor that becomes much simpler upon differentiation
2. A not-quite-a product that we turn into a product
3. An integrand that reappears after applying IBP

We illustrate each of these methods with an example.

2.2.1 A Product with One Factor That Becomes Much Simpler Upon Differentiation

We let u be whichever factor becomes simpler when it is differentiated. The other factor by default must then be set equal to $\mathtt{d}v$.

Example 2.2.1.1. Integrating a Product

Suppose we wish to find an antiderivative for the function $x \cdot \cos(x)$. We can either choose $u = x$ or $u = \cos(x)$. Since $u = x$ has lovely little constant function 1 as its derivative, whereas $u = \cos(x)$ would produce just another trig function as its derivative, we conclude $u = x$ is the better choice.

Choice of u and $\mathtt{d}v$	
$u = x$	$v = \sin(x)$
$\mathtt{d}u = \mathtt{d}x$	$\mathtt{d}v = \cos(x)\,\mathtt{d}x$

We are now ready to calculate the antiderivative via IBP:

$$\int x \cdot \cos(x)\,\mathtt{d}x = \int u\,\mathtt{d}v = uv - \int v\,\mathtt{d}u = x \cdot \sin(x) - \int \sin(x)\,\mathtt{d}x = x \cdot sin(x) + \cos(x) + C$$

Exercise 2.2.1.2. Checking Our Work ☕

Take the derivative of our result, $x\sin(x) + \cos(x) + C$, to verify that it is in fact the correct antiderivative!

Exercise 2.2.1.3. An Integral via both u-sub and IBP ☕☕☕

Consider the integral

$$\int x\sqrt{x+1}\,\mathtt{d}x$$

- Evaluate the integral using the u-sub $u = x + 1$.

- Evaluate the integral using IBP, choosing $u = x$ and $\mathtt{d}v = \sqrt{x+1}\,\mathtt{d}x$.

- Your answers will appear very different! Is one incorrect? Or are they compatible?

2.2.2 A Not-Quite-a Product That We Turn into a Product

Often, an integrand that does not appear to be a product can be rewritten as product in a helpful way. This often includes **rewriting the integrand as the integrand times one**. We let u be the entire integrand, leaving $\mathtt{d}v$ to just be the invisible 1 times $\mathtt{d}x$.

Example 2.2.2.1. Multiplying by 1 in an IBP

Suppose we wish to find an antiderivative for the function $\arccos(x)$. We identify $u = \arccos(x)$ which leaves $\mathtt{d}v = 1 \cdot \mathtt{d}x$. Thus we make the following declarations:

Choice of u and $\mathtt{d}v$	
$u = \arccos(x)$	$v = x$
$\mathtt{d}u = -\frac{1}{\sqrt{1-x^2}}\,\mathtt{d}x$	$\mathtt{d}v = 1 \cdot \mathtt{d}x$

We are now ready to calculate the antiderivative via IBP:

$$\int \arccos(x) \cdot 1 \cdot \mathtt{d}x = \int u\,\mathtt{d}v = uv - \int v\,\mathtt{d}u = x \cdot \arccos(x) - \int x \left(-\frac{1}{\sqrt{1-x^2}} \right) \mathtt{d}x$$

$$= x\arccos(x) + \int \frac{x}{\sqrt{1-x^2}} \mathtt{dx} = x\arccos(x) - \sqrt{1-x^2} + C$$

Exercise 2.2.2.2. Filling in the Details ☕☕

Notice that the very last step of the above example was in fact a u-substitution! Show the details of how that antiderivative was carried out.

$$\int \frac{x}{\sqrt{1-x^2}}\,\mathtt{d}x =$$

Exercise 2.2.2.3. The Antiderivative of the Natural Logarithm ☕☕

Find an antiderivative for the function $\ln(x)$.

2.2.3 An Integrand that Reappears After Applying IBP

Sometimes, we can get the original expression to come back after applying integration by parts one or more times. Once this occurs, you can **give some name to the integral (we will use I)** and solve for it as you would solve any equation in algebra!

Example 2.2.3.1. An Integrand that Reappears After IBP

Suppose we wish to find an antiderivative for the function $e^{2x}\cos(x)$. Call I the desired antiderivative. That is:

$$I = \int e^{2x}\cos(x)\,\mathtt{d}x$$

We now wish to apply IBP, so we make the following declarations:

Choice of u and $\mathtt{d}v$	
$u = e^{2x}$	$v = \sin(x)$
$\mathtt{d}u = 2e^{2x}\,\mathtt{d}x$	$\mathtt{d}v = \cos(x)\,\mathtt{d}x$

We are now ready to calculate the antiderivative via IBP:

$$\begin{aligned} I &= \int u\,\mathtt{d}v \\ &= uv - \int v\,\mathtt{d}u \\ &= e^{2x}\sin(x) - \int \sin(x)\cdot 2e^{2x}\,\mathtt{d}x \\ &= e^{2x}\sin(x) - 2\int e^{2x}\sin(x)\,\mathtt{d}x \end{aligned}$$

We notice now that the new integral is again a product of functions (and does not appear to be doable via u-sub) so we apply IBP once again with the following declarations (using new u and v):

Choice of u and $\mathtt{d}v$	
$u = e^{2x}$	$v = -\cos(x)$
$\mathtt{d}u = 2e^{2x}\,\mathtt{d}x$	$\mathtt{d}v = \sin(x)\,\mathtt{d}x$

We now proceed with the previous expression, using the new IBP setup and notice that the original integral I reappears:

$$
\begin{aligned}
I &= e^{2x}\sin(x) - 2\int u\,\mathtt{d}v \\
&= e^{2x}\sin(x) - 2\left(uv - \int v\,\mathtt{d}u\right) && (\bowtie) \\
&= e^{2x}\sin(x) - 2\left(e^{2x}(-\cos(x)) - \int(-\cos(x))2e^{2x}\,\mathtt{d}x\right) && (\bowtie) \\
&= e^{2x}\sin(x) + 2e^{2x}\cos(x) - 4\int e^{2x}\cos(x)\,\mathtt{d}x \\
&= e^{2x}\sin(x) + 2e^{2x}\cos(x) - 4I
\end{aligned}
$$

At first glimpse this seems troubling; we have reduced the problem we are trying to solve to solving the exact same problem that we are trying to solve! Yet upon further inspection, it becomes clear that this is in fact an equation involving I, and thus we can solve for it! Proceeding:

$$
\begin{aligned}
I &= e^{2x}\sin(x) + 2e^{2x}\cos(x) - 4I \\
5I &= e^{2x}\sin(x) + 2e^{2x}\cos(x) \\
I &= \frac{e^{2x}\sin(x) + 2e^{2x}\cos(x)}{5}
\end{aligned}
$$

and we are done, concluding that

$$\int e^{2x}\cos(x)\,\mathtt{d}x = \frac{e^{2x}\sin(x) + 2e^{2x}\cos(x)}{5} + C$$

Exercise 2.2.3.2. Carefulness ☕

In the example above, there are two lines labeled with bowties ($\bowtie$). Explain briefly in a sentence or two why those giant parentheses are present. What would go wrong if those parentheses were not there?

Example 2.2.3.3. Another Reappearing IBP Integral

Suppose we wish to find an antiderivative for the function $\tan(x)\sec^2(x)$. We identify $\mathtt{d}v = \sec^2(x)\,\mathtt{d}x$ as having a nice clean antiderivative, which leaves $u = \tan(x)$ by default. Thus we make the following declarations:

Choice of u and $\mathtt{d}v$	
$u = \tan(x)$	$v = \tan(x)$
$\mathtt{d}u = \sec^2(x)\,\mathtt{d}x$	$\mathtt{d}v = \sec^2(x)\,\mathtt{d}x$

We are now ready to calculate the antiderivative via IBP:

$$\int \tan(x)\sec^2(x)\,\mathtt{d}x = \tan(x)\tan(x) - \int \tan(x)\sec^2(x)\,\mathtt{d}x$$

We notice that the original integral has reappeared! We give it the name I and solve. The equation becomes $I = \tan^2(x) - I$, which implies that $2I = \tan^2(x)$. Dividing by two produces the following result:

$$\int \tan(x)\sec^2(x)\,\mathtt{d}x = \frac{1}{2}\tan^2(x) + C$$

Exercise 2.2.3.4. Alternate Solutions ☕☕

Find the antiderivative of $\tan(x)\sec^2(x)$ yet again but by two different methods! In particular, try...

- ...a u-sub with $u = \tan(x)$.

- ...an IBP with $u = \sec(x)$ and $\mathtt{d}v = \tan(x)\sec(x)\,\mathtt{d}x$.

Exercise 2.2.3.5. A Tricky but Important One: Secant Cubed ☕☕☕

Find an antiderivative for the function $\sec^3(x)$. (**Hint:** Split the cube as $\sec^3(x) = \sec^2(x)\sec(x)$. Also, the Pythagorean Identity $\tan^2(x) = \sec^2(x) - 1$ will be useful.)

2.3 Mixed Practice

Sometimes it is not obvious which technique to use in solving a particular problem. One must often use more than one technique of integration in combination. Try the following:

Exercise 2.3.0.1. Practice on u-sub and/or IBP ☕☕☕

- Find an antiderivative for the function $\cos(\sqrt{x})$.

- Evaluate $\int e^{\sqrt{2x}}\,\mathtt{d}x$.

- Evaluate $\int \arcsin(5x)\,\mathtt{d}x$.

- Evaluate $\int e^{2x} \sin(2x)\, \mathtt{d}x$.

Exercise 2.3.0.2. Who is u vs Who is $\mathtt{d}v$?

Suppose we wish to find an antiderivative for the function $x^{2.5}\ln(x)$. There are two natural choices for u. We can let $u = x^{2.5}$ and $\mathtt{d}v = \ln(x)\,\mathtt{d}x$, or we can let can let $u = \ln(x)$ and $\mathtt{d}v = x^{2.5}\,\mathtt{d}x$.

- Apply just the first step of IBP with $u = x^{2.5}$ and $\mathtt{d}v = \ln(x)\,\mathtt{d}x$.

$$\int x^{2.5}\ln(x)\,\mathtt{d}x =$$

- Apply just the first step of IBP with $u = \ln(x)$ and $\mathtt{d}v = x^{2.5}\,\mathtt{d}x$.

$$\int x^{2.5}\ln(x)\,\mathtt{d}x =$$

- Write a short explanation regarding which choice of u will be easier to use to evaluate the integral and why.

- Carry out the integral using whichever choice you decided was easier.

$$\int x^{2.5}\ln(x)\,\mathtt{d}x =$$

- Differentiate your answer to check that your antiderivative is correct.

2.4 Integrating Products of Powers of Sine and Cosine

In this section, we give an algorithm to find an antiderivative of the form

$$\int \sin^n(x)\cos^m(x)\,\mathtt{d}x$$

for $n, m \in \mathbb{N}$.

Exercise 2.4.0.1. Knowledge is Power ☕

There are two exponents in the integrand above.

- What symbol above is the exponent of sine?
- What symbol above is the exponent of cosine?

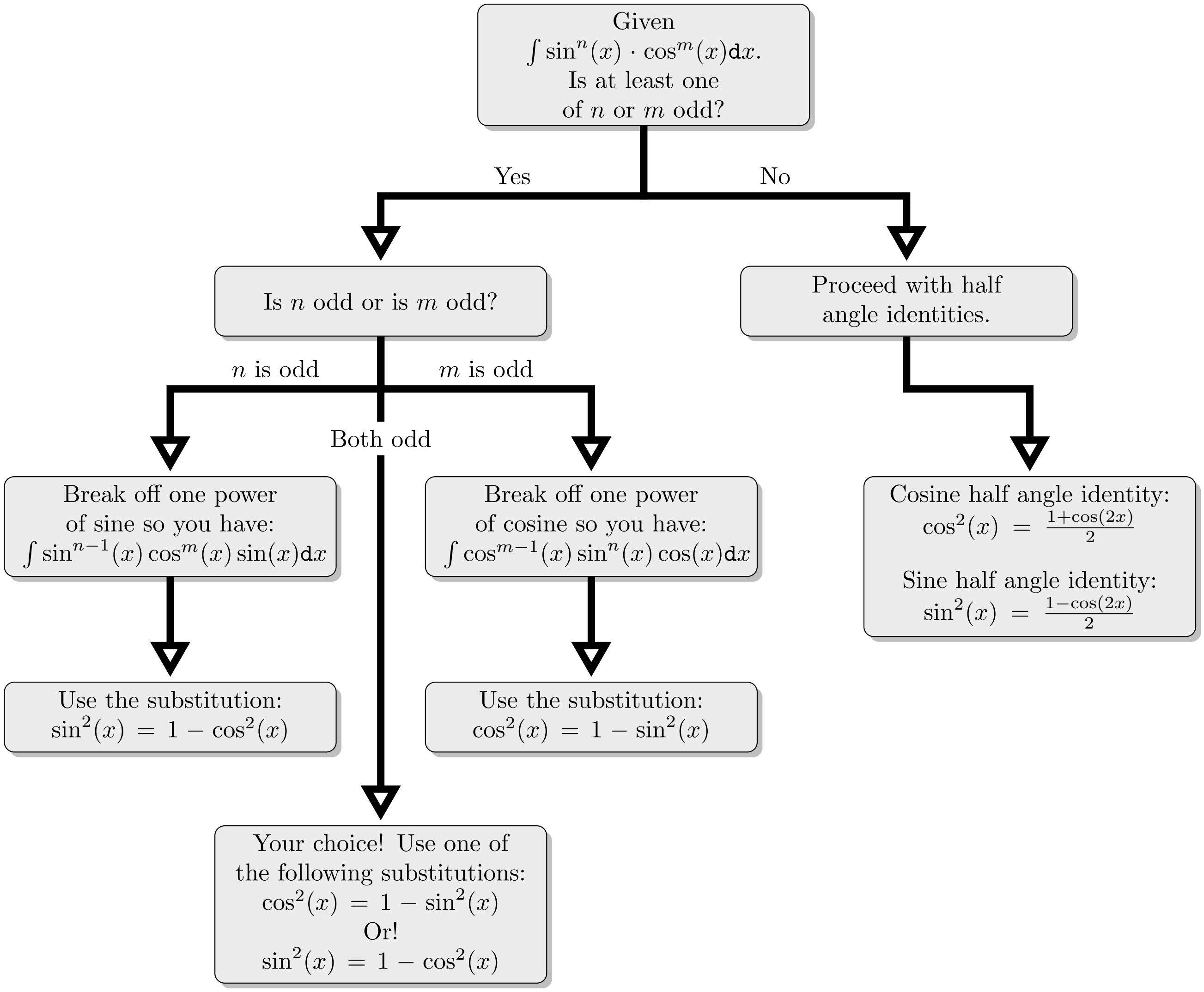
Note that some sine-cosine integrals can be done by techniques you have already learned. For example, n or m is equal to 1, ordinary u-substitution will work just fine!

Exercise 2.4.0.2. u-sub with Sines and Cosines ☕☕

Evaluate the following integral using the substitution $u = \sin(x)$:

$$\int \sin^2(x)\cos(x)\,\mathtt{d}x$$

There are two types of integrals containing powers of sine and cosine. The first type is the case where we have at least one odd exponent; the second type is where both exponents are even. We show an overview of how to handle each case in the following awesome flow chart:



2.4.1 At Least One Odd Power

Recall the Pythagorean identity for sine and cosine (written in two useful forms here):

Pythagorean Theorem Slightly Rewritten	
$\cos^2(x) = 1 - \sin^2(x)$	$\sin^2(x) = 1 - \cos^2(x)$

If at least one exponent is odd, we pull one of those functions out for the "$\mathtt{d}u$" and perform u-sub. We then use the Pythagorean trig identity to rewrite sine and cosine in terms of each other as needed.

Example 2.4.1.1. Odd Power Case

Here we compute the integral

$$\int \sin^7(x)\cos^2(x)\,\mathtt{d}x$$

. In this case, we proceed using the substitution $u = \cos(x)$, so $\mathtt{d}x = \frac{1}{-\sin(x)}\,\mathtt{d}u$.

$$\begin{aligned}
\int \sin^7(x)\cos^2(x)\,\mathtt{d}x &= \int \sin^6(x)\cos^2(x)\sin(x)\,\mathtt{d}x \\
&= \int \left(\sin^2(x)\right)^3 \cos^2(x)\sin(x)\frac{1}{-\sin(x)}\,\mathtt{d}u \\
&= \int \left(1-\cos^2(x)\right)^3 \cos^2(x)(-1)\,\mathtt{d}u \\
&= -\int \left(1-u^2\right)^3 u^2\,\mathtt{d}u \\
&= -\int \left(1-3u^2+3u^4-u^6\right) u^2\,\mathtt{d}u \\
&= -\int \left(u^2-3u^4+3u^6-u^8\right)\mathtt{d}u \\
&= -\left(\frac{1}{3}u^3-\frac{3}{5}u^5+\frac{3}{7}u^7-\frac{1}{9}u^9\right)+C \\
&= -\frac{1}{3}\cos^3(x)+\frac{3}{5}\cos^5(x)-\frac{3}{7}\cos^7(x)+\frac{1}{9}\cos^9(x)+C
\end{aligned}$$

Exercise 2.4.1.2. Why Odd Mattered ☕

In Example 2.4.1.1, the exponent of sine (in this case, the number 7) being odd really mattered. If that 7 were replaced by an even number instead, why would this approach have failed? Answer in a few short sentences below.

Exercise 2.4.1.3. Try a Few with Odd Exponents ☕☕

1. Find an antiderivative for the function $\sin^5(x)\cos^2(x)$.

2. Evaluate $\int \cos^9(x)\, \mathtt{d}x$. (**Hint:** Pascal's Triangle will be extremely helpful!)

Exercise 2.4.1.4. Two Different Options ☕☕

1. Consider $\int \cos(x)\sin^3(x)\, \mathtt{d}x$.

(a) Compute this integral using $u = \cos(x)$.

(b) Compute this integral using $u = \sin(x)$.

(c) Your two answers will appear very different! Show that they are in fact compatible.

2. Consider $\int \cos^3(x) \sin^{11}(x) \, \mathtt{d}x$.

   (a) Can you compute this integral using $u = \cos(x)$? Explain.

   (b) Can you compute this integral using $u = \sin(x)$? Explain.

(c) Which of the two above substitutions will be easier to use? Carry out the integration, using the easier of the two.

2.4.2 Both Even Powers

Recall the Half-Angle Identities!

Half-Angle Identities	
$\cos^2(x) = \frac{1+\cos(2x)}{2}$	$\sin^2(x) = \frac{1-\cos(2x)}{2}$

If the powers of sine and cosine are both even, we use the half-angle identities for both sine and cosine. This can get quite messy, but it works!

Exercise 2.4.2.1. Just Cosines without Sine ☕

Consider the following integral:

$$\int \cos^6(x)\,\mathtt{d}x$$

Here the exponent on cosine is the even number 6. What is the exponent of sine in that integrand? Is that an even number?

Example 2.4.2.2. Carrying Out Antidifferentiation with the Half-Angle Identities

We now show how the half-angle identities help antidifferentiate the sixth power of cosine.

$$\begin{aligned}\int \cos^6(x)\,\mathtt{d}x &= \int \left(\cos^2(x)\right)^3 \mathtt{d}x\\ &= \int \left(\frac{1+\cos(2x)}{2}\right)^3 \mathtt{d}x\\ &= \frac{1}{8}\int 1 + 3\cos(2x) + 3\cos^2(2x) + \cos^3(2x)\,\mathtt{d}x\\ &= \frac{1}{8}\left(\int 1\,\mathtt{d}x + \int 3\cos(2x)\,\mathtt{d}x + \int 3\cos^2(2x)\,\mathtt{d}x + \int \cos^3(2x)\,\mathtt{d}x\right)\end{aligned}$$

Notice that we now have four integrals. The first is easy, the second is a u-substitution, and the third is another even power of cosine (where we again use the half-angle identity). Finally, the fourth is an odd power of cosine, so we can use the technique from Section 2.4.1.

Exercise 2.4.2.3. Finishing the Example ☕☕

Carry out each of these processes to compute the four integrals:

- $\int 1\,\mathtt{d}x$

- $\int 3\cos(2x)\,\mathtt{d}x$

- $\int 3\cos^2(2x)\,\mathtt{d}x$

- $\int \cos^3(2x)\,\mathtt{d}x$

Add your antiderivatives together and combine like terms to produce your final answer for the integral! Oh and remember that one-eighth.

$$\int \cos^6(x)\, \mathtt{d}x =$$

Exercise 2.4.2.4. Checking the Previous Example ☕☕☕

Differentiate your answer and verify you get the original integrand back.

Exercise 2.4.2.5. Practice with the Even Case ☕☕

1. Find an antiderivative for the function $\sin^2(3x)$.

2. Find an antiderivative for the function $\sin^4(x)$.

3. Find an antiderivative for the function $\sin^2(x)\cos^2(x)$.

2.5 Trigonometric Substitution

Though in theory you could use any trigonometric function, the three commonly used trigonometric substitutions are sine, tangent, and secant. The substitutions are motivated by the Pythagorean Identities from trigonometry.

Exercise 2.5.0.1. Recalling the Pythagorean Identities ☕

- Start with the Pythagorean Identity for sine and cosine:

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

- Subtract $\sin^2(\theta)$ from both sides. Write the resulting equation below.

- Again, start with the Pythagorean Identity for sine and cosine. What would we have to divide both sides by in order to get the corresponding identity for tangent and secant (written below)?

$$1 + \tan^2(\theta) = \sec^2(\theta)$$

- How would you then obtain the identity below?

$$\sec^2(\theta) - 1 = \tan^2(\theta)$$

You can take any of the identities above and multiply both sides by a^2 (where a represents an arbitrary positive real constant) to produce a more general identity. This is what results in the commonly used trigonometric substitutions for integrals, summarized in the table below.

Trigonometric Substitutions

If you see...	...make the substitution...	...because...
$a^2 - x^2$	$x = a\sin(\theta)$	$a^2 - a^2\sin^2(\theta) = a^2\cos^2(\theta)$
$a^2 + x^2$	$x = a\tan(\theta)$	$a^2 + a^2\tan^2(\theta) = a^2\sec^2(\theta)$
$x^2 - a^2$	$x = a\sec(\theta)$	$a^2\sec^2(\theta) - a^2 = a^2\tan^2(\theta)$

Exercise 2.5.0.2. Why Only Three Cases? ☕

In the table above, we have cases for how to clean up expressions of the form $a^2 - x^2$, $a^2 + x^2$, and $x^2 - a^2$. Why is there not a fourth case for $x^2 + a^2$?

2.5.1 Sine Substitution

When we see an expression of the form $a^2 - x^2$ in the integrand, we think of the identity $1 - \sin^2(\theta) = \cos^2(\theta)$. This motivates the following substitution:

Sine Substitution

$$x = a \cdot \sin(\theta)$$

The next example will require use of the Double-Angle Identities for sine and cosine. We recall these before we dive in!

Exercise 2.5.1.1. Recalling the Double-Angle Formulas ☕

- The double-angle formula for sine is $\sin(2\theta) =$
- The double-angle formula for cosine is $\cos(2\theta) =$
- What do you get if you apply the sine double-angle identity to $\sin(4\theta)$? Specifically, think of $\sin(4\theta)$ as $\sin(2 \cdot 2\theta)$.

We now put our sine substitution to use to evaluate an antiderivative!

Example 2.5.1.2. Using a Sine Substitution

Suppose we wish to evaluate

$$\int \left(4 - x^2\right)^{3/2} \mathtt{d}x$$

We use the substitution suggested above, specifically

$$x = 2 \cdot \sin(\theta).$$

We then differentiate both sides to find the conversion between the differentials and then multiply both sides by $\mathtt{d}\theta$:

$$\frac{\mathtt{d}x}{\mathtt{d}\theta} = 2 \cdot \cos(\theta)$$

We now use the above equations to substitute for x and $\mathtt{d}x$ in the integral:

$$
\begin{aligned}
\int \left(4-x^2\right)^{3/2} \mathtt{d}x &= \int \left(4-(2\cdot\sin(\theta))^2\right)^{3/2} 2\cdot\cos(\theta)\mathtt{d}\theta \\
&= 2\int \left(4-4\cdot\sin^2(\theta)\right)^{3/2} \cos(\theta)\mathtt{d}\theta \\
&= 2\int \left(4\left(1-\sin^2(\theta)\right)\right)^{3/2} \cos(\theta)\mathtt{d}\theta \\
&= 2\int \left(4\left(\cos^2(\theta)\right)\right)^{3/2} \cos(\theta)\mathtt{d}\theta \\
&= 2\int (4)^{3/2}\left(\cos^2(\theta)\right)^{3/2} \cos(\theta)\mathtt{d}\theta \\
&= 16\int \cos^3(\theta)\cdot\cos(\theta)\mathtt{d}\theta \\
&= 16\int \cos^4(\theta)\mathtt{d}\theta
\end{aligned}
$$

Recall the previous section where we learned how to antidifferentiate even powers of sine and cosine! Accordingly, we use the half-angle identities.

$$
\begin{aligned}
\int \left(4-x^2\right)^{3/2} \mathtt{d}x &= 16\int \left(\cos^2(\theta)\right)^2 \mathtt{d}\theta \\
&= 16\int \left(\frac{1+\cos(2\theta)}{2}\right)^2 \mathtt{d}\theta \\
&= 16\int \frac{1+2\cdot\cos(2\theta)+\cos^2(2\theta)}{4}\mathtt{d}\theta \\
&= 4\int 1+2\cdot\cos(2\theta)+\cos^2(2\theta)\mathtt{d}\theta \\
&= 4\int 1+2\cdot\cos(2\theta)+\frac{1+\cos(4\theta)}{2}\mathtt{d}\theta \\
&= 4\int \frac{3}{2}+2\cdot\cos(2\theta)+\frac{1}{2}\cos(4\theta)\mathtt{d}\theta \\
&= 4\left(\frac{3}{2}\theta+\sin(2\theta)+\frac{1}{8}\sin(4\theta)\right)+C \\
&= 6\theta+4\cdot\sin(2\theta)+\frac{1}{2}\sin(4\theta)+C
\end{aligned}
$$

We have successfully taken the antiderivative! However, it still remains to unwind the trigonometric substitution back in terms of x rather than θ. Our original substitution argument is θ, whereas currently we have 2θ and 4θ as arguments. In order to resolve this, we use the sine and cosine double angle formulas and the Pythagorean identity. Proceeding:

$$\begin{aligned}
\int \left(4-x^2\right)^{3/2} \,\mathtt{d}x &= 6\theta + 4\cdot\sin(2\theta) + \frac{1}{2}\sin(4\theta) + C \\
&= 6\theta + 4\cdot 2\cdot\sin(\theta)\cos(\theta) + \sin(2\theta)\cos(2\theta) + C \\
&= 6\theta + 4\cdot 2\cdot\sin(\theta)\cos(\theta) + 2\cdot\sin(\theta)\cos(\theta)\left(\cos^2(\theta) - \sin^2(\theta)\right) + C \\
&= 6\theta + 8\cdot\sin(\theta)\sqrt{1-\sin^2(\theta)} + 2\cdot\sin(\theta)\sqrt{1-\sin^2(\theta)}\left(1-2\sin^2(\theta)\right) + C \\
&= 6\cdot\arcsin\left(\frac{x}{2}\right) + 8\frac{x}{2}\sqrt{1-\frac{x^2}{4}} + 2\frac{x}{2}\sqrt{1-\frac{x^2}{4}}\left(1-2\frac{x^2}{4}\right) + C \\
&= 6\cdot\arcsin\left(\frac{x}{2}\right) + 4x\sqrt{1-\frac{x^2}{4}} + \sqrt{1-\frac{x^2}{4}}\left(x-\frac{x^3}{2}\right) + C
\end{aligned}$$

Exercise 2.5.1.3. Checking Our Work ☕☕☕

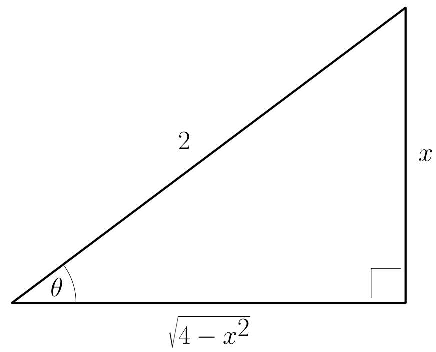
Verify the result of the previous example by differentiating!

An Alternate Approach

In the above example, we made it back from θ to x by just bashing it to bits with trig identities. Sometimes a cleaner approach can be to use a little geometry. Since we had the substitution $x = 2\sin(\theta)$, we can divide both sides by 2 to obtain the following:

$$\sin(\theta) = \frac{x}{2}$$

Since sine is the ratio of the opposite side to the hypotenuse in a right triangle, we can label the opposite side as x and the hypotenuse as 2.



This makes it easier to know what to substitute for other trig functions of θ. For example, cosine is the ratio of the adjacent side length to the hypotenuse, so we have:

$$\cos(\theta) = \frac{\sqrt{4-x^2}}{2}$$

Exercise 2.5.1.4. Try One on your own! ☕☕☕

Evaluate the following antiderivative:

$$\int x^3 \left(16 - x^2\right)^{5/2} \, \mathtt{d}x$$

(**Hint:** Recall our methods for integrating powers of sines and cosines!)

2.5.2 Secant Substitution

When we see an expression of the form $x^2 - a^2$ in the integrand, we think of the identity $\sec^2(\theta) - 1 = \tan^2(\theta)$, so we use the following substitution:

Secant Substitution

$x = a \cdot \sec(\theta)$

Example 2.5.2.1. A Secant Substitution

Suppose we wish to evaluate the following integral:

$$\int \frac{1}{x^4 - 9x^2} \, \mathtt{d}x$$

Since $x^4 - 9x^2 = x^2\left(x^2 - 9\right)$, we use the following substitution:

$$\begin{aligned} x &= 3\sec(\theta) \\ \mathtt{d}x &= 3\sec(\theta)\tan(\theta)\,\mathtt{d}\theta \end{aligned}$$

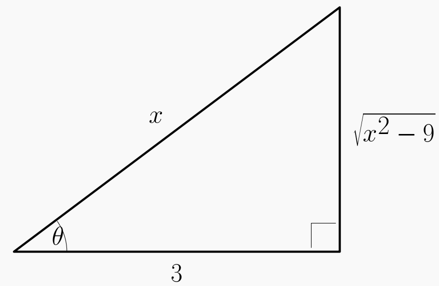
We now apply these substitutions to rewrite the integral in terms of θ.

$$\begin{aligned}
\int \frac{1}{x^4 - 9x^2}\,\mathtt{d}x &= \int \frac{1}{x^2\left(x^2-9\right)}\,\mathtt{d}x \\
&= \int \frac{3\sec(\theta)\tan(\theta)}{9\sec^2(\theta)\left(9\sec^2(\theta)-9\right)}\,\mathtt{d}\theta \\
&= \int \frac{3\sec(\theta)\tan(\theta)}{81\sec^2(\theta)\tan^2(\theta)}\,\mathtt{d}\theta \\
&= \frac{1}{27}\int \frac{1}{\sec(\theta)\tan(\theta)}\,\mathtt{d}\theta \\
&= \frac{1}{27}\int \frac{\cos^2(\theta)}{\sin(\theta)}\,\mathtt{d}\theta \\
&= \frac{1}{27}\int \frac{1-\sin^2(\theta)}{\sin(\theta)}\,\mathtt{d}\theta \\
&= \frac{1}{27}\int \frac{1}{\sin(\theta)}\,\mathtt{d}\theta - \frac{1}{27}\int \frac{\sin^2(\theta)}{\sin(\theta)}\,\mathtt{d}\theta \\
&= \frac{1}{27}\int \csc(\theta)\,\mathtt{d}\theta - \frac{1}{27}\int \sin(\theta)\,\mathtt{d}\theta \\
&= -\frac{1}{27}\ln\left|\csc(\theta)+\cot(\theta)\right| + \frac{1}{27}\cos(\theta) + C
\end{aligned}$$

Here we have successfully taken the antiderivative, and now need to just get back to x from θ. We draw a triangle and label the sides according to our substitution. In particular,

$$\sec(\theta) = \frac{x}{3} = \frac{\text{hypotenuse}}{\text{adjacent}}$$

so we can let the hypotenuse be x and the adjacent side be 3.



This enables us to compute the other trig functions using this triangle.

Exercise 2.5.2.2. Getting from θ back to x

Complete the above example by using the triangle to find the values of the other trig functions.

$$\begin{aligned}\cos(\theta) &= \\ \cot(\theta) &= \\ \csc(\theta) &= \end{aligned}$$

Then plug these expressions back into our antiderivative to get a final answer in terms of x rather than θ. Make these substitutions and then simplify to verify the final answer shown below. Show your work below.

$$\begin{aligned}\int \frac{1}{x^4-9x^2}\,\mathtt{d}x &= -\frac{1}{27}\ln\left|\csc(\theta)+\cot(\theta)\right| + \frac{1}{27}\cos(\theta) + C \\ &= \\ &= \\ &= \frac{1}{9x} - \frac{1}{27}\ln\left|\frac{x+3}{\sqrt{x^2-9}}\right| + C\end{aligned}$$

Exercise 2.5.2.3. Yes You Can! Take the Cant Out of Secant!

Evaluate the following antiderivative:

$$\int \sqrt{x^2-4}\,\mathtt{d}x$$

(**Hint:** Exercise 2.2.3.5 will be helpful!)

2.5.3 Tangent Substitution

When we see an expression of the form $a^2 + x^2$ or $x^2 + a^2$ (which are the same) in the integrand, we think of the identity $\tan^2(\theta) + 1 = \sec^2(\theta)$, so we use the following substitution:

Tangent Substitution

$x = a \cdot \tan(\theta)$

Exercise 2.5.3.1. Revisiting an Old Friend ☕☕

- Recall the derivative of arctangent:

$$\frac{\mathtt{d}}{\mathtt{d}x}\left(\arctan(x)\right) =$$

- We should be able to reverse the above by taking the antiderivative of the right-hand side. Perform this antiderivative using the substitution $x = \tan(\theta)$:

$$\int \frac{1}{1+x^2}\,\mathtt{d}x =$$

2.5.4 Preprocessing with Algebra or u-sub

Often we need to do a little algebra and/or u-sub to get the integrand into a form where we can then perform trig sub.

Exercise 2.5.4.1. A Bit of Algebra to Help Us ☕☕☕

- Explain why the two following expressions are equal:

$$(4x^2+1)^2 = 16\left(x^2+\left(\frac{1}{2}\right)^2\right)^2$$

- Use the equality above along with a tangent substitution to evaluate the following antiderivative:

$$\int \frac{1}{(4x^2+1)^2}\,\mathrm{d}x$$

A trick from algebra that is often used with trigonometric substitution is completing the square. You might need to complete the square to get it into a form where a trig sub will work.

Example 2.5.4.2. Completing the Square

Suppose we wish to find an antiderivative for the function $\left(x^2+x-1\right)^{-2}$. We begin by completing the square on the quadratic polynomial:

$$\begin{aligned} x^2+x-1 &= x^2+x+\frac{1}{4}-\frac{1}{4}-1 \\ &= \left(x+\frac{1}{2}\right)^2-\frac{5}{4} \end{aligned}$$

We now use the substitution that this quadratic motivates. Namely, we pick $a=\frac{\sqrt{5}}{2}$ since we want its square to be five-fourths. Where x used to go in the problems above, we now have an $x+\frac{1}{2}$. Thus our substitution is $x+\frac{1}{2}=\frac{\sqrt{5}}{2}\sec(\theta)$, or more explicitly:

$$x=\frac{\sqrt{5}}{2}\sec(\theta)-\frac{1}{2}$$

Taking the derivative of both sides with respect to θ shows that

$$\mathtt{d}x = \frac{\sqrt{5}}{2}\sec(\theta)\tan(\theta)\mathtt{d}\theta$$

Exercise 2.5.4.3. Completing the Example ☕☕☕

Use the substitutions suggested in the example above to find the antiderivative.

$$\int \frac{1}{(x^2+x-1)^2}\mathtt{d}x =$$

Exercise 2.5.4.4. Try One On Your Own ☕☕☕

Evaluate the following antiderivative:

$$\int \frac{x}{\sqrt{2x^2-4x-7}}\,\mathtt{d}x$$

2.6 Partial Fraction Decomposition

In this section, we will combine the techniques of all previous sections and learn how to antidifferentiate rational functions!

Exercise 2.6.0.1. What is a Rational Function Again? ☕

What is the definition of a *rational function*?

A partial fraction decomposition (PFD) is a way to decompose a rational function (a polynomial divided by a polynomial) as a sum of simpler rational functions. This is purely an algebraic trick that fundamentally does not involve calculus. It is useful in many contexts! Here we apply it to (of course) finding antiderivatives. Typically, a given rational function is too challenging to antidifferentiate as is. Once we break it up into smaller pieces via PFD, it becomes manageable.

The fundamental idea is simple. If we have a fraction that has more than one factor in the denominator, we can rewrite it as a sum of fractions whose denominators have the original denominator as their least common multiple.

Exercise 2.6.0.2. Trying This with Integers Before We Go to Polynomials ☕

Consider the fraction $\frac{1}{6}$. We notice that the denominator, six, is equal to two times three. Thus, we attempt to write one-sixth as a sum of fractions whose denominators are two and three.
Find integers A and B such that:

$$\frac{1}{6} = \frac{A}{2} + \frac{B}{3}$$

Check your answer by adding the fractions on the right hand side back together to verify you get one-sixth.

2.6.1 Warming Up with a Small Example

Partial fraction decomposition is the same idea, except we are working with polynomials rather than just integers.

Example 2.6.1.1. Our First Decomposition!

To decompose the fraction $\frac{1}{x^2-1}$, we first factor the denominator into $x^2 - 1 = (x-1)(x+1)$. Thus, we look for an expression of the form

$$\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$$

for some numbers A and B. To find such A and B, we multiply both sides by $x^2 - 1$ to produce the polynomial equation

$$1 = A(x+1) + B(x-1)$$

Since we want the expressions to be equal for all values of x, we pick convenient values of x to plug in to solve for A and B.
Set $x = 1$:

$$1 = A \cdot (2) + B \cdot (0) \implies A = \frac{1}{2}$$

Set $x = -1$:

$$1 = A \cdot (0) + B \cdot (-2) \implies B = -\frac{1}{2}$$

At last, we have obtained the partial fraction decomposition!

$$\frac{1}{x^2-1} = \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}$$

Exercise 2.6.1.2. Checking Our Work ☕

Take the right-hand side of the above equation and add the two fractions together by finding a common denominator. Verify that their sum is the original rational function $\frac{1}{x^2-1}$.

Example 2.6.1.3. Finding the Same PFD by Expanding and Equating Coefficients

We repeat the above example but demonstrate an alternate way to find our coefficients. Recall the equation

$$1 = A(x+1) + B(x-1)$$

In the previous example, we proceeded by plugging in numerical values for x. Instead, we could fully multiply out the polynomials and combine like terms. This produces

$$1 = (A+B)x + (A-B)$$

We can pad the left-hand side with a degree one term with coefficient zero to put both sides in the form “number times x plus number”.

$$0x + 1 = (A+B)x + (A-B)$$

Now we can construct a system of two equations in two unknowns by equating one coefficient at a time. Specifically, we build it as:

Degree zero coefficient of LHS = Degree zero coefficient of RHS	$\Longrightarrow$	$1 = A - B$
Degree one coefficient of LHS = Degree one coefficient of RHS	$\Longrightarrow$	$0 = A + B$

The resulting linear system in two equations and two unknowns can then be solved via any applicable method (substitution, elimination, matrices, etc).

Exercise 2.6.1.4. Solve the System ☕

Solve the linear system of two equations and two unknowns in the example above. Verify you obtain the same values for A and B that we found in Example 2.6.1.1.

Exercise 2.6.1.5. Using the PFD to Find an Antiderivative ☕☕

- Find an antiderivative of $\frac{1}{x^2-1}$ by antidifferentiating

$$\frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}$$

- Verify the answer is the same as what you would get if you had taken the antiderivative of $\frac{1}{x^2-1}$ using the trigonometric substitution $x = \sec(\theta)$.

It turns out there are three strange things that can happen when finding a PFD, namely:

1. The degree of the numerator is greater than or equal to the degree of the denominator.
2. The denominator has one or more irreducible quadratic factors (where irreducible quadratic means a degree two polynomial that has no real roots).
3. The denominator has one or more repeated factors.

Each has a particular workaround. Below, we describe these methods and show a corresponding hideous example that demonstrates all of these steps.

Exercise 2.6.1.6. Reminding Ourselves of Some Language ☕

- What exactly does *irreducible quadratic* mean?

- Give an example of a quadratic polynomial that is irreducible.

- Give an example of a quadratic polynomial that is not irreducible.

- Is the polynomial x^2 an irreducible quadratic? Explain why or why not.

2.6.2 The General Method of PFD

The process for performing a partial fraction decomposition of $\frac{p(x)}{q(x)}$ is as follows:

1. **Polynomial Long Division:** If the degree of $p(x)$ is not strictly smaller than the degree of $q(x)$, start by performing polynomial long division to split the fraction into a quotient and remainder. In the remainder term, the numerator will now have degree less than the denominator.

2. **Factor Denominator:** Factor the denominator into a product of powers of linear and irreducible quadratic polynomials.

3. **Set Up Terms in the Summation:**

   (a) **Linear Factors:** If the denominator is divisible by $(x-r)^n$ for some real number r and positive natural number n, we build terms that look like

   $$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \frac{A_2}{(x-r)^3} + \cdots + \frac{A_n}{(x-r)^n}$$

   where the A_i represent unknown real constants. That is, you use all consecutive powers of a linear factor as denominators and have arbitrary constants as numerators.

   (b) **Irreducible Quadratic Factors:** Let b and c be real numbers and suppose x^2+bx+c is an irreducible quadratic. If the denominator is divisible by $(x^2+bx+c)^n$ for some positive natural number n, we build terms that look like

   $$\frac{A_1x+B_1}{x^2+bx+c} + \frac{A_2x+B_2}{(x^2+bx+c)^2} + \frac{A_2x+B_2}{(x^2+bx+c)^3} + \cdots + \frac{A_nx+B_n}{(x^2+bx+c)^n}$$

   where the A_i and B_i represent unknown real constants. That is, you use all consecutive powers of a linear factor as denominators and have arbitrary constants as numerators.

4. **Clear Denominators:** Multiply each side of your equation by the denominator $q(x)$ to clear all fractions.

5. **Solve for Unknowns:** Solve for the unknown constants by plugging in convenient values of x (since we want the expression to be true for all values of x). The roots of $q(x)$ are always good choices for x values, but other friendly numbers like zero or one are also often helpful.

6. **Plug Values Back into the Previously Unknown Numerators:** Plug your constants back in to conclude the equality of your original rational expression with its PFD.

Example 2.6.2.1. An Epic PFD

We now find the partial fraction decomposition of the rational function

$$r(x) = \frac{x^7 + 8x^6 + 25x^5 + 52x^4 + 79x^3 + 13x^2 - 61x + 81}{x^6 + 9x^5 + 28x^4 + 36x^3 + 27x^2 + 27x}$$

This rational function has quotient $x - 1$ and remainder $6x^5 + 44x^4 + 88x^3 + 13x^2 - 34x + 81$ upon long division. So, for our first step in the decomposition we have

$$r(x) = x - 1 + \frac{6x^5 + 44x^4 + 88x^3 + 13x^2 - 34x + 81}{x^6 + 9x^5 + 28x^4 + 36x^3 + 27x^2 + 27x}$$

We now ignore the quotient and work on breaking up the fractional piece. The denominator is divisible by x, so we factor that out. Next, we use the Rational Root Theorem to form a list of possible roots and divide off the corresponding factors as we find them. Working out all the algebra, we conclude the denominator factors as

$$x^6 + 9x^5 + 28x^4 + 36x^3 + 27x^2 + 27x = x(x + 3)^3(x^2 + 1)$$

In this particular setting, x, $x + 3$, $(x + 3)^2$, and $(x + 3)^3$ are the relevant powers of linear factors. The factor $x^2 + 1$ is the only irreducible quadratic. (**Note:** $(x+3)^2$ is not an irreducible quadratic term; it is a common mistake to consider it so. It is a power of a linear term and should be treated as such.) We now set up our sum.

$$\frac{6x^5 + 44x^4 + 88x^3 + 13x^2 - 34x + 81}{x(x + 3)^3(x^2 + 1)} = \frac{A}{x} + \frac{B}{x + 3} + \frac{C}{(x + 3)^2} + \frac{D}{(x + 3)^3} + \frac{Ex + F}{x^2 + 1}$$

Since fractions are a pain, we get rid of them! Multiplying both sides by $x(x + 3)^3(x^2 + 1)$, our equation becomes

$$6x^5 + 44x^4 + 88x^3 + 13x^2 - 34x + 81$$
$$= A(x+3)^3(x^2+1) + Bx(x+3)^2(x^2+1) + Cx(x+3)(x^2+1) + Dx(x^2+1) + (Ex+F)x(x+3)^3$$

We now solve for our unknown coefficients. It is highly convenient to set $x = 0$. This produces the equation $81 = A(3)^3$ which implies $A = 3$. Similarly, we set $x = -3$. This produces the equation

$$6(-3)^5 + 44(-3)^4 + 88(-3)^3 + 13(-3)^2 - 34(-3) + 81 = D(-3)((-3)^2 + 1)$$

which simplifies to $30 = D(-30)$ which implies $D = -1$. We have now run out of the most convenient values to choose for x, namely the roots of the denominator. At this point, we unfortunately need to do something messy! We can either plug in less than optimal values of x, for example $x = 1$, then $x = -1$, then $x = 2$, etc, and solve the resulting simultaneous system of equations that results. Or, we can multiply out the polynomials and equate coefficients one degree at a time (the method of Example 2.6.1.3). Carrying out either of these methods will produce

$$B = 2, C = 1, E = 1, F = -5$$

At last, we plug the values for the constants A, B, C, D, E, and F back into the original decomposition (with quotient). Our final PFD is

$$\frac{x^7 + 8x^6 + 25x^5 + 52x^4 + 79x^3 + 13x^2 - 61x + 81}{x^6 + 9x^5 + 28x^4 + 36x^3 + 27x^2 + 27x} = x - 1 + \frac{3}{x} + \frac{2}{x + 3} + \frac{1}{(x + 3)^2} - \frac{1}{(x + 3)^3} + \frac{x - 5}{x^2 + 1}$$

Exercise 2.6.2.2. Identifying the Steps of PFD ☕

In the example above, label each of the six steps of partial fraction decomposition. Where exactly does each step occur?

Exercise 2.6.2.3. Which Type of Numerator Goes Where? ☕

In the above example, notice that the factor $(x+3)^2$ corresponded to a term of the form

$$\frac{C}{(x+3)^2}$$

and not a term of the form

$$\frac{Cx+D}{(x+3)^2}.$$

Why was this the case?

Well, that's the process of partial fraction decomposition! Why are we doing it in a calculus course? Because a generic rational function is really hard to integrate, but the partial fraction decomposition is made up of simpler terms that are much easier to integrate. Let's find the antiderivative of that beast above!

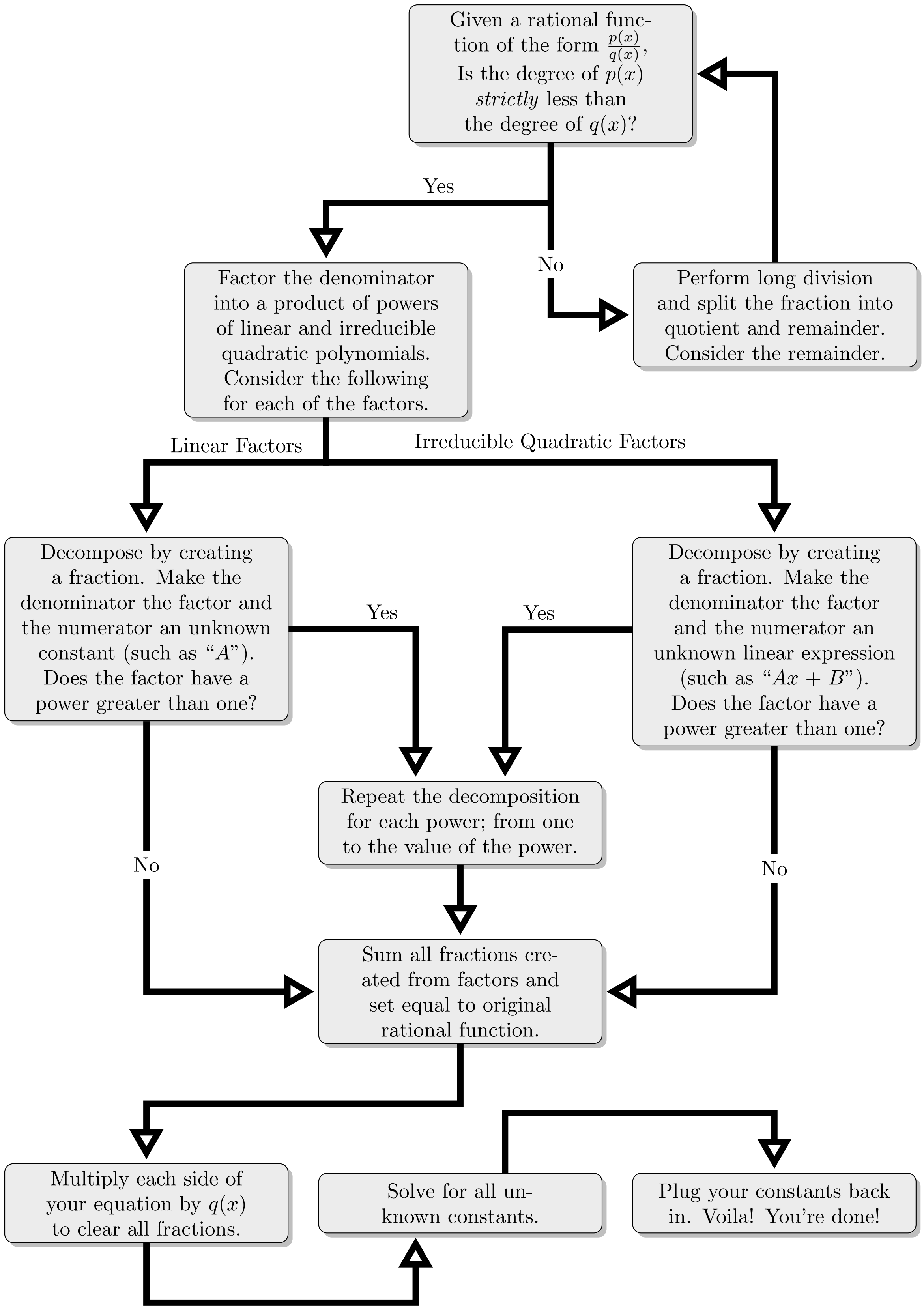
Example 2.6.2.4. Return of the Son of *Using a PFD to Find an Antiderivative*

We apply our PFD to compute the following antiderivative:

$$\begin{aligned}
&\int \left(\frac{x^7+8x^6+25x^5+52x^4+79x^3+13x^2-61x+81}{x^6+9x^5+28x^4+36x^3+27x^2+27x} \right) \mathtt{d}x \\
&= \int \left(x - 1 + \frac{3}{x} + \frac{2}{x+3} + \frac{1}{(x+3)^2} - \frac{1}{(x+3)^3} + \frac{x-5}{x^2+1} \right) \mathtt{d}x \\
&= \frac{x^2}{2} - x + 3\ln(x) + 2\ln(x+3) + -\frac{1}{x+3} + \frac{1}{2(x+3)^2} + \int \frac{x}{x^2+1}\,\mathtt{d}x + \int \frac{-5}{x^2+1}\,\mathtt{d}x \\
&= \frac{x^2}{2} - x + 3\ln(x) + 2\ln(x+3) + -\frac{1}{x+3} + \frac{1}{2(x+3)^2} + \frac{1}{2}\ln(x^2+1) - 5\arctan(x)
\end{aligned}$$

Oh, and um, plus C.

2.6.3 Sweet PFD Flow Chart



Exercise 2.6.3.1. Now you cry! I mean, try! ☕☕☕

Find the following antiderivatives. Keep in mind that not every step of PFD will necessarily occur in every problem!

- $\displaystyle\int \frac{1}{x^2 - 9x + 20}\,\mathtt{d}x$

- $\displaystyle\int \frac{1}{x^4 - 9}\,\mathtt{d}x$

- $\displaystyle\int \frac{x^4}{x^2+1}\,\mathrm{d}x$

- $\displaystyle\int \frac{2}{x^5+2x^3+x}\,\mathrm{d}x$

- $\displaystyle\int \frac{x-2}{x^3+x^2+3x-5}\,\mathtt{d}x$

Exercise 2.6.3.2. Revisiting an Old Friend ☕☕☕

Recall Example 2.5.2.1, where we found the antiderivative of

$$\frac{1}{x^4-9x^2}$$

via trig sub. Find this antiderivative again but via PFD! Verify your answer is compatible with what trig sub produced.

Chapter 3

Geometric Applications of Integrals

Now that we are much better at the process of antidifferentiation, we apply integrals to the classic problems of geometry. We find lengths, areas, volumes, and centers of mass. Before we begin, we state L'Hospital's Rule, which will assist in computing areas of unbounded regions.

3.1 L'Hospital's Rule

L'Hospital's Rule (LHR) allows us to evaluate indeterminate limits of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. It says that in either of these cases, we can simply differentiate the numerator and the denominator and try again.

Theorem 3.1.0.1. L'Hospital's Rule

Let c be a real number, ∞, or $-\infty$. If $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ or $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = \infty$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

For the moment, we will just accept LHR and use it. In Section 5.8, we will prove LHR using power series. Notice that here we do not differentiate with a Quotient Rule. We instead simply differentiate the top and differentiate the bottom.

Example 3.1.0.2. Sine of a Small Angle

Consider the following limit:

$$\lim_{x\to 0} \frac{\sin(\theta)}{\theta}$$

It is indeterminate of the form $\frac{0}{0}$. Thus it is valid to apply LHR.

$$\begin{aligned}\lim_{x\to 0} \frac{\sin(\theta)}{\theta} &= \lim_{x\to 0} \frac{(\sin(\theta))'}{(\theta)'} \\ &= \lim_{x\to 0} \frac{\cos(\theta)}{1} \\ &= 1\end{aligned}$$

Exercise 3.1.0.3. Interpreting the Above Example ☕

Since the ratio of $\sin(\theta)$ to θ approaches 1 as θ gets small, it would be appropriate to say the following (fill in the blanks):

For small values of θ, ______________$\approx$______________.

Note that this property comes up frequently in physics! For example, when modeling the motion of a mass hanging from a spring, Hooke's Law tells us that force is proportional to displacement. We use the same model to describe motion of a pendulum, even though in that case force is not technically proportional to displacement, but rather the *sine* of displacement. Why can we throw away the sine? Well because for small displacements, the sine of the displacement is roughly equal to the displacement!

Exercise 3.1.0.4. Practice with LHR ☕☕

Evaluate the following limits using L'Hospital's Rule. In each case, justify why it is ok to use it!

- $\lim_{x\to\infty} \frac{x^2}{2^x}$
- $\lim_{x\to 2} \frac{x-2}{\sin(\pi x)}$
- $\lim_{x\to\infty} \frac{\arctan(x)-\pi/2}{\sin(1/x)}$

3.1.1 Other Indeterminate Forms

There are many other indeterminate forms besides just $\frac{0}{0}$ and $\frac{\infty}{\infty}$. Others that come up include:

- $0 \cdot \infty$
- 0^0

- 1^∞
- $\infty - \infty$

Often these other forms can be rearranged algebraically to become $\frac{0}{0}$ or $\frac{\infty}{\infty}$. After this rearrangement, they can then be evaluated with LHR (or perhaps the algebra itself resolves the indeterminate form and LHR will not be needed). Common helpful strategies include:

- Multiplying the top and bottom of the limit by the same expression (especially the conjugate of an expression involving a radical).
- Taking e to the ln of the limit.
- Rewriting a product as a fraction via $a \cdot b = \frac{b}{\frac{1}{a}}$.

Example 3.1.1.1. Rewriting a Different Indeterminate Form

Consider the function $\sin(x)^{\tan(x)}$. As x approaches $\frac{\pi}{2}$ from the left, the function takes on the indeterminate form 1^∞. Thus, we try the second strategy described above, where we take e to the ln of the limit. Proceeding:

$$\begin{aligned}
\lim_{x\to\pi/2^-} \sin(x)^{\tan(x)} &= \lim_{x\to\pi/2^-} e^{\ln\left(\sin(x)^{\tan(x)}\right)} \\
&= \lim_{x\to\pi/2^-} e^{\tan(x)\ln(\sin(x))} \\
&= \lim_{x\to\pi/2^-} e^{\tan(x)\ln(\sin(x))}
\end{aligned}$$

Notice the exponent is now the indeterminate form $0 \cdot \infty$. Since the exponential function is continuous, we can move the limit inside and use LHR!

$$\begin{aligned}
\lim_{x\to\pi/2^-} \sin(x)^{\tan(x)} &= \lim_{x\to\pi/2^-} e^{\tan(x)\ln(\sin(x))} \\
&= e^{\lim_{x\to\pi/2^-}(\tan(x)\ln(\sin(x)))} \\
&= e^{\lim_{x\to\pi/2^-}\left(\frac{\ln(\sin(x))}{\cot(x)}\right)} \\
&= e^{\lim_{x\to\pi/2^-}\left(\frac{(\ln(\sin(x)))'}{(\cot(x))'}\right)} \\
&= e^{\lim_{x\to\pi/2^-}\left(\frac{\frac{\cos(x)}{\sin(x)}}{-\csc^2(x)}\right)} \\
&= e^{\lim_{x\to\pi/2^-}(-\cos(x)\sin(x))} \\
&= e^{-0\cdot 1} \\
&= 1
\end{aligned}$$

Exercise 3.1.1.2. Identifying LHR ☕

In the example above, circle the exact step where LHR was applied. Why was it ok to use LHR on that step? Write a short sentence to explain.

Exercise 3.1.1.3. Rewriting ☕☕☕

Utilize these strategies to rewrite the limits below as $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and then evaluate. Note that some of these may need LHR after rewriting and some may not!

1. $\lim_{x\to\infty} x \cdot \sin(1/x)$

2. $\lim_{x\to\infty} x - \sqrt{x^2+4x+3}$

3. $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$ (**Note:** This limit is often taken as the definition of the constant you get here!)

Exercise 3.1.1.4. Polyexposaurus ☕☕☕

- Any positive real number raised to the zero is...
- Zero raised to any positive real number is...
- So, what is $\lim_{x \to 0^+} x^x$?

Be careful when using LHR to only apply it in the two indeterminate forms specified above. Applying LHR to an expression that is not either $\frac{0}{0}$ or $\frac{\infty}{\infty}$ will most likely produce incorrect results.

Exercise 3.1.1.5. L'Urgent Care ☕☕

Consider the following limit.

$$\lim_{x \mapsto \pi} \frac{\sin(x)}{x}$$

- Why would it be wrong to apply LHR to the above limit?
- What do you get if you blindly apply LHR?
- What should the limit actually be?

3.1.2 Growth Orders

LHR is often used for comparing growth orders of functions. To compare the growth orders of functions, we take the limit of their ratio as x approaches infinity and then see if the ratio approaches zero, a nonzero constant, or infinity to see which is growing faster. More formally:

Definition 3.1.2.1. Growth Order

Let $f(x)$ and $g(x)$ be functions on the real numbers.

- If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$, then $g(x)$ has *larger growth order* than $f(x)$.
- If $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ is a nonzero constant, then $f(x)$ and $g(x)$ have the *same growth order*.
- If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$, then $f(x)$ has *larger growth order* than $g(x)$.

Exercise 3.1.2.2. Comparing the Growth Orders of Two Lines ☕☕

Consider the following two linear functions:

$$f(x) = 6x + 1$$
$$g(x) = 2x - 1$$

Fill out the table below to study some of their values and corresponding ratios. Use decimal approximations for values that aren't integers.

x	1	10	100	1,000	10,000
$f(x)$					
$g(x)$					
$f(x)/g(x)$					

- From the table, does it appear that the ratio $f(x)/g(x)$ is approaching zero, infinity, or a nonzero constant?

- Use LHR to compute the limit
$$\lim_{x\to\infty} \frac{f(x)}{g(x)}$$
How does it relate to the values in the data table?

- See the above definition of Growth Order. In this case, would you say f and g have the same growth order, or does one function have larger growth order than the other?

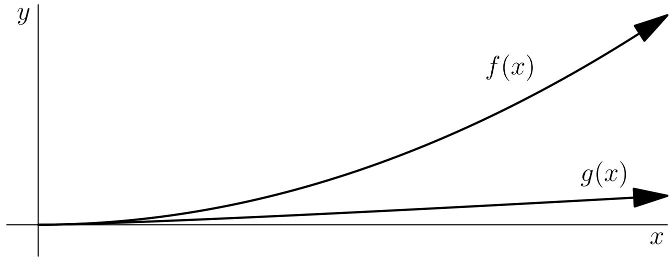
The above calculation justifies why it is ok to talk about something having *linear growth order* or *growing linearly*. Any two lines have the same growth order (ignoring vertical and horizontal), so it is perfectly well-defined to talk about linear growth even if the slope or intercepts of the lines being discussed are unknown.

This concept comes up frequently in computer science when you try to measure the runtime of algorithms. If $f(x)$ is the number of operations performed by an algorithm that is handed an input of size x, then a logarithmic growth order of $f(x)$ is generally more desirable than a linear growth order, which is more desirable than quadratic growth order, and so on.

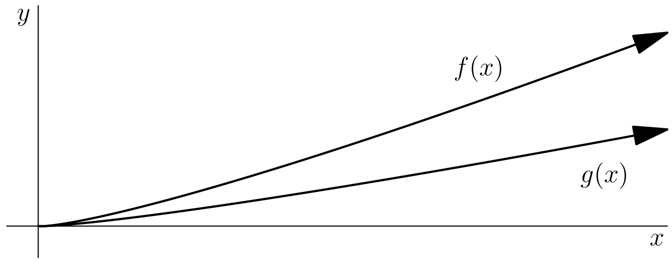
Exercise 3.1.2.3. A Visual Representation of Growth Order ☕☕☕

In each of the graphs below, there is a graph of $f(x)$ and a graph of $g(x)$. Based on the graphs, do you expect that f and g have the same growth order, or is one larger?

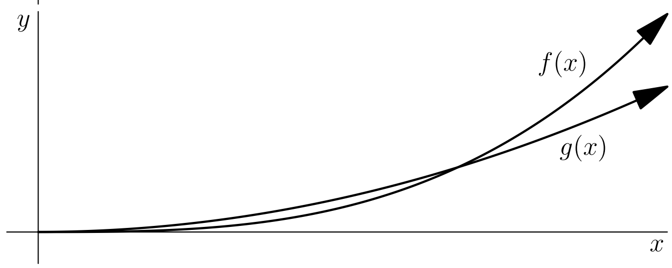
1.



2.



3.



Exercise 3.1.2.4. Race of the Turtles ☕☕

Rank these functions by growth order from slowest to fastest:

$$x\ln(x), x^{1.1}, x\,(\ln(x))^2$$

Exercise 3.1.2.5. Exponential vs Polynomial ☕☕☕

Explain why an exponential function will always have larger growth order than an polynomial function.

3.2 Improper Integrals

In Calculus I, all of our definite integrals corresponded to the area of a bounded region. A definite integral over an unbounded region is called *improper*.

3.2.1 Vertically Unbounded Regions

If the integrand has a vertical asymptote between the limits of integration, we must proceed by approximating the unbounded region with a bounded region and then taking a limit.

Exercise 3.2.1.1. Analyzing a Vertical Asymptote ☕☕

Consider the following integral:

$$\int_0^1 \ln(x)\,\mathtt{d}x$$

- Explain why the above integral would be called improper.

- Find the antiderivative of the function $\ln(x)$.

- Fill out the following table. For each definite integral, include a rough sketch of the region whose signed area it corresponds to.

c	$\int_c^1 \ln(x)\,\mathtt{d}x$	Graph of Region
0.1		
0.01		
0.001		
0.0001		

- As c approaches 0 from the right, what does the area seem to be approaching?

The above calculation motivates the definition of an improper integral.

Definition of Improper Integral I

If $f(x)$ has a vertical asymptote at $x = a$

but is continuous on the interval $(a, b]$,

then the improper integral is defined as

$\int_a^b f(x)\,\mathtt{d}x = \lim_{c\to a^+} \int_c^b f(x)\,\mathtt{d}x$

If this limit converges to a number, then we say the improper integral *converges*. Otherwise, we say the improper integral *diverges*. We apply the definition to finish off the above exercise.

Example 3.2.1.2. Area Between the Axes and Natural Log

To calculate the area between $y = 0$, $x = 0$, and $y = \ln(x)$, we use the definition of improper integral. We proceed with the following calculation:

$$\begin{aligned}\int_0^1 \ln(x)\,\mathtt{d}x &= \lim_{c\to 0^+} \int_c^1 \ln(x)\,\mathtt{d}x \\ &= \lim_{c\to 0^+} x\ln(x) - x\Big]_{x=c}^{x=1} \\ &= \lim_{c\to 0^+} (1\ln(1) - 1) - (c\ln(c) - c)\end{aligned}$$

In the above limit, all terms are harmless except for $\lim_{c\to 0} c\ln(c)$, which is indeterminate of the form $0\cdot\infty$. We can rewrite as

$$\lim_{c\to 0^+} c\ln(c) = \lim_{c\to 0^+} \frac{\ln(c)}{\left(\frac{1}{c}\right)}$$

to get it in the form $\frac{\infty}{\infty}$ (up to a minus sign which is harmless), where we can apply LHR.

Exercise 3.2.1.3. We'll Actually Finish This Problem Here, Promise ☕☕

Finish evaluating the limit above and verify the area matches your estimations from Exercise

3.2.1.1.

Let us now construct the analogous definition for a vertical asymptote occurring at the right-hand endpoint rather than the left-hand endpoint.

Exercise 3.2.1.4. Completing the Definition

Suppose a function $f(x)$ is continuous on $[a, b)$ but had a vertical asymptote at $x = b$. Use the diagram below to help complete the definition of such an improper integral. Fill in the boxes below.

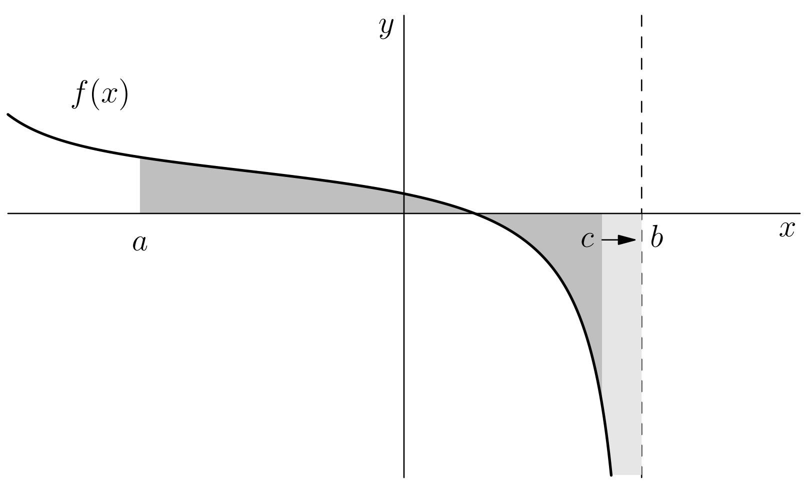
Definition of Improper Integral II

If $f(x)$ has a vertical asymptote at $x = a$

but is continuous on the interval $(a, b]$,

then the improper integral is defined as

$\int_a^b f(x)\,\mathtt{d}x = \lim_{c\to\square} \int_{\square}^{\square} f(x)\,\mathtt{d}x$

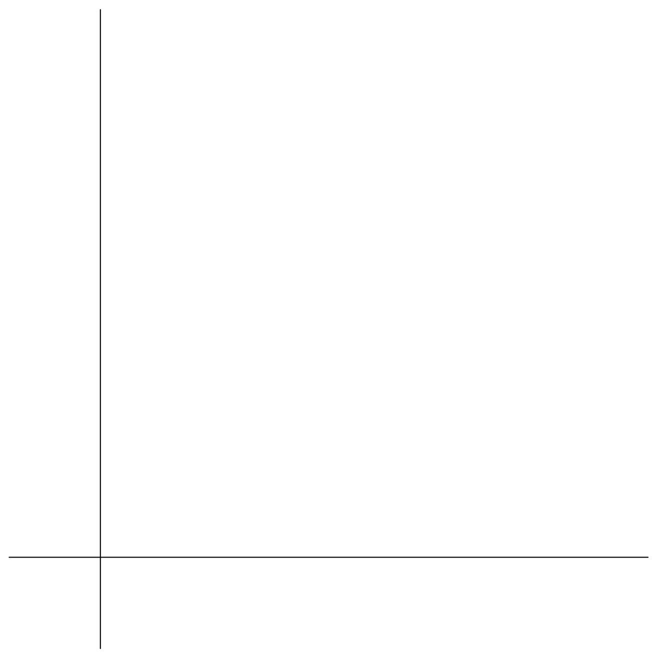


Once again, if this limit converges to a number, then we say the improper integral *converges*. Otherwise, we say the improper integral *diverges*.

Exercise 3.2.1.5. Illustrate the Computation

Here we demonstrate an example of computing an improper integral with the vertical asymptote at the right-hand endpoint. Illustrate the computation on the axes below. Show the graph of the integrand and the locations of the bounds a, b, and c.

$$\begin{aligned}\int_{x=0}^{x=\pi/2} \tan(x)\,\mathtt{d}x &= \lim_{c\to\pi/2^-} \int_{x=0}^{x=c} \tan(x)\,\mathtt{d}x \\ &= \lim_{c\to\pi/2^-} -\ln\left(\cos(x)\right)\Big]_{x=0}^{x=c} \\ &= \lim_{c\to\pi/2^-} -\ln\left(\cos\left(c\right)\right) + \ln\left(\cos(0)\right) \\ &= \infty\end{aligned}$$



If the vertical asymptote is in the interior (rather than at an endpoint) of the interval over which you are integrating, it may be necessary to split it into several integrals.

Example 3.2.1.6. A Particular Unbounded Region

Suppose we wish to calculate the area bounded by the x-axis, the line $x = 1$, the line $x = -1$, and the graph of $f(x) = \frac{1}{\sqrt{|x|}}$. Notice the function has a vertical asymptote at $x = 0$, which is in the interior of the interval over which we wish to integrate. Thus, we split the integral into two integrals. The first has a vertical asymptote at the right-hand endpoint and the second has a vertical asymptote at the left-hand endpoint. We handle each accordingly.

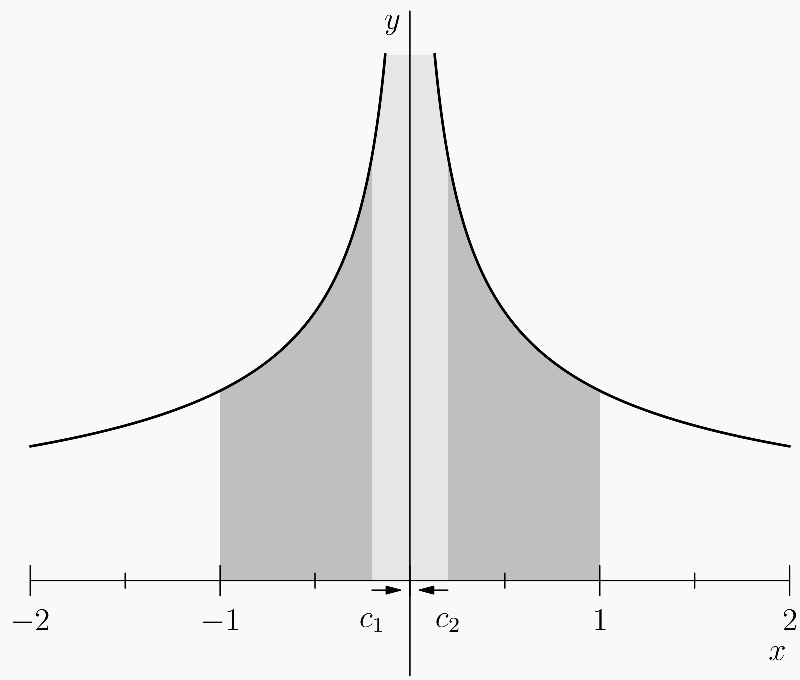
$$\begin{aligned}\int_{-1}^{1}\frac{1}{\sqrt{|x|}}\,\mathtt{d}x &= \int_{-1}^{0}\frac{1}{\sqrt{|x|}}\,\mathtt{d}x + \int_{0}^{1}\frac{1}{\sqrt{|x|}}\,\mathtt{d}x \\ &= \lim_{c_1\to 0^-}\int_{-1}^{c_1}\frac{1}{\sqrt{|x|}}\,\mathtt{d}x + \lim_{c_2\to 0^+}\int_{c_2}^{1}\frac{1}{\sqrt{|x|}}\,\mathtt{d}x\end{aligned}$$

We now evaluate each of those integrals separately and add their totals.

$$\begin{aligned}\lim_{c_1\to 0^-}\int_{-1}^{c_1}\frac{1}{\sqrt{|x|}}\,\mathtt{d}x &= \lim_{c_1\to 0^-} -2\sqrt{|x|}\Big|_{x=-1}^{x=c_1} \\ &= \lim_{c_1\to 0^-} -2\sqrt{-c_1}+2\sqrt{1} \\ &= 0+2 \\ &= 2\end{aligned}$$

The other region is just a reflection across the y-axis and thus must also have area 2. We conclude

$$\int_{-1}^{1}\frac{1}{\sqrt{|x|}}\,\mathtt{d}x = 4$$



Exercise 3.2.1.7. Some Subtle Sign Business ☕

In the above computation, why is there a negative sign on the antiderivative, producing $-2\sqrt{|x|}$

instead of just $2\sqrt{|x|}$? (**Hint:** Graph the function $2\sqrt{|x|}$!)

Ok, now you give it a shot!

Exercise 3.2.1.8. Improper Integral Practice ☕☕

Evaluate the following integrals and draw graphs similar to the figure above. Show how you are evaluating the improper integral as a limit of integrals of bounded regions.

- $\int_2^4 \frac{1}{\sqrt{x-2}} \, \mathtt{d}x$

- $\int_2^4 \frac{1}{x^2-4} \, \mathtt{d}x$

- $\int_0^{\pi/2} \sec(x)\, \mathtt{d}x$

- $\int_{-\pi/2}^{\pi/2} \csc^2(x)\, \mathtt{d}x$

3.2.2 Horizontally Unbounded Regions

If the integral is over an interval that includes plus or minus infinity as one of the endpoints, we must proceed by approximating via a bounded interval and then taking the limit as the endpoint goes to plus or minus infinity.

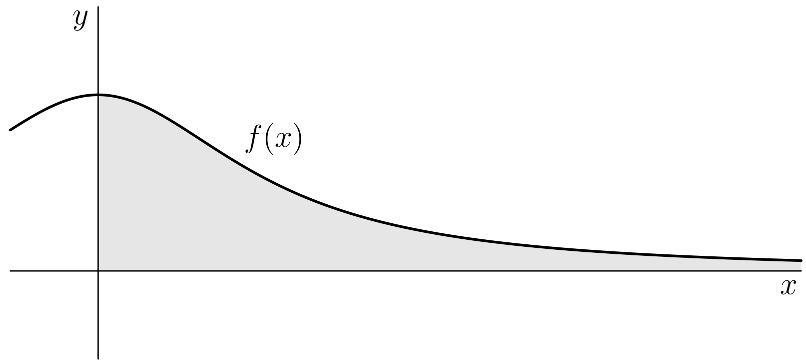
Definition of Improper Integral III

Let $f(x)$ be continuous on the interval $[a, \infty)$ for some real number a.

Then we define $\int_a^\infty f(x)\, \mathtt{d}x = \lim_{c\to\infty} \int_a^c f(x)\, \mathtt{d}x$.

An integral to negative infinity is defined analogously via the corresponding limit.

Exercise 3.2.2.1. Finding c



Here is a graph that represents the above definition for $a = 0$. Interpret the definition by labeling c on the graph and explaining the role it plays.

Example 3.2.2.2. Area Under $f(x) = \frac{1}{x^2}$

Suppose we wish to compute the area under the curve $f(x) = \frac{1}{x^2}$ over the interval $[1, \infty)$. We apply the definition of the improper integral as a limit of bounded integrals.

$$\begin{aligned}\int_{x=1}^{x=\infty} \frac{1}{x^2}\,\mathtt{d}x &= \lim_{c\to\infty} \int_{x=1}^{x=c} \frac{1}{x^2}\,\mathtt{d}x \\ &= \lim_{c\to\infty} -\frac{1}{x}\Big]_{x=1}^{x=c} \\ &= \lim_{c\to\infty} -\frac{1}{c} + \frac{1}{1} \\ &= 1\end{aligned}$$

Thus, the area under the curve is 1.

Exercise 3.2.2.3. Area Under $1/x^p$ ☕☕☕

- Calculate the improper integral $\int_{x=1}^{x=\infty} \frac{1}{x^3}\, \mathtt{d}x$.

- Calculate the improper integral $\int_{x=1}^{x=\infty} \frac{1}{x^1}\, \mathtt{d}x$.

- Calculate the improper integral $\int_{x=1}^{x=\infty} \frac{1}{x^{1/2}}\, \mathtt{d}x$.

- For what real numbers p will $\int_{x=1}^{x=\infty} \frac{1}{x^p}\, \mathtt{d}x$ converge? For what p will it diverge?

If the integral is across the entire real number line, one must split into two separate integrals, similar to how we handled a vertical asymptote in the interior of our interval.

Definition of Improper Integral III

Let $f(x)$ be continuous on the entire real number line.

For any real number a, we define $\int_{-\infty}^{\infty} f(x)\, \mathtt{d}x = \int_{-\infty}^{a} f(x)\, \mathtt{d}x + \int_{a}^{\infty} f(x)\, \mathtt{d}x$.

For the following problems, you may spot yourself the following fact we will prove in Calc 3:

$$\int_{-\infty}^{\infty} e^{-x^2}\, \mathtt{d}x = \sqrt{\pi}$$

Exercise 3.2.2.4. More Practice with Improper Integrals ☕☕

Now, try the following integrals and for each draw a graph like the above figure that represents your integral as a limit of integrals of bounded regions.

- $\int_0^\infty xe^{-x^2}\,\mathtt{d}x$

- $\int_{-\infty}^\infty xe^{-x^2}\,\mathtt{d}x$

- $\int_{-\infty}^\infty x^2e^{-x^2}\,\mathtt{d}x$

- $\int_2^\infty \frac{1}{x\ln(x)}\,\mathtt{d}x$

- $\int_2^\infty \frac{1}{x(\ln(x))^2} \, \mathtt{d}x$

- Consider the integral $\int_0^\infty \sin(x) \, \mathtt{d}x$. Explain why it would be incorrect to say that all the positive and negative area cancel each other out to be zero. In particular, cite the definition of the improper integral as a limit in your explanation.

3.3 Area Between Curves

Recall that a definite integral calculates the signed area under a curve. Thus, we can find the signed area between two curves by taking their difference and integrating.

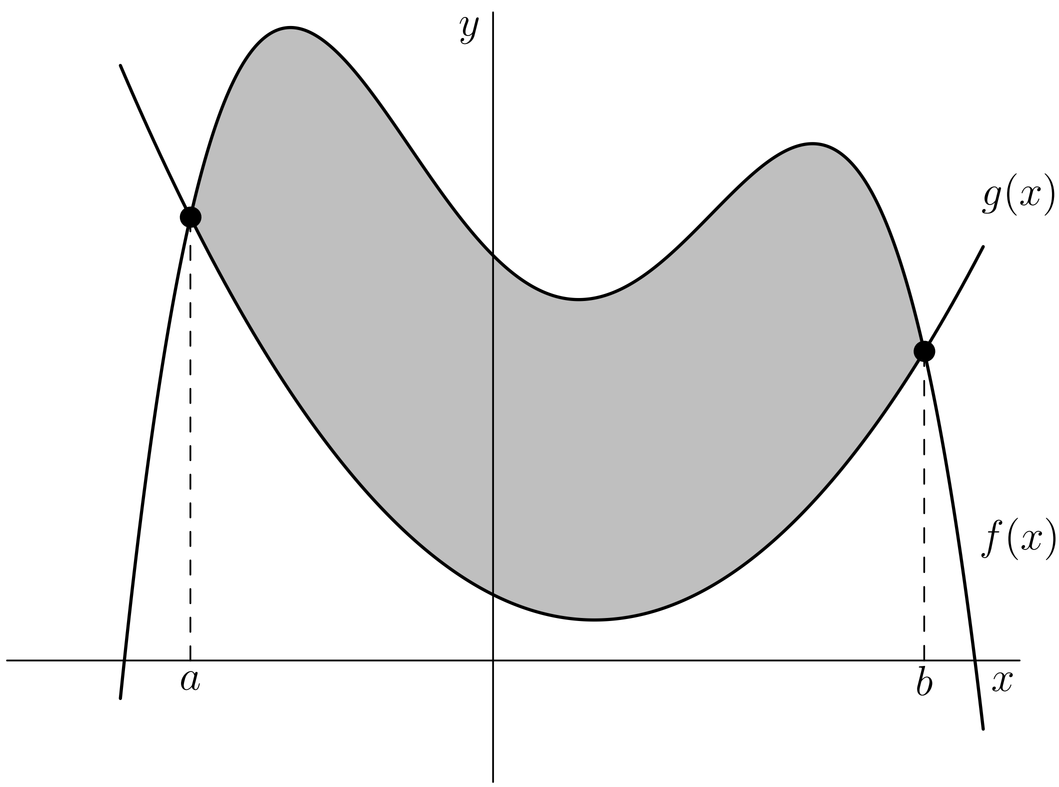
Area Between Curves

Let $g(x) \leq f(x)$ for all x in an interval $[a, b]$.

Then the area bounded by the graphs

$x = a$, $x = b$, $y = f(x)$, and $y = g(x)$ is

$A = \int_a^b (f(x) - g(x))\,\mathrm{d}x$.

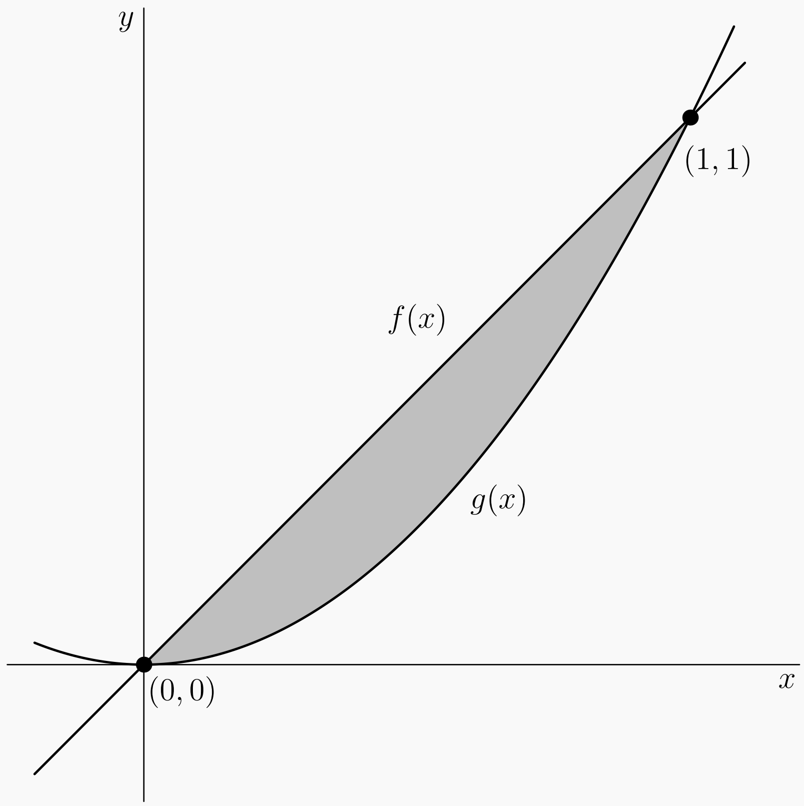


Example 3.3.0.1. Quadrature of a Parabola

Suppose we wish to find the area between curves $f(x) = x$ and $g(x) = x^2$. To accomplish this, we set the two formulas equal to each other to solve for the points of intersection. The line and parabola meet where

$$x^2 = x \implies x^2 - x = 0 \implies x(x-1) = 0 \implies x = 0 \text{ or } x = 1.$$

Thus the points of intersection are at $(0, 0)$ and $(1, 1)$.



Thus the area between curves is

$$\begin{aligned}\int_{x=0}^{x=1} \left(x - x^2\right) \mathtt{d}x &= \left.\frac{x^2}{2} - \frac{x^3}{3}\right]_{x=0}^{x=1} \\ &= \frac{1}{2} - \frac{1}{3} \\ &= \frac{1}{6}\end{aligned}$$

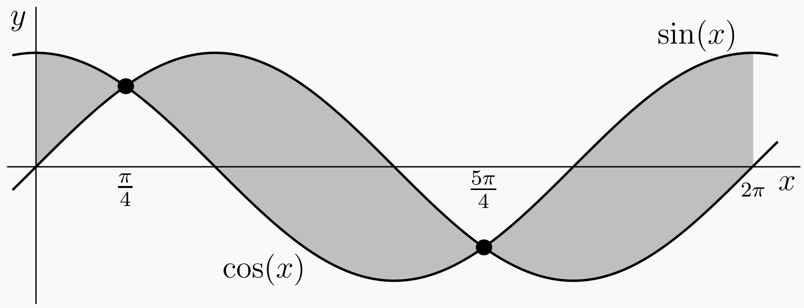
Note that if the curves intersect multiple times, you might have to split the integral onto the corresponding intervals.

Example 3.3.0.2. A Region with More Crossings

Find the area between the graphs of sine and cosine between $x = 0$ and $x = 2\pi$.
Again, to accomplish this, we set the two formulas equal to each other to solve for the points of intersection.

$$\sin(x) = \cos(x) \implies \tan(x) = 1 \implies x = \pi/4 \text{ or } x = 5\pi/4$$

Thus the points of intersection are at $\left(\pi/4, \sqrt{2}/2\right)$ and $\left(5\pi/4, -\sqrt{2}/2\right)$.



We now compute the area.

$$\begin{aligned} A &= \int_0^{\pi/4} (\cos(x) - \sin(x))\, \mathtt{d}x + \int_{\pi/4}^{5\pi/4} (\sin(x) - \cos(x))\, \mathtt{d}x + \int_{5\pi/4}^{2\pi} (\cos(x) - \sin(x))\, \mathtt{d}x \\ &= (\sin(x) + \cos(x))\,\big|_0^{\pi/4} + (-\cos(x) - \sin(x))\,\big|_{\pi/4}^{5\pi/4} + (\sin(x) + \cos(x))\,\big|_{5\pi/4}^{2\pi} \end{aligned}$$

Exercise 3.3.0.3. Complete the Example ☕☕☕

Finish the computation and verify the area is $4\sqrt{2}$.

Exercise 3.3.0.4. A Common Mistake ☕

Briefly write in words, why would simply evaluating

$$\int_{x=0}^{x=2\pi} \cos(x) - \sin(x)\, \mathtt{d}x$$

in the example above not give the area of the shaded region?

3.3.1 Area of a Circle

Let's now prove an old friend, the formula for the area of a circle!

Exercise 3.3.1.1. Area of a Circle ☕☕

- Recall the equation for a circle of radius r is $x^2 + y^2 = r^2$. Draw a diagram that shows that this equation is a consequence of the Pythagorean Theorem.
- Solve for y and note that the square root requires a "plus or minus". To get the top curve $f(x)$, choose the positive square root. To get the bottom curve $g(x)$, choose the negative square root. Write your formulas for $f(x)$ and $g(x)$ below:
  - Top Half: $f(x) =$
  - Bottom Half: $g(x) =$
- Use an integral to find the area between f and g to obtain the formula for the area of a circle of radius r.

3.3.2 Some Other Regions for Practice

Find the area between the following curves. Graph the curves and shade the region!

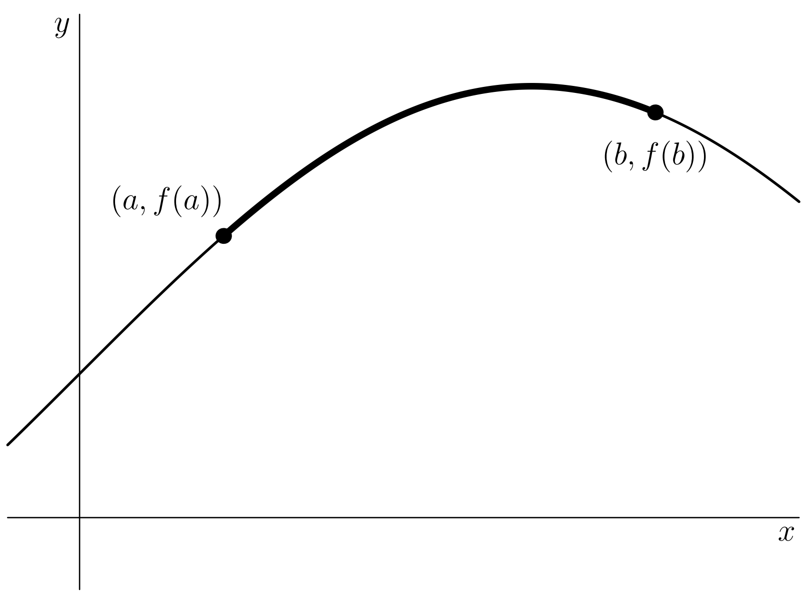
Exercise 3.3.2.1. Other Regions ☕☕

- $f(x) = x^3 - x^2 - x + 1$ and $g(x) = x^3 + x^2 - x - 1$
- $y = \sqrt{1 - x^2}$ and $y = 1/2$
- $y = \tan(x)$ restricted to the domain $(-\pi/2, \pi/2)$ and $y = \frac{4}{\pi}x$

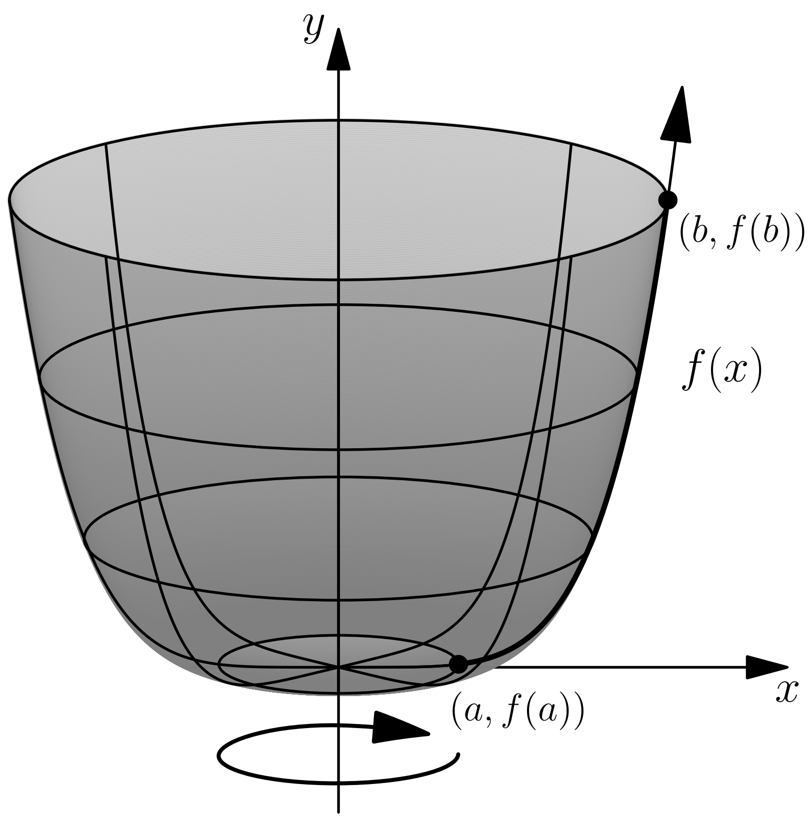
3.4 Arc Length, Surface Area, Volume

In this section, we will look at four formulas built out of integrals. Let's just get them all down here, and we'll play with them and explain them as we go!

1. **Arc Length:** The length of the graph of a function $f(x)$ from the point $(a, f(a))$ to the point $(b, f(b))$ is given by $L = \int_{x=a}^{x=b} \sqrt{1 + (f'(x))^2}\, \mathtt{d}x$.

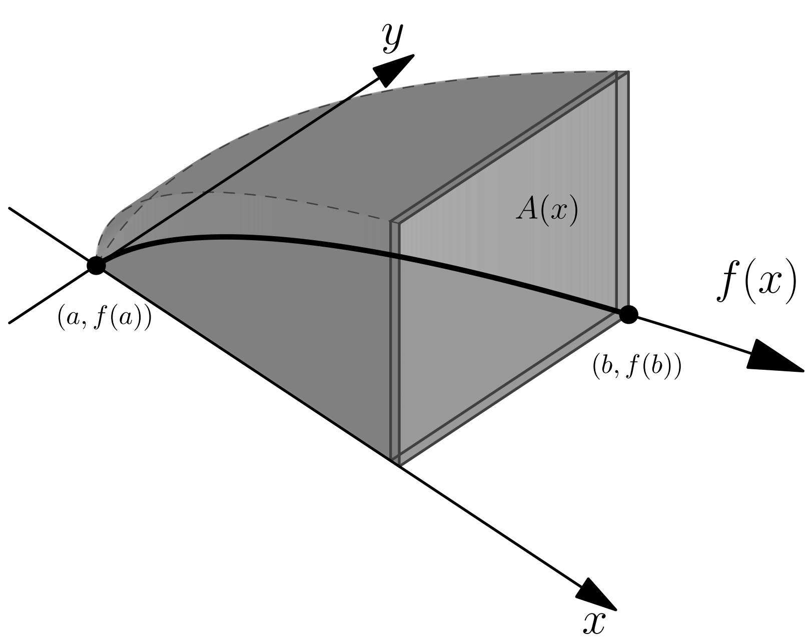


2. **Surface Area of a Surface of Revolution:** If the graph of $f(x)$ from the point $(a, f(a))$ to the point $(b, f(b))$ is revolved around the y-axis, the surface area is given by $SA = \int_{x=a}^{x=b} 2\pi x\sqrt{1 + (f'(x))^2}\, \mathtt{d}x$

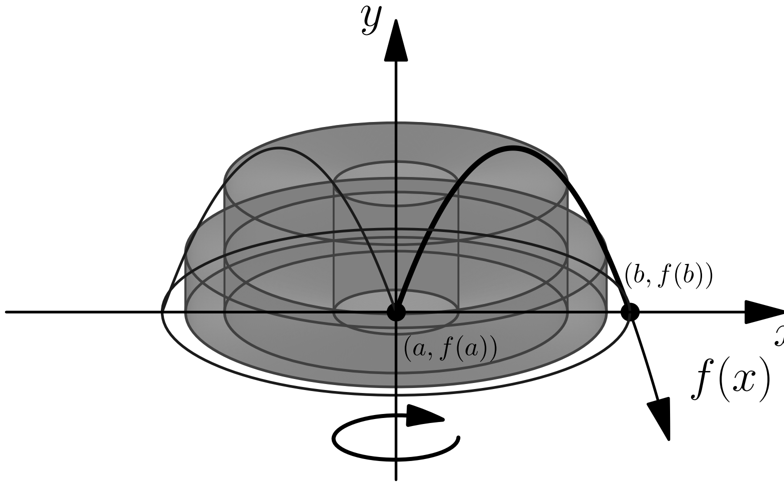


3. **Volume:** We have two distinct methods for volume.

   - **Cross Sections:** Suppose a 3D solid starts at $x = a$ and ends at $x = b$ and the function $A(x)$ represents the area of the cross-section at location x, then the solid has volume $V = \int_{x=a}^{x=b} A(x)\, \mathtt{d}x$.



- **Cylindrical Shells:** If the region under of $f(x)$ from the point $(a, f(a))$ to the point $(b, f(b))$ is revolved around the y-axis, the volume is given by $V = \int_{x=a}^{x=b} 2\pi x f(x)\, \mathtt{d}x$



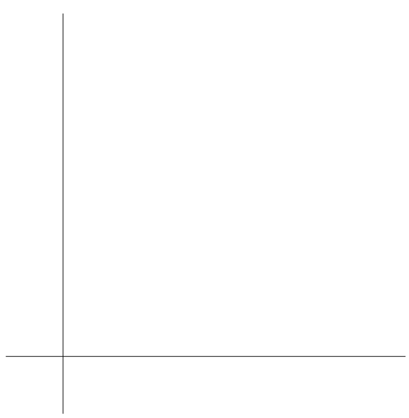
Let's now do a little example of each and throughout analyzing the example, convince ourselves that each formula is correct.

3.4.1 The Arc Length Formula

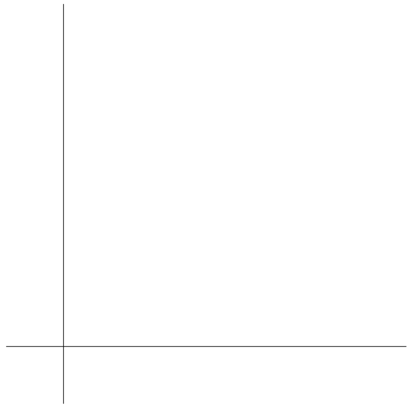
The arc length formula is in essence the following idea: to approximate the length of a curve, let's split it up into line segments, compute each of the line segment lengths using the Pythagorean Theorem, and then take the limit as the number of line segments goes to infinity. Time to carry this out on a good old vanilla parabola!

Exercise 3.4.1.1. Approximating the Length of a Parabola with Line Segments

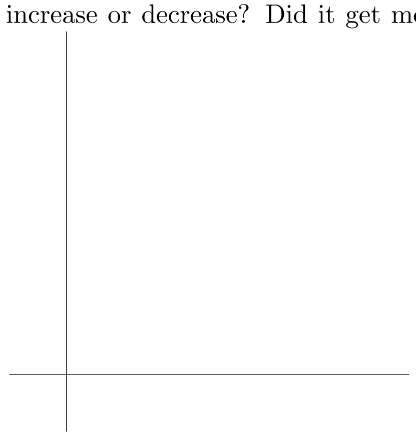
- On the axes below, graph the function $f(x) = x^2$ from the point $A = (0, 0)$ to $E = (1, 1)$. Estimate the length of this arc by just connecting those two endpoints with a straight line and calculating its length via the Pythagorean Theorem.



- Again graph the function $f(x)$ from the point $A = (0, 0)$ to $E = (1, 1)$, but on this graph also include a label for the point $C = (\frac{1}{2}, \frac{1}{4})$. This time lets estimate the length of this arc using two line segments. Specifically, calculate the lengths of $\overline{AC}$ and $\overline{CE}$ and add their lengths to estimate the arc length of the parabola. Did your estimate go up or down by using two segments instead of just one? Does this make sense?



- Define A, C, and E as before. Define B to be the point on the parabola with x-coordinate one-fourth and D to be the point on the parabola with x-coordinate three-fourths. Again estimate the length of the curve by adding the lengths of the segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DE}$. What happens to the estimate? Did it increase or decrease? Did it get more or less



accurate as compared to the true arc length?

Ok so at this point we get the idea. More line segments will result in a more accurate approximation, and the limit of these approximations as the number of line segments goes to infinity will produce the exact result. Here is a derivation that shows this process will result in exactly the right formula.

Let $x_0, x_1, x_2, \ldots, x_n$ be equally spaced points along the x-axis from a to b. That is, $x_0 = a$, $x_n = b$, and for each $i \in \{0, 1, 2, \ldots, n-1\}$, $\Delta x = x_{i+1} - x_i = \frac{b-a}{n}$.

With this setup, if we want the length of a line segment connecting points x_{i+1} and x_i, we would use the Pythagorean Theorem to obtain:

$$\sqrt{(\Delta x)^2 + (f(x_{i+1}) - f(x_i))^2}$$

as the length.

$$\begin{aligned}
L &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \sqrt{(\Delta x)^2 + (f(x_{i+1}) - f(x_i))^2} \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} \sqrt{1 + \frac{(f(x_{i+1}) - f(x_i))^2}{(\Delta x)^2}} \sqrt{(\Delta x)^2} \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} \sqrt{1 + \left(\frac{f(x_{i+1}) - f(x_i)}{\Delta x}\right)^2} \Delta x \\
&= \int_{x=a}^{x=b} \sqrt{1 + (f'(x))^2}\, \mathtt{d}x
\end{aligned}$$

Exercise 3.4.1.2. Exact Arc Length of the Parabola ☕☕☕

- We now apply this integral formula to calculate the exact length of our parabolic segment. Plugging in the formula $f(x) = x^2$ into the arc length integral, we obtain:

$$L = \int_{x=0}^{x=1} \sqrt{1 + (2x)^2}\, \mathtt{d}x$$

  Finish the evaluation of this integral. (**Hint:** The arc length integral will be quite difficult! See Example 2.2.35 for help.)

- How does the exact value of the arc length compare to the approximations?

3.4.2 Circumference of a Circle

Now let's use the integrals to compute the circumference of a circle. Though one can take the formula for the circumference of a circle as the definition of π, it is still nice to go through this as proof of concept.

Exercise 3.4.2.1. Circumference of a Circle ☕☕

- Use the arc length formula to calculate the circumference of a circle with radius r.

- Take the derivative of your formula for area of a circle with respect to r. How does it relate to your formula for the circumference of a circle? Draw a picture and indicate why this freakish coincidence actually geometrically makes sense.

3.4.3 Lengths of Some Other Fun Arcs

Draw the graph and compute the length of each of the following arcs using our arc length formula.

Exercise 3.4.3.1. Other Arcs ☕☕☕

1. Line segment from a point (x_0, y_0) to a point (x_1, y_1). (**Hint:** To you get your function, use the point-slope form of a line and then solve for y.) How does this compare to the length

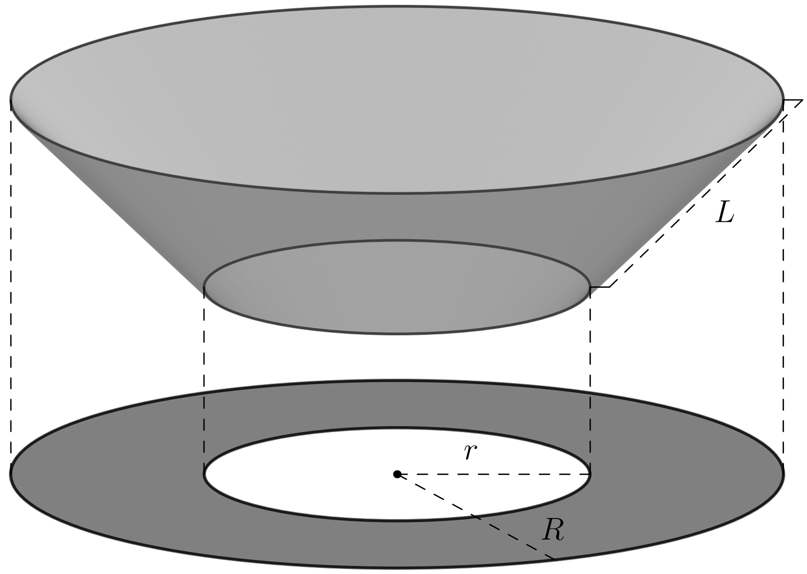
the Pythagorean Theorem would give you?

2. The graph of $f(x) = \ln(x)$ from $x = 1$ to $x = e$. Take an approximation using two line segments, and then get the exact length using an arc length integral. How do they compare?

3. The graph of $f(x) = e^x$ from $x = 0$ to $x = 1$.

3.4.4 Surface Area

For the surface area of a surface of revolution, we use the frustum of a right circular cone as the object that approximates the surface, much as we used line segments for arc length. Thus, to get off the ground, we must figure out what the surface area of a frustum is. Suppose it has inner radius r and outer radius R and let L be the diagonal length, top to bottom, of the frustum. Project the shape onto a flat plane sitting under it to produce a washer whose outer radius is R and inner radius is r.



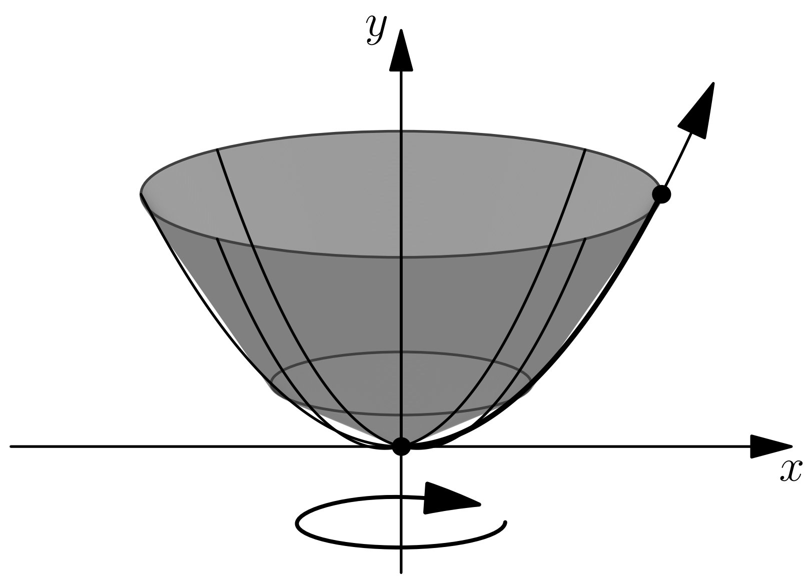
Notice that the washer in the plane has area $\pi(R^2 - r^2)$. Also, the radial lengths got uniformly scaled by a factor of $\frac{L}{R-r}$, since every segment of length $R - r$ got stretched into a segment of length L. Thus we have that the area of the frustum is $\frac{L}{R-r} \cdot \pi(R^2 - r^2)$, which simplifies to:

$$SA = \pi L(R + r)$$

Exercise 3.4.4.1. Parabolic Bowl ☕☕☕

To test this out, consider the parabola $y = x^2$ from $x = 0$ to $x = 1$. Create a surface by revolving this curve around the y-axis.

- Approximate the surface area of this region using two frusta, one that goes from $x = 0$ to $x = 1/2$, and one that goes from $x = 1/2$ to $x = 1$.



- As in the previous cases, we can see that chopping it up into more approximating regions will increase the accuracy of our approximation as the number of these goes to infinity. However, actually adding up the surface areas by hand would become frusta-ting, so instead we just set up a limit of a summation. Draw a diagram and set up the limit of a sum of surface areas of frusta, similarly to how we did for the arc length formula. Fill in the middle part of the derivation below:

$$\begin{aligned} L &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \pi \sqrt{(x_{i+1}-x_i)^2 + (f(x_{i+1})-f(x_i))^2}\,(x_{i+1}+x_i) \\ &= \\ &= \\ &= \\ &= \\ &= \\ &= \int_{x=a}^{x=b} 2\pi x \sqrt{1+(f'(x))^2}\,\mathrm{d}x \end{aligned}$$

- Use this integration formula to find the exact surface area of the parabolic bowl. How does it compare to your approximation?

Exercise 3.4.4.2. Surface Area of a Sphere

We now ask what the surface area of a sphere is. Since the right-hand side of a circle is not the graph of a function, we will just use the top right quarter circle to revolve about the y-axis and then multiply by 2 to pick up the bottom half of the sphere.

- In particular, use the function $f(x) = \sqrt{r^2 - x^2}$ for $x = 0$ to $x = r$ in our surface area integral to find the formula for the surface area for a sphere of radius r:

- Take the derivative of the formula for sphere volume with respect to r. How does it relate to the formula for surface area? Again, draw a diagram explaining this geometrically.

3.4.5 Volume

We have two fundamentally different methods: volume by cross sections and volume by cylindrical shells.

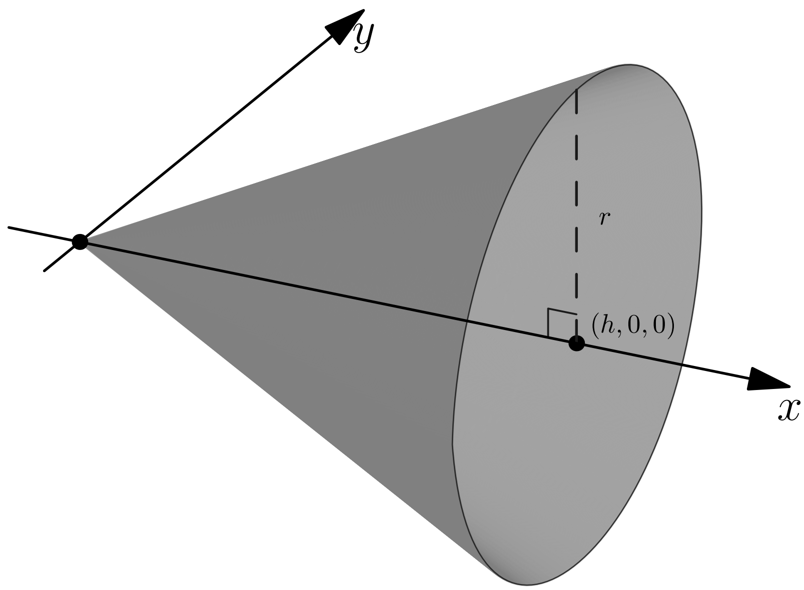
Volume by Cross-sectional Area

Recall that we calculate the area of a planar region by integrating the height at each x-coordinate; here we compute volume of a 3D solid by integrating the area at each x-coordinate. More formally, we say that the volume of a 3D figure that starts at $x = a$ and ends at $x = b$ with the function $A(x)$ returning the area of the cross-section at coordinate x is given by:

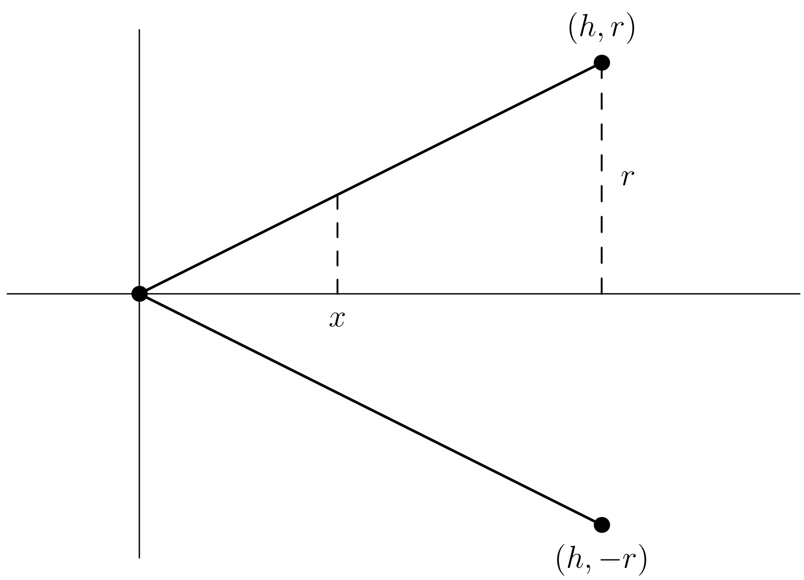
$$V = \int_{x=a}^{x=b} A(x)\,\mathtt{d}x$$

The Cone, Pyramid, and Tetrahedron

Let's try this out on a cone! Suppose we have a right circular cone of height h and radius r. Place the cone so that the vertex lies at the origin and the center of the base lies at the point $(h, 0, 0)$.



Any cross section parallel to the base is clearly a circle. Thus to compute the area of each circle we just need to find the radius of an arbitrary cross section at location x. To help us, we will imagine a 2D "side view" of the middle of the cone.



Notice the top boundary of this shape is the graph of the linear function $f(x) = \frac{r}{h}x$. This height is exactly the radius of the circular cross section of the cone at location x.

Exercise 3.4.5.1. Check the Boundary ☕

Briefly explain why this formula is the correct formula for the top boundary!

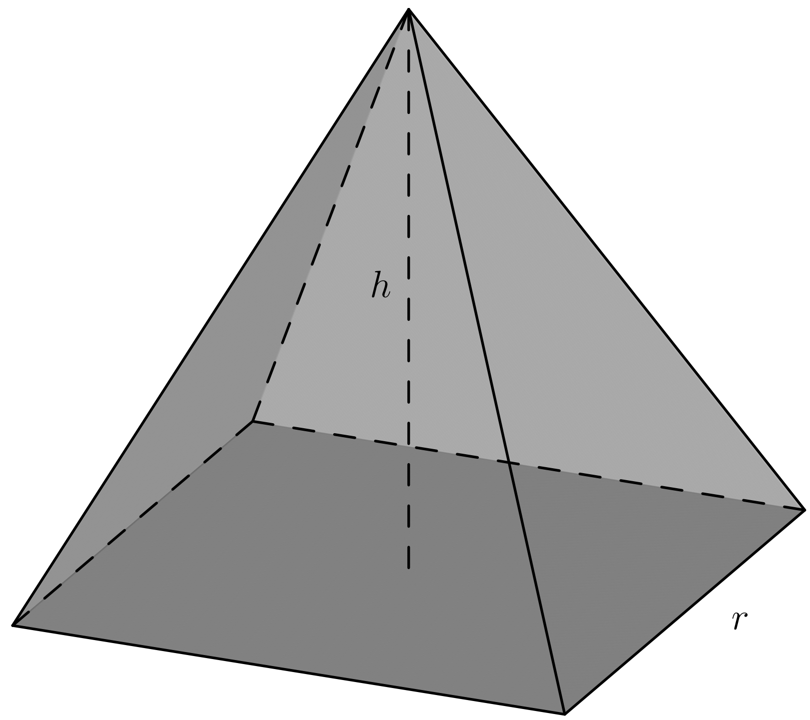
We can now set up and evaluate our volume integral.

$$\begin{aligned} V &= \int_{x=0}^{x=h} A(x)\, \mathtt{d}x \\ &= \int_{x=0}^{x=h} \pi \left(\frac{r}{h}x\right)^2 \mathtt{d}x \\ &= \int_{x=0}^{x=h} \pi \frac{r^2}{h^2} x^2 \, \mathtt{d}x \\ &= \pi \frac{r^2}{3h^2} x^3 \Big|_{x=0}^{x=h} \\ &= \pi \frac{r^2}{3h^2} h^3 - 0 \\ &= \frac{1}{3}\pi r^2 h \end{aligned}$$

Note that this is actually a very clean formula; it says the area of a cone is one-third times the area of the base times the height.

Exercise 3.4.5.2. The Pyramid ☕☕

- Consider a square base pyramid of side length s and height h. What would you conjecture for the volume of this solid based on our cone computation above?



- Use integration of cross sectional area to verify your conjecture and formally compute the volume of the pyramid.

Exercise 3.4.5.3. A Tetrahedron ☕☕

- In three dimensions, plot a tetrahedron that has vertices $(0,0,0),(a,0,0),(0,b,0)$, and $(0,0,c)$. Based on how the volumes came out for the cone and pyramid, what would you suspect for the volume of this figure?

- Use integration of cross sections to find the volume of that tetrahedron.

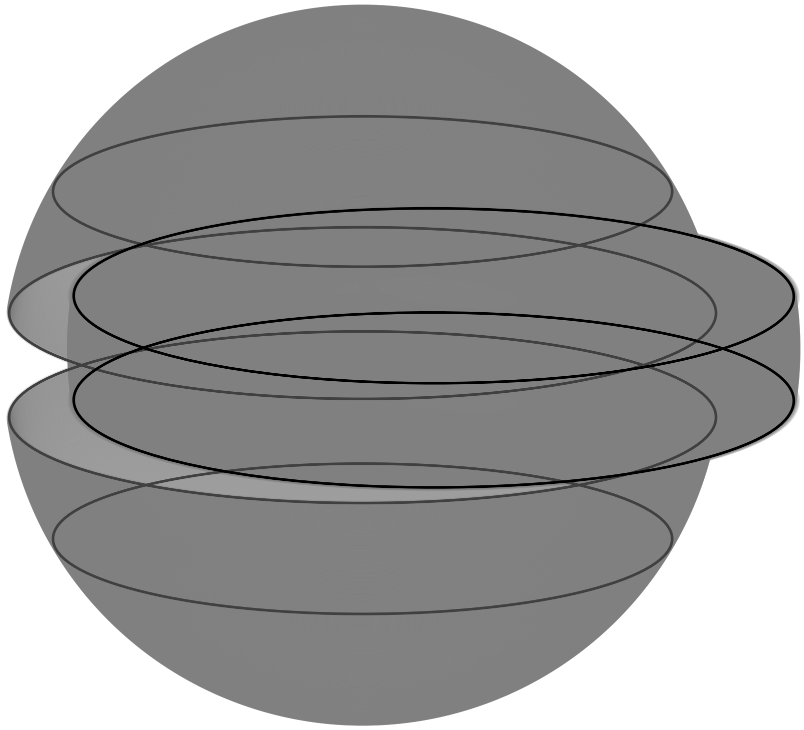
Exercise 3.4.5.4. Other Bases ☕☕☕☕

- What happens if you start with other shapes as the base of your figure? If you form a solid by connecting the boundary of the base to a point with line segments, do you always get just one-third times the area of the base times the height as the volume? Or, can you find some bases for which this formula does not hold?

The Sphere

Note the sphere is the solid of revolution constructed by rotating a circle centered at the origin about either the x or y axis.

First, let's use cross-sectional area, which many sources call the "disc method" since a cross-section of a sphere is a circular disc.



Exercise 3.4.5.5. The Volume of a Sphere ☕☕

- Draw a "side view" of the sphere much like we did for the cone. What is the formula for the top boundary curve?

- Compute the volume of a sphere via an integral of cross-sectional area:

Cylindrical Shells

Another technique that could have been used to compute the volume of the sphere is volume by cylindrical shells. If R is a region bounded by $x = a$ and $x = b$ on the sides, and bounded by $f(x)$ and $g(x)$ above and below respectively, the the volume of the surface of revolution of R is:

$$V = 2\pi \int_{x=a}^{x=b} r(x)(f(x) - g(x))\,\mathtt{d}x$$

where $r(x)$ is the radius of revolution at coordinate x.

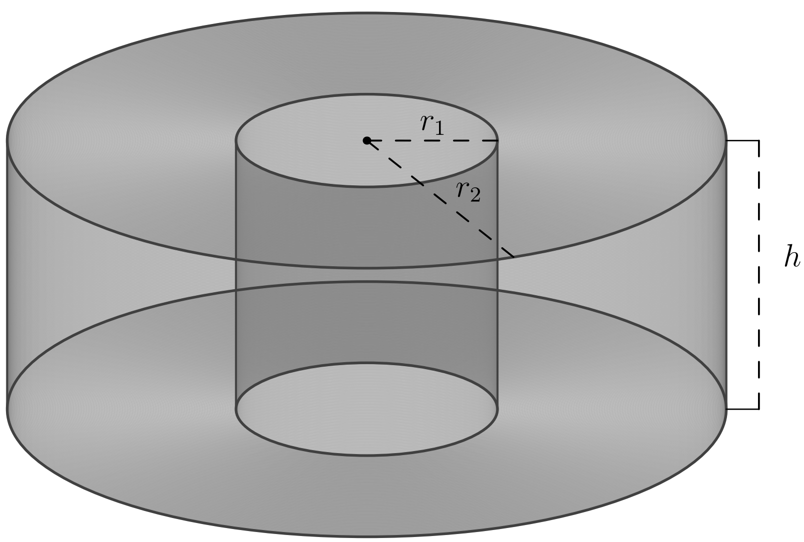
Note that this comes from approximating the volume of the region by using nested cylinders with smaller cylinders deleted from their middle (hence *cylindrical shells*). In particular, we are cutting the region into shells that approximate the volume, and then taking the limit as the thickness of these shells goes to zero (and the number of shells goes to infinity).

Let us play with an analogous example to what we did for the arc length of the parabola to see where the above formula comes from.

Cylindrical Shells in a Small Example

Definition 3.4.5.6. Cylindrical Shells

A *cylindrical shell* is a cylinder with a second cylinder of equal height but smaller radius deleted out of the middle of it.

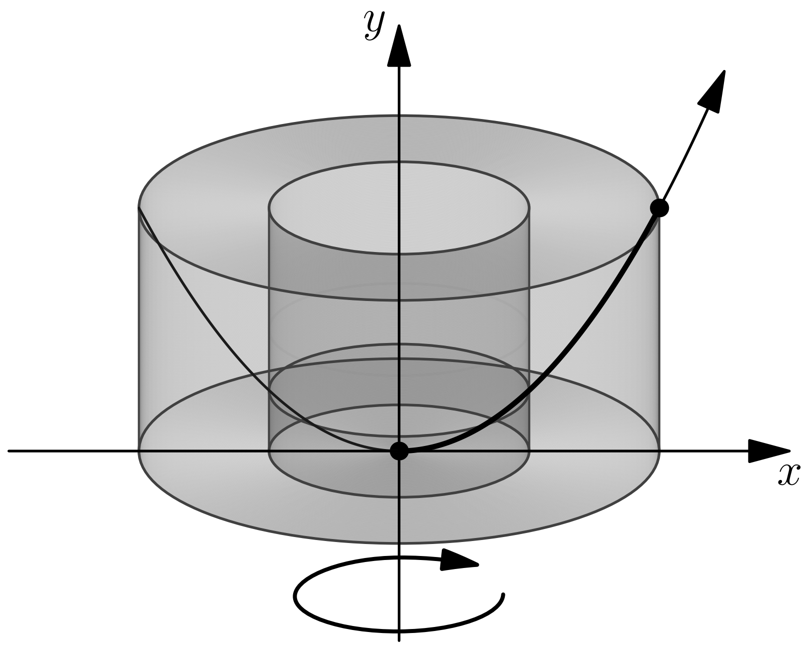


Exercise 3.4.5.7. Playing with Cylindrical Shells ☕

See the diagram above of a cylindrical shell with height h, outer radius of r_2, and inner radius r_1.

Show the volume of that shell is given by $V = h\pi\left(r_2^2 - r_1^2\right)$.

Suppose we take the region under $f(x) = x^2$ between $x = 0$ and $x = 1$ and revolve it around the y-axis. We could estimate the volume using two cylindrical shells as follows:



- One cylinder of height one-fourth and radius one-half, centered at the y-axis. (We can consider this to be a shell where the inner deleted cylinder had radius zero.)
- One cylinder of height one and radius one, centered at the y-axis, but with a cylinder of height one and radius one-half deleted out of the middle of it.

Exercise 3.4.5.8. Volumes Approximated by Shells

- Compute the approximate volume of that region by adding the volumes of the cylindrical shells described above.

- Draw the same region but this time split it into four cylindrical shells with x-coordinates at zero, one-quarter, one-half, three-quarters, and one. Draw a diagram showing the shells and compute the approximate volume. How does this compare to the previous approximation?

Once again, we get the idea. Chopping it up into more shells will produce a more accurate approximation, and taking the number of shells to infinity will then make the approximations approach the exact answer. We now build up the cylindrical shells volume formula much in the same manner we did for the arc length integral.

Suppose we wish to find the volume of the region under the graph of $f(x)$ from the point $(a, f(a))$ to the point $(b, f(b))$ revolved around the y-axis. We begin by splitting into n cylindrical shells. Specifically, let $x_0, x_1, x_2, \ldots, x_n$ be equally spaced points along the x-axis from a to b. That is, $x_0 = a$, $x_n = b$, and for each $i \in \{0, 1, 2, \ldots, n-1\}$, $\Delta x = x_{i+1} - x_i = \frac{b-a}{n}$.

With this setup, if we want the volume of the cylindrical shells between points x_{i+1} and x_i, we would use our volume of a cylindrical shell formula to obtain

$$f(x_{i+1})\pi\left(x_{i+1}^2 - x_i^2\right)$$

as the volume. We then add up the volumes of all shells and take the limit as the number of shells goes to infinity:

$$\begin{aligned}
L &= \lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_{i+1})\pi\left(x_{i+1}^2 - x_i^2\right)\\
&= \lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_{i+1})\pi\left(x_{i+1} - x_i\right)\left(x_{i+1} + x_i\right)\\
&= \lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_{i+1})\pi\left(x_{i+1} - \Delta x + x_{i+1}\right)\Delta x\\
&= \lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_{i+1})\pi\left(2x_{i+1} - \Delta x\right)\Delta x\\
&= \lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_{i+1})\pi\left(2x_{i+1}\right)\Delta x - \lim_{n\to\infty}\Delta x\sum_{i=0}^{n-1} f(x_{i+1})\pi\Delta x\\
&= \int_{x=a}^{x=b} 2\pi x f(x)\,\mathtt{d}x - \lim_{n\to\infty}\frac{b-a}{n}\int_{x=a}^{x=b} f(x)\pi\,\mathtt{d}x\\
&= \int_{x=a}^{x=b} 2\pi x f(x)\,\mathtt{d}x - 0
\end{aligned}$$

Thus, the exact volume is given by

$$V = \int_{x=a}^{x=b} 2\pi x f(x)\,\mathtt{d}x$$

Exercise 3.4.5.9. Parabolic Bowl ☕☕☕

- Use this formula to compute the exact volume of the region under the parabola, revolved about the y-axis. Specifically, evaluate the integral $V = \int_{x=0}^{x=1} 2\pi x(x^2)\, \mathrm{d}x$. How does the exact volume compare to the approximations?

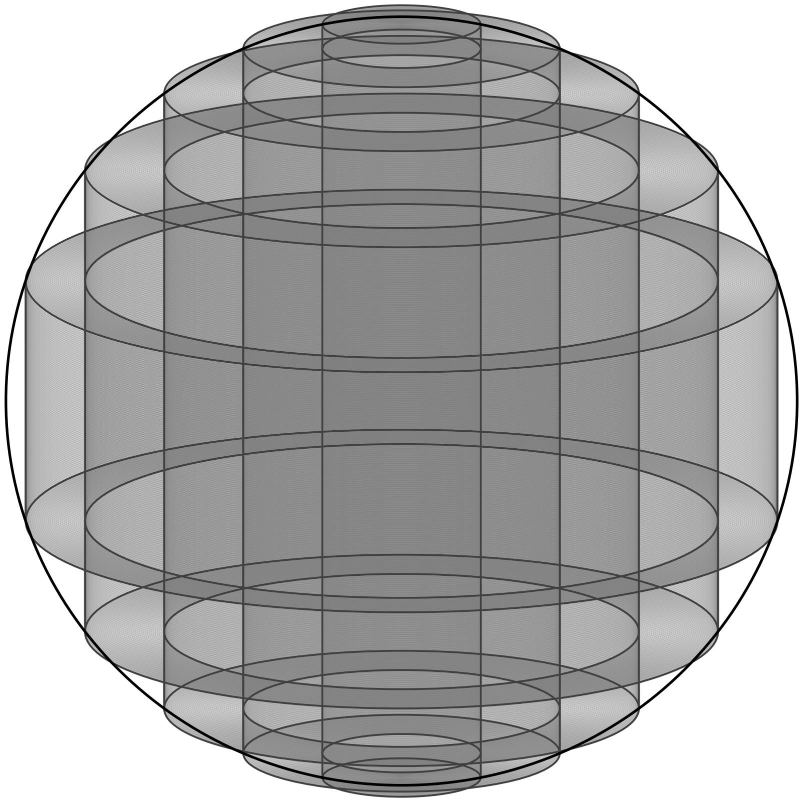
- Compute the exact volume of the same region using integration of cross sections. Verify you get the same result!

Exercise 3.4.5.10. Volume of a Cone, Again! ☕☕

Use integration by cylindrical shells to again compute the volume of a cone with circular base of radius r and height h. Verify you get the same result! (**Hint:** To set up this region, this time place the center of the circular base at the origin and then obtain your $f(x)$ from slope-intercept form of the line connecting the points $(0, h)$ and $(r, 0)$.)

Exercise 3.4.5.11. Volume of a Sphere, Again! ☕☕

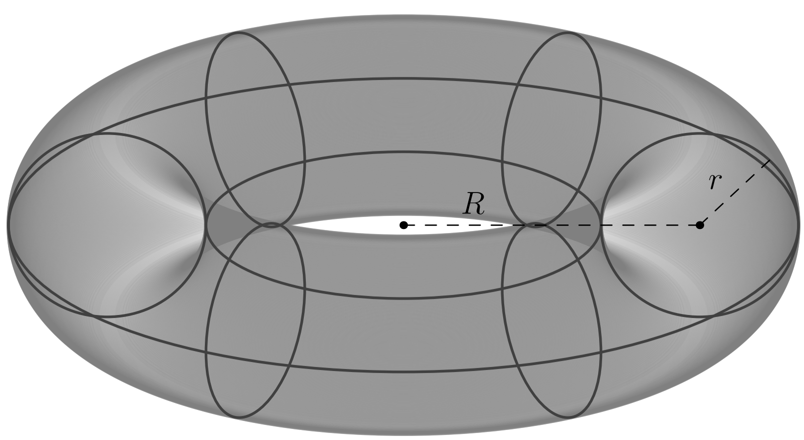
- Notice that a sphere can be obtained by taking the right half of a disc and revolving it about the y-axis. Thus we can use shells to produce an alternate derivation of the volume of a sphere. Draw a side view of the sphere. What are the top and bottom boundary curves?



- Compute the volume of a sphere of radius r via shells below. Verify it is the same as what you got via cross-sections.

3.4.6 The Torus

Torus is the formal name for a doughnut. Note that we can define a torus via two radii: R, the distance from the center of the doughnut hole to the center of the part you eat, and r, the radius of the circle that is the cross section of the doughnut if you cut it vertically.



We now play the same game we played with the sphere and the circle! Let's find the volume and the surface area and then determine how they relate via the derivative.

Exercise 3.4.6.1. Volume and Surface Area of a Torus ☕☕☕

1. First, use cross sections to find the volume of the torus. (Note here your cross-sections are large circles with small circles deleted, so this is what many references will call the *washer*

method though it is really just a special case of cross sections.):

2. Second, use shells to find the volume of a torus. Verify you get the same answer as via cross-sections.

3. Finally, find the surface area of a torus via our surface area formula.

4. If you take the derivative of volume a the torus do you get the surface area of a torus? Do you need to differentiate with respect to R, r, or does it not matter? What does all of this have to do with glazing a doughnut?

3.5 Center of Mass

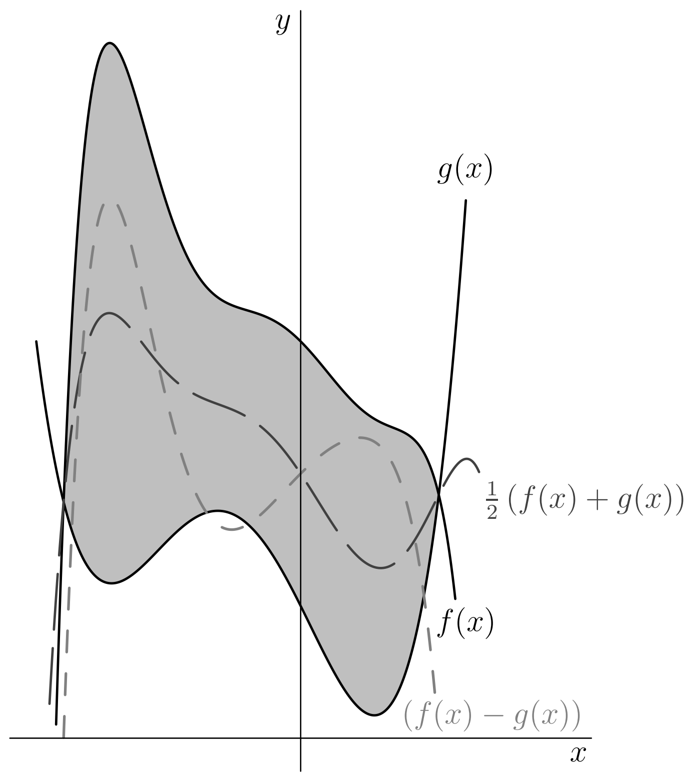
Let R be a region bounded above by $f(x)$, below by $g(x)$, on the left by $x = a$, and on the right by $x = b$. Let $(\bar{x}, \bar{y})$ be the center of mass of R. The formulas for the x- and y- coordinates of the center of mass of a 2D plate in the shape of R:

$$\bar{x} = \frac{1}{m} \int_{x=a}^{x=b} x\,(f(x) - g(x))\,dx$$

$$\bar{y} = \frac{1}{m} \int_{x=a}^{x=b} \frac{1}{2}\,(f(x) + g(x))\,(f(x) - g(x))\,dx$$

The integrals above are often referred to as *moments*, a term physicists use to describe the turning effect of a force. The physical interpretation of the center of mass is as follows: this is in theory the point where we could perfectly balance our 2D figure on a post. Also in most physics or engineering applications, one can replace the entire figure with a point mass located at the center of mass. This simplifies many otherwise difficult problems. (If you are ever going to take the *Fundamentals of Engineering* exam, this is a huge trick people use!)

Note: Here we make the simplifying assumption that the unit of mass we are using is 1 unit of mass per unit of area. Thus, m can be computed as the area of R.

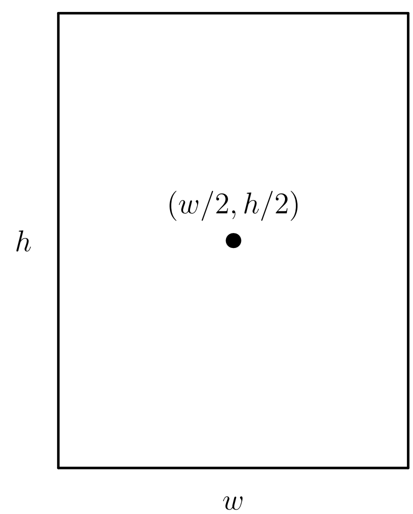


Comment: These formulas are plausible, as in some sense we are measuring the tendency to rotate about an axis. We can see the first integral is accumulating x (the length of the torque arm) times $(f(x) - g(x))$, which is measuring the height of the figure at location x. The second integral is much less intuitive, though we can see it again as accumulating the heights times the average of the y coordinate of the boundaries, which in aggregate gives us a measure of the central tendency of the vertical component of the figure. These formulas will be proven in Calc III via double-integration.

We now play with center of mass for a few of our common shapes.

3.5.1 The Rectangle

Suppose we have a rectangle of width w and height h. If we coordinatize the rectangle by placing the lower-left corner at the origin and the bottom side on the positive x-axis, we would expect that the center of mass should come out to be the point $(w/2, h/2)$ since that intuitively is the "middle" of our rectangle. Let's try out the integrals and verify this is indeed what we get. Specifically:

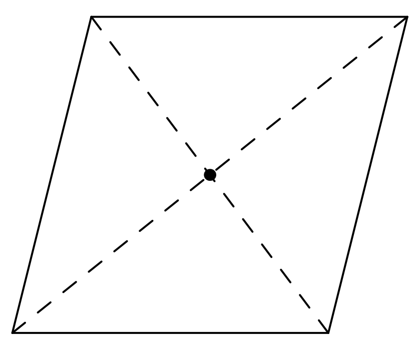


Exercise 3.5.1.1. A Rectangle

1. Let $f(x) = h$ the constant function whose graph is a horizontal line at height h, be the top of the rectangle. Let $g(x) = 0$, the constant function whose graph is a horizontal line at height zero, be the bottom of the rectangle. Let the lines $x = 0$ and $x = w$ be the left and right boundaries of the rectangle. Sketch this figure below.

2. Use our center of mass formulas to compute $(\bar{x}, \bar{y})$ and verify that it is in fact $(w/2, h/2)$.

3.5.2 The Parallelogram



Again by intuition, we would imagine the center of mass of a parallelogram should be at the intersection of the diagonals. Let's see if this is in fact where the center of mass of a parallelogram must lie. We will do this in three steps:

Exercise 3.5.2.1. Parallelogram ☕☕

1. Coordinatize the parallelogram. Choosing "nice" coordinates for our figure is the first step. Without loss of generality, we can make one corner of our parallelogram to be the origin and choose one side to lie on the positive y-axis, much like the rectangle. The beauty of this is then the parallelogram is fully determined by only three arbitrary parameters, the horizontal and vertical coordinates of the upper right corner, and the vertical coordinate of

the upper left corner. Call these a, b, and c respectively.

2. Compute linear equations for the diagonals in terms of a, b, and c. Solve a simultaneous system of linear equations to find the coordinates of the intersection of these lines.

3. Use the integrals for center of mass to compute the actual center of mass. See if it agrees with the coordinates of the point of intersection computed above.

Ok nice, so it did work out to be the same! Well it kind of had to there, right? I mean what else could the center of mass of a parallelogram have been?

3.5.3 The Triangle

And now for a case where the end of the story is far less predictable! Consider the triangle. For the triangle there are *four* completely reasonable geometric guesses as to what the "center" of a triangle could be.

Exercise 3.5.3.1. Different Notions of Center of a Triangle ☕

Sketch corresponding diagrams for four of Euclid's most beautiful theorems:

1. The altitudes of a triangle intersect in a point. (This point is called the *orthocenter*.)

2. The medians of a triangle intersect in a point. (This point is called the *barycenter*.)

3. The perpendicular bisectors of a triangle intersect in a point. (This point is called the

circumcenter.)

4. The angle bisectors of a triangle intersect in a point. (This point is called the *incenter.*)

So we ask again... which one is actually the center of mass? Or is it something different entirely that is not on our list of guesses above? Well, let's figure it out. The steps for determining this are essentially the same as for the parallelogram.

Exercise 3.5.3.2. The Triangle ☕☕☕

1. Coordinatize the triangle. Choosing "nice" coordinates for our figure is the first step. Without loss of generality, we can make one corner of our triangle to be the origin and choose one side to lie on the positive y-axis, much like the rectangle and parallelogram. Now the triangle is fully determined by only three arbitrary parameters, the horizontal and vertical coordinates of the only corner not on the y-axis, and the y-coordinate of the point on the y-axis but not at the origin. Similar to the parallelogram, respectively call these a,b, and c. Note how helpful this is... a generic triangle in the plane would be determined by six

parameters! (two per corner)

2. Note that for extreme cases, the circumcenter and orthocenter can actually lie outside of the triangle. This means these are likely to be incorrect guesses, as we would intuitively think the center of mass of the triangle should always lie inside the triangle itself. Thus, we won't expend effort trying to find coordinates for the orthocenter or circumcenter. To confirm this, draw one example of a triangle below that has orthocenter outside and one example of a triangle that has circumcenter outside.

Going down the list, let's find the barycenter. The process is the same as above: let's find equations for two of the medians and find their point of intersection. (Note that thanks to Euclid's theorem that the three medians intersect in a point, it is not necessary to use the equation for the third median as it is guaranteed to pass through the same point of intersection.)

3. Use the integrals for center of mass to compute the actual center of mass. See if it agrees with the coordinates of the point of intersection computed above.

4. Explain why at this point we do not need to try the incenter.

3.5.4 The Half Disc

Ok, now for the half disc, where there isn't even a reasonable geometric basis for a guess!

Exercise 3.5.4.1. The Half Disc ☕☕

Find the center of mass of an upper-half circle of radius r. Use $f(x) = \sqrt{r^2 - x^2}$ as your upper boundary and $g(x) = 0$ as the lower. Sketch your figure and its center of mass below. (**Hint:** You can determine one coordinate of the center of mass by symmetry. So you only need to compute one of the moments.)

Exercise 3.5.4.2. Center Outside ☕☕☕☕

Can you come up with a shape whose center of mass lies outside the shape? Find such an example, or explain why this is not possible.

3.6 Pappus' Theorem

The next result beautifully combines two of our previous topics, center of mass and surface areas/volumes of figures of revolution.

Let's revisit the torus and notice a too-good-to-be-true-but-it-is kind of fact.

Exercise 3.6.0.1. Surface Area as Perimeter Times Length of Revolution! ☕

Recall our construction of the torus as the revolution of the circle given by

$$(x - R)^2 + y^2 = r^2$$

about the y-axis.

- Consider the center of the circle, $(R, 0)$. What is the length of the path this point takes as it completes one revolution about the y-axis?

- What is the perimeter of the circle?

- Multiply the two above quantities together. How does this product compare to the surface area of the torus as computed in Exercise 3.4.6.1?

Part II

Sequences and Series

Chapter 4

Sequences and Series: Commas and Plus Signs Run Amok

4.1 Definition of Sequences

Definition 4.1.0.1. Sequence

A *sequence* is a function whose domain is $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, the set of natural numbers.

Typically for $n \in \mathbb{N}$, we write a_n as the output corresponding to the input n. Technically the sequence itself is the map $n \mapsto a_n$, though since this is a bit cumbersome to write, we often write just a_n to refer to the entire sequence, similar to how we write $f(x)$ for a function on the real numbers.

Less formally, a sequence is simply a list of objects. The correspondence between sequences as maps and sequences as lists of objects is that the k^{th} object in the list is the output corresponding to $k-1$ under the map. That is, the map $n \mapsto a_n$ corresponds to the list $a_0, a_1, a_2, a_3, \ldots$.

Example 4.1.0.2. The Sequence of Even Natural Numbers

Consider the list of nonnegative even numbers: $0, 2, 4, 6, 8, 10, \ldots$. We can view this as the map $n \mapsto 2n$, which we can see as a function on the naturals with the following inputs and outputs:

$$\begin{gathered} 0 \mapsto 0 \\ 1 \mapsto 2 \\ 2 \mapsto 4 \\ 3 \mapsto 6 \\ 4 \mapsto 8 \\ \vdots \end{gathered}$$

4.2 Explicit Formulas

Often times we define a sequence via what is called an *explicit formula*. This is a formula given in terms of n that shows explicitly how to compute the output corresponding to an input n.

Example 4.2.0.1. The Sequence of Even Natural Numbers: Explicit Formula

The sequence of even natural numbers defined above has

$$a_n = 2n$$

as its explicit formula.

Exercise 4.2.0.2. Practice with Explicit Formulas

Find an explicit formula for the sequence of...

- ...odd natural numbers.

$$1, 3, 5, 7, \ldots$$

- ...even integers starting at -4 and counting upwards, two at a time.

$$-4, -2, 0, 2, \ldots$$

- ...all multiples of 5, starting from 20 and counting downwards.

$$20, 15, 10, 5, 0, -5, \ldots$$

- ...all natural numbers that are one more than a multiple of 3.

$$1, 4, 7, 10, \ldots$$

- ...consecutive powers of 2, starting from 1.

$$1, 2, 4, 8, \ldots$$

- ...terms that alternate forever between positive and negative one.

$$1, -1, 1, -1, \ldots$$

- ...fractions whose numerators are all even natural numbers starting from zero and whose denominators are all odd natural numbers starting from 1.

$$\frac{0}{1}, \frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \ldots$$

4.3 Recursive Formulas

Recursion is a beautiful, powerful, and useful concept. It is the idea of defining a structure in terms of smaller instances of that same type of structure. In the case of sequences, we want to define a later term a_n as a formula given in terms of a_k for some k values strictly less than n. This definition of later terms built out of previous terms is called the *recursion*. Additionally, a recursive definition requires the definition of some initial term or terms to get the process rolling. These early terms are called the *base cases* or *initial terms*.

Returning to our favorite little example once again, we ask how we can find a recursive formula for the sequence of even numbers. Notice how the later terms relate to the earlier terms; each term is exactly two more than the previous term. We build a recursive formula out of this observation.

Example 4.3.0.1. The Sequence of Even Natural Numbers: Recursive Formula

$$\begin{aligned} a_0 &= 0 \\ a_n &= 2 + a_{n-1} \text{ for } n \geq 1 \end{aligned}$$

Exercise 4.3.0.2. Absorbing the Language ☕

In the formula above, which part is the base case and why? Which part is the recursion and why?

Exercise 4.3.0.3. Practice with Recursive Formulas ☕☕

Find a recursive formula for the sequence of...

- ...odd natural numbers.
$$1, 3, 5, 7, \ldots$$
- ...even integers starting at -4 and counting upwards, two at a time.
$$-4, -2, 0, 2, \ldots$$
- ...all multiples of 5, starting from 20 and counting downwards.
$$20, 15, 10, 5, 0, -5, \ldots$$
- ...all natural numbers that are one more than a multiple of 3.
$$1, 4, 7, 10, \ldots$$
- ...consecutive powers of 2, starting from 1.
$$1, 2, 4, 8, \ldots$$
- ...terms that alternate forever between positive and negative one.
$$1, -1, 1, -1, \ldots$$
- ...fractions whose numerators are all even natural numbers starting from zero and whose denominators are all odd natural numbers starting from 1.
$$\frac{0}{1}, \frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \ldots$$

4.4 Factorials

Some sequences have no simple explicit formula and are most easily thought of recursively. The sequence of factorials is a famous example of this type.

4.4.1 Recursive Formula for Factorials

Example 4.4.1.1. Factorials, Defined Recursively

Consider the following recursively defined sequence:

$$\begin{aligned} a_0 &= 1 \\ a_n &= n \cdot a_{n-1} \text{ for } n \geq 1 \end{aligned}$$

We can unwind this recursion a bit to obtain a more accessible expression for factorials. Observe the following calculations based on the base case and recursion given above:

$$\begin{aligned} a_0 &= 1 = 1 \\ a_1 &= 1 \cdot a_0 = 1 \\ a_2 &= 2 \cdot a_1 = 2 \cdot 1 = 2 \\ a_3 &= 3 \cdot a_2 = 3 \cdot 2 \cdot 1 = 6 \\ a_4 &= 4 \cdot a_3 = 4 \cdot 3 \cdot 2 \cdot 1 = 24 \\ a_5 &= 5 \cdot a_4 = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \end{aligned}$$

This sequence comes up so frequently that we give it its own symbol, the exclamation point! Since the factorial of n always amounts to the product of all natural numbers greater than or equal to 1 but less than or equal to n, we write the following:

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$$

Note that almost any expression involving a shady "$\cdots$" is truthfully a recursion in disguise!

Exercise 4.4.1.2. Why is the Factorial of Zero Equal to One? ☕

Looking carefully at the above definition, you will notice that

$$0! = 1$$

It is a common mistake to compute 0! as 0 instead. Here is one way to see why it should in fact be 1.

- If you compute 2^2, how many numbers are you multiplying together?
- If you compute 2^1, how many numbers are you multiplying together?

- If you compute 2^0, how many numbers are you multiplying together?
- Right, zero. A product like this is called an *e*mpty product and is always defined to be one, since that is the multiplicative identity.
- If you compute 3! as the product of all natural numbers greater than or equal to 1 and but less than or equal to 3, how many numbers are you multiplying together?
- If you compute 2! as the product of all natural numbers greater than or equal to 1 and but less than or equal to 2, how many numbers are you multiplying together?
- If you compute 1! as the product of all natural numbers greater than or equal to 1 and but less than or equal to 1, how many numbers are you multiplying together?
- If you compute 0! as the product of all natural numbers greater than or equal to 1 and but less than or equal to 0, how many numbers are you multiplying together?
- Since 0! is also an empty product (much like 2^0), what should we define it to be?

The following type of simplification will occur frequently throughout the our adventures in infinite series and power series. They all follow directly from the recursive definition of factorials.

Exercise 4.4.1.3. Simplifying Factorials ☕☕

Let n be a natural number greater than or equal to 1. Reduce the following fractions! (Are they fractorials?)

- $\frac{n!}{(n+1)!}$
- $\frac{(n+1)!}{n!}$
- $\frac{(n+2)!}{n!}$
- $\frac{(2n+2)!}{(2n)!}$

4.4.2 Explicit Formula for Factorials: The Gamma Function

The gamma function, denoted Γ, is the most common smooth interpolation of the factorial function on the positive integers. The gamma function is defined using improper integrals.

Definition 4.4.2.1. The Gamma Function

For $n \in \mathbb{R}$, define

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, \mathtt{d}x$$

That is, $\Gamma(n)$ is the area under the graph of $x^{n-1}e^{-x}$ in the first quadrant.

Exercise 4.4.2.2. Computing Values of Gamma ☕☕☕

- To see the manner in which the gamma function provides a continuous analog of the factorial function, fill out the values in the following table:

n	$n!$	$\Gamma(n)$
1		
2		
3		
4		

  Computing the values of the gamma function will require quite a bit of work. You don't have to show all details of the integrals above, but make sure you are comfortable doing such manipulations by hand.

- Given the table above, conjecture an explicit formula for the factorials. It won't be a nice little algebraic formula, but express it in terms of an improper integral. Write the conjecture below.

$$n! = \int_0^\infty \qquad\qquad\qquad\qquad \mathtt{d}x$$

Observe that you were be able to use the smaller instances of the gamma function to help you compute the larger instances! That is, when you apply integration by parts to compute $\Gamma(n)$, it will produce an expression that involves the integral you computed for $\Gamma(n-1)$.

It turns out this relationship is exactly what shows that the Γ function will *always* match the values of the factorial function. For factorial, we have:

$$n! = n \cdot (n-1)!$$

Exercise 4.4.2.3. Gamma Recursion ☕☕☕

What is the corresponding relationship for the Gamma function? Specifically, how does $\Gamma(n)$ relate to $\Gamma(n-1)$? Write your answer below.

4.4.3 An Approximation for the Factorials

As mentioned above, there does not exist a simple algebraic explicit formula for the factorial function. However **Stirling's Formula** gives a very nice explicit asymptotic formula.

Stirling's Formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

That is to say, as n approaches infinity, $n!$ approaches $\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. We don't have the tools to fully prove Stirling's Formula in this course, though it will be occasionally helpful to have a rough measure of the growth order of a factorial function.

Exercise 4.4.3.1. Comparing Factorial Growth Orders ☕☕

Rank the following functions in growth order from smallest to largest:

$$a_n = n^n$$
$$b_n = e^n$$
$$c_n = n^2$$
$$d_n = n!$$

4.5 Arithmetic and Geometric Sequences

Here we provide the definitions for two particularly famous families of sequences, arithmetic and geometric.

Definition 4.5.0.1. Arithmetic Sequence

A sequence a_n is called *arithmetic* if and only if there exists some real constant d such that $a_{n+1} - a_n = d$ for all natural numbers n. In such a sequence, the number d is called the *common difference.*

Definition 4.5.0.2. Geometric Sequence

A sequence a_n is called *geometric* if and only if there exists some real constant r such that $a_{n+1}/a_n = r$ for all natural numbers n. In such a sequence, the number r is called the *common ratio*.

Exercise 4.5.0.3. Playing with the Definition ☕☕

- Return to Exercise 4.2.2. Which of those sequences are arithmetic? For those that are, what is the common difference?

- Return again to Exercise 4.2.2. Which of those sequences are geometric? For those that are, what is the common ratio?

- Give an informal definition of an arithmetic sequence. (Think of what you would say if you had to explain what it was to a fifth grader).

- Give an informal definition of a geometric sequence. (Think again of what you would say if you had to explain what it was to a fifth grader).

- Give an example of a sequence that is arithmetic but not geometric.
- Give an example of a sequence that is geometric but not arithmetic.
- Can a sequence simultaneously be both arithmetic and geometric? If it is possible, give an example of such a sequence. If it is not possible, explain why it is not possible.

Exercise 4.5.0.4. Converting Between Recursive and Explicit Definitions ☕☕

- Write a sentence that explains the difference between defining a sequence recursively vs defining a sequence explicitly.
- Consider the following recursively-defined sequence:

$$a_0 = 5$$
$$a_n = 2 \cdot a_{n-1}$$

Write out the first five terms of this sequence. Can you find an explicit formula?

- Consider the following explicitly-defined sequence:

$$a_n = 3n - 2$$

Write out the first five terms of this sequence. Can you find an recursive formula?

4.6 Convergence of Sequences

4.6.1 Intuitive and Formal Definitions

Consider the sequence $a_n = \frac{1}{2^n}$. Listing out a few terms, we see that a_n looks like:

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$$

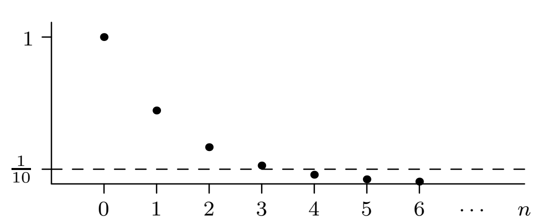
We would like a way to describe the long-term behavior of such a sequence. Intuitively, we see that the numbers are becoming arbitrarily close to zero.

Exercise 4.6.1.1. The Idea of Convergence

- How far into the sequence would you have to travel to find only terms that are no more than one-tenth from zero? (See diagram!)

- How far into the sequence would you have to travel to find only terms that are no more than one-hundredth from zero?

- How far into the sequence would you have to travel to find only terms that are no more than one-thousandth from zero?



No matter how small of a measurement we choose (one-tenth, one-hundredth, one-thousandth, etc), we could always find that after a certain point, all of our sequence terms are no further than that measurement from zero. This is exactly the notion we will reformulate in a more formal manner to define sequential convergence.

Recall the mathematical shorthands often used to help concisely state messy definitions: the symbol "$\forall$" means "for all" and the symbol "$\exists$" means "there exists". Also recall that for any real numbers a and b, the distance between a and b can be written as $|a-b|$.

We now define the limit of a sequence!

Definition 4.6.1.2. Convergence of a Sequence

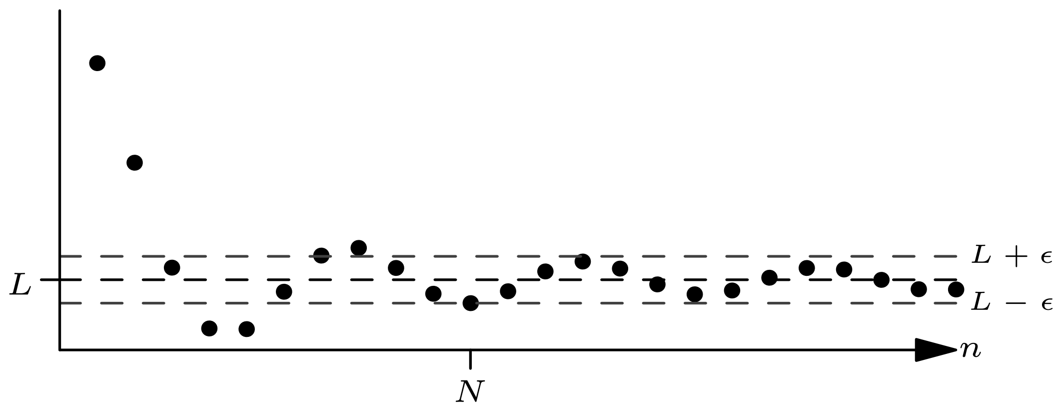
We say the sequence a_n converges to a limit $L \in \mathbb{R}$ and write

$$\lim_{n\to\infty} a_n = L$$

if and only if

$$\forall \epsilon > 0, \exists N \in \mathbb{R}, \forall n \in \mathbb{N}, n > N \implies |a_n - L| < \epsilon$$

If no such L exists, we say the sequence *diverges*.



Exercise 4.6.1.3. Digesting the Definition ☕☕

- In the definition of convergence, what role does ϵ play? Specifically, what is it bounding the distance between?

- In the definition of convergence, what role does N play? What role does n play?

- Restate the formal definition of sequential convergence in words rather than symbols. The statement

$$\lim_{n\to\infty} a_n = L$$

means...

Exercise 4.6.1.4. The Fibonacci Numbers ☕☕

Define the sequence of Fibonacci numbers F_n via the following recursive formula:

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_n &= F_{n-1} + F_{n-2} \end{aligned}$$

1. Compute the first eight terms of the Fibonacci sequence using the above recursion. That is, compute F_0 through F_7.

2. Compute the following quantities:

$$\begin{aligned} F_2/F_1 &= \\ F_3/F_2 &= \\ F_4/F_3 &= \\ F_5/F_4 &= \\ F_6/F_5 &= \\ F_7/F_6 &= \end{aligned}$$

3. What would you conjecture about

$$\lim_{n\to\infty} F_{n+1}/F_n$$

Does it seem to be going to infinity, zero, or stabilizing at something inbetween?

It is difficult to tell exactly what that limit of ratios is without knowing an explicit formula for the Fibonacci numbers. Stay tuned, as we will find this in a later chapter!

4.6.2 $N - \epsilon$ Proofs

This complicated definition can be unwound into a to-do list for what one must do to prove that a sequence converges to a particular limit. In particular, to show that the limit of a_n is equal to a number L, one must:

- Let ϵ be an arbitrary positive real number.
- Choose N, typically defined as a function of ϵ, since smaller values of ϵ will usually require a larger N to be chosen.
- Let n represent an arbitrary natural number greater than N.
- Using the definition of N and the assumption that $n > N$, prove that any corresponding a_n satisfies $|a_n - L| < \epsilon$.

Figuring out exactly what N should be in terms of ϵ usually requires a bit of algebra before the proof is written up. If the formula for a_n is clean enough, you might be able to just work backwards from the inequality $|a_n - L| < \epsilon$. If you solve it for n, you will find an expression that n must be larger than. Note here we are essentially just finding an inverse function for a_n.

Example 4.6.2.1. Solving for N

Let us solve for N with regards to our sequence $a_n = \frac{1}{2^n}$. Since here we suspect $L = 0$, we solve for n in the following inequality:

$$\begin{aligned}
\left|\frac{1}{2^n} - 0\right| &< \epsilon \\
\frac{1}{2^n} &< \epsilon \\
\frac{1}{\epsilon} &< 2^n \\
\ln\left(\frac{1}{\epsilon}\right) &< \ln\left(2^n\right) \\
\ln\left(\frac{1}{\epsilon}\right) &< n\ln(2) \\
\frac{\ln\left(\frac{1}{\epsilon}\right)}{\ln(2)} &< n
\end{aligned}$$

Thus we determined our choice of N, namely

$$N = \frac{\ln\left(\frac{1}{\epsilon}\right)}{\ln(2)}$$

Exercise 4.6.2.2. Justifying Our Work ☕

In words, annotate the above example to indicate why each line follows from the previous.

Now that we found our value for N, we are ready to follow the steps described above and construct our proof.

Example 4.6.2.3. Writing an $N - \epsilon$ Proof

Prove that

$$\lim_{n\to\infty} \frac{1}{2^n} = 0$$

Proof. Let ϵ be an arbitrary positive real number. Choose $N = \frac{\ln\left(\frac{1}{\epsilon}\right)}{\ln(2)}$. Let n be a natural number such that $n > N$. Under these circumstances, we wish to show that a corresponding a_n will be less than ϵ away from 0. Proceeding:

$$\begin{aligned}
\left|\frac{1}{2^n} - 0\right| &= \frac{1}{2^n} \\
&< \frac{1}{2^N} \\
&< \frac{1}{2^{\left(\frac{\ln\left(\frac{1}{\epsilon}\right)}{\ln(2)}\right)}} \\
&= \frac{1}{2^{\left(\log_2\left(\frac{1}{\epsilon}\right)\right)}} \\
&= \frac{1}{\frac{1}{\epsilon}} \\
&= \epsilon
\end{aligned}$$

Thus, for indices n that are larger than our choice of N, the corresponding terms in our sequence are less than ϵ away from zero as desired. □

Exercise 4.6.2.4. Justifying Our Work ☕

Once again in words, annotate the above example to indicate why each line follows from the previous. Pay particular attention to identify where we used the starting assumption that $n > N$.

As this course does not go through a general treatment of what constitutes a proof or how to come up with one, the example above could be taken as a template for how an $N-\epsilon$ proof should be written. In a more in-depth study of analysis, you will encounter more complicated situations where the above template may be too simplistic. It will be expanded upon when you have the right tools! For now, follow the above proof template for the following exercises:

Exercise 4.6.2.5. Verifying a Limit ☕☕☕

Consider the sequence given by the following explicit formula:

$$a_n = \frac{2n}{n+1}$$

- List the first five terms of the sequence. What do the terms appear to be converging to as n goes to ∞?
- If you choose $\epsilon = 0.1$, what would the corresponding N be?
- If you choose $\epsilon = 0.05$, what would the corresponding N be?

- Write an $N-\epsilon$ proof that verifies your guess above is correct.

Exercise 4.6.2.6. Writing $N-\epsilon$ Proofs ☕☕☕

Write $N-\epsilon$ proofs for each of the following limits:

1. $\lim_{n\to\infty} \frac{n}{3n+1} = \frac{1}{3}$

2. $\lim_{n\to\infty} \sqrt{9 + 1/n} = 3$

4.7 Series

While a sequence is a list of numbers, a series is a sum of a list of numbers. That is, a sequence is a list of numbers with commas inbetween; a series is a list of numbers with plus signs inbetween. We often use the very compact *sigma notation* to represent series.

Definition 4.7.0.1. Sigma Notation for Series

If a_n is a sequence and j, k are both natural numbers, then we define the series:

$$\sum_{n=j}^{k} a_n = a_j + a_{j+1} + a_{j+2} + \cdots + a_k$$

That is, we add up all consecutive terms of the sequence a_n, starting at index j and stopping at index k.

If the starting index is greater than the stopping index, we consider the sum to be empty. Since it has no terms, we define the total to be zero.

The sequence a_n that is being totaled is called the *summand*, much as the function $f(x)$ is referred to as the integrand in the expression $\int f(x)\,\mathtt{d}x$.

Exercise 4.7.0.2. Sigma Notation ☕

Evaluate the following sums:

- $\sum_{n=0}^{3} 2n$
- $\sum_{n=0}^{3} n^2$
- $\sum_{n=0}^{3} 2^n$

Exercise 4.7.0.3. Properties of Summations ☕☕☕

Let c be an arbitrary real number, j and k natural numbers with $j < k$, and a_n and b_n be arbitrary sequences. For each of the following properties, explain why it is true, or come up with a counterexample that shows it is not.

- $\sum_{n=j}^{k} c \cdot a_n = c \sum_{n=j}^{k} a_n$

- $\sum_{n=j}^{k} (a_n + b_n) = \left(\sum_{n=j}^{k} a_n\right) + \left(\sum_{n=j}^{k} b_n\right)$

- $\sum_{n=j}^{k} (a_n \cdot b_n) = \left(\sum_{n=j}^{k} a_n\right) \cdot \left(\sum_{n=j}^{k} b_n\right)$

- $\sum_{n=0}^{k} a_n = \sum_{n=1}^{k+1} a_{n-1}$

- $\sum_{n=0}^{k} c = ck$

- $\sum_{n=1}^{k} c = ck$

4.8 Arithmetic Series

If the summand a_n is an arithmetic sequence, the summation is called an *arithmetic series*. In this case, we have a nice formula for the sum!

Theorem 4.8.0.1. Arithmetic Series Formula

Let a_n be an arithmetic sequence with initial term a_0 and common difference d. Then the following sum has closed form

$$\sum_{n=0}^{N} a_n = (N+1) \cdot \frac{a_0 + (a_0 + Nd)}{2}$$

A nice short way to state the arithmetic series formula is: "*The sum of an arithmetic series is equal to the number of terms times the average of the first term and the last term.*"

Exercise 4.8.0.2. Lining Up the Formal and the Informal ☕

In the more formal statement of the arithmetic series formula, what expression represents...

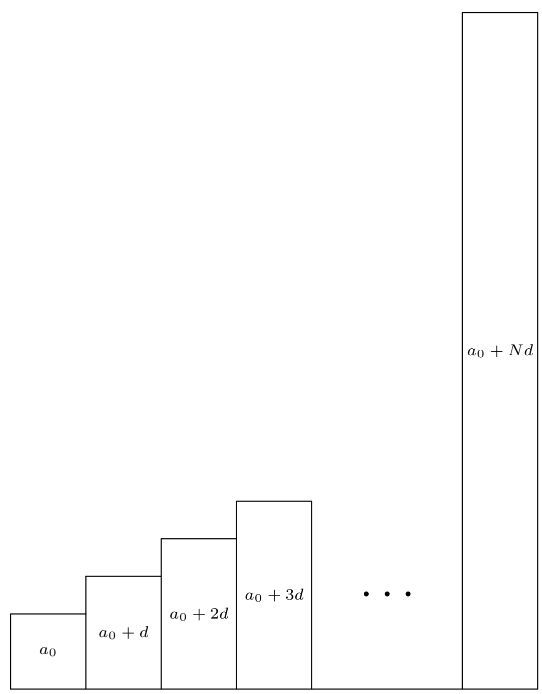
- ..."number of terms"?
- ..."first term"?
- ..."last term"?
- ..."average"?

Exercise 4.8.0.3. A Visual Argument for the Arithmetic Series Formula ☕☕☕

Here we draw a diagram to show why the Arithmetic Series Formula works. Consider the arithmetic sum

$$(a_0) + (a_0 + d) + (a_0 + 2d) + \cdots + (a_0 + Nd)$$

- For each term in the sum, we draw a corresponding rectangle. Specifically, a one by a_0 rectangle represents the first term, a one by $a_0 + d$ rectangle represents the second term, and so on. These rectangles are stacked in order in the first quadrant, next to each other on the x-axis, with sides of width one all on the x-axis. Explain why the area of the region is equal to the sum.



- Duplicate the entire region in the opposite order to build one giant rectangle. Draw a one by $a_0 + Nd$ rectangle on top of the leftmost, then a one by $a_0 + (N-1)d$ rectangle on top of the second, and so on until the last rectangle gets topped with a one by a_0 rectangle. In this new giant rectangle that is formed...
  - ...what is the width?
  - ...what is the height?
  - ...what is the total area?

- Explain why the total area of that rectangle must be exactly double the value of the arithmetic sum.

- Divide the total area by two to arrive at the arithmetic series formula!

Note that you have seen something similar in Calculus I in the context of evaluating Riemann Sums. In particular, Gauss's Formula states:

$$\sum_{n=1}^{N} n = \frac{N(N+1)}{2}$$

Exercise 4.8.0.4. Gauss's Formula as an Arithmetic Series ☕☕

Pretend for a second (or thirty) that you do not know Gauss's Formula. Evaluate $\sum_{n=1}^{N} n$ using the Arithmetic Series Formula. Verify this produces the right-hand side of Gauss's Formula. (**Hint:** What is the first term a_0? What is the common difference d?)

Exercise 4.8.0.5. Practice with Arithmetic Series ☕☕

- Add up all the whole numbers from 1 to 1000 inclusive.

- Add up all the whole numbers from 1000 to 2000 inclusive.

- What is the sum of all multiples of seven between 1000 and 2000?

- Compute the following summation:

$$\sum_{n=4}^{13}(3n-1)$$

4.9 Geometric Series

If the summand a_n is a geometric sequence, the summation is called an *geometric series*. In this case, we again have a nice formula for the sum!

Theorem 4.9.0.1. Finite Geometric Series Formula

Let a_n be a geometric sequence with initial term a_0 and common ratio r. Then the following sum has closed form

$$\sum_{n=0}^{N} a_n = a_0 \cdot \frac{1-r^{N+1}}{1-r}$$

In words, you can state the geometric series formula as: "*The sum of a geometric series is equal to the first term times one minus the common ratio raised to the number of terms, divided by one minus the common ratio.*"

Exercise 4.9.0.2. An Algebraic Argument for the Geometric Series Formula ☕☕☕

Here we use algebra to demonstrate why the Geometric Series Formula is valid. Consider the following geometric series and call it S for sum:

$$S = (a_0) + (a_0 r) + \left(a_0 r^2\right) + \cdots + \left(a_0 r^N\right)$$

- Explain why the following equality holds:

$$rS = (a_0 r) + \left(a_0 r^2\right) + \left(a_0 r^3\right) + \cdots + \left(a_0 r^{N+1}\right)$$

- Subtract the two above equations. Fill in the right hand side below.

$$S - rS =$$

- Solve for S in the equation above to construct the Geometric Series Formula!

Exercise 4.9.0.3. Trying Out the Geometric Series Formula ☕☕

Consider the summation

$$1 + 10 + 10^2 + 10^3 + 10^4 + 10^5$$

- Find the total by just doing the arithmetic. Evaluate the powers of ten and then add them up.

- Find the total by using the Geometric Series Formula. Verify that your answers match!

Exercise 4.9.0.4. Powers of Two ☕☕

Consider the summation

$$1 + 2 + 2^2 + 2^3 + 2^4 + 2^5$$

- Find the total by just doing the arithmetic. Evaluate the powers of two and then add them up.

- Find the total by using the Geometric Series Formula. Verify that your answers match!

- Use the Geometric Series Formula to evaluate

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^N$$

- Write in words the answer to the following: "*A finite sum of consecutive powers of two, starting at one, is equal to...* "

Economists often use Geometric Series when studying economic activity. The example below is related to the idea of the *velocity of money*, a measurement of how quickly money gets re-spent as it is received.

Exercise 4.9.0.5. Velocity of Money ☕☕

Suppose in an economy, people on average spend 80% of what income they receive. For example, say contractor earns \$100,000 for a building. He then spends 80% of this on a fancy automobile. Collectively, the car salesman, dealership, and auto manufacturer receive \$80,000. They in turn go spend 80% of that \$80,000 on goods and services that are then received as income by others.
Suppose also that on average, money changes hands roughly once a month. That is, there is a one-month delay between receiving money as income and going out to spend it.
The government invests \$5 billion in public infrastructure. How much total economic activity is actually generated by this investment in one year? Use the Geometric Series formula to evaluate your answer!

Here is an example that will look out of place in the Geometric Series Section. We will later see why this is in fact Geometric series in disguise!

Exercise 4.9.0.6. Partial Sums of Fibonacci Numbers ☕☕☕☕

Recall F_n, the sequence of Fibonacci numbers.

- Compute the following quantities:

$$\begin{aligned} F_0 &= \\ F_0 + F_1 &= \\ F_0 + F_1 + F_2 &= \\ F_0 + F_1 + F_2 + F_3 &= \\ F_0 + F_1 + F_2 + F_3 + F_4 &= \\ F_0 + F_1 + F_2 + F_3 + F_4 + F_5 &= \\ F_0 + F_1 + F_2 + F_3 + F_4 + F_5 + F_6 &= \end{aligned}$$

- Explain why $F_0 + F_1 + \cdots + F_N$ cannot be evaluated by directly applying the Arithmetic or

Geometric Series Formula.

- Do you notice any patterns in the sums above? Venture a guess for a closed form for

$$\sum_{i=0}^{n} F_i$$

Can you prove your guess is correct?

4.10 The Sequence of Partial Sums

Given a sequence a_n, we build a new sequence A_N called the *sequence of partial sums* by keeping a running total of all terms in a_n from 0 to N. We state this definition more formally.

Definition 4.10.0.1. Sequence of Partial Sums

Let a_n be a sequence. Define the *sequence of partial sums* of a_n to be

$$A_N = \sum_{n=0}^{N} a_n$$

When studying a sequence and its partial sums, it can be helpful to organize your data in a table.

Example 4.10.0.2. From a Sequence to Partial Sums

Consider the sequence of odd natural numbers $a_n = 2n + 1$. We compute a few partial sums and see if we can notice a pattern.

n	a_n	A_n	Total
0	1	1	1
1	3	1+3	4
2	5	1+3+5	9
3	7	1+3+5+7	16
4	9	1+3+5+7+9	25
5	11	1+3+5+7+9+11	36

We notice the column of totals contains all perfect squares. In particular, the number in row n is always exactly $(n + 1)^2$. Thus, the pattern suggests that

$$A_N = \sum_{n=0}^{N} (2n + 1) = (N + 1)^2$$

Exercise 4.10.0.3. Computing a Partial Sum with the Arithmetic Series Formula

Notice the sum A_N above is in fact an arithmetic series! Use the arithmetic series formula to evaluate

$$A_N = \sum_{n=0}^{N} (2n + 1)$$

and confirm it matches our conjectured formula from the table.

Notice also that given partial sums, we can uncover the sequence from which it came. The difference of two consecutive partial sums will be a single term in the sequence, since

$$\begin{aligned} A_N - A_{N-1} &= \sum_{n=0}^{N} a_n - \sum_{n=0}^{N-1} a_n \\ &= (a_0 + a_1 + a_2 + \cdots + a_N) - (a_0 + a_1 + a_2 + \cdots + a_{N-1}) \\ &= a_N \end{aligned}$$

Example 4.10.0.4. From Partial Sums to a Sequence

Let's try to undo the previous example! Suppose we start with $A_N = (N+1)^2$. We draw a table to see what terms a_n would have been added together to obtain those totals.

N	A_N	a_N	Difference
0	1	1-0	1
1	4	4-1	3
2	9	9-4	5
3	16	16-9	7
4	25	25-16	9
5	36	36-25	11

We see that sure enough, the last column is the sequence of odd numbers and is always one more than twice N. Thus, we have that $a_N = 2N + 1$.

Exercise 4.10.0.5. Taking a Difference of Partial Sums ☕☕

Using the formula $A_N = (N+1)^2$, try taking the difference $A_N - A_{N-1}$ and verify you get the same a_N.

Note that we have two different indices, as we are taking the convention that n indexes the sequence a_n and N indexes the partial sums A_N. Thus, depending on which we start with, it looks like we have the "wrong" index for the other (A_n vs A_N or a_n vs a_N). This is nothing to worry about, as the sequence is really just the mapping from the natural numbers to the reals. This is similar to how $f(x) = x^2$ and $f(t) = t^2$ are the same function on the reals, but just listed with different independent variables.

Thus, we have two forms, a sequence and a sequence of partial sums. We also have a way to go back and forth between the two forms. This calls for a game of... TELEPHONE!

Break into groups of four and play telephone with one of the following pages. If you are handed a sequence a_n, find the sequence of partial sums A_N, fold over the original a_n, and pass it along. If you are handed a sequence of partial sums A_N, find the sequence it came from a_n, fold over the original A_N, and pass it along.

$$A_N = 5 \cdot \frac{1 - \frac{1}{2^{(N+1)}}}{1 - \frac{1}{2}}$$

$\boxed{a_n = \qquad\qquad\qquad\qquad\qquad\qquad}$

$\boxed{A_N = \qquad\qquad\qquad\qquad\qquad\qquad}$

$\boxed{a_n = \qquad\qquad\qquad\qquad\qquad\qquad}$

$\boxed{A_N = \qquad\qquad\qquad\qquad\qquad\qquad}$

$$a_n = 2n + 3$$

$\boxed{A_N =}$

$\boxed{a_n =}$

$\boxed{A_N =}$

$\boxed{a_n =}$

$$a_n = 2/3^n$$

$$\boxed{A_N = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$\boxed{a_n = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$\boxed{A_N = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$\boxed{a_n = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$A_N = 1$$

$\boxed{a_n = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$

$\boxed{A_N = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$

$\boxed{a_n = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$

$\boxed{A_N = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$

Often the study of the real numbers and related objects is called *continuous* mathematics while the study of the natural numbers and related objects is called *discrete* mathematics. In this course, we encounter many interesting parallels between the two pursuits!

Exercise 4.10.0.6. Discrete/Continuous Analogy ☕☕☕☕

- In what ways is taking the partial sums of a sequence similar to taking the integral of a function over the real numbers? (**Hint:** Plot your sequence and draw rectangles with height a_n and width one.)

- In what ways is taking the difference of consecutive terms in a sequence $(a_n - a_{n-1})$ similar to taking the derivative of a function over the real numbers? (**Hint:** Plot your sequence and think of how you could obtain $a_n - a_{n-1}$ as the slope of a secant line.)

- Suppose you start with a sequence a_n. You add up terms to create the sequence of partial sums A_N. You then take the difference of consecutive terms in the partial sums and find that $A_N - A_{N-1} = a_N$. What theorem of calculus is this analogous to and why?

4.11 Infinite Series

Well here's an interesting question.

What does it mean to add up infinitely many numbers?
-Lots of people

We provide the most commonly used modern definition.

Definition 4.11.0.1. Infinite Series, Convergence, and Divergence

Let a_n be a sequence of real numbers. Then the *infinite sum* of all terms of a_n is defined to be the limit of partial sums A_N. That is,

$$\sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} A_N = \lim_{N\to\infty} \sum_{n=0}^{N} a_n$$

If the limit exists, we say the infinite series *converges* to the value of the limit. If the limit is infinity or does not exist, then we say the infinite series *diverges*.

The idea is simple; if you want to add up infinitely many numbers, a good place to start is by just adding up finitely many of them. However, if you only add up finitely many, your answer has some error to it. If you want that error to go down, add up more and more of them! The limit of the values of these partial sums will be the exact answer.

Exercise 4.11.0.2. The Return of the Discrete/Continuous Analogy ☕☕☕

In what way is the definition of an infinite series analagous to the definition of a horizontally unbounded improper integral?

Exercise 4.11.0.3. The Definitions in Words ☕☕

We have defined three very important interconnected structures:

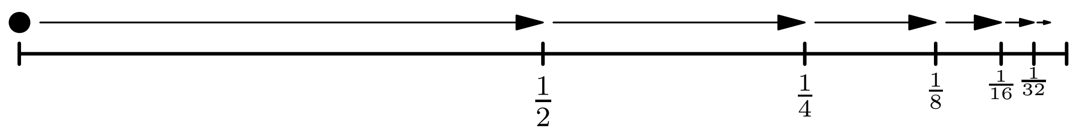
1. A sequence a_n.
2. A sequence of partial sums A_N.
3. An infinite series $\sum_{n=0}^{\infty} a_n$

Describe in words how the three structures are related and are built from one another.

4.11.1 Zeno's Paradox, Resolution, and Consequences

The next example is traceable back to the writings of Aristotle in the third century BC! Specifically, he states Zeno's Paradox of *Dichotomy* as:

> *That which is in locomotion must arrive at the half-way stage before it arrives at the goal.*



This was meant to be a "proof" that an object (say an arrow in flight) could never reach its target. This paradox is resolved with our notion of infinite series.

Exercise 4.11.1.1. A Classic Infinite Series

Consider the sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$. This can be interpreted as the sequence of distances the arrow must travel in the Dichotomy paradox (if say it were fired one meter from its target and all lengths are measured in meters). Since it were fired from one meter away, we expect that the total distance traveled is one.

- Find an explicit formula a_n that describes the sequence above.

- Compute the corresponding sequence of partial sums A_N.

- Evaluate the infinite sum
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$
by taking the limit of the sequence of partial sums. Verify the total is in fact one.

In the above case, we were able to provide a resolution for the paradox and verify the total with our geometric series formula, but the answer was not particularly surprising. Here is a more interesting example!

Exercise 4.11.1.2. An Alternating Geometric Series ☕☕☕

Suppose a bug moves forward half a meter. It then moves backwards one-fourth of a meter. It then moves forward one-eighth of a meter. It then moves backwards one-sixteenth of a meter. This pattern of moving forwards, then backwards, by half the previous distance each time, continues forever. At the end of time, where does the bug end up?
To solve this problem, we notice that it is equivalent to adding up the sequence:

$$a_n = \frac{1}{2}\left(-\frac{1}{2}\right)^n$$

- Let $A_N = \sum_{n=0}^{N} a_n$ be the sequence of partial sums. Find a formula for A_N.

- Compute $\sum_{n=0}^{5} a_n$

- Compute $\sum_{n=0}^{10} a_n$

- Compute $\sum_{n=0}^{\infty} a_n$ from the definition of an infinite series.

So, where does the bug end up?

The notion of an infinite series can be used to give a rigorous interpretation to the infinite decimal expansions as well!

Exercise 4.11.1.3. Repeating Decimal Expansion ☕☕

- Write 0.333 as a geometric series with three terms and common ratio 1/10. Compute its value via the finite geometric series formula.

- Write 0.3333 as a geometric series with four terms and common ratio 1/10. Compute its value via the finite geometric series formula.

- Write 0.33333 as a geometric series with five terms and common ratio 1/10. Compute its value via the finite geometric series formula.

- Write
$$\underbrace{0.3333\ldots3}_{n \text{ threes}}$$
as a geometric series with n terms and common ratio 1/10. Compute its value in terms of n via the finite geometric series formula.

- Take the limit as n approaches infinity of your formula from the previous part to prove that "point three repeating" really does equal one-third.

We can generalize the previous examples. Notice in all cases, the geometric series formula let us calculate an explicit formula for the sequence of partial sums. As long as the common ratio $|r| < 1$, the limit as $n \to \infty$ will exist. This brings us to the *infinite geometric series* formula (also sometimes just referred to as the geometric series formula).

Theorem 4.11.1.4. Infinite Geometric Series Formula

If a and r are real numbers and $|r| < 1$, then

$$a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}$$

Exercise 4.11.1.5. Using the Geometric Series Formula ☕☕

Consider the following series:

$$\sum_{n=5}^{14} \frac{3^n}{2^{2n+1}}$$

- Write out the terms of the above series.

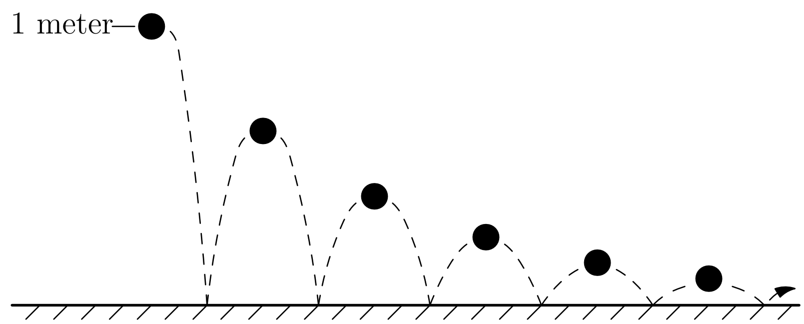
- Is the above series geometric? Explain why or why not. If so, what is the common ratio r? What is the first term a?

- Find the value of the above series.

- Compute the following infinite series:

$$\sum_{n=5}^{\infty} \frac{3^n}{2^{2n+1}}$$

Exercise 4.11.1.6. The Bouncing Ball ☕☕



A magical bouncy ball is bounced from a height of 1 meter. On each bounce, it always rebounds to exactly five-eighths of the height it fell from. What is the ball's total vertical distance traveled from now until the end of time?

Exercise 4.11.1.7. Evaluating Another Infinite Series ☕☕

Consider the sequence given by the following formula:

$$a_n = 2$$

Now consider the infinite sum:

$$\sum_{n=0}^{\infty} a_n$$

- Write out the first five terms of the sequence a_n. Also write out the first five terms of the sequence of partial sums for the corresponding series.

- Find an explicit formula for the sequence of partial sums.

- Does the infinite series converge? If so, what value does it converge to?

Exercise 4.11.1.8. A Telescoping Sum ☕☕☕

Consider the following sequence:

$$a_n = \frac{2}{n^2 + 5n + 6}$$

- Compute the first five terms of the sequence.

- Compute the first five partial sums of the sequence.

- Based on your data, conjecture a formula for

$$A_N = \sum_{n=0}^{N} \frac{2}{n^2 + 5n + 6}$$

- Prove your answer is correct via a partial fraction decomposition. Specifically, perform a PFD on $a_n = \frac{2}{n^2+5n+6}$ and then notice that when you add the terms in a partial sum, all but two terms cancel! (This lucky happening is what is referred to as a series *telescoping*, as it is collapsing in on itself much like a retractable telescope would.)

- Use your formula for the partial sums and the definition of an infinite series to write an $N-\epsilon$ proof for the value of

$$\sum_{n=0}^{\infty} \frac{2}{n^2+5n+6}$$

Exercise 4.11.1.9. Practice with Infinite Series ☕☕☕

For each of the following sequences a_n, carry out the following steps:

- Write out the first five terms of the sequence a_n. Also write out the first five terms of the sequence of partial sums A_N for the corresponding series.
- Find a formula for the sequence of partial sums $A_N = \sum_{n=0}^{N} a_n$.
- Does the infinite series $\sum_{n=0}^{\infty} a_n$ appear to converge? If so, what value does it appear to converge to?

And now, the sequences:

1. The sequence defined by

$$a_n = 2n$$

2. The sequence defined by
$$a_n = 2^n$$

3. The sequence defined by
$$a_n = \left(\frac{2}{3}\right)^n$$

4. The sequence defined by
$$a_n = \left(\frac{-1}{2}\right)^n$$

5. The sequence defined by

$$a_n = (-1)^n$$

6. The sequence defined by

$$a_0 = 3$$
$$a_n = \frac{-1}{3} a_{n-1}$$

7.

$$a_0 = 5$$
$$a_n = a_{n-1} + 1$$

8. The sequence defined by

$$a_0 = 1$$
$$a_n = -a_{n-1}$$

4.12 Absolute Convergence and Rearrangements

When we add up finitely many numbers, we take properties like commutativity and associativity for granted. We add up numbers in whatever order is most convenient. With infinite series, we cannot be quite so cavalier!

Example 4.12.0.1. Did You Know that Zero Equals One?

Here we use the fact that 0=-1+1.

$$\begin{aligned}
1 &= 1 + 0 + 0 + 0 + 0 + \cdots \\
&= 1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + (-1 + 1) + \cdots \\
&= 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - \cdots \\
&= (1 - 1) + (1 - 1) + (1 - 1) + (1 - 1) + (1 + -1) + \cdots \\
&= 0 + 0 + 0 + 0 + 0 + \cdots \\
&= 0
\end{aligned}$$

Exercise 4.12.0.2. What? ☕☕☕

Let us now correctly analyze the infinite sum

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - \cdots$$

- Consider the sequence $a_n = (-1)^n$. Find the corresponding partial sums A_N.

- What is the limit of the sequence of partial sums?

- Thus, what is the correct value of the infinite series $1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - \cdots$?

It turns out that the key lies in the distinction between a series being convergent vs being *absolutely convergent*, a stronger type of convergence.

Definition 4.12.0.3. Absolute Convergence

An infinite series $\sum_{n=0}^{\infty} a_n$ is *absolutely convergent* if and only if $\sum_{n=0}^{\infty} |a_n|$ is convergent.

Absolute convergence is the idea that it wasn't just some sort of cancellation of positive and negative terms that let the partial sums stabilize. Rather, the magnitudes of the terms were going to zero quickly enough.

Example 4.12.0.4. An Absolutely Convergent Series

The infinite series

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} - \frac{1}{16} + \cdots$$

is absolutely convergent because

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots$$

still converges. In particular, its total is 2.

Contrast this concept with the following definition, a weaker form of convergence called conditional.

Definition 4.12.0.5. Conditional Convergence

An infinite series $\sum_{n=0}^{\infty} a_n$ is *conditionally convergent* if and only if it converges but $\sum_{n=0}^{\infty} |a_n|$ diverges.

Example 4.12.0.6. A Conditionally Convergent Series

The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \frac{1}{5} + \cdots$$

is conditionally convergent because

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

diverges. In particular, it is the harmonic series which totals to ∞.

It turns out that for absolutely convergent series, rearranging of terms and any sort of normal algebraic manipulation is fine. This is a theorem that is rather difficult to prove and will be saved for a later mathematical adventure. For this course, we will just use it!

4.13 Convergence Tests

It is often too difficult to determine the exact value of an infinite series (if it converges at all). Thus, we usually settle for the knowledge *that* a series converges (or diverges) as opposed to finding *what* number it converges to. By "settling" as such, we are not actually giving up too much. If we can guarantee that a series converges, it means it is safe to approximate it by just taking a partial sum with lots of terms.

Exercise 4.13.0.1. Why Convergence is So Critical ☕☕

Why would it not make sense to approximate a divergent infinite series using a partial sum with lots of terms?

Below, we detail eight commonly used tests for convergence.

1. No Hope Test
2. Geometric Series Test
3. Direct Comparison Test
4. Integral Test
5. p Test
6. Alternating Series Test
7. Limit Comparison Test
8. Ratio Test

4.13.1 No Hope Test and the Harmonic Series

The No Hope Test is sometimes also referred to as the Divergence Test or the n^{th} Term Test. Intuitively, this test says that if the terms of the sequence a_n do not go to zero, then their sum has no hope of converging. We give a more formal statement here.

Theorem 4.13.1.1. No Hope Test

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=0}^{\infty} a_n$ diverges.

What it does *not* say:

$$\text{If } \lim_{n\to\infty} a_n = 0, \text{ then } \sum_{n=0}^{\infty} a_n \text{ converges.} \qquad (☹)$$

This is a fallacy and a *very* common mistake. If the terms of the sequence go to zero, then the series has some chance of converging, but it is no guarantee. Here is a classic counterexample, the harmonic series! The following divergence proof by Johann Bernoulli comes from the mid-seventeenth century.

Example 4.13.1.2. Divergence of the Harmonic Series

Here we show that

$$\sum_{n=0}^{\infty} \frac{1}{n} = \infty$$

We accomplish this by showing the partial sums will exceed any sum of one-half added to itself again and again. Proceeding:

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \cdots \\
&> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \cdots \\
&= 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \frac{8}{16} + \cdots \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots \\
&= \infty
\end{aligned}$$

Since the harmonic series is greater than a sum of infinitely many one-half's, it is infinite.

Exercise 4.13.1.3. Justifying the Steps ☕☕

Annotate the above proof with a short comment justifying each line of equality or inequality.

Exercise 4.13.1.4. Revisiting a Convergent Series ☕☕

In the previous section, we showed that

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$$

If you attempt to use Bernoulli's argument to show that it diverges, where does it break down? Why can't you just group terms together into batches that are at least size one-half?

Exercise 4.13.1.5. No Hope Test Backwards ☕

Explain why the Harmonic Series is a counterexample to the claim tagged with ☹.

Now try a few on your own!

Exercise 4.13.1.6. Practice with the No Hope Test ☕☕

1. Use the No Hope Test to prove that the series $\sum_{n=1}^{\infty} \frac{\sqrt{2n^2+2}}{n}$ diverges.

2. Use the No Hope Test to prove that the series $\sum_{n=1}^{\infty} \cos(1/n)$ diverges.

3. What does the No Hope Test tell you about the convergence/divergence of the series $\sum_{n=1}^{\infty} \sin(1/n)$?

4.13.2 Geometric Series Test

This is essentially just a restatement of the Geometric Series Formula. Recall that a geometric series is a series of the form

$$\sum_{n=0}^{\infty} a \cdot r^n$$

for some real numbers a and r. That is to say, it is a series that has starting term a and common ratio r.

Theorem 4.13.2.1. Geometric Series Test

A geometric series converges if $|r| < 1$ (and in fact converges to the value $\frac{a}{1-r}$) and diverges otherwise.

Exercise 4.13.2.2. Practice with the Geometric Series ☕☕☕

1. Explain why $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is not a geometric series (and thus this test would be inapplicable).

2. Explain why the series $18 - 6 + 2 - \frac{2}{3} + \frac{2}{9} - \cdots$ converges absolutely.

3. Give an example of a geometric series that converges conditionally, or explain why it is not possible to construct such a series.

Exercise 4.13.2.3. Why Not Arithmetic? ☕

Why is there *not* another result in this section called the "Arithmetic Series Test"? Why is only the geometric series getting to have all the fun?

4.13.3 Integral Test

Here we use integrals to test convergence of infinite series! Intuitively this test says:

An infinite series converges if and only if the corresponding improper integral converges.

Theorem 4.13.3.1. Integral Test

Let $a \in \mathbb{N}$ and $f(n)$ be a decreasing function on $[a, \infty)$. If $\int_{x=a}^{\infty} f(x)dx$ converges, then $\sum_{n=a}^{\infty} f(n)$ converges as well. Likewise if $\int_{x=a}^{\infty} f(x)dx$ diverges, then $\sum_{n=a}^{\infty} f(n)$ diverges as well.

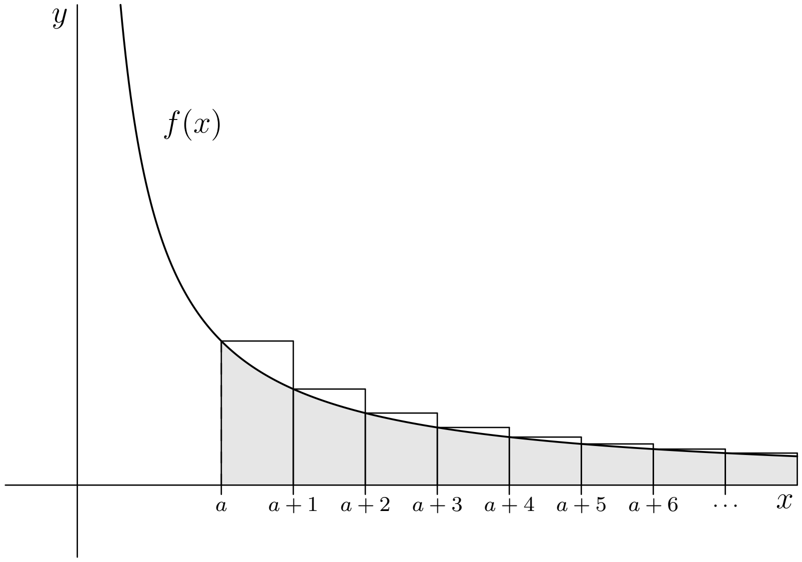
Here is a "proof by picture" to justify the Integral Test.

Exercise 4.13.3.2. Explaining the Integral Test ☕☕☕

Study the following diagrams and use them to determine why the series converges if and only if the series converges. Specifically:

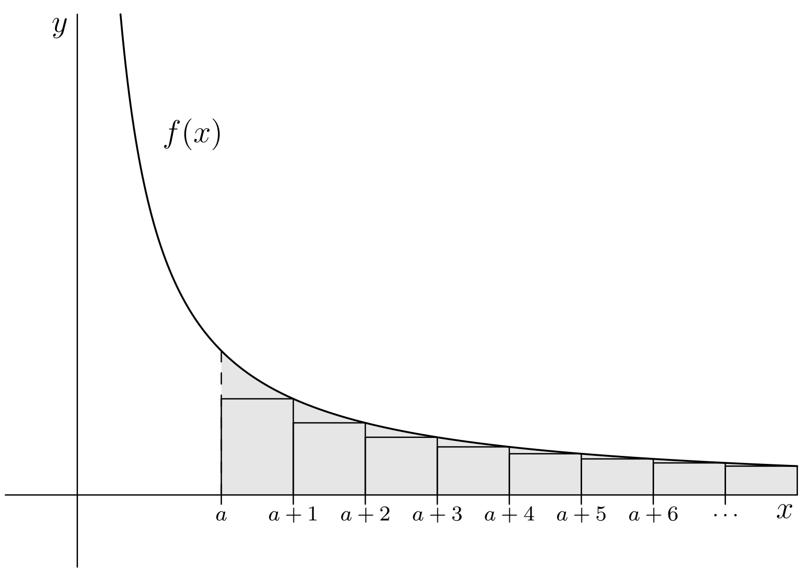
- Explain why the diagram below justifies the inequality

$$\int_{x=a}^{\infty} f(x)\,\mathrm{d}x \leq \sum_{n=a}^{\infty} f(n)$$



- Explain why the diagram below justifies the inequality

$$\sum_{n=a+1}^{\infty} f(n) \le \int_{x=a}^{\infty} f(x)\, \mathrm{d}x$$



- Add $f(a)$ to both sides of the previous inequality to conclude

$$\sum_{n=a}^{\infty} f(n) \leq f(a) + \int_{x=a}^{\infty} f(x)\,\mathtt{d}x$$

- Putting both inequalities together, we now have that

$$\int_{x=a}^{\infty} f(x)\,\mathtt{d}x \leq \sum_{n=a}^{\infty} f(n) \leq f(a) + \int_{x=a}^{\infty} f(x)\,\mathtt{d}x \tag{$\bowtie$}$$

  Explain why this inequality shows that the infinite series converges if and only if the corresponding improper integral converges.

Use the Integral Test to decide if the following infinite series converge or diverge.

Exercise 4.13.3.3. Practice with the Integral Test ☕☕

1. $\sum_{n=2}^{\infty} \frac{1}{n\,\mathtt{ln}(n)}$

2. $\sum_{n=2}^{\infty} \frac{1}{n(\mathtt{ln}(n))^2}$

3. $\sum_{n=2}^{\infty} \frac{1}{n^2+1}$

The next example shows why the assumption of $f(x)$ being a decreasing function is necessary for the Integral Test.

Exercise 4.13.3.4. An Interesting Example ☕☕☕

Consider the function

$$f(x) = |\sin(\pi x)|$$

- Graph the function $f(x)$ over the positive x-axis.
- Explain why the integral $\int_{x=0}^{\infty} f(x)\,\mathtt{d}x$ is equal to infinity.
- Compute the infinite sum $\sum_{n=a}^{\infty} f(n)$.
- In this case, the integral diverged, while the infinite sum converged. Why does this example not contradict the Integral Test?

4.13.4 p-Test

Note that the Integral Test shows that the p-Test still works for series, since it worked for improper integrals.

Theorem 4.13.4.1. The p-Test

Let p be a real number. Then the sum $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges otherwise.

Exercise 4.13.4.2. Practice with the p-Test ☕

1. Give an example of a series that converges by the p-test.

2. Give an example of a series that diverges by the p-test.

3. Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ by the p-test.

4.13.5 Alternating Series Test

Theorem 4.13.5.1. Alternating Series Test (AST)

If a sequence of positive numbers a_n monotonically approaches zero as n approaches infinity, then

$$\sum_{n=0}^{\infty} (-1)^n a_n$$

converges.

Intuitively, this test says that the positives and negatives will cancel each other out and the partial sum pendulum will eventually stabilize at some well-defined limit.

Check that the two series below converge by the Alternating Series Test. For each, check if it converges absolutely or converges conditionally!

Exercise 4.13.5.2. The AST in Action ☕☕

- $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$

- $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n}$

In a convergent alternating series, the partial sums always "leapfrog" back and forth over the limiting value the series converges to. This implies that the value of the infinite series is always no further away than the next unused term in any partial sum.

Theorem 4.13.5.3. Error Bound

Let a_n be a sequence of positive decreasing terms. Let $\sum_{n=0}^{\infty}(-1)^n a_n$ be a convergent series. Then

$$\left|\sum_{n=0}^{\infty}(-1)^n a_n - \sum_{n=0}^{N}(-1)^n a_n\right| < |a_{N+1}|$$

Exercise 4.13.5.4. Alternating Series Error Bug

Consider again the bug described in Exercise 4.11.1.2. After the bug has reversed course five times, how close does the error bound guarantee the bug is to its final location? Use the actual final location to confirm this error bound is correct.

4.13.6 Limit Comparison Test

This test says that if two sequences have the same growth order, then the corresponding infinite series either both converge or both diverge. This works because if two sequences have the same growth order, then in the long term they are just a nonzero constant factor apart, and multiplying by a nonzero constant factor cannot change convergence or divergence. Furthermore, having larger magnitude terms than a divergent series implies divergence, and smaller magnitude terms than a convergent series implies convergence.

Theorem 4.13.6.1. Limit Comparison Test (LCT)

Let a_n and b_n be sequences of nonnegative terms.

- Suppose a_n has larger growth order than b_n, and $\sum_{n=0}^{\infty} b_n$ diverges. Then $\sum_{n=0}^{\infty} a_n$ also diverges.
- Suppose a_n has smaller growth order than b_n, and $\sum_{n=0}^{\infty} b_n$ converges. Then $\sum_{n=0}^{\infty} a_n$ also converges.
- Suppose a_n and b_n have the same growth order. Then $\sum_{n=0}^{\infty} b_n$ and $\sum_{n=0}^{\infty} b_n$ either both converge or both diverge.

LCT is particularly useful if a_n is expressed as a fraction with the numerator and denominator both algebraic (expressed just with polynomials and radicals). In this case, one can build a comparison sequence by taking the ratio of leading terms from the numerator and denominator.

Example 4.13.6.2. Using Leading Terms from the Numerator and Denominator

Lets determine the convergence/divergence of

$$\sum_{n=1}^{\infty} \frac{7n+3}{n\sqrt{n^2+n+1}}$$

Call $a_n = \frac{7n+3}{n\sqrt{n^2+n+1}}$. The idea is that in the numerator, the "plus three" is insignificant as n approaches infinity. Thus, we keep only the $7n$ in the numerator. In the denominator, we observe that the $n+1$ is insignificant compared to the n^2 it is being added to. Again, we keep only the

$n\sqrt{n^2} = n \cdot n = n^2$, the leading term of the denominator. We have built our comparison function

$$b_n = \frac{7n}{n^2} = \frac{7}{n}$$

Next, we verify that a_n and b_n have the same growth order. Proceeding:

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{7n+3}{n\sqrt{n^2+n+1}} \frac{n}{7} \\
&= \lim_{n\to\infty} \frac{7n+3}{7\sqrt{n^2+n+1}} \\
&= \lim_{n\to\infty} \frac{7n^2+3n}{7n\sqrt{n^2+n+1}} \\
&= \lim_{n\to\infty} \frac{7n^2+3n}{7\sqrt{n^2}\sqrt{n^2+n+1}} \\
&= \lim_{n\to\infty} \frac{7n^2+3n}{7\sqrt{n^4+n^3+n^2}} \\
&= \lim_{n\to\infty} \frac{7n^2+3n}{7\sqrt{n^4+n^3+n^2}} \frac{\frac{1}{n^2}}{\frac{1}{\sqrt{n^4}}} \\
&= \lim_{n\to\infty} \frac{7+\frac{3}{n}}{7\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}} \\
&= \frac{7}{7\sqrt{1}} \\
&= 1
\end{aligned}$$

Thus, a_n and b_n have the same growth order, since their ratio is a nonzero constant. Furthermore,

$$\sum_{n=1}^{\infty} \frac{7}{n} = 7 \sum_{n=1}^{\infty} \frac{1}{n}$$

which diverges by Bernoulli's argument in Example 4.13.2.
By LCT, $\sum_{n=1}^{\infty} \frac{7n+3}{n\sqrt{n^2+n+1}}$ diverges as well.

Exercise 4.13.6.3. Practice with LCT

Use LCT to prove the following series converge or diverge:

- $\sum_{n=1}^{\infty} \frac{n+2}{n^3+1}$

- $\sum_{n=1}^{\infty} \frac{n^5}{n\sqrt{n^7+3n+1}}$

4.13.7 Ratio Test

This test is essentially a LCT against a geometric series.

Exercise 4.13.7.1. Ratio of Consecutive Terms ☕

If a_n is a geometric sequence, what is a_{n+1}/a_n?

For a sequence that is not geometric, there may not be a constant ratio of consecutive terms, but we can still look at the limit of ratios of successive terms!

Theorem 4.13.7.2. Ratio Test

Consider the series

$$\sum_{n=0}^{\infty} a_n$$

- If $\lim_{n\to\infty} a_{n+1}/a_n < 1$ then the series converges absolutely.
- If $\lim_{n\to\infty} a_{n+1}/a_n > 1$ then the series diverges.
- If $\lim_{n\to\infty} a_{n+1}/a_n = 1$ then the ratio test gives no information.

The Ratio Test is particularly helpful in analyzing series involving factorials, since so much cancellation will occur when computing the ratio of consecutive terms.

Example 4.13.7.3. A Series with Factorials ☕☕

Here we analyze the convergence of the series

$$\sum_{n=0}^{\infty} \frac{2^n}{n!}$$

Call $a_n = \frac{2^n}{n!}$. We now compute the limit of the ratio of consecutive terms:

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{\frac{2^{(n+1)}}{(n+1)!}}{\frac{2^n}{n!}} \\
&= \lim_{n\to\infty} \frac{2^{(n+1)}}{2^n} \frac{n!}{(n+1)!} \\
&= \lim_{n\to\infty} 2 \frac{n \cdots 3 \cdot 2 \cdot 1}{(n+1) \cdot n \cdots 3 \cdot 2 \cdot 1} \\
&= \lim_{n\to\infty} \frac{2}{n+1} \\
&= 0 \\
&< 1
\end{aligned}$$

Thus, the series converges by the Ratio Test!

Exercise 4.13.7.4. Practice with Ratio Test ☕☕

Use the Ratio Test to prove the following converge absolutely or diverge, or explain why the Ratio Test provides no information in that case:

1. $\sum_{n=1}^{\infty} \frac{n+1}{n!}$

2. $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$

3. $\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$

4. $\sum_{n=1}^{\infty} \frac{2^n}{n^{10}}$

4.13.8 Direct Comparison Test

The Direct Comparison Test (DCT) is similar to LCT, except we are directly comparing magnitudes of terms instead of growth orders. We already used the idea of DCT in both Bernoulli's analysis of the Harmonic series and the Integral Test.

Theorem 4.13.8.1. Direct Comparison Test

Let a_n and b_n be sequences. If for all natural numbers n, $|a_n| \leq |b_n|$, then

- $\sum_{n=0}^{\infty} b_n$ converges implies that $\sum_{n=0}^{\infty} a_n$ also converges.
- $\sum_{n=0}^{\infty} a_n$ diverges implies that $\sum_{n=0}^{\infty} b_n$ also diverges.

The trick when using DCT is to pick an easy series to compare to, for example a p-series.

Exercise 4.13.8.2. Practice with DCT ☕☕

Use the Direct Comparison Test to prove the following converge or diverge:

1. $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$

2. $\sum_{n=1}^{\infty} \frac{2+\cos(n)}{n}$

3. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

4.13.9 Mixed Practice

In practice, when you encounter a series, there are usually many tests that will apply. Try the following, using any valid applicable test you like!

Exercise 4.13.9.1. Mixed Practice ☕☕☕

Determine if each of the following infinite series converges absolutely, converges conditionally, or diverges. In each case explain what tests you used and how!

- $\sum_{n=0}^{\infty} \frac{n}{n+2}$

- $\sum_{n=0}^{\infty} \frac{n}{n^2+2}$

- $\sum_{n=0}^{\infty} \frac{(-1)^n n}{n^2+2}$

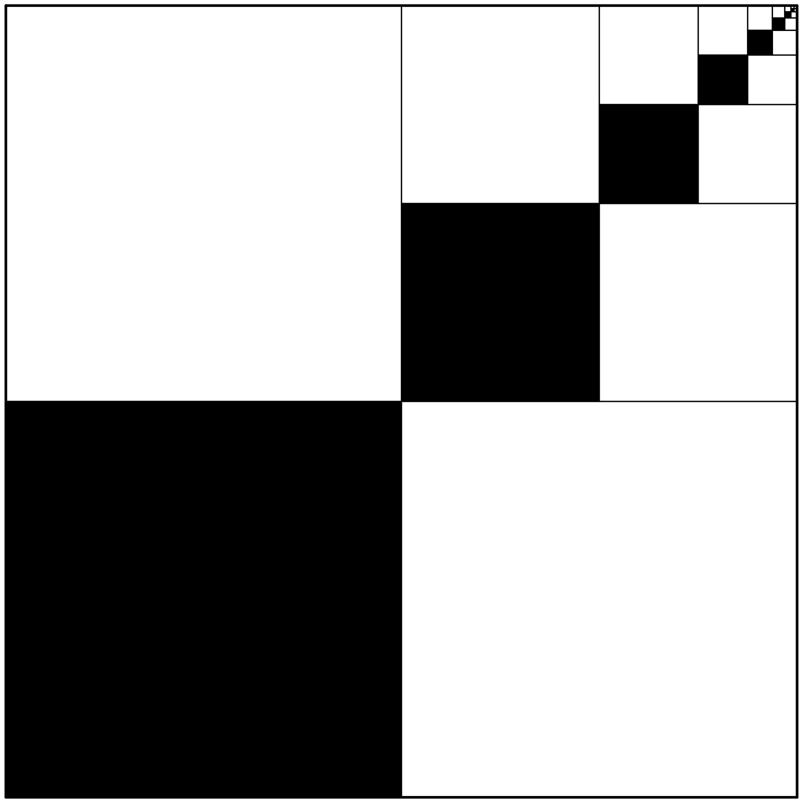
- $\sum_{n=0}^{\infty} \frac{(-1)^n n}{n^3+2}$

4.13.10 A Classical Infinite Series

Having played with formalism for far too long at this point, it is time to visit an infinite series coming from the geometry of Archimedes!

Exercise 4.13.10.1. An Infinite Series of Archimedes ☕☕☕

Consider the following diagram.



Assume the entire square has side length 1 and that each further subdivision into squares uses side lengths that are half the previous.

- What proportion of the whole large square is colored black? As a consequence, what is the total area of all the black squares added up?

- Write the area of each individual black square and build an infinite series. What kind of series is this?

- Find the sum of the series using the infinite geometric series formula. Verify it agrees with your total for the area above.

- Can you show the series converges using the Geometric Series Test?

- Can you show the series converges using the Ratio Test?

- Can you show the series converges using the Alternating Series Test?

- Can you show the series converges using the Limit Comparison Test? (**Hint:** Compare to $1/n^2$.)

- Can you show that the series converges using the Integral Test?

Chapter 5

Power Series

5.1 Wouldn't It Be Nice If All Functions Were Polynomials?

Think about say, differentiating and antidifferentiating. It becomes difficult when rational functions, trigonometric functions, logarithms, and exponentials are involved. If every function were just polynomial, calculus would be much easier!

Power series is an attempt to make this dream a reality: turn these non-polynomial functions into polynomials! There is just one slight hangup; it is mathematically impossible. For example:

Theorem 5.1.0.1. Cosine Cannot Be Written as a Polynomial

Cosine cannot be written as a polynomial.

Proof. Let n be a natural number. By the Fundamental Theorem of Algebra, a degree n polynomial has at most n roots. However, $\cos(x)$ has infinitely many roots. Thus, the cosine function cannot be equal to any polynomial, since no polynomial has infinitely many roots. $\square$

Unphased by minor complications like something being impossible, we proceed anyway. If polynomials can only have as many roots as their degree and cosine needs infinitely many roots, maybe we just need to allow polynomials to have infinite degree! This is exactly the definition of a *power series*. A power series is simply a polynomial whose degree is allowed to be infinite. More formally:

Definition 5.1.0.2. Power Series

A *power series* is an expression of the form

$$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$$

for real or complex numbers a_i.

Many other sources refer to the above as a *Maclaurin Series*, or a *Taylor Series Centered at Zero*. Since it is a sum of powers of x, we stick with the simple descriptive name, power series.

Example 5.1.0.3. The Power Series for Cosine, One Coefficient at a Time

We return to our original goal! Since we cannot find a finite degree polynomial equal to the cosine function, we instead find an infinite degree polynomial (aka power series) for cosine. We desire a_i

such that:

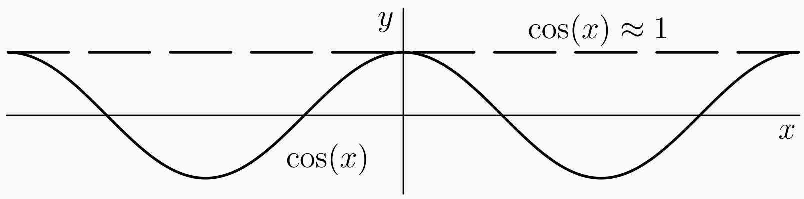
$$\cos(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$$

We solve for the coefficients a_i one at a time.

- **Solving for a_0:** We certainly want this formula to be true for $x = 0$, so let's plug $x = 0$ into both sides and see what happens.

$$\begin{aligned} \cos(0) &= a_0 + a_1 \cdot 0 + a_2 \cdot 0^2 + a_3 \cdot 0^3 + a_4 \cdot 0^4 + a_5 \cdot 0^5 + \cdots \\ 1 &= a_0 \end{aligned}$$

Thus, we found our first coefficient of our power series. Notice that this shows us a degree zero polynomial approximation $P_0(x) = 1$ to cosine! In a certain sense, it is giving us the best horizontal line approximation to the graph of cosine near the origin.



- **Solving for a_1:** To solve for a_1, we can't just plug in $x = 0$ because a_1 will get multiplied by zero. This causes a_1 to disappear, and we can't solve for it. So, we need some operation which will keep the a_1 around but get rid of the x attached to it. Differentiation fits this description perfectly! So, we take the derivative of both sides.

$$\begin{aligned} (\cos(x))' &= \left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots\right)' \\ -\sin(x) &= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots \end{aligned}$$

Now if we plug $x = 0$ into both sides, we will be able to solve for a_1, rather than deleting it.

$$\begin{aligned} -\sin(0) &= a_1 + 2a_2 \cdot 0 + 3a_3 \cdot 0^2 + 4a_4 \cdot 0^3 + 5a_5 \cdot 0^4 + \cdots \\ -0 &= a_1 \\ a_1 &= 0 \end{aligned}$$

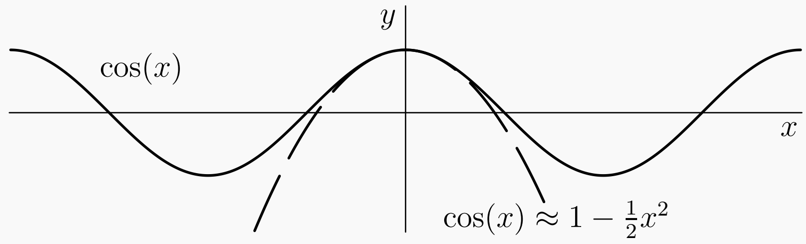
Thus, the best degree one polynomial approximation to cosine is $P_1(x) = 1 + 0x = 1$. Notice this is no different from the degree zero approximation, and notice this is also identical to our definition of tangent line from Calculus I!

- **Solving for a_2:** To solve for a_2, we take another derivative of both sides, to strip away the x that it was being multiplied by. After differentiating, we plug in $x = 0$.

$$\begin{aligned} (-\sin(x))' &= \left(a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots\right)' \\ -\cos(x) &= 2a_2 + 3 \cdot 2a_3x + 4 \cdot 3a_4x^2 + 5 \cdot 4a_5x^3 + \cdots \\ -\cos(0) &= 2a_2 + 3 \cdot 2a_3 \cdot 0 + 4 \cdot 3a_4 \cdot 0^2 + 5 \cdot 4a_5 \cdot 0^3 + \cdots \\ -1 &= 2a_2 \\ a_2 &= -\frac{1}{2} \end{aligned}$$

We now have the best degree-two polynomial approximation for cosine near zero!

$$P_2(x) = 1 + 0x - \frac{1}{2}x^2 = 1 - \frac{1}{2}x^2$$



- **Solving for a_3:** Yet again we differentiate and then plug in zero to find the coefficient a_3.

$$\begin{aligned}
(-\cos(x))' &= \left(2a_2 + 3\cdot 2a_3x + 4\cdot 3a_4x^2 + 5\cdot 4a_5x^3 + \cdots\right)' \\
\sin(x) &= 3\cdot 2a_3 + 4\cdot 3a_4x + 5\cdot 4\cdot 3a_5x^2 + \cdots \\
\sin(0) &= 3\cdot 2a_3 + 4\cdot 3a_4\cdot 0 + 5\cdot 4\cdot 3a_5\cdot 0^2 + \cdots \\
0 &= 3\cdot 2a_3 \\
a_3 &= 0
\end{aligned}$$

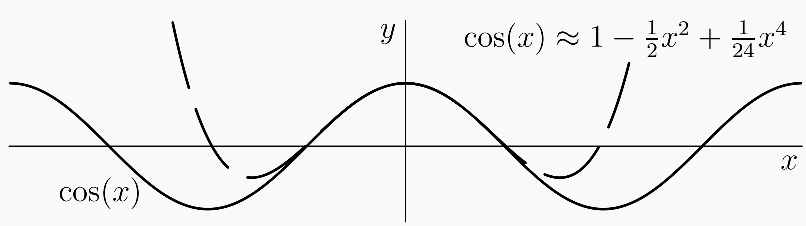
Graphically this makes sense; it is essentially saying that we couldn't represent cosine more accurately with a cubic polynomial function than we could have with a quadratic. Near zero, the graph of cosine looks a lot more like a parabola than it does like the graph of a cubic.

$$P_3(x) = 1 + 0x - \frac{1}{2}x^2 + 0x^3 = 1 - \frac{1}{2}x^2$$

- **Solving for a_4:** Yet again we differentiate and then plug in zero to find the coefficient a_4.

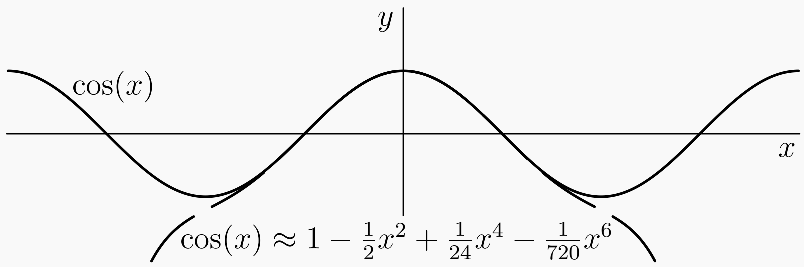
$$\begin{aligned}
(\sin(x))' &= \left(3\cdot 2a_3 + 4\cdot 3a_4x + 5\cdot 4\cdot 3a_5x^2 + \cdots\right)' \\
\cos(x) &= 4\cdot 3\cdot 2a_4 + 5\cdot 4\cdot 3\cdot 2a_5x + \cdots \\
\cos(0) &= 4\cdot 3\cdot 2a_4 + 5\cdot 4\cdot 3\cdot 2a_5\cdot 0 + \cdots \\
1 &= 4\cdot 3\cdot 2a_4 \\
a_4 &= \frac{1}{4!}
\end{aligned}$$

$$P_4(x) = 1 + 0x - \frac{1}{2}x^2 + 0x^3 + \frac{1}{4!}x^4 = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$$



- **Solving for a_5 and a_6:** By a similar argument, we find $a_5 = 0$ and $a_6 = -\frac{1}{6!}$. Thus, we have the best sixth-degree polynomial approximation.

$$P_6(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{6!}x^6$$



From here we see the pattern:

- **Left-hand Side:** The derivatives will continue cycling through the four functions $\cos(x), -\sin(x), -\cos(x)$, and $\sin(x)$. When we plug in $x = 0$, these functions give us the numbers $1, 0, -1$, and 0 respectively.

- **Right-hand Side:** After n derivatives, the only term that will not have a power of x attached to it is of the form $n!a_n$. We then must divide both sides by $n!$ to solve for a_n.

Extrapolating this pattern, we can state the full power series for cosine.

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \cdots$$

Often, we condense our notation by writing the power series in sigma notation rather than in expanded form:

$$\cos(x) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n)!}x^{2n}$$

Ok, you know what has to happen next.

Exercise 5.1.0.4. Time for Sine ☕☕

Repeat the above process to find the power series for sine!

And again.

Exercise 5.1.0.5. The Exponential Function

Repeat the above process to find the power series for the natural exponential function, $f(x) = e^x$.

And again.

Exercise 5.1.0.6. A Familiar Function ☕☕

- Repeat the above process to find the power series for the function $f(x) = \frac{1}{1-x}$.

- How does the above power series relate to the geometric series formula?

This method can be considered the brute force method of finding a power series. Later, we will develop more efficient methods, but this is how we get off the ground! Let us sum up (heh) this method below:

Brute Force Method for Finding a Power Series

To find a power series for a function $f(x)$:

• Write down the form of an unknown power series:

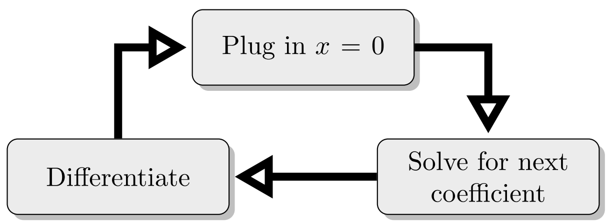
$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$

• Plug in $x = 0$ to solve for a_0.

• Differentiate both sides and plug in $x = 0$ to solve for a_1.

• Differentiate both sides and plug in $x = 0$ to solve for a_2.

• Repeat this until you have as many terms as you need!



Often this method is called *Taylor's Formula*. The key idea is to notice that to solve for the coefficient a_n, we must differentiate exactly n times and then divide by $n!$. We represent this algebraically:

Theorem 5.1.0.7. Taylor's Formula

If $f(x)$ and all of its derivatives are defined at $x = 0$, then

$$\begin{aligned} f(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \end{aligned}$$

for any value of x for which the sum converges.

While the method above works for many functions, it sometimes fails. In particular, it was no coincidence that the natural logarithm was absent from our examples above!

Exercise 5.1.0.8. Natural Log

Try to find a power series by our brute force method on the function $f(x) = \ln(x)$. Where does it fail?

To work around this, we adopt a more flexible view of power series. Rather than attempting to write our function as a sum of powers of x, which requires the function and its derivatives to be defined at $x = 0$, we express it as a sum of powers of $(x - a)$ for some real number a. It is essentially the same process, but to solve for the coefficients, you would set $x = a$ instead of $x = 0$. This produces a power series centered at a. We state this method below:

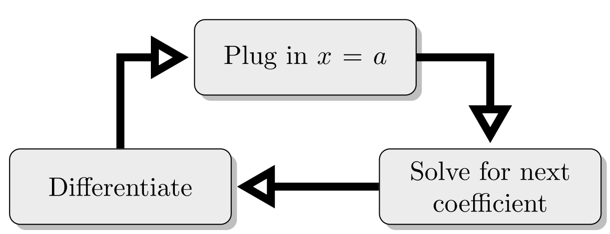
Finding a Power Series Centered at a

To find a power series centered at a for a function $f(x)$:

- Write down the form of an unknown power series:

$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + a_4(x-a)^4 + \cdots$

- Plug in $x = a$ to solve for a_0.
- Differentiate both sides and plug in $x = a$ to solve for a_1.
- Differentiate both sides and plug in $x = a$ to solve for a_2.
- Repeat this until you have as many terms as you need!



Exercise 5.1.0.9. Natural Log, Again ☕☕

Use the upgraded brute force method to find a power series for $f(x) = \ln(x)$ centered at 1.

The next exercise constructs the binomial series, an extremely important series in the fields of combinatorics, probability, and statistics.

Exercise 5.1.0.10. Binomial Series ☕☕☕

Let $m \in \mathbb{R}$. A function of the form

$$f(x) = (1+x)^m$$

is called a *binomial*, since it has two terms in the polynomial inside parentheses. Let us run the brute force method on this function to find its power series, the *binomial series*.

- Use the brute force method to find the first five terms in the series.

- Since that is cumbersome to write down, we define the *binomial coefficient* "m choose n" to be the following:

$$\binom{m}{n} = \frac{m \cdot (m-1) \cdot (m-2) \cdots (m-n+1)}{n!}$$

For example, $\binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3!} = 35$.

Demonstrate that with this notation, the terms you found via brute force are equivalent to the series

$$(1+x)^m = \sum_{n=0}^{\infty} \binom{m}{n} x^n$$

Exercise 5.1.0.11. Understanding Binomial Coefficient Notation ☕

In the formula for the binomial coefficient $\binom{m}{n}$, how many numbers are multiplied together in the numerator?

5.2 Interval of Convergence

With power series, be warned that not every infinite series will converge for all values of x.

Exercise 5.2.0.1. A Convergent Series

Let us consider what happens when we evaluate the power series

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \cdots$$

at $x = 1$.
We obtain the claim that $\cos(1) = 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} - \cdots$. Let us test this numerically, remembering that an infinite series is really just a limit of partial sums. Calculate decimal representations of the partial sums below.

Partial Sum	Decimal Approximation
1	
$1 - \frac{1}{2!}$	
$1 - \frac{1}{2!} + \frac{1}{4!}$	
$1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!}$	
$1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!}$	

Does it appear that the partial sums are converging to the true value of $\cos(1)$?

Exercise 5.2.0.2. A Not-So-Convergent Series

Let us consider what happens when we evaluate the power series

$$\ln(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \cdots$$

at $x = 3$.
We obtain the claim that $\ln(3) = 2 - \frac{1}{2}2^2 + \frac{1}{3}2^3 - \frac{1}{4}2^4 + \frac{1}{5}2^5 - \cdots$. Let us again test this numerically, remembering that an infinite series is really just a limit of partial sums. Calculate decimal representations of the partial sums below.

Partial Sum	Decimal Approximation
2	
$2 - \frac{1}{2}2^2$	
$2 - \frac{1}{2}2^2 + \frac{1}{3}2^3$	
$2 - \frac{1}{2}2^2 + \frac{1}{3}2^3 - \frac{1}{4}2^4$	
$2 - \frac{1}{2}2^2 + \frac{1}{3}2^3 - \frac{1}{4}2^4 + \frac{1}{5}2^5$	

Does it appear that the partial sums are converging to the true value of $\ln(3)$?

5.2.1 Definition of IOC

As you can see, power series expansions may be valid for some values of x but not others. The set of all the "good" x-values is called the interval of convergence.

Definition 5.2.1.1. Interval of Convergence

Given a power series, the set of all x values for which the infinite sum converges is called the *interval of convergence* (IOC). The midpoint of the interval is called the *center* of the IOC and the distance from the center to endpoints is called the *radius*.

On the interior of the interval, the power series will not only be convergent, but it will be absolutely convergent. This is of enormous convenience, because it means we are free to rearrange terms and do algebra as we like with our power series.

The fact that such a set is always an interval (as opposed to say just scattered points or a union of two disjoint intervals) is not obvious, but it turns out to be true. To find the interval of convergence, it is typically easiest to use the Ratio Test. Since power series always have powers of x, these will cancel nicely when we apply the Ratio Test.

Example 5.2.1.2. Interval of Convergence for Cosine

Consider again our power series

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}$$

Let us apply the Ratio Test to this series. Our goal is to get the absolute value of the ratios of consecutive terms to be less than 1, in which case we are certain the series converges.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+2}\frac{1}{(2(n+1))!}x^{2(n+1)}}{(-1)^{n+1}\frac{1}{(2n)!}x^{2n}}\right| \\
&= \lim_{n\to\infty}\left|\frac{x^{2n+2}\,(2n)!}{(2n+2))!x^{2n}}\right| \\
&= \lim_{n\to\infty}\left|\frac{x^2\,(2n)\,(2n-1)\cdots 3\cdot 2\cdot 1}{(2n+2)\,(2n+1)\,(2n)\,(2n-1)\,(2n-2)\cdots 3\cdot 2\cdot 1}\right| \\
&= \lim_{n\to\infty}\left|\frac{x^2}{(2n+2)\,(2n+1)}\right| \\
&= \left|x^2\right|\lim_{n\to\infty}\left|\frac{1}{(2n+2)\,(2n+1)}\right| \\
&= 0\left|x^2\right| \\
&= 0
\end{aligned}$$

The ratio of consecutive terms, being zero, is always strictly less than one no matter what x is. Since it converges for every x value, the interval of convergence is (∞, ∞).

Exercise 5.2.1.3. Careful with Algebra! ☕

Annotate each line above. Provide a short phrase indicating the reason each simplification was valid.

Exercise 5.2.1.4. IOC for Natural Log ☕☕☕

Repeat the process above for the series

$$\ln(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}(x-1)^n$$

For what x-values is the limit of ratios of consecutive terms less than one?

Notice in the above example, if $x = 2$ or $x = 0$, the limit of ratios of consecutive terms is equal to 1. This is the case in which the ratio test gives us no information. Thus, we must test these series for convergence separately. We can use any of the tests from Section 4.13.

Exercise 5.2.1.5. Checking Endpoints ☕☕

- Set $x = 2$ in the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}(x-1)^n$ and test it for convergence/divergence.

- Set $x = 0$ in the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}(x-1)^n$ and test it for convergence/divergence.

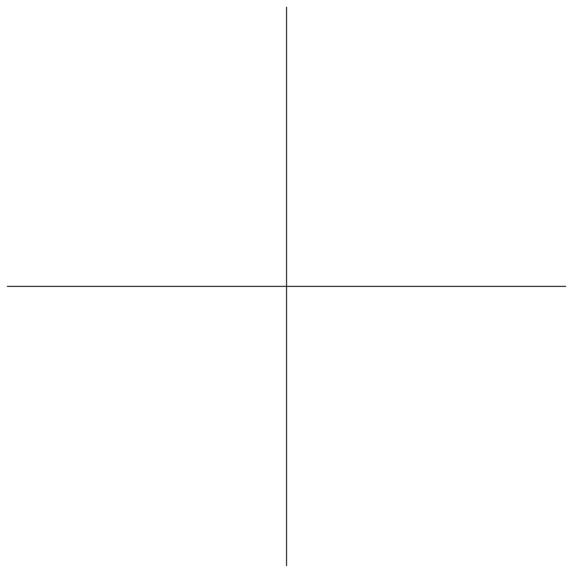
- Explain why the interval of convergence for $\ln(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}(x-1)^n$ is $(0, 2]$.

5.2.2 Looking at IOC Graphically

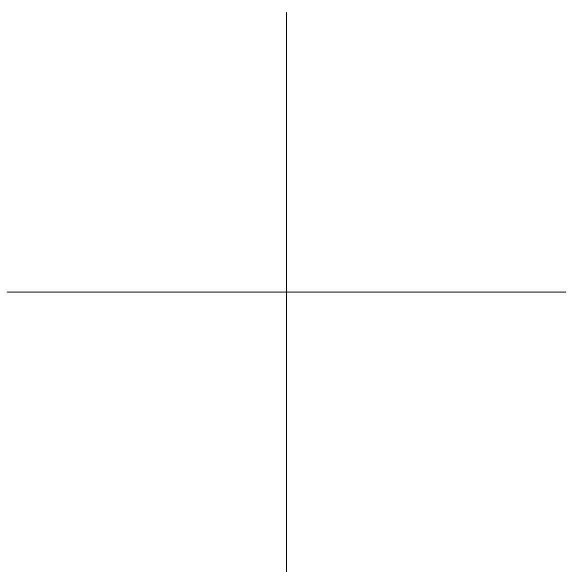
So far we found two IOC's. Our series for cosine had IOC $(-\infty, \infty)$. Our series for the natural logarithm had IOC $(0, 2]$. Let us compare these situations graphically!

Exercise 5.2.2.1. I See the IOC

- Sketch the graph of cosine, and on the same axes plot the graphs of $P_n(x)$ for $n = 0, 1, 2, 3, 4, 5,$ and 6. You may use a CAS to help come up with the graphs of the polynomials.



- Sketch the graph of natural log, and on the same axes plot the graphs of $P_n(x)$ for $n = 0, 1, 2, 3, 4, 5,$ and 6. You may use a CAS to help come up with the graphs of the polynomials.



- What feature of those graphs shows you the IOC? Explain!

5.3 Divide and Conquer with IOC

Recall our general framework for power series. To find a power series centered at $x = a$ for a function $f(x)$, we write down an equation of the form

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_2(x-a)^3 + a_2(x-a)^4 + \cdots$$

and then repeatedly plug in $x = a$ and differentiate in order to solve for the coefficients, one at a time.

Though this method will work to find the coefficients as long as the function and its derivatives exist at $x = a$, there remains the question: for what x-values will the infinite series on the right actually converge to the corresponding value of $f(x)$? In this activity, we investigate this both numerically (just using a table of values) and theoretically (using the ratio test). We call the set of all x-values for which a given series converges the *interval of convergence*.

Here we work through this framework in a variety of examples.

Exercise 5.3.0.1. Natural Logarithm ☕☕☕

Start with the function $f(x) = \ln(x)$ and do the following:

- Set up a power series centered at $a = 2$ for $\ln(x)$. Solve for the degree 2, degree 3, degree 4, degree 5, and degree 6 power series approximations for $\ln(x)$ centered at 2. Accomplish this by just repeatedly plugging in $x = 2$ and differentiating both sides. Call these functions $P_2(x), P_3(x), P_4(x)$, and $P_5(x)$, respectively.
- Make a small table of values where you list out the values of each of your P_i evaluated at $x = 1/2$. Compare this to the true value of $\ln(1/2)$.
- Make a small table of values where you list out the values of each of your P_i evaluated at $x = 2$. Compare this to the true value of $\ln(2)$.
- Perform the Ratio Test on your series expansion for $\ln(x)$. For what x would you have a ratio less than one?
- How do your results of the ratio test compare to the numerical evidence you found above?
- Notice that there will be two x-values that cause the series to have ratio exactly equal to one when Ratio Test is applied. Since the Ratio Test gives no info in that case, try a different test for each of those two series.
- At last, state the power series you came up with and the Interval of Convergence.

Exercise 5.3.0.2. Arctangent ☕☕☕

Start with the function $f(x) = \arctan(x)$ and do the following:

- Set up a power series centered at $x = 0$ for $\arctan(x)$. Solve for the degree 3, degree 5, degree 7, and degree 9 power series approximations. Accomplish this by just repeatedly plugging in $x = 0$ and differentiating both sides. Call these functions $P_3(x), P_5(x), P_7(x)$, and $P_9(x)$, respectively.
- Make a small table of values where you list out the values of each of your P_i evaluated at $x = 1/2$. Compare this to the true value of $\arctan(1/2)$.

- Make a small table of values where you list out the values of each of your P_i evaluated at $x = 2$. Compare this to the true value of $\arctan(2)$.
- Perform the Ratio Test on your series expansion for $\arctan(x)$. For what x would you have a ratio less than one?
- How do your results of the ratio test compare to the numerical evidence you found above?
- Notice that there will be two x-values that cause the series to have ratio exactly equal to one when Ratio Test is applied. Since the Ratio Test gives no info in that case, try a different test for each of those two series.
- At last, state the power series you came up with and the Interval of Convergence.

Exercise 5.3.0.3. Exponential ☕☕☕

Start with the function $f(x) = e^x$ and do the following:

- Set up a power series centered at $x = 0$ for e^x. Solve for the degree 2, degree 3, degree 4, and degree 5 power series approximations. Accomplish this by just repeatedly plugging in $x = 0$ and differentiating both sides. Call these functions $P_2(x), P_3(x), P_4(x)$, and $P_5(x)$, respectively.
- Make a small table of values where you list out the values of each of your P_i evaluated at $x = 1/2$. Compare this to the true value of $e^{1/2}$.
- Make a small table of values where you list out the values of each of your P_i evaluated at $x = 2$. Compare this to the true value of e^2.
- Perform the Ratio Test on your series expansion for e^x. For what x would you have a ratio less than one?
- How do your results of the ratio test compare to the numerical evidence you found above?
- At last, state the power series you came up with and the Interval of Convergence.

Exercise 5.3.0.4. Cosine ☕☕☕

Start with the function $f(x) = \cos(x)$ and do the following:

- Set up a power series centered at $x = \pi$ for $\cos(x)$. Solve for the degree 2, degree 4, degree 6, and degree 8 power series approximations. Accomplish this by just repeatedly plugging in $x = \pi$ and differentiating both sides. Call these functions $P_2(x), P_4(x), P_6(x)$, and $P_8(x)$, respectively.
- Make a small table of values where you list out the values of each of your P_i evaluated at $x = \pi/2$. Compare this to the true value of $\cos(\pi/2)$.
- Make a small table of values where you list out the values of each of your P_i evaluated at $x = 2$. Compare this to the true value of $\cos(2)$.

- Perform the Ratio Test on your series expansion for $\cos(x)$. For what x would you have a ratio less than one?
- How do your results of the ratio test compare to the numerical evidence you found above?
- At last, state the power series you came up with and the Interval of Convergence.

Exercise 5.3.0.5. Square Root ☕☕☕

Start with the function $f(x) = \sqrt{x}$ and do the following:

- Explain why you can't do a power series centered at zero for $\sqrt{x}$. (**Hint:** Try it.)
- Instead, set up a power series centered at $x = 1$ for $\sqrt{x}$. Solve for the degree 2, degree 3, degree 4, degree 5, and degree 6 power series approximations for $\sqrt{x}$ centered at $x = 1$. Accomplish this by just repeatedly plugging in $x = 1$ and differentiating both sides. Call these functions $P_2(x), P_3(x), P_4(x)$, and $P_5(x)$, respectively.
- Make a small table of values where you list out the values of each of your P_i evaluated at $x = 1/2$. Compare this to the true value of $\sqrt{1/2}$.
- Make a small table of values where you list out the values of each of your P_i evaluated at $x = 2$. Compare this to the true value of $\sqrt{2}$.
- Perform the Ratio Test on your series expansion for $\sqrt{x}$. For what x would you have a ratio less than one?
- How do your results of the Ratio Test compare to the numerical evidence you found above?
- Notice that there will be two x-values that cause the series to have ratio exactly equal to one when Ratio Test is applied. Since the Ratio Test gives no info in that case, try a different test for each of those two series.
- At last, state the power series you came up with and the Interval of Convergence.

Exercise 5.3.0.6. Reciprocal ☕☕☕

Start with the function $f(x) = \frac{1}{1+x}$ and do the following:

- Set up a power series centered at $x = 0$ for $\frac{1}{1+x}$. Solve for the degree 2, degree 3, degree 4, and degree 5 power series approximations. Accomplish this by just repeatedly plugging in $x = 0$ and differentiating both sides. Call these functions $P_2(x), P_3(x), P_4(x)$, and $P_5(x)$, respectively.
- Make a small table of values where you list out the values of each of your P_i evaluated at $x = 1/2$. Compare this to the true value of $\frac{1}{1+\frac{1}{2}}$.
- Make a small table of values where you list out the values of each of your P_i evaluated at $x = 2$. Compare this to the true value of $\frac{1}{1+2}$.

- Perform the Ratio Test on your series expansion for $\frac{1}{1+x}$. For what x would you have a ratio less than one?
- How do your results of the ratio test compare to the numerical evidence you found above?
- Notice that there will be two x-values that cause the series to have ratio exactly equal to one when Ratio Test is applied. Since the Ratio Test gives no info in that case, try a different test for each of those two series.
- At last, state the power series you came up with and the Interval of Convergence.

5.4 New Series from Old

Although the brute force method is a great starting point, we don't want to have to do that every time we want a power series for a function. Now that we have a library of known power series to draw from, we wish to manipulate these to construct new series rather than starting from scratch every time. We have four loose categories for finding new series from old:

- Substitution
- Algebra
- Differentiation
- Antidifferentiation

5.4.1 Substitution

Replacing x in a known series by another expression is often useful.

Exercise 5.4.1.1. Practice with Substitution ☕☕☕

1. Find a power series and the IOC for $\sin(x-1)$ centered at 1.

2. Find a power series and the IOC for $\sin(2x)$ centered at 0.

5.4.2 Algebra

An enormous plethora of algebraic tricks are useful with regards to finding power series for functions. Try the following:

Exercise 5.4.2.1. Practice with Algebra ☕☕☕

- Find a power series and the IOC for $\frac{1}{x^2-x-12}$ centered at 0. (**Hint:** Use PFD and geometric series.)

- Find a power series and the IOC for e^x centered at 2. (**Hint:** Replace x by $(x-2)+2$ and then pull out an e^2.)

- Find a power series and the IOC for $\sin(x)\cos(x)$ centered at 0. (**Hint:** Avoid taking the nasty product of those two series by applying the sine double angle identity!)

- Find a power series and the IOC for $\sin^2(x)$ centered at 0. (**Hint:** Avoid taking the nasty product of sine with itself by applying the sine half angle identity!)

- Find a power series and IOC for $\frac{1}{x}$ centered at 5.

5.4.3 Differentiation

Often we differentiate a known power series term-by-term to find a new series.

Exercise 5.4.3.1. Practice with Differentiation ☕☕☕

- Find a power series and IOC for $\frac{1}{x^2}$ centered at 5. (**Hint:** Use the answer from the last problem.)

- Find a power series and IOC for $\frac{1}{x^3}$ centered at 5. (**Hint:** Use the answer from the last problem.)

- Take the term-by-term derivative of the power series for e^x centered at 0. Verify that you do in fact get e^x back!

5.4.4 Antidifferentiation

Often we antidifferentiate a power series term-by-term to find a new power series. You will have a "$+C$" to solve for by plugging in the center for x after taking the antiderivative. Notice that this C is really just a_0, and plugging in the center for x is just the first step of the brute force method.

Exercise 5.4.4.1. Practice with Antidifferentiation ☕☕☕

- Take the term-by-term antiderivative of the power series for e^x centered at 0. Verify that you do in fact get e^x back!

- Find a power series and IOC for $\ln(1-x)$ centered at zero.

Exercise 5.4.4.2. Four Different Methods for Finding the Same Power Series ☕☕

Compute the power series for the function

$$f(x) = \frac{1}{(1-x)^2}$$

by following the four different methods outlined below:

1. Finding a power series using differentiation!
   - Write out the power series for $\frac{1}{1-x}$.

- Differentiate both sides.

2. Finding a power series by multiplying together two known series!
   - Write out the power series for $\frac{1}{1-x}$.
   - Square both sides. Note that this will involve a gigantic infinite FOIL on the right-hand side!

3. Finding a power series using long division!
   - Expand the denominator of $f(x)$, rewriting our function as $f(x) = \frac{1}{1-2x+x^2}$.
   - Perform polynomial long division using the lowest degree term on each step to identify

your quotient.

4. Finding a power series via brute force!
   - Write a general unknown series $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$.
   - Plug in zero and differentiate to repeatedly solve for the coefficients one at a time (our brute force method).

5.5 Error Bounds

Here we state a theorem (without proof) regarding just how accurate a finite degree polynomial approximation is.

Theorem 5.5.0.1. Taylor's Error Theorem

Let $f(x)$ be a function, n a natural number, and $P_n(x)$ the degree n power series approximation centered at a real number a. Let M be an upper bound for $\left|f^{(n+1)}(z)\right|$ where z is any number between x and a. Then the error (also called the remainder) in the approximation

$$f(a) \approx P_n(a)$$

is no worse than the quantity

$$\frac{M|x-a|^{n+1}}{(n+1)!}$$

That is,

$$|f(a) - P_n(a)| \leq \frac{M|x-a|^{n+1}}{(n+1)!}$$

Note that in the above theorem, the expression $f^{(n+1)}$ represents $n+1$ derivatives applied to the function f. It is not an exponent. Notice also that we can be a bit careless when choosing a value of M. If M is exactly the max value of $f^{(n+1)}$ between x and a, that will give us the tightest error bound. Often, for simplicity's sake, we will intentionally choose a M value that is a bit too large. This still provides a valid error bound, just not one that is as tight as it could have been.

Example 5.5.0.2. Taylor's Error Theorem Applied to Sine

Suppose we wish to compute $\sin(0.1)$ *by hand*! Here is a feasible approach.
Consider the third degree polynomial approximation $P_3(x)$ of sine, centered at zero. We have that for x values near zero,

$$\sin(x) \approx x - \frac{1}{6}x^3$$

Suppose we wish to compute $\sin(0.1)$ *by hand* using this approximation. We evaluate

$$\begin{aligned}
\sin(0.1) &\approx 0.1 - \frac{1}{6} \cdot (0.1)^3 \\
&= 0.1 - (0.16666\ldots) \cdot (0.001) \\
&= 0.1 - 0.00016666\ldots \\
&= 0.099833333\ldots
\end{aligned}$$

Taylor's Error Theorem allows us to analyze how accurate this approximation is. We list all the components to plug into the error formula below:

- The function $f(x)$ is $\sin(x)$, the function that we took the power series approximation of.
- The value of n is 3, the degree of the polynomial approximation. Notice though here since the degree four coefficient in the power series for sine is zero. Thus, $P_3(x) = P_4(x)$, so we can get away with using $n = 4$ which will give a better error bound.
- The center of the power series, a, is zero in this case since we used just powers of x, not powers of $x - a$ for some nonzero a.

- The value of x is 0.1, since that is the input value to the function.
- The upper bound $M = 1$ will suffice, since any derivative of a sine or cosine is again a sine or cosine (plus or minus) and thus has outputs of magnitude less than or equal to one.

Plugging all of this information into our error bound, we get that our error is no worse than

$$\frac{M|x-a|^{n+1}}{(n+1)!} = \frac{1|0.1-0|^5}{(5)!} = (0.000001)\cdot(0.008333\ldots) = 0.00000008333\ldots$$

This tells us that our approximation of $\sin(0.1)$ is incredibly accurate! The difference between the true value of $\sin(0.1)$ and our approximation $P_4(0.1) = 0.099833333\ldots$ is less than $0.00000008333\ldots$. Another way to say this is that if we were writing out the digits of $\sin(0.1)$ as a decimal, our approximation would have the correct first seven digits past the decimal point (up to rounding).

Exercise 5.5.0.3. Checking Our Work ☕

Compute $\sin(0.1)$ on a calculator or CAS. Verify that the first seven digits after the decimal are correct, and verify that the difference between the true and approximate values is less than $0.00000008333\ldots$ as claimed.

5.6 What the Cosine Button on Your Calculator Does

It is worth noting that the definitions of common non-polynomial functions (roots, logs, trig functions, etc) are often very useful for intuition, understanding, and proving theoretical results. However, they are often atrocious for practical computation! For example, consider (in year 1 BC, 1 year Before Computers) trying to compute the quantity $\sqrt{4.1}$. What are you going to do, guess and check? Will you play the high-low game?

What is needed is a method for expressing less computationally tractable functions as polynomials (which can be evaluated using good old arithmetic). We describe our method below!

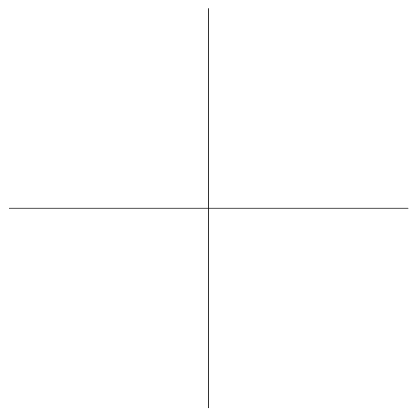
To compute $f(x)$ for a non-polynomial function f :

1. Find a power series expansion for f, preferably centered near x.
2. Take a finite degree approximation to this power series (or if the full power series is too difficult to obtain, maybe only compute finitely many terms in the first place).
3. Plug the value for x into your finite degree approximation.
4. Use Taylor's Error Theorem to be sure that your approximation is sufficiently accurate for your purposes.

This is the idea behind calculators and computer programs that can so efficiently compute so many wacky functions. Let's try it by hand together a bit to get a feel for the method.

Exercise 5.6.0.1. Seeing Taylor's Error Theorem on a Graph

- What is the degree 2 power series for $\cos(x)$ centered at zero? Write the function and plot the graph of both cosine and this parabolic approximation.



- On your graph, label y-coordinates for both functions (cosine and the parabolic approximation) at $x = 1$. What is the gap between the two y-coordinates at $x = 1$? That is, how far is the true value of $\cos(1)$ from the estimated value using the degree two polynomial approximation for cosine? (This gap is what we refer to as the error.)
- Apply Taylor's Error Theorem to this situation. What bound does it give on the error for approximating $\cos(1)$ via the degree two power series centered at zero? How does this relate to your measurement of the error above?
- Suppose we had a calculator that displays ten digits on the screen. What degree power series *would* you need to guarantee ten digits of accuracy in computing $\cos(1)$? Show explicitly how you used Taylor's Error Theorem to get your result. (**Hint:** Since any derivative of

cosine will just again be plus or minus cosine or sine, you can take $M = 1$.)

Exercise 5.6.0.2. The Error in a Square Root Calculation ☕☕☕

Suppose we wish to compute the number $\sqrt{4.1}$.

- Compute a degree one power series for the function $f(x) = \sqrt{x}$ centered at $x = 4$.

- Plug 4.1 into your degree one power series to get an approximation for $\sqrt{4.1}$. (Note that this is *exactly* what you did in Calc 1 when you looked at tangent lines and linearization as an approximation tool. The only difference here is we're not stuck at degree 1; we can crank up the degree as much as we like to improve accuracy!)

- Compute a degree two power series for the function $f(x) = \sqrt{x}$ centered at $x = 4$.

- Plug 4.1 into your degree two power series to get an approximation for $\sqrt{4.1}$.

- Compute a degree three power series for the function $f(x) = \sqrt{x}$ centered at $x = 4$.

- Plug 4.1 into your degree three power series to get an approximation for $\sqrt{4.1}$.

- Apply Taylor's Error Theorem in all three cases above, the degree one, two, and three cases. How many digits of accuracy do we get in each case?

Exercise 5.6.0.3. Clear-Cut Logging ☕☕☕

Compute ln(0.9) *by hand* accurate to three decimal places. Really. Just by hand. Show all work (including all arithmetic!) below. (**Hint:** Use Taylor's Error Theorem to make sure you aren't working harder than you need and accidentally using too many terms in the power series.)

It is worth noting that for most of the history of mathematics, logarithms were computed in essentially this manner, by hand, and then stored in massive tables which then people would use to look them up. A particularly successful 19th Century French mathematician Gaspard de Prony led a group which compiled a table of logarithms of integers between 1 and 200,000 accurate to 19 decimal places! His name is one of only 72 engraved onto the Eiffel Tower.

Exercise 5.6.0.4. You're Still Smarter Than the Machines ☕☕☕

Complete the following entirely by hand with no calculator use whatsoever!

- Compute $\sqrt[3]{1001}$ by using a degree two power series approximation for the function $f(x) = \sqrt[3]{x}$ centered at $a = 1000$.

- How many digits of accuracy does Taylor's Error Theorem guarantee in this case?

5.7 Graphing Using Power Series

5.7.1 Second Derivative Test via Power Series

Recall the Second Derivative Test from Calculus I.

Exercise 5.7.1.1. Go Ahead, Recall It ☕

State the Second Derivative Test.

How does this relate to looking at the degree two power series of a function? Specifically:

Exercise 5.7.1.2. Power Series Interpretation of the Second Derivative Test ☕☕

Let c be a critical point of the function $f(x)$.

- If $f''(c) < 0$, what does that tell you about the degree two power series for $f(x)$ centered at c? Explain.

- If $f''(c) > 0$, what does that tell you about the degree two power series for $f(x)$ centered at c? Explain.

- When $f''(c) = 0$, in Calculus I we would say the Second Derivative Test gave no information. With power series, how can you get around this situation and figure out the graph's behavior at that point?

Let us now put this to work, analyzing a complicated function!

Exercise 5.7.1.3. An Ugly Polynomial ☕☕☕

Graph the function by hand $f(x) = 36x - 30x^2 + \frac{28}{3}x^3 - x^4$ using the following steps:

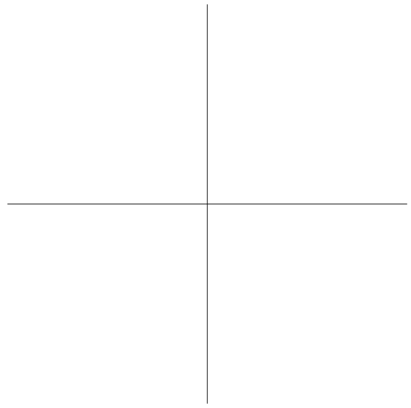
- Determine the end behavior of the function. That is, figure out the limits as x approaches infinity and minus infinity.

- Compute $f(x)$ at the x-values $x = 0, 2, 4, 6$ and plot those points to get started.

- Compute $f'(x)$ and set it equal to zero to find any critical points.

- Find the power series of $f(x)$ centered at each of those critical points. From these series conclude max/min/saddle at each critical point.

- Sketch the graph!



5.7.2 Symmetry via Power Series

One powerful technique in mathematics is the exploitation of symmetry. In the context of graphing, two commonly used types of symmetry are even symmetry and odd symmetry. Recall the definitions of even and odd symmetry below.

Exercise 5.7.2.1. Definitions of Even and Odd

Complete the definitions.

- We say a function $f(x)$ has *even* symmetry if and only if...

- We say a function $f(x)$ has *odd* symmetry if and only if...

Exercise 5.7.2.2. Symmetry of Sine and Cosine

- Use the power series for cosine to prove that cosine has even symmetry.

- Use the power series for sine to prove that sine has odd symmetry.

As the above exercise demonstrates, the key to a function having even (odd) symmetry is for the power series to only have terms of even (odd) degree! This not only justifies the names of the types of symmetry, but also gives us a quick and easy way to determine the symmetry of a function.

Exercise 5.7.2.3. Testing Our Known Series for Symmetry ☕☕

Scan through our list of known power series. Classify each function as odd, even, or neither. List any symmetries that you found below!

5.8 Evaluating Limits Using Power Series

Often when faced with an indeterminate form of a limit, it can be resolved by replacing functions with power series. Specifically, since a limit is trying to study a function as x approaches some value a, we use a power series centered at a for the function causing us trouble (or a series centered at whatever value the input is approaching).

Example 5.8.0.1. Evaluating a Limit Using Power Series

Consider the limit

$$\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$$

First, notice this is of the indeterminate form $0\cdot\infty$. This is a situation we would have typically handled via LHR, but now we have different tools! As x approaches infinity, $\frac{1}{x}$ approaches zero, so it makes sense to replace sine by its power series centered at zero. This will allow us to resolve the indeterminate form using ordinary algebra!

$$\begin{aligned}\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) &= \lim_{x\to\infty} x\left(\frac{1}{x}-\frac{1}{3!}\frac{1}{x^3}+\frac{1}{5!}\frac{1}{x^5}+\cdots\right)\\ &= \lim_{x\to\infty} 1-\frac{1}{3!}\frac{1}{x^2}+\frac{1}{5!}\frac{1}{x^4}+\cdots\\ &= 1-\frac{1}{3!}0^2+\frac{1}{5!}0^4+\cdots\\ &= 1\end{aligned}$$

Exercise 5.8.0.2. Checking Against LHR

Verify the previous limit calculation using LHR.

Exercise 5.8.0.3. Practice with Limits via Power Series

Evaluate each of the following limits two ways:

1. Using a power series centered at the x value that x is approaching.

2. Via L'Hospital's Rule.

Notice that the results of the second are justified by the first!

- $\lim_{x\to 0} \frac{1-\cos(x)}{x}$

- $\lim_{x\to 0} \frac{1-\cos(x)}{x^2}$

- $\lim_{x\to 1} \frac{\ln(x)}{x-1}$

- $\lim_{x\to 1} \frac{\ln(x)}{(x-1)^2}$

Also, as promised in Section 3.1, we provide a LHR justification using power series!

Exercise 5.8.0.4. Seeing LHR through a Power Series Lens ☕☕☕☕

Here we analyze the case where $f(x)$ and $g(x)$ both are functions with convergent power series expansions at a real number c. Assume also

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$$

We now write out the power series:

$$f(x) = a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$
$$g(x) = b_1(x-c) + b_2(x-c)^2 + b_3(x-c)^3 + \cdots$$

If $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ and $b_1 \neq 0$, then

$$\begin{aligned}
\lim_{x\to c} \frac{f(x)}{g(x)} &= \lim_{x\to c} \frac{a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots}{b_0 + b_1(x-c) + b_2(x-c)^2 + b_3(x-c)^3 + \cdots} \\
&= \lim_{x\to c} \frac{0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots}{0 + b_1(x-c) + b_2(x-c)^2 + b_3(x-c)^3 + \cdots} \\
&= \lim_{x\to c} \frac{(x-c)\left(a_1 + a_2(x-c) + a_3(x-c)^2 + \cdots\right)}{(x-c)\left(b_1 + b_2(x-c) + b_3(x-c)^2 + \cdots\right)} \\
&= \lim_{x\to c} \frac{a_1 + a_2(x-c) + a_3(x-c)^2 + \cdots}{b_1 + b_2(x-c) + b_3(x-c)^2 + \cdots} \\
&= \frac{a_1}{b_1} \\
&= \lim_{x\to c} \frac{a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots}{b_1 + 2b_2(x-c) + 3b_3(x-c)^2 + \cdots} \\
&= \lim_{x\to c} \frac{f'(x)}{g'(x)}
\end{aligned}$$

- How can we handle the case where $a_1 = b_1 = 0$?

- How can we handle the case where $b_1 = 0$ but $a_1 \neq 0$?

- What if $c = \infty$ instead of a real number?

- What if the indeterminate limit is of the form $\frac{\infty}{\infty}$ instead of $\frac{0}{0}$?

5.9 The Fibonacci Numbers via Power Series

Recall the Fibonacci numbers are the recursively defined sequence given by:

$$F_0 = 0$$

$$F_1 = 1$$

$$F_{n+2} = F_{n+1} + F_n$$

Exercise 5.9.0.1. Listing a Few Terms ☕

Use the above recursion to compute the first five Fibonacci numbers. List these below.

We now compare these numbers to the coefficients of a particular power series.

Exercise 5.9.0.2. Coefficients of a Particular Power Series ☕☕☕

- Use long division to find the first five coefficients of the power series of the following function:

$$f(x) = \frac{x}{1 - x - x^2}$$

- Whoa. What do you notice about the Fibonacci numbers vs the coefficients in that power series?

We had a recursively defined sequence and a sequence of power series coefficients. We now compare these to an explicitly defined sequence.

Exercise 5.9.0.3. A Particular Explicitly Defined Sequence ☕☕

Define a sequence a_n via the following explicit formula:

$$a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

- Compute the first five terms of this sequence by simply plugging in n values and crunching numbers on a calculator or CAS. List your answers below.

- What? Yes, really. Right?

Ok let's figure out what the heck is going on.

Let the function $f(x) = F_0 + F_1x + F_2x^2 + F_3x^3 + \cdots$. That is, f is defined to be a function whose power series has the Fibonacci sequence as its coefficients. This is called the *generating function* for the Fibonacci numbers.

Exercise 5.9.0.4. Studying the Generating Function ☕☕☕

1. Find a power series for the function $xf(x)$.

2. Find a power series for the function $x^2f(x)$.

3. Use the above to find a power series for the function $f(x) - xf(x) - x^2f(x)$.

4. Solve the above equation for $f(x)$ to get $f(x) = \frac{x}{1-x-x^2}$.

We now treat the function $f(x) = \frac{x}{1-x-x^2}$ as a "New Series from Old" style exercise. Once we find a formula its coefficients, we will have a formula for the Fibonacci numbers!

Exercise 5.9.0.5. Finding an Explicit Formula for the Coefficients ☕☕☕

1. Factor the polynomial $1 - x - x^2$ via the quadratic formula.

2. Use this factorization to find the partial fraction decomposition of $f(x) = \frac{x}{1-x-x^2}$.

3. Use the geometric series formula to find a power series for each term in the PFD, then add them together to get a power series for the Fibonacci generating function $f(x) = \frac{x}{1-x-x^2}$.

4. Equate coefficients by degree with our original expression for $f(x)$ to obtain the above explicit formula for the Fibonacci numbers!

The explicit formula for the Fibonacci Numbers is known as *Binet's Formula*. We state it again here, just because it is so nice to look at.

Theorem 5.9.0.6. Binet's Formula

For all $n \in \mathbb{N}$,

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

Exercise 5.9.0.7. Ratio of Consecutive Fibonacci Numbers ☕☕

We now revisit Exercise 4.6.1.4, armed with Binet's Formula! Use Binet's Formula to compute the limit of the ratio of consecutive Fibonacci numbers. That is, compute an exact value for:

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n}$$

5.10 Evaluating Infinite Sums Using Power Series

In Section 4.13, we developed many ways to determine if an infinite series converges, but we had almost no methods for determining what value a series converges to. Unless the series was geometric or telescoping, we were stuck.

Armed with our list of power series, we have much better tools! Given an infinite series we wish to evaluate, we can now:

- Look for key identifying features of the series that remind us of some known power series.
- See what x-value we can plug into our known power series to obtain the infinite series (or something close enough to it).
- Check that x-value was in the IOC of the power series, to be sure we were using a valid input.

Example 5.10.0.1. Evaluating an Infinite Series

Consider the infinite series

$$-\frac{3}{2!}+\frac{3^2}{3!}-\frac{3^3}{4!}+\frac{3^4}{5!}-\frac{3^5}{6!}+\cdots$$

We can convince ourselves that it converges using the Ratio Test. But what value does it converge to? We begin by noticing the similarity to the power series for e^x based on the consecutive factorials in the denominator. We then notice the ascending powers of 3 and the alternating signs in the infinite sum; this motivates $x=-3$ as a good choice for input. We write this infinite series down and then play with the equation until we obtain the infinite series above:

$$e^{-3}=1-\frac{3}{1!}+\frac{3^2}{2!}-\frac{3^3}{3!}+\frac{3^4}{4!}-\frac{3^5}{5!}+\cdots$$

Notice that the powers of three in our infinite sum are one less than the corresponding factorial, and the signs are off. Dividing both sides by negative three fixes this.

$$-\frac{1}{3}e^{-3}=-\frac{1}{3}+\frac{1}{1!}-\frac{3^1}{2!}+\frac{3^2}{3!}-\frac{3^3}{4!}+\frac{3^4}{5!}-\cdots$$

The first two terms can be moved over to left-hand side.

$$-\frac{1}{3}e^{-3}+\frac{1}{3}-\frac{1}{1!}=-\frac{3}{2!}+\frac{3^2}{3!}-\frac{3^3}{4!}+\frac{3^4}{5!}-\cdots$$

We combine those terms with arithmetic, and we have our total for the infinite series!

$$-\frac{3}{2!}+\frac{3^2}{3!}-\frac{3^3}{4!}+\frac{3^4}{5!}-\cdots=-\frac{1}{3}e^{-3}-\frac{2}{3}$$

Exercise 5.10.0.2. Checking with Ratio Test and Some Numerics ☕

- In the above example, it is claimed that the series converges by the Ratio Test. Verify by applying the Ratio Test and showing all details of the computation below.

- Evaluate $-\frac{1}{3}e^{-3} - \frac{2}{3}$ in a calculator or CAS. Compute a large partial sum of terms from the infinite series and verify that the answer is reasonable. (A quick numeric check like this is very valuable for catching minus sign mistakes or other algebra errors!)

Ok, try a few!

Exercise 5.10.0.3. Evaluating Infinite Series Using Power Series ☕☕☕

Evaluate each of the following to a number in closed form. Or, if the series does not converge, simply say "diverges" and give a brief explanation why.

1. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

2. $1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \cdots$

3. $3 - \frac{3}{2!} + \frac{3}{3!} - \frac{3}{4!} + \frac{3}{5!} - \cdots$

4. $3 - \frac{3^2}{2!} + \frac{3^3}{3!} - \frac{3^4}{4!} + \frac{3^5}{5!} - \cdots$

5. $3 - \frac{3^2}{2} + \frac{3^3}{3} - \frac{3^4}{4} + \frac{3^5}{5} - \cdots$

6. $1 - \frac{1}{2} + \frac{1}{3 \cdot 2} - \frac{1}{4 \cdot 3} + \frac{1}{5 \cdot 4} - \cdots$

7. $6 + \frac{6}{4} + \frac{6}{9} + \frac{6}{16} + \frac{6}{25} + \cdots$

8. $\binom{40}{0} + \binom{40}{1} + \binom{40}{2} + \binom{40}{3} + \binom{40}{4} + \cdots$

9. $\binom{40}{0} - \binom{40}{1} + \binom{40}{2} - \binom{40}{3} + \binom{40}{4} - \cdots$

Notice also that if you simply skip the step of plugging in a number for x, you can often evaluate a power series to a closed form. This will be particularly useful in Chapter 7.

Exercise 5.10.0.4. Finding Closed Forms for Power Series ☕☕☕

Evaluate each of the following into a closed form.

1. $5x - \frac{5}{2!}x^2 + \frac{5}{3!}x^3 - \frac{5}{4!}x^4 + \frac{5}{5!}x^5 - \cdots$

2. $-\frac{5}{2!}x^2 + \frac{5}{3!}x^3 - \frac{5}{4!}x^4 + \frac{5}{5!}x^5 - \cdots$

3. $1 + 5x - \frac{5^2}{2!}x^2 + \frac{5^3}{3!}x^3 - \frac{5^4}{4!}x^4 + \frac{5^5}{5!}x^5 - \cdots$

4. $5 + 5^2x - \frac{5^3}{2!}x^2 + \frac{5^4}{3!}x^3 - \frac{5^5}{4!}x^4 + \frac{5^6}{5!}x^5 - \cdots$

5. $5 + 5^2x - \frac{5^3}{2!}x^2 - \frac{5^4}{3!}x^3 + \frac{5^5}{4!}x^4 + \frac{5^6}{5!}x^5 - \frac{5^7}{6!}x^6 - \frac{5^8}{7!}x^7 \cdots$

6. $5x + 5^2x^2 - \frac{5^3}{2!}x^3 - \frac{5^4}{3!}x^4 + \frac{5^5}{4!}x^5 + \frac{5^6}{5!}x^6 - \frac{5^7}{6!}x^7 - \frac{5^8}{7!}x^8 \cdots$

We have two forms, closed form functions and their power series. We also have a way to go back and forth between the two forms. This calls for another game of... TELEPHONE!

Break into groups of four and play telephone with one of the following pages. If you are handed an explicit formula for a function $f(x)$, find the power series centered at one for that same function fold over the original $f(x)$, and pass it along. If you are handed a power series, find the function $f(x)$ it evaluates to, fold over the original power series, and pass it along.

$$\sum_{n=0}^{\infty}(x-1)^n$$

$\boxed{f(x) = \qquad\qquad\qquad\qquad}$

$\boxed{\sum_{n=0}^{\infty} \qquad\qquad\qquad\qquad}$

$\boxed{f(x) = \qquad\qquad\qquad\qquad}$

$\boxed{\sum_{n=0}^{\infty} \qquad\qquad\qquad\qquad}$

$$\sum_{n=0}^{\infty} \binom{1/2}{n} (x-1)^n$$

$\boxed{f(x) = \qquad\qquad\qquad\qquad}$

$\boxed{\sum_{n=0}^{\infty} \qquad\qquad\qquad\qquad}$

$\boxed{f(x) = \qquad\qquad\qquad\qquad}$

$\boxed{\sum_{n=0}^{\infty} \qquad\qquad\qquad\qquad}$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}(x-1)^n$$

$f(x) =$

$\sum_{n=0}^{\infty}$

$f(x) =$

$\sum_{n=0}^{\infty}$

$$\sum_{n=0}^{\infty}(-1)^n(x-1)^n$$

$f(x) =$

$\sum_{n=0}^{\infty}$

$f(x) =$

$\sum_{n=0}^{\infty}$

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)!}(x-1)^n$$

$f(x) =$

$\sum_{n=0}^{\infty}$

$f(x) =$

$\sum_{n=0}^{\infty}$

Bhaskara's Approximation of Sine

Kenneth M Monks

It's important to note that Euler's impressive use and development of power series in the 18th century was far from the first attempt by mathematicians worldwide to try to approximate trig functions via algebraic functions.

Let's compare a far earlier method by Bhaskara during the seventh century in India. Bhaskara proposed using the following (surprising!) algebraic approximation for sine:

$$sin(x) \approx \frac{16x(\pi - x)}{5\pi^2 - 4x(\pi - x)}$$

a) Plot both functions using a graphing utility, $sin(x)$ and the approximation. How do the graphs compare? What is the worst the error ever is on the interval $[0, \pi]$? (You can just estimate the error visually from the graph, nothing too technical here.)

b) How crazy is it how crazy good that is? Right? I know.

Historians are unsure of how he came upon this formula as he only published the result and not his derivation. Here we will together go through a plausible line of reasoning which could lead to such a formula, and we will analyze the results and compare them to our power series approximations.

So suppose we want to approximate sine. Realistically, we don't gain anything by approximating it outside of the interval $[0, \pi]$ since any other value of sine could be computed via a reference angle to something between zero and π. Let's think of some nice properties that sine has on that interval that we would want our approximation to also have:

-Certainly we want our approximation to be zero at 0 and π, and for those to be the only zeroes on the interval.

- Additionally, we'd like our approximation to satisfy the symmetry that sine has on that interval. In particular, $sin(x) = sin(\pi - x)$.

Thinking in this manner, a plausible first attempt at approximating sine via something algebraic could be:

$$sin(x) \approx x(\pi - x)$$

a) Verify this approximation satisfies the two properties listed above.

b) How good of an approximation is this? Plot both below and describe what is good about the approximation, as well is what is not good about it. How large does the error get on that interval?

c) As you probably suspect from your work in b), it seems it would be wise to scale our approximation down to get the heights closer. Scale it by whatever constant is needed to get the y-value correct at $\pi/2$. That is, find a real number a such that the approximation $ax(\pi - x)$ is perfect at the point $\pi/2$.

d) How does your new approximation from c) compare? Does it still satisfy the desired properties from a)? How large does the error get using the new approximation?

One way we could improve upon what happened in d) is to scale by different amounts at different parts of the interval, rather than just scaling by a constant factor across the whole interval where the ideal scaling factor may be very different from point to point. We want to scale by something algebraic though to keep our approximation algebraic. So let's think of different polynomials we could scale by.

e) Scaling by a constant was essentially scaling by a degree zero polynomial. The next natural thing to try would be a degree 1 polynomial. Why would scaling it by a degree 1 polynomial not work? (HINT: one of our properties from a) would fail!)

f) Thus, it is reasonable to next try scaling it by a degree 2 polynomial. In interest of preserving symmetry, we'll scale by something of the form $b + cx(\pi - x)$. Thus the form of our approximation will be:

$$sin(x) \approx \frac{ax(\pi - x)}{b + cx(\pi - x)}$$

Why can we assume $b = 1$? (HINT: If b was not 1, how could you reduce the above fraction to get it to be 1?)

g) Use the two known rational values that $\sin(\pi/2) = 1$ and $\sin(\pi/6) = 1/2$ to solve for a and c.

h) Simplify your expression with a and c plugged in to get Bhaskara's formula.

i) How many terms would you need in the power series for sine to achieve the same accuracy as Bhaskara's formula on that interval? Use Taylor's Error Theorem to get your result.

5.11 Rational and Irrational Numbers

One of the oldest questions in mathematics is the following:

Which real numbers are rational, and which are irrational?

5.11.1 Rational Numbers

Recall that the set of rational numbers is the set of all numbers that can be expressed as the ratio of two integers. More formally:

$$\mathbb{Q} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0 \right\}$$

So to show a number is rational is usually not terribly hard... we simply need to find some integers a and b whose ratio is your number.

Exercise 5.11.1.1. Definition of Rational Numbers ☕

Show that the real number 2.1 is rational. What is your a and what is your b?

If the decimal expansion is repeating and not terminating, it is slightly more tricky, as we need an infinite geometric series.

Exercise 5.11.1.2. Repeating Decimals are Rational ☕☕

1. Use a geometric series to show that .0131313131313... is rational:

2. The infinite geometric series formula in fact proves that every number with a repeating

decimal expansion is rational. Explain why this is the case:

5.11.2 Irrational Numbers

What is *much* harder is to show that a number is irrational. To accomplish this, we must show that it is not possible to express a number as a ratio of integers for *any* choice of integers. Since there are infinitely many, we obviously cannot run through all choices of a and b to verify that none of them work. Rather, an irrationality argument typically follows a pattern of logic known as "proof by contradiction" (*Reductio ad absurdum*). The basic idea is to assume the opposite of what you are trying to prove and deduce an absurd conclusion, thus implying your working assumption was false. In this case, to show that a number $r \in \mathbb{R}$ is irrational via proof by contradiction:

1. Assume r is rational.
2. Thus there exist some integers a and b with $r = \frac{a}{b}$.
3. Use the equation $r = \frac{a}{b}$ and known properties of the number r to deduce a statement we know is false.
4. Conclude that our assumption of r being rational must have been false, so r is in fact irrational.

5.11.3 The Square Root of Two

Here is the classic proof that the square root of two is irrational. It relies on the simple fact that a number is even if and only if it can be written as a multiple of two.

Exercise 5.11.3.1. Irrationality of $\sqrt{2}$ ☕☕☕

Fill in the blanks in the following proof:

Proof: Assume that $\sqrt{2}$ is rational. Then

$$\sqrt{2} = \frac{a}{b}$$

for some $a, b \in \mathbb{Z}$. We may assume that a and b are relatively prime. That is, if a and b had a common factor greater than one, we could cancel it out of the fraction, so we may as well assume a and b were chosen to have no common factors greater than one.
By squaring both sides and clearing denominators in the above equation, we get

____

Thus a^2 is even, since it is two times an integer. But since a^2 is even, a must be even as well. Therefore we can write $a = 2m$ for some number $m \in$ ______. Plugging this into the above equation produces the following:

$$2b^2 = (2m)^2 \implies 2b^2 = 4m^2 \implies b^2 = \_\_\_\_\_\_$$

Thus b^2 is even so ______ is also even.
However, a and b were taken to be relatively prime! If a and b are both even than the initial fraction $\frac{a}{b}$ was not a reduced fraction as assumed since we could cancel a two out of the top and bottom. Thus we have reached a contradiction, so our initial assumption must have been false.
We conclude the square root of two is irrational! **QED**

There is a lot of interesting history behind this result. The square root of two first came up as a result of the Pythagorean Theorem being used to measure the diagonal of a square. The exact history of this result is sketchy! Different authors tell different versions of the story.

Exercise 5.11.3.2. A Brief Literature Search ☕☕

Regardless of which account is correct, history indicates that the result is at least how old? Who are two other Greeks who may have discovered the result? Cite your sources below.

5.11.4 Euler's Constant e

Significantly harder than proving the irrationality of the square root of two is proving that the number e is irrational. The reason e is fundamentally harder to deal with is because e turns out to be **transcendental** whereas $\sqrt{2}$ is not. This means that $\sqrt{2}$ is a root of a polynomial equation with integer coefficients, whereas e is not the root of any such polynomial.

Exercise 5.11.4.1. The Square Root of 2 is not Transcendental ☕

Justify the above claim by finding a polynomial with integer coefficients that has $\sqrt{2}$ as a root.

Notice that in our proof of the irrationality of $\sqrt{2}$, our very first step was exactly the condition that the number is a root of a polynomial with integer coefficients. Since we have no such polynomial for

e, we must start our proof based on something else. It turns out this "something" is the infinite series expansion for e! ← *excitement, not factorial* Let us step through this together.

Exercise 5.11.4.2. Proof That e is Irrational ☕☕☕

Fill in the missing parts of the argument below:

Proof: First let's write e as an infinite series. To do this, recall the power series for the exponential function:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Set $x = 1$ to get an infinite series for e:

$$e = e^1 = \rule{6cm}{0.4pt}$$

Now we proceed by the classic proof technique: proof by ______________. Accordingly, we assume e is in fact rational and then show that it leads to an absurd statement.
Thus, assume e is rational. Then there exist some $a, b \in \mathbb{Z}$ such that

$$e = \rule{2cm}{0.4pt}$$

For any n we can multiply both sides of the above equation by ______________ to obtain

$$n!be = n!a$$

Notice that the right-hand side is an integer because $n!$ and a both are. Thus the left-hand side must also be an integer. Notice however, the left-hand-side can be decomposed as follows:

$$n!be = bn!\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right) + bn!\left(\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots\right)$$

The term $bn!\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right)$ is an integer because

__________________________________.

We now proceed to show that the second term is not an integer for sufficiently large n. This will produce a contradiction since the left-hand side was an integer for all n. In particular, we will show that $bn!\left(\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots\right) = b\left(\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots\right)$ is between $\frac{b}{n+1}$ and $\frac{b}{n}$, which for $n > b$ will be nonintegral. Proceeding, we have:

$$\begin{aligned}
\frac{1}{n+1} &< \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots && (5.1)\\
&< \frac{1}{n+1} + \frac{1}{(n+1)(n+1)} + \frac{1}{(n+1)(n+1)(n+1)} + \cdots && (5.2)\\
&= \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots && (5.3)\\
&= \frac{\frac{1}{n+1}}{1 - \frac{1}{n+1}} && (5.4)\\
&= \frac{1}{n} && (5.5)
\end{aligned}$$

The above steps are justified as follows. The inequality on line (1) is true because ______________________________. To get from line (1) to line (2) we notice that

____. To get from line (2) to line (3) is simply algebra. To get from line (3) to line (4) is an infinite geometric series with common ratio _______________ and initial term _______________. To get from line (4) to line (5) is again just algebra (check this step).
Multiplying all sides by b, we get

$$\frac{b}{n+1} < b\left(\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots\right) < \frac{b}{n}$$

as desired, which completes the proof.

QED

This proof is actually not the original proof this fact; Euler first proved it using continued fractions. The proof above is due to Fourier. It was again proved in 1873 by Charles Hermite. It appeared in his landmark paper *Sur la fonction exponentielle*. Hermite's method worked not only for e but also lead to the proof that π is irrational. This proof however is quite a bit harder than the proof for e, and will be saved for later in your mathematical career, along with the proofs that π and e are transcendental!

5.12 Partition Counting via Power Series

It turns out that power series are one of the best tools for counting that which is hard to count. Ready to play?

Definition 5.12.0.1. Partition

A **partition** of a positive natural number n is an expression of n as a sum of positive integers. The terms in that sum are called **parts**.

For example, 1+1+2+3 is a partition of 7 with parts 1,1,2, and 3. We do not count different orderings of the same numbers as different partitions. So 1+1+2+3 and 1+2+1+3 would be counted as the same partition.

It often becomes of interest to know how many partitions a number has. For example, the number 4 has five partitions because:

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$$

Note that writing 4 as a sum of just the one number 4 itself counts as a valid partition.

$$5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1$$

shows that 5 has seven partitions.

In case you aren't tickled pink by the inherent mathematical challenge of counting such a thing, know that partitions come up in innumerable (ok, numerable but large) applications outside of mathematics, as you are often faced with decomposing a quantity into smaller quantities. For example, they come up when simulating nuclear fission; when an atom is smashed, the nucleus of protons and neutrons is broken into a set of smaller clusters of subatomic particles. The sum of the particles in the set of clusters must equal the original size of the nucleus. As such, the number of partitions of the original number of protons counts all the possible ways to smash the atom.

Thus, we seek to find how many partitions of a natural number n are there? If we were to program a computer or graphing calculator to find such a quantity, how could we do it? It turns out that one answer lies with power series!

Consider the following function:

$$f(x) = \frac{1}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^3}\frac{1}{1-x^4}\frac{1}{1-x^5}\frac{1}{1-x^6}\cdots$$

We claim that in expanded form,

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots$$

then a_n is the number of partitions of the integer n. Work through the following exercise to see why!

Exercise 5.12.0.2. Partition Generating Function ☕☕

- Begin by using the geometric series to replace each of the factors in the above infinite product with a power series. Write this expression below.

- Begin to expand out the series by multiplying out the factors above. Simply proceed one degree at a time. Get all the coefficients out to at least degree 5. How do the terms that arise in this product correspond to the partitions we listed for 5 above? (This will be a bit computationally intensive, but it will be worth it.)

- Use the above to find the number of partitions of 20. You may use a CAS to do algebra or differentiation for you, but indicate below what instructions you gave the software.

It turns out the beauty of this method is that it is highly robust! It is easy to modify if we want to find the partitions using only particular positive integers instead of having any part size at our disposal.

Example 5.12.0.3. Only Certain Part Sizes

Find all partitions of 10 using only parts of size 1, 4, and 5. Here we would have

$$\begin{aligned} 10 &= 5+5 \\ &= 5+4+1 \\ &= 5+1+1+1+1+1 \\ &= 4+4+1+1 \\ &= 4+1+1+1+1+1+1 \\ &= 1+1+1+1+1+1+1+1+1+1 \end{aligned}$$

which is six partitions of 10 using only parts of size 1,4, and 5.

Exercise 5.12.0.4. Restricted Partition Generating Functions ☕☕☕

1. Expand the following function out to get a power series of degree ten. Explain how the number of partitions with restricted part size above corresponds to the degree 10 coefficient.

$$f(x) = \frac{1}{1-x}\frac{1}{1-x^4}\frac{1}{1-x^5}$$

2. Suppose we wanted to figure out how many ways we can make change for a dollar using pennies, nickels, dimes, and quarters. How could we accomplish this? Find the number and explain your solution. Again you may use mathematica, WolframAlpha, or another

computer algebra system to multiply polynomials for you.

5.13 Mixed Practice

Here is a wide spread of practice problems using power series to help absorb all of the concepts and techniques!

Exercise 5.13.0.1. Nothing New, Just Practice ☕☕

1. - Find a degree two power series for the function $\sqrt[3]{x}$ centered at 8.
   - Use your approximation from part a) to estimate the value of $\sqrt[3]{8.05}$.
   - Use Taylor's Error Theorem to give a bound on how bad the error could be in your estimation of the value of $\sqrt[3]{8.005}$. Type the exact value into a calculator or CAS and confirm that you have obtained the desired accuracy.

2. Evaluate each of the following infinite series to a closed form. Explain your reasoning.

- $\sum_{n=0}^{\infty} \frac{3^n}{n!2^{n+1}}$

- $\sum_{n=3}^{\infty} \frac{(-1)^n}{n}$

- $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!2^{2n}}$

3. - List by hand how many ways you could make change for 35 cents using nickels, dimes, and quarters.

   - Show how you would use geometric series as generating functions to reach this same conclusion. Verify your answers match.

4. Find a rational number that approximates the square root of e accurate to three decimal places. (**Hint:** Think of the function e^x evaluated at $x = 1/2$. Then use Taylor's Error Theorem to guarantee sufficient accuracy.)

5. Find power series and Interval of Convergence for the following functions:

   - $f(x) = \frac{1}{x}$ centered at -3

- $f(x) = \frac{1}{1-x}$ centered at -3

- $f(x) = \frac{1}{e^x}$ centered at 0

- $f(x) = \frac{1-x-x^2}{1-x}$ centered at 0

- $f(x) = 2^x$ centered at 1

- $f(x) = x^2 + x + 1$ centered at 5

6. Find a degree three power series centered at zero for the function $\frac{1}{4-x^2}$ in five different ways:

- Brute force.

- Via geometric series with a substitution for x.

- Via a multiplication of the series for $\frac{1}{2-x}$ with $\frac{1}{2+x}$.

- Via a sum of the series that result in a partial fraction decomposition of $\frac{1}{4-x^2}$.

- Via long division, dividing the numerator 1 by the denominator $4 - x^2$.

5.14 Power Series Reference Sheet

Here we compile our list of known series. Write each below, both listing terms and in sigma notation, indicating the center and the interval of convergence in each case (except Binomial which will not fit in that little IOC block):

Function	Series in Sigma Notation	Series in Expanded Form	IOC
Geometric:			
Exponential:			
Sine:			
Cosine:			
Natural Log:			
Arctangent:			
Binomial:			*Long Story*
Fibonacci:			

Part III

Coming Attractions

Chapter 6

Introduction to Calculus III: Parametric and Polar

We begin by briefly thinking about the word *dimension*.

Exercise 6.0.0.1. Dimension ☕

One intuitive notion of dimension comes from the idea of how you would assign units to measure it. If an object has length, you would call it one-dimensional. If an object has area, it is called two-dimensional. If it has volume, it is called three-dimensional. State the dimension of each of the following objects:

- $\{x \in \mathbb{R} : x < 2\}$
- $\{(x, y) \in \mathbb{R}^2 : x < 2\}$
- The closed interval $[2, 3]$
- The circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$
- The disc $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$

In Calculus III, you will redo all of the key concepts of Calculus I and II but in three (or more) dimensions. Often the difficultly of higher-dimensional calculus is notational more than anything! In three or more dimensions, it becomes messier to write down the same concepts. To make this cleaner, we develop better languages for points and curves beyond our standard coordinate system.

6.1 Parametric Curves

Many of the objects we study, like circles or graphs of functions, are one-dimensional objects even though we usually view them as embedded in a two-dimensional plane. Thus, we can represent both x and y (the two dimensions) in terms of the same parameter t.

Definition 6.1.0.1. Parametric Curve

Let $x(t)$ and $y(t)$ be functions of t and let $D \subset \mathbb{R}$. Then a *parametric curve* is a set of points

$$\{(x(t), y(t)) : t \in D\}$$

Typically, D is an interval or union of intervals. We can graph most curves by just selecting t values from the domain D and plotting the corresponding points.

Exercise 6.1.0.2. A Warm-up Parametric Curve

Consider the parametric curve

$$\{(2t, 3t + 1) : t \in [-1, 3]\}$$

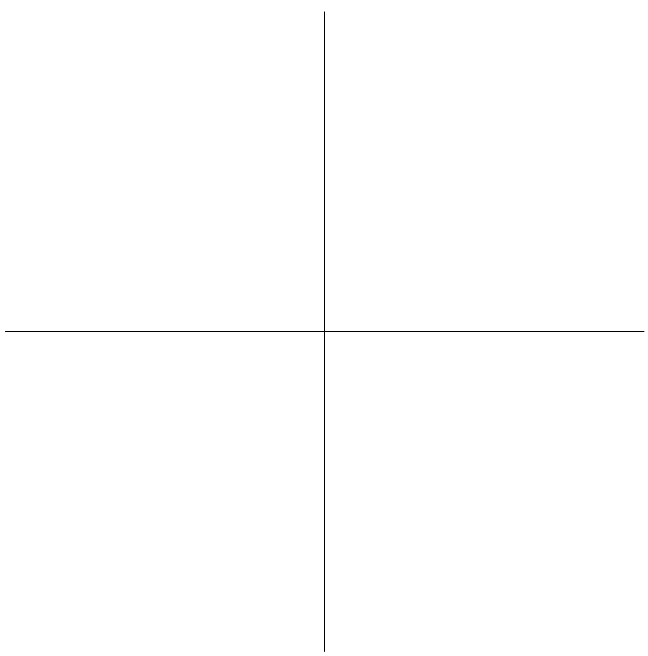
That is,

$$\begin{aligned} x(t) &= 2t \\ y(t) &= 3t + 1 \\ -1 \leq\ &t \leq 3 \end{aligned}$$

- Use the above formulas for $x(t)$ and $y(t)$ and the following t-values selected from $D = [-1, 3]$ to fill out the following table:

t	-1	0	1	2	3
$x(t)$					
$y(t)$					

- Plot those five points on the axes below. What type of shape does it appear to be?



- Solve the equation $x = 2t$ for t. Substitute this expression for t into the equation $y = 3t + 1$. What does this new equation tell you about the parametric curve?

Here is an example of a parametric curve used in Trigonometry (though not called so at the time).

Exercise 6.1.0.3. The Unit Circle ☕☕

- Explain why the parametric curve

$$C_1 = \{(\cos(t), \sin(t)) : t \in [0, 2\pi]\}$$

is the familiar unit circle from trigonometry.

- Consider the curve

$$C_2 = \{(\sin(t), \cos(t)) : t \in [0, 2\pi]\}$$

How are the curves C_1 and C_2 similar? How are they different?

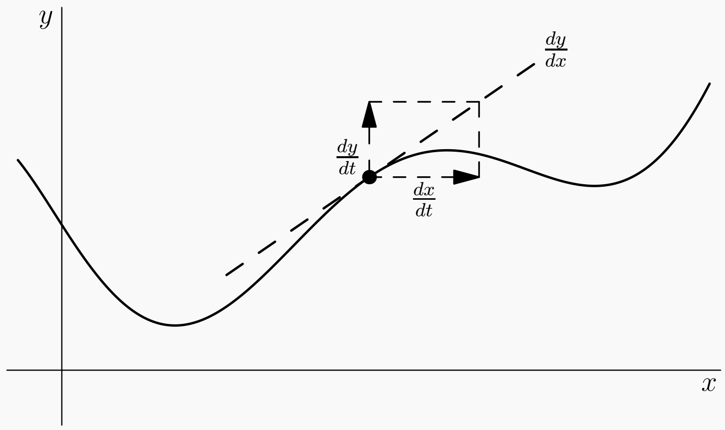
6.2 Derivatives of Parametric Curves: Slopes of Tangent Lines

To compute the derivative of a parametric curve, we recall that the slope of a line is the change in y-coordinate divided by the change in x-coordinate. In the context of parametric curves, these can be computed as rates of change with respect to the parameter t.

Definition 6.2.0.1. Parametric Derivatives

Let $(x(t), y(t))$ be a parametric curve. Then the slope of the tangent line can be computed as

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$



Notice that the above formula is just a slightly rearranged version of the chain rule. In particular, if we consider a portion of the graph of $(x(t), y(t))$ that passes the Vertical Line Test, then we can consider y as a function of x, which x in turn is a function of t. So if we wanted to ask how y changes with respect to t, we would have to take the rate of change of y with respect to x and multiply it by the rate of change of x with respect to t (by the chain rule). Expressing this Chain Rule in symbols instead of words, we have:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Exercise 6.2.0.2. Understanding the Definition ☕

How would you get from the chain rule application shown above to our definition of parametric derivatives?

Example 6.2.0.3. Parametric Derivatives on a Parabola

Consider the parametric curve given by

$$\{\left(t^2, t\right) : t \in [0, \infty)\}$$

To find the slope of a tangent line to this parabola, we can use the parametric derivative formula:

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \\ &= \frac{\frac{d}{dt}(t)}{\frac{d}{dt}(t^2)} \\ &= \frac{1}{2t}\end{aligned}$$

Alternately, we could convert the curve to a cartesian equation and differentiate with respect to y. Proceeding, we notice this curve is contained in the graph of $y = \sqrt{x}$, since the formulas $x = t^2$ and $y = t$ satisfy that relationship. Thus, we can differentiate y with the power rule:

$$\begin{aligned}\frac{dy}{dx} &= \left(\sqrt{x}\right)' \\ &= \frac{1}{2\sqrt{x}}\end{aligned}$$

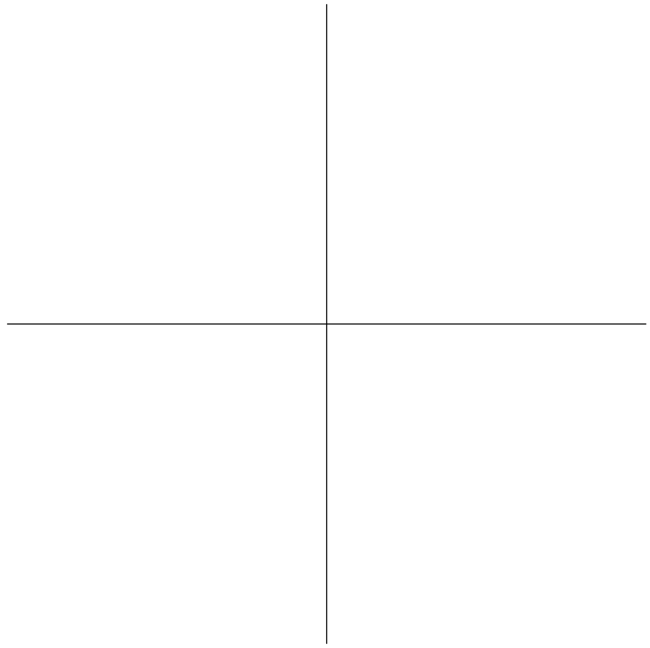
Exercise 6.2.0.4. Equivalence of the Results ☕☕

In the above example, we have two distinct expressions for $\frac{dy}{dx}$. Explain why they are in fact equivalent.

Exercise 6.2.0.5. The Tangent Line to an Ellipse ☕☕

- Plot the parametric curve given by

$$\{(2\cos(t), \sin(t)) : t \in [0, 2\pi]\}$$



- Find the point on the graph located at $t = \pi/4$, and find the slope of the tangent line at that point using the parametric derivative formula. Sketch the tangent line on your graph above.

- Verify the above curve is in fact the ellipse given by $\frac{x^2}{4} + y^2 = 1$.

- Use implicit differentiation on the equation $\frac{x^2}{4} + y^2 = 1$ to find $\frac{dy}{dx}$ at that same point and verify your answers match!

Exercise 6.2.0.6. Finding a Parameterization ☕☕☕

Find a parameterization of the path that consists of two full clockwise laps around the ellipse given by

$$\frac{(x-3)^2}{4} + (y-3)^2 = 1$$

starting from the point (3,2).

Exercise 6.2.0.7. A Hyperbola ☕☕

Consider the parametric curve given by:

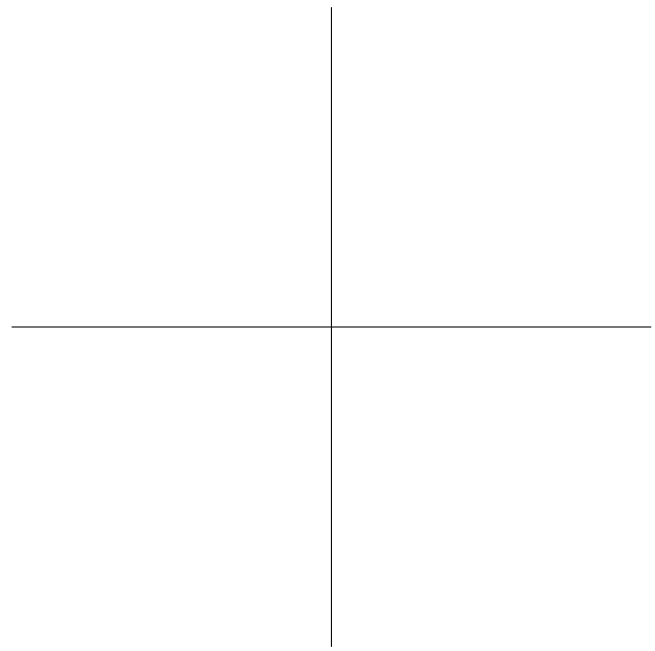
$$x(t) = e^t - e^{-t}$$

$$y(t) = e^t + e^{-t}$$

$$t \in [0, \infty)$$

1. Show that the above curve is contained in the hyperbola $y^2 - x^2 = 4$.

2. Graph the parametric curve.



3. Find dy/dx using the parametric formula for derivatives. Take the limit as t approaches infinity and interpret on your graph.

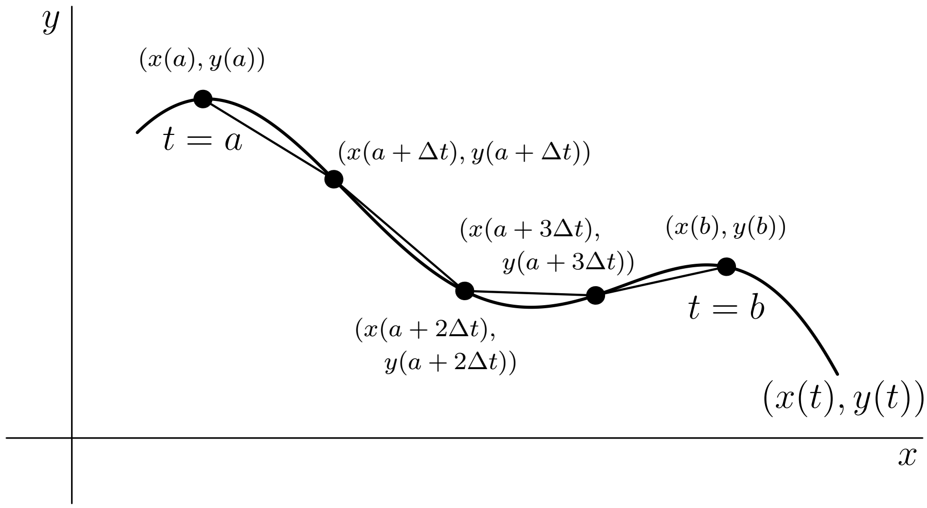
6.3 Integrals of Parametric Curves: Arc Length

The length of a parametric curve is given by the following formula.

Theorem 6.3.0.1. Parametric Arc Length

Let a parametric curve C be given by $(x(t), y(t))$ for $a \leq t \leq b$. Then the arc length is computed via

$$\int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, \mathtt{d}t$$



The construction here is nearly identical to the construction of the arc length of the graph of a function in Section 3.4.1. We select points corresponding to t-values along the curve, compute the sum of the lengths of the line segments connecting them, and take the limit as the number of line segments goes to infinity.

Exercise 6.3.0.2. Fill in the Blanks! Derivation of the Arc Length Formula

Let $t_0, t_1, t_2, \ldots, t_n$ be equally spaced points in the interval $[a, b]$. That is, $t_0 = a$, $t_n = b$, and for each $i \in \{0, 1, 2, \ldots, n-1\}$, $\Delta t =$ ____________________.
With this setup, if we want the length of a line segment connecting points $(x(t_{i+1}), y(t_{i+1}))$ and $(x(t_i), y(t_i))$, we would use the Pythagorean Theorem to obtain

$$\sqrt{\left(\qquad\qquad\qquad\qquad\right)}$$

as the length.

$$\begin{aligned} L &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \sqrt{\qquad\qquad\qquad\qquad} \\ &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \sqrt{\qquad\qquad\qquad\qquad} \sqrt{(\Delta t)^2} \\ &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \sqrt{\qquad\qquad\qquad\qquad} \Delta t \\ &= \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, \mathrm{d}t \end{aligned}$$

As usual, when first trying out a new tool, it is best to use it in a case where you already know the answer.

Exercise 6.3.0.3. Checking the Circumference of a Circle ☕☕

Consider the following parametrization:

$$x(t) = r\cos(t)$$

$$y(t) = r\sin(t)$$

for $t \in [0, 2\pi]$.

- Explain why this is a parameterization of a circle of radius r.

- Use the parametric arc length formula to compute the length of the curve. Compare it to your known formula for the circumference of a circle. Does the answer make sense?

Exercise 6.3.0.4. A Familiar Conic in Disguise ☕☕☕

Consider the following parametrization:

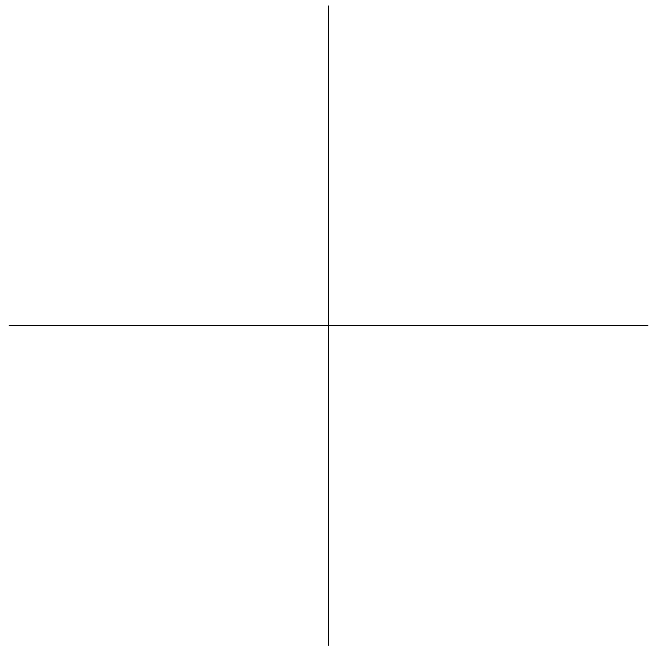
$$x(t) = \sqrt{|t|}$$

$$y(t) = 3t - 1$$

for $t \in [-2, 2]$.

- Convert this to a cartesian equation. What kind of shape is it?

- Sketch the curve. Indicate any vertical or horizontal tangent lines and where they occur.



- Use the parametric arc length formula to compute the length of the curve. (**Hint:** To handle the absolute value, just find the arc length on the interval [0,2] where you can ignore the absolute value and then apply symmetry.) Does the answer make sense?

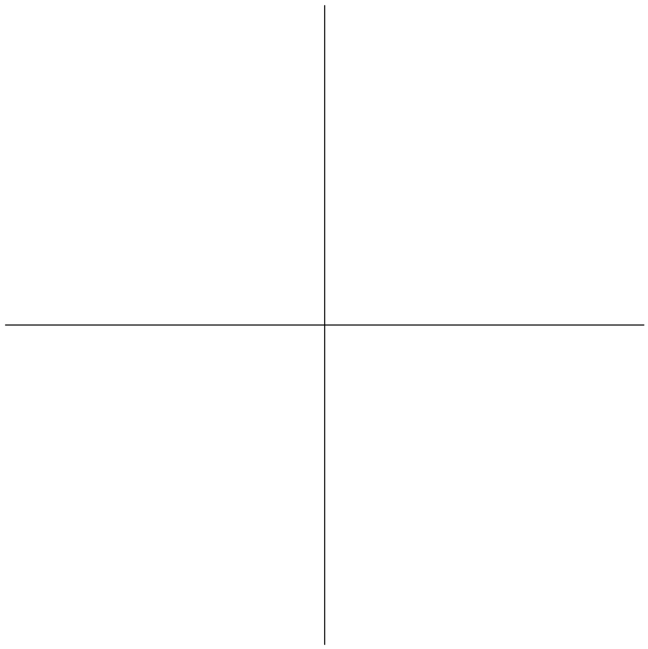
Ok, time to finally play with a curve that is not just a conic.

Exercise 6.3.0.5. Analyzing a Stranger Curve

- Sketch the graph of the following parametric curve C:

$$C = \{\big(e^t \cos(t), e^t \sin(t)\big) : 0 \leq t \leq 2\pi\}$$

Include labels of points on the graph at $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.



- Where does the above graph have vertical tangent lines? Where does the above graph have horizontal tangent lines? Mark them on your graph.

- What is the length of C?

6.4 Hyperbolic Sine and Cosine

You may have seen in a previous course (or if not, then here they are!) the definitions of the *hyperbolic sine* and *hyperbolic cosine* functions. They are typically defined as follows:

- $\cosh(t) = \frac{e^t + e^{-t}}{2}$
- $\sinh(t) = \frac{e^t - e^{-t}}{2}$

This of course prompts the question: "why do these e things get called sine or cosine?" We answer this question below.

Exercise 6.4.0.1. Power Series for Hyperbolic Sine and Cosine ☕☕

- Find a power series for cosh by using what we know about the series for the exponential function. How does the resulting series relate to cosine?

- Find a power series for sinh by using what we know about the series for the exponential function. How does the resulting series relate to sine?

And of course there are more questions prompted here: "Why do these e things get called hyperbolic? What do these hyperbolic functions have to do with hyperbolas?"

Exercise 6.4.0.2. Parametric Curve Generated by Hyperbolic Sine and Cosine ☕☕☕

Consider the parametric curve

$$\{(\cosh(t), \sinh(t)) : t \in \mathbb{R}\}$$

- Verify this parametric curve satisfies the cartesian equation for a hyperbola given by:

  $$x^2 - y^2 = 1$$

  by plugging the exponential definitions for our hyperbolic trig functions in for x and y.

- Again, verify this parametric curve satisfies the cartesian equation for the same hyperbola given by:

  $$x^2 - y^2 = 1$$

  by plugging the power series formulas for our hyperbolic trig functions in for x and y.

6.5 Polar Coordinates

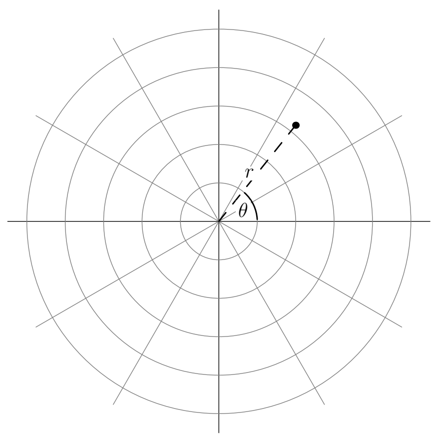
We use the *(horizontal,vertical)* coordinate system so much that it is easy to think that it is somehow inherent to a plane. However, a plane is just a geometric object; a coordinate system is an arbitrary system of labels that we slap on after the fact. Here we explore a different commonly used coordinate system, *polar coordinates*.

6.5.1 Points in Polar Coordinates

Assume we chose an origin in the plane, a direction that we call the positive x-axis, and some point along that ray that marks off unit distance. We now define coordinates for the rest of the plane based on these choices.

Definition 6.5.1.1. Plotting Points in Polar Coordinates

The point (θ, r) is the point located at an angle θ radians counterclockwise from the positive x-axis, a distance of r units from the origin.



Notice the angles are measured in the same manner as on the unit circle in trigonometry. The difference here is we allow any real number r as radius, rather than only radius one. We do allow r to be a negative number, in which case we travel "backwards" along the ray given by θ.

Example 6.5.1.2. Polar Coordinates are not Unique!

Be warned that any given point will have many different representations in polar coordinates. For example, consider the cartesian point (1,-1). In polar coordinates, we have many ways to represent this point. We can think of the angle as $\theta = -\pi/4$ and the radius as $r = \sqrt{2}$. We can also think of the angle as $\theta = 7\pi/4$ and the radius as $r = \sqrt{2}$. Yet another valid way to reach that same point is to use angle $\theta = 3\pi/4$ and the radius $r = -\sqrt{2}$. Thus, in polar coordinates we have that

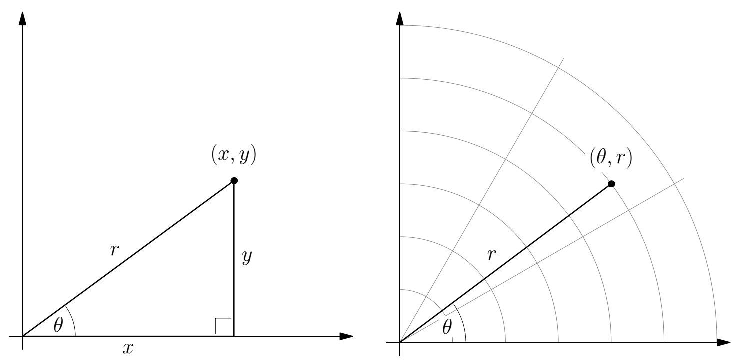
$$\left(-\pi/4, \sqrt{2}\right) = \left(7\pi/4, \sqrt{2}\right) = \left(3\pi/4, -\sqrt{2}\right)$$

all represent the same point.

We now see how right-triangle trigonometry allows us to convert between polar coordinates and cartesian coordinates.

Exercise 6.5.1.3. Converting Between Polar and Cartesian ☕☕☕

1. See the diagram below, with a point in QI labeled with both cartesian and polar measurements. For each of the conversion formulas listed below, write a short sentence afterwards explaining how it comes from the diagram.



   - $x^2 + y^2 = r^2$
   - $x = r\cos(\theta)$
   - $y = r\sin(\theta)$
   - $\tan(\theta) = \frac{y}{x}$
   - $\theta = \arctan\left(\frac{y}{x}\right)$

2. If the point of interest were in a different quadrant, do the above formulas still hold? Do any of them require adjustment? Explain.

Exercise 6.5.1.4. Plotting in Polar ☕☕

- Plot the polar point $(5\pi/4, 4)$. What are its cartesian coordinates?

- Consider the cartesian point (2,0). What are *all* possible ways of writing that point in polar coordinates?

Exercise 6.5.1.5. Do Any Points Have the Same Name? ☕☕☕

Do any points happen to have the same label in both polar and cartesian coordinates? Find all points that do, and explain why there are no more!

It is worth noting why "polar coordinates" are called what they are called. Cartesian coordinates look like a grid of horizontal and vertical lines. This is a great approximation of what latitude and longitude lines look like if you are standing at a random point on earth and think of your surroundings as approximated by a plane. But, if you are standing at the north or south pole, the latitude and longitude lines do not in any way look like a grid!

Exercise 6.5.1.6. Justifying the Name ☕☕

What do the lattitude and longitutde lines look like if you are standing at the north or south pole? Draw a small graph below.

Exercise 6.5.1.7. The Idea of Coordinate Systems ☕☕☕☕

Create another coordinate system for the plane that is not cartesian and is not polar! Describe your system of labeling all the points!

6.5.2 Graphs of Equations and Functions in Polar Coordinates

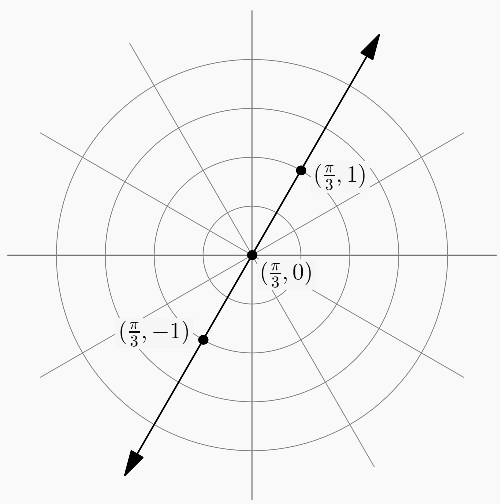
An equation in polar coordinates is an equality between expressions involving r and θ. We often wish to view the solutions visually by plotting all points (θ, r) in the plane that make the equations true (just as one would in cartesian).

Example 6.5.2.1. Graphing a Polar Equation

Suppose we wish to graph the equation

$$\theta = \pi/3$$

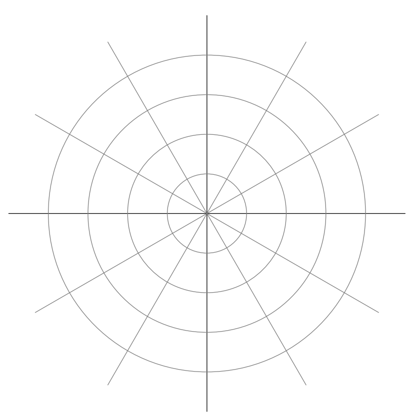
A point satisfies that equation if and only if the angle is $\pi/3$. The radius r is free to be any real number -positive, negative, or zero. For example, points that satisfy the equation include $(\pi/3, 1), (\pi/3, 0)$, and $(\pi/3, -1)$. Thus the graph is a line through the origin at $60°$ to the positive x-axis.



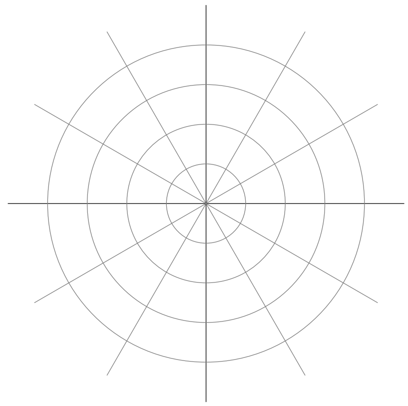
Exercise 6.5.2.2. Graphing Equations ☕☕

Graph the following equations.

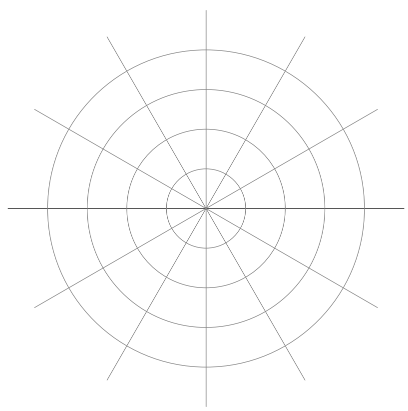
- $r = 2$



- $r = -2$



- $\theta^2 = \pi^2/4$



If the equation can be solved for r, we can consider r as a function of the independent variable θ. To graph a function, we simply make an input-output table of θ values and corresponding $r(\theta)$ values and plot the corresponding points $(\theta, r(\theta))$.

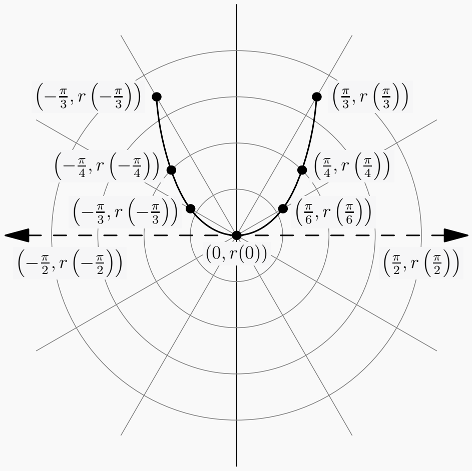
Example 6.5.2.3. Graphing a Polar Function

Plot the polar function

$$r(\theta) = \tan(\theta)$$

over the domain $-\pi/2 < \theta < \pi/2$. We select input values for θ that are clean unit circle values to plot.

θ	$-\pi/2$	$-\pi/3$	$-\pi/4$	$-\pi/6$	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$r(\theta)$	DNE	$-\sqrt{3}$	-1	$-\sqrt{3}/3$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	DNE



Exercise 6.5.2.4. Analyzing the Graph

Does the graph appear to have any asymptotes? If so, where?

Looking at the graph prompts the question "can we find a Cartesian equation that describes the same set of points"? Here we use the formulas from Exercise 6.5.3 to rewrite all instances of r and θ in terms of x and y. Also, perhaps the cartesian equation can confirm our asymptote suspicions above!

Example 6.5.2.5. Converting to Cartesian

Let's find a cartesian equation for the graph of $r(\theta) = \tan(\theta)$ from the previous example. Since we do not have a particularly clean conversion formula for r itself but rather for r^2, it can be helpful to either multiply both sides by r or square both sides. In this case, squaring both sides will be cleaner so we take that path. Proceeding:

$$\begin{aligned} r &= \tan(\theta) \\ r^2 &= (\tan(\theta))^2 \\ x^2 + y^2 &= \left(\frac{y}{x}\right)^2 \\ x^2 + y^2 &= \left(\frac{y}{x}\right)^2 \\ x^4 + x^2y^2 &= y^2 \\ x^4 + \left(x^2 - 1\right) y^2 &= 0 \\ y^2 &= \frac{x^4}{1 - x^2} \\ y &= \pm \frac{x^2}{\sqrt{1 - x^2}} \end{aligned}$$

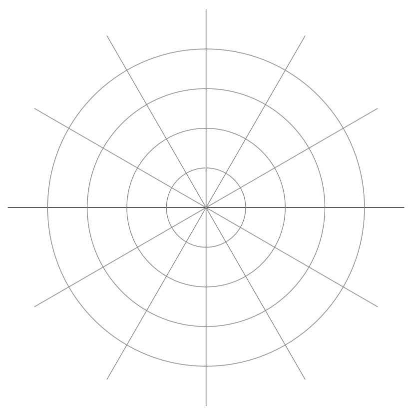
Exercise 6.5.2.6. Analyzing the Graph, Round II ☕☕

Does the cartesian formula tell you anything further about the apparent asymptotes on the graph?

The next exercise shows why converting a polar graph to cartesian coordinates can help analyze the geometry of the graph.

Exercise 6.5.2.7. Graphing a Function and Converting ☕☕☕

- Graph the function $r(\theta) = \sin(\theta)$. Does it look like a circle?



- Is it a circle? If so, what is the center and radius? Convert the equation to cartesian coordinates to confirm!

6.6 Derivatives in Polar Coordinates

Suppose we have the graph of a polar function $r(\theta)$, and we would like to find the slope of the tangent line at a point. We can consider this graph to be a parameterized curve by treating $t = \theta$ as the parameter. Specifically, the parameterization is:

$$x(t) = r(t)\cos(t)$$
$$y(t) = r(t)\sin(t)$$

Exercise 6.6.0.1. Deriving the Derivative ☕☕

Use the formula for the derivative of a parametric curve to find the formula for the derivative of a polar graph.

Exercise 6.6.0.2. Using the Formula ☕☕

Use the polar derivative formula above to find the slope of the graph of $r(\theta) = \sec(\theta)$. What does this let you conclude about that graph?

6.7 Area in Polar Coordinates

To compute area in polar coordinates, we essentially repeat the process of taking a Riemann sum. Rather than using rectangles however, we use sectors of circles.

Exercise 6.7.0.1. Area of a Single Sector ☕☕

- What is the area of an entire circle with radius r? Draw the circle.

- Within your circle, draw a sector of that circle with angle θ. What proportion of the area of the entire circle does that sector occupy?

- Explain why the area of that sector is:

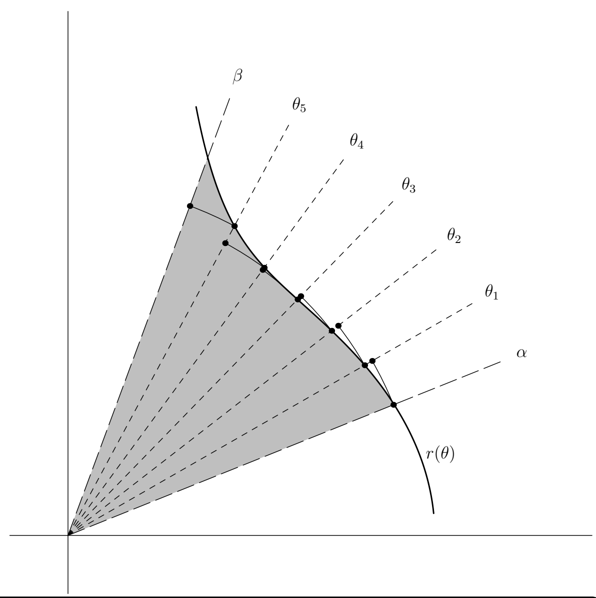
$$A = \frac{1}{2}r^2\theta$$

We now repeat the process of taking a Riemann sum using sectors of circles. In particular, say we wish to find the area under the graph of $r(\theta)$ between two rays specified by angles $\theta = \alpha$ and $\theta = \beta$.

Let $\theta_0, \theta_1, \theta_2, \ldots, \theta_n$ be equally spaced angles from α to β. That is, $\theta_0 = \alpha$, $\theta_n = b$, and for each $i \in \{0, 1, 2, \ldots, n-1\}$, $\Delta\theta = \theta_{i+1} - \theta_i = \frac{\beta-\alpha}{n}$.

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{2} r^2(\theta_i)\,\Delta\theta \\ &= \frac{1}{2} \lim_{n\to\infty} \sum_{i=0}^{n-1} r^2(\theta_i)\,\Delta\theta \\ &= \frac{1}{2} \int_{\theta=\alpha}^{\theta=\beta} r^2(\theta)\,\mathrm{d}\theta \end{aligned}$$

Thus, we have the formula for polar area!



Theorem 6.7.0.2. Polar Area

The area under a polar curve $r(\theta)$ between $\theta = \alpha$ and $\theta = \beta$ is

$$A = \frac{1}{2} \int_{\theta=\alpha}^{\theta=\beta} r^2(\theta)\,\mathrm{d}\theta$$

Sometimes a helpful way to remember the above formula is to write it as:

$$A = \int_{\theta=\alpha}^{\theta=\beta} \pi r^2(\theta) \frac{\mathrm{d}\theta}{2\pi}$$

This way you can think of the integrand as the area of a circle being multiplied by what ratio of 2π radians the change in θ is occupying. Canceling the two π's and pulling the $\frac{1}{2}$ outside of the integral lands you back at the Polar Area formula.

Exercise 6.7.0.3. Looking for Patterns ☕☕☕

Fill out the table! For each of the following,

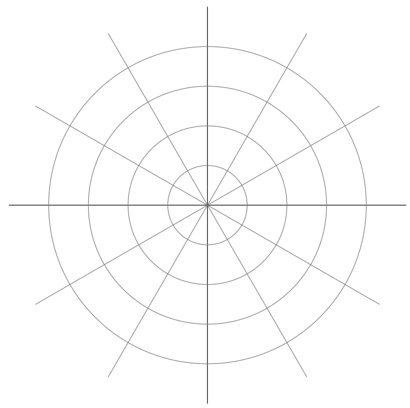
- Plot the graph of $r(\theta)$
- Find the area inside just one "petal".
- What patterns in n do you see? What can you say about the percent of the unit circle that lies inside rather than outside the graph?

n	Graph of $r(\theta) = \sin(n\theta)$	Area of One Petal
2		
3		
4		
5		
n		

n	Graph of $r(\theta) = \cos(n\theta)$	Area of One Petal
2		
3		
4		
5		
n		

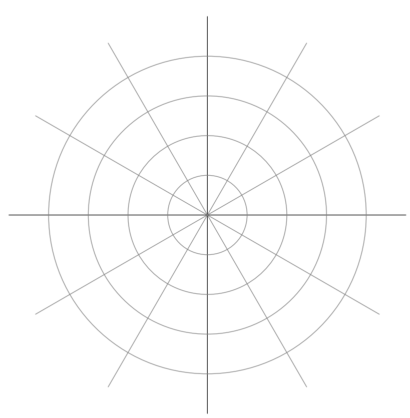
Exercise 6.7.0.4. Area Bounded by Two Polar Curves ☕☕☕

Plot both $r_1(\theta) = \frac{1}{2}\sec(\theta)$ and $r_2(\theta) = \cos(\theta)$ on the same set of axes. Find the area of the region that is to the right of r_1 but inside r_2.



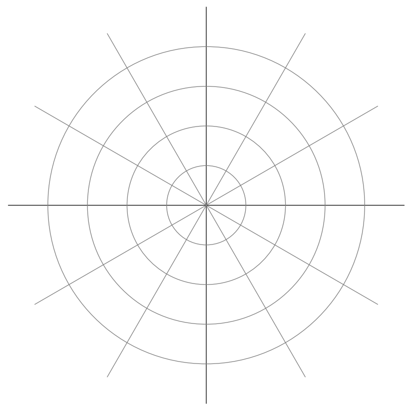
Exercise 6.7.0.5. Mixed Practice with Polar Curves ☕☕☕

1. (a) Sketch the graph of $r(\theta) = 2\cos(2\theta)$.



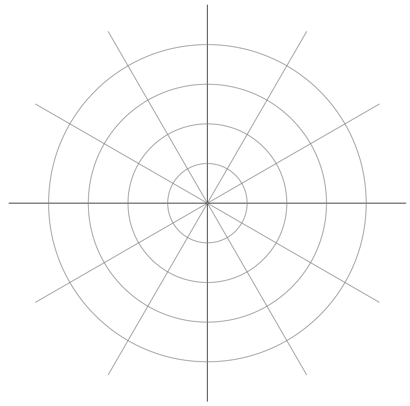
(b) Convert the above curve to cartesian. That is, find a polynomial equation in x and y whose solution set describes the same set of points. (**Hint:** Begin by applying the cosine double-angle identity!)

2. (a) Sketch the graph of $r(\theta) = \frac{1}{2} + \cos(\theta)$.



(b) Find the area enclosed by the inner loop of the graph.

3. (a) Plot the graphs of both $r_1(\theta) = 1 + \cos(\theta)$ and $r_2(\theta) = 1 - \cos(\theta)$ on the same axes.



(b) Shade the region contained inside both curves. Find its area.

6.8 Microphone Design

A microphone is a device that picks up sound (variations in air pressure) and produces an electrical signal. For any microphone, sound engineers want what is called the *polar pattern*, a graph indicating all locations from which sound is picked up with equal intensity. Microphones that are physically designed differently will have different polar patterns.

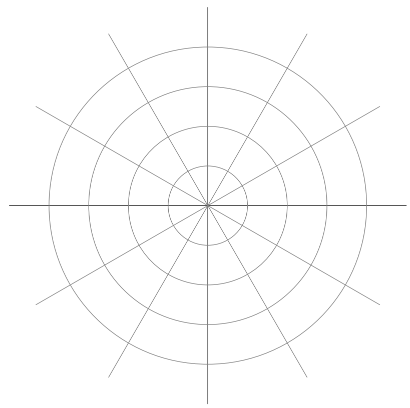
The key element to a microphone is some mechanical device that the waves of air pressure can compress. There are two basic types of devices:

6.8.1 Diaphragm

A spherical diaphragm responds equally to changes in air pressure from any side. Thus given a sound of a particular volume, the response in the microphone sensitivity is proportional to the distance from the diaphragm. A microphone with such a diaphragm is called an *omnidirectional microphone* and is represented by the polar pattern $r(\theta) = 1$, since the microphone has equal sensitivity to all points on a circle.

Exercise 6.8.1.1. Diaphragm Microphones ☕

Plot the polar pattern for a diaphragm microphone function below.

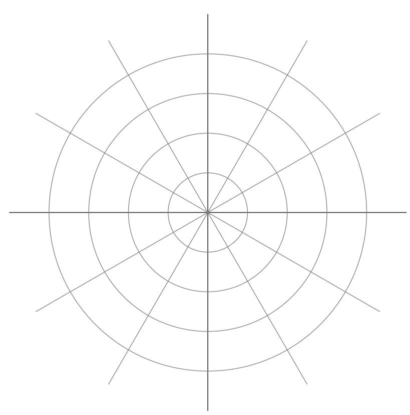


6.8.2 Ribbon

The other main type of device is a ribbon that floats in a magnetic field. Since it is a horizontal ribbon, it picks up changes in air pressure proportion to the sine of the angle to the source. (Imagine for example in physics a force pushing on a wall at an angle... the force that goes into the wall is not equal to the magnitude of the whole force but rather the magnitude times sine of the angle.) A microphone equipped with such a device is called a *ribbon microphone* or a *figure eight microphone* and has polar pattern given by $r(\theta) = |\sin(\theta)|$. Here we are taking absolute values because we are just denoting sensitivity, not the wave itself.

Exercise 6.8.2.1. Ribbon Microphones ☕☕

Plot the polar pattern for a ribbon microphone function below.



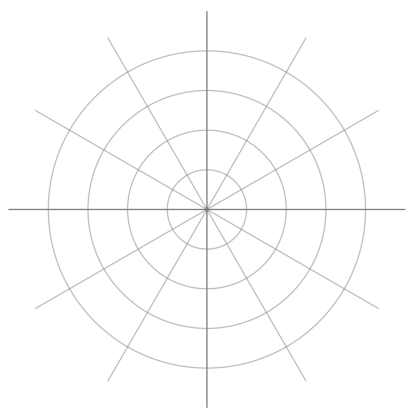
6.8.3 Cardioid

There are many situations where one of the above microphones is perfect for the purpose at hand. However, when a band is playing live music on a stage, the above two microphones do not work. The basic setup is the following: if a singer sings into the microphone, the main speakers are pointed towards the audience and not towards the singer. Thus, it is necessary to have monitors (smaller speakers pointing the opposite direction) so that the singer can hear herself. However, if the microphone picks up the sound coming out of the monitor, it's going to be again reproducing the same sound it just heard. The waves combine amplitude again and again, and this leads to that horrible high-pitched screeching noise known as feedback.

The solution to this is to design a microphone that picks up sound from one side but not from the other. The ingenious way engineers figured out how to do this was to simply make a microphone with *both* a ribbon and a diaphragm inside! The waves produced add to each other to make a single signal. Thus the sine of the ribbon will combine with the diaphragm's signal on one side, but cancel it out on the other! The polar pattern is given by the function $r(\theta) = 1 + \sin(\theta)$ (adding the waves together). Such a mic is called a *cardioid microphone* and is the standard mic for onstage live sound. The Shure 57 and Shure 58 are cardioid microphones and have been the standard mic used onstage for about 40 years now!

Exercise 6.8.3.1. Cardiod Microphones ☕☕

1. Plot the polar pattern corresponding to the above described cardioid microphone.



2. Find the area of the region where sounds are at least as sensitive as they are on the boundary of that cardioid. (That is, find the area enclosed by the above polar curve.)

Chapter 7

Introduction to Differential Equations

In this course, we got really good at two things: finding antiderivatives and using power series. It is no accident that the study of differential equations relies primarily on those two techniques! Here we show just two methods for solving differential equations: separation of variables, based on antidifferentiation, and power series solutions, based on power series (really!).

7.1 What is a Differential Equation?

Definition 7.1.0.1. Differential Equation

A *differential equation* (DE) is an equation involving a variable (say y) that stands for some unknown function, and also involving one or more derivatives of y. The *solution* to a differential equation is the set of all functions y that make the equation true.

We begin with a nice bridge troll riddle. We ask "What functions are equal to their own derivative?".

Example 7.1.0.2. Functions Equal to Their Own Derivative

To state this question in the language of differential equations, we say that we wish to solve the DE

$$y' = y$$

Exercise 7.1.0.3. Guess and Check ☕

Can you think of any functions that are equal to their own derivative? Do you think you have all of them, or are some likely still out there?

As you can see, guess and check is not a good method for solving even the simplest of differential equations. We now take a more structured approach.

7.2 Separable Equations

Definition 7.2.0.1. Separable

Let x be the independent variable and let y represent an unknown function of x. A differential equation is *separable* if and only if it can be written in the form

$$\frac{dy}{dx} = F(x)G(y)$$

for some functions F and G.

Our method for solving a separable differential equation is as follows:

1. Write right-hand side of the differential equation in factored form, one function of x times one function of y.

2. Separate variables by multiplying both sides by $\frac{1}{G(y)}\,\mathtt{d}x$.

3. Antidifferentiate both sides.

4. Solve for y, if possible. (If not, we at least have an implicit solution.)

We try out this method on the previous example.

Example 7.2.0.2. Separation of Variables

Notice the differential equation

$$y' = y$$

is separable because it can be rewritten as

$$\frac{\mathtt{d}y}{\mathtt{d}x} = (1)(y)$$

That is, our factored form uses the functions $F(x) = 1$ and $G(y) = y$. We now perform separation of variables and antidifferentiate both sides.

$$\begin{aligned}
\frac{1}{y}\,\mathtt{d}y &= 1\,\mathtt{d}x \\
\int \frac{1}{y}\,\mathtt{d}y &= \int 1\,\mathtt{d}x \\
\ln|y| &= x + C \\
|y| &= e^{x+C} \\
y &= \pm e^{C} e^{x} \\
y &= Ce^{x}
\end{aligned}$$

Notice that on the last line for simplicity, we clean up the constant $\pm e^C$ by just calling it C.

Exercise 7.2.0.3. Analyzing the Example ☕

- Why were we able to just put a $+C$ on one side when we integrated? What would have happened if we put it on both sides?
- When we renamed $\pm e^C$ as C, we technically introduced a new solution. The expression $\pm e^C$ is incapable of being equal to zero, but C can be. Verify that the $C = 0$ solution is valid to include as a solution to the differential equation.

Exercise 7.2.0.4. More Complicated DEs ☕☕☕

1. Solve the following differential equation via separation of variables:

$$\frac{dy}{dx} = xy + x$$

2. Solve the following Initial Value Problem via separation of variables:

$$\frac{dy}{dx} = e^{y-x}\sec(y)(1+x^2)$$

Note that you will not be able to obtain an explicit formula for y in terms of x but rather an implicit solution. Use the initial condition $y(0) = 0$ to solve for C.

7.3 Power Series Solutions

Power series provide a very effective method for solving differential equations. The steps are simple:

- Set the unknown function y equal to an unknown power series.
- Plug the power series in for all occurrences of y. Expand and combine like terms.
- Equate coefficients one degree at a time (much like we do when solving for unknowns in a PFD).
- Solve for the coefficients a_0, a_1, $a_2, \ldots$ one at a time in terms of a_0.
- Plug those coefficients back into the power series expansion for y to obtain a power series solution.
- Try to recognize that power series as a known function or variant thereof.

The process is thus very mechanical, but sometimes working through the details becomes a bit messy. We repeat the previous example with this new method.

Example 7.3.0.1. Revisiting Our Frist DE

Here we solve

$$y' = y$$

using power series. First, let y be written as a generic unknown power series:

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots$$

Plug this expression into the differential equation and expand.

$$\begin{aligned}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots\right)' &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots \\ a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots\end{aligned}$$

We now equate coefficients one degree at a time, and solve for every coefficient in terms of a_0.

$$\begin{array}{lrcl} \text{Degree 0:} & a_1 = a_0 & \Longrightarrow & a_1 = a_0 \\ \text{Degree 1:} & 2a_2 = a_1 & \Longrightarrow & a_2 = \frac{1}{2}a_0 \\ \text{Degree 2:} & 3a_3 = a_2 & \Longrightarrow & a_3 = \frac{1}{3!}a_0 \\ \text{Degree 3:} & 4a_4 = a_3 & \Longrightarrow & a_4 = \frac{1}{4!}a_0 \\ \quad\vdots & \vdots & & \vdots \\ \text{Degree } n-1\text{:} & na_n = a_{n-1} & \Longrightarrow & a_n = \frac{1}{n!}a_0 \end{array}$$

We can now plug all coefficients back into our expression for y and simplify until we obtain a closed form for y.

$$\begin{aligned} y &= a_0 + a_0x + \frac{1}{2!}a_0x^2 + \frac{1}{3!}a_0x^3 + \frac{1}{4!}a_0x^4 + \cdots \\ &= a_0\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots\right) \\ &= a_0e^x \end{aligned}$$

Notice we have obtained the same solution as via separation of variables! Clearly, the power series solution was way more work. The reason it is so valuable though is that there are many DEs which are not separable but for which the power series method works just fine.

Exercise 7.3.0.2. Comparing the Methods ☕☕☕

1.
   - Show the differential equation $\frac{dy}{dx} = yx$ is separable and use this to separate variables and solve the differential equation.
   - Solve the same differential equation via power series. Confirm you get the same answer.

2.
   - Explain why the differential equation $\frac{dy}{dx} = yx + x + 1$ is not separable.

- Solve the same differential equation via power series.

- Check your answer is correct by plugging it back into the original DE.

3. Consider the DE given by:

$$y(0) = 1$$
$$y'(0) = 0$$
$$y'' = -y$$

Solve this DE via power series (use the initial conditions to solve for a_0 and a_1).

7.4 Modeling with Differential Equations

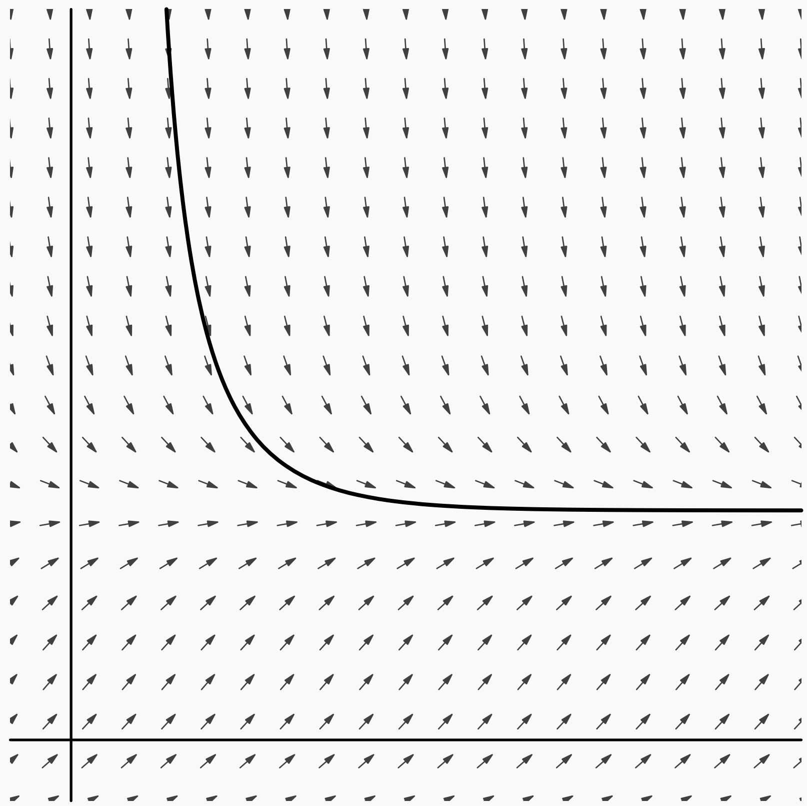
Differential equations are used extensively in applied mathematics and the sciences to describe models, which are then solved using mathematics to find explicit formulas for the quantities of interest.

Example 7.4.0.1. Newton's Law of Cooling

Newton's Law of Cooling states that the rate of change of the temperature of a small object in a room is proportional to the difference between room temperature and the temperature of the object. If A is the constant that represents the ambient temperature (room temperature), $T(t)$ represents the temperature of the room at time t, and k is the constant of proportionality, then this situation can be modeled by:

$$\frac{dT}{dt} = k(T - A)$$

Here we introduce the idea of a *slope field*: a grid of small dashes that indicate the slope $\frac{\mathrm{d}T}{\mathrm{d}t}$ at every point (t, T) in the plane. Here we draw a slope field that governs solution curves to this model and show one sample solution curve.



Exercise 7.4.0.2. Newton's Law of Cooling ☕☕

- Label the above diagram. What variables do the axes correspond to? Can you find where the horizontal line $T = A$ is located?

- In this model, would it make sense that the proportionality constant k is positive or negative? Why?

- Solve the differential equation by separation of variables.

- Solve the differential equation by power series.

Note that those solutions give explicit formulas for the solution curves above, which is significantly more useful than just thinking of it intuitively as "following the arrows".

Exercise 7.4.0.3. Malthusian Population Model

A simple intuitive population model goes as follows:

If there are more individuals in a population, there will be more babies produced.

Here is a slightly more technical restatement of the same idea:

The rate of growth of a population is proportional to the size of the population.

- Let $P(t)$ be the size of the population at time t. Rewrite the above growth principal in the language of differential equations.

- Solve your differential equation using power series.

- Solve your differential equation using separation of variables and confirm that your answers match.

- Use your formula to find the limit of $P(t)$ as t approaches infinity.

- Under what real life conditions might this model be realistic? Under what conditions might this model be unrealistic?

As you probably noticed, the above model is slightly ridiculous for large time values since it would claim that eventually any species would fill up the entire visible universe with bodies. So let's adjust it to fix that unrealistic assumption. Here's an upgrade:

Exercise 7.4.0.4. Logistic Population Model

If there are more individuals in a population, there will be more babies produced, but then it slows down as it approaches some sort of maximum possible population (a limit perhaps based on food supply, available habitat, etc).

Here is a slightly more technical restatement of the same idea:

The rate of growth of a population is jointly proportional to both the size of the population and the distance from some maximum possible population.

- Let $P(t)$ be the size of the population at time t and let M for maximum be a constant that the population cannot exceed. Rewrite the above growth principal in the language of differential equations.

- Solve your differential equation using power series out to a degree two approximation (this will be much too difficult to solve the whole thing using power series!).

- Solve your differential equation using separation of variables and confirm that your answers match out to the degree two approximation.

- Use your formula to find the limit of $P(t)$ as t approaches infinity.

- Suppose you started with population $P(0) = 2M$. What would your model predict would happen to the population?

Chapter 8

Introduction to Complex Numbers

The extension from the real numbers to the complex numbers has far-reaching affects. In this chapter, we give a brief introduction to complex numbers and then show how they interact with almost every topic in the course!

8.1 Complex Numbers

The complex numbers arise out of the fact that the simple little equation $x^2 + 1 = 0$ has no solution over the reals. Thus, we create the number i to represent a root of that polynomial. That is, $i^2 + 1 = 0$.

Definition 8.1.0.1. Complex Numbers

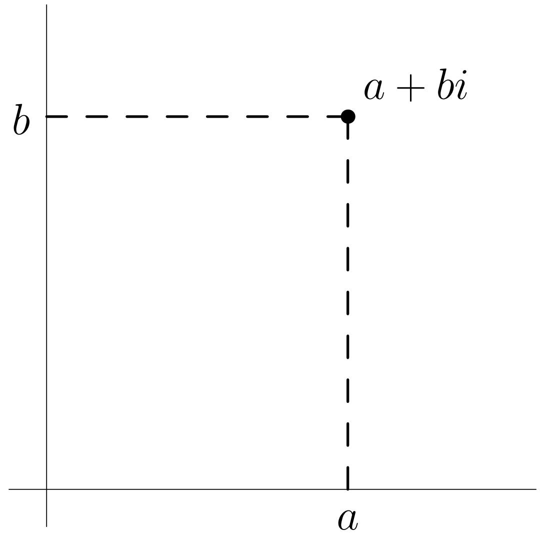
The set of *complex numbers* is the set of all numbers that can be written in the form $a + bi$ for real numbers a and b.

We perform arithmetic in the complex numbers using the usual rules of arithmetic and algebra along with the extra identity $i^2 = -1$.

Exercise 8.1.0.2. Containment of the Reals ☕

- Is 3 a complex number? Can you write 3 in the form $a + bi$ for real numbers a and b?
- Does the set of complex numbers contain all real numbers?

We can visualize complex numbers in the complex plane, where a (the *real part*) is the horizontal component and b (the *imaginary part*) is the vertical.



8.2 Euler's Identity and Consequences

Look again at the power series for the exponential function, sine, and cosine:

$$\begin{aligned} e^x &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 + \cdots \\ \cos(x) &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots \\ \sin(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots \end{aligned}$$

You have to wonder if there is some way to add together sine and cosine to get the exponential function! Sure the signs are off, but otherwise things seem so right. Sine has all the odd factorial denominators, cosine has all the even factorial denominators, and the exponential function has all of them! It turns out that i is exactly the constant we need to fix those minus signs!

Exercise 8.2.0.1. Proof of Euler's Identity ☕☕

- Write out a power series for $e^{i\theta}$.

- Write out a power series for $\cos(\theta) + i\sin(\theta)$.

- Verify the two are equal!

The fact that there is any relationship whatsoever between sine, cosine, and e is very surprising when you think of how differently those quantities are defined! We again state this incredible theorem, Euler's Identity!

Theorem 8.2.0.2. Euler's Identity

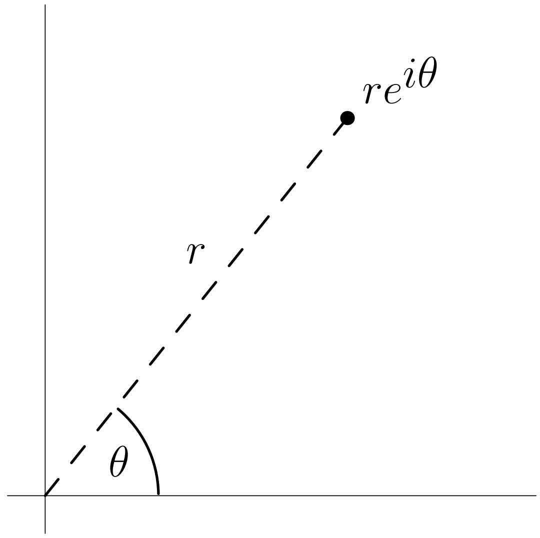
For any real number θ,

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

If we multiply both sides by a real number r, we then obtain

$$re^{\theta} = r\cos(\theta) + ir\sin(\theta)$$

We notice that the horizontal component $r\cos(\theta)$, is in fact the conversion for x into polar coordinates. Likewise, $r\cos(\theta)$, is the conversion for y into polar coordinates. This means that the complex number $re^{i\theta}$ is in fact the point located at angle θ and radius r in the complex plane.



8.2.1 Complex Roots

One interesting fact about the complex numbers is that the number of n^{th} roots of *every* real number is exactly n. So every number has two square roots, three cubed roots, and so on. We use $re^{i\theta}$ form to find these complex roots.

Example 8.2.1.1. The Cubed Roots of Two

To find all cubed roots of two, we solve the equation

$$z^3 = 2$$

We begin by putting both z and 2 in complex polar form. We write $z = re^{i\theta}$ and $2 = 2e^{i0}$. We plug these into the equation, expand the powers.

$$\begin{aligned} z^3 &= 2 \\ \left(re^{i\theta}\right)^3 &= 2e^{i0} \\ r^3e^{i3\theta} &= 2e^{i0} \end{aligned}$$

We now equate the radius and the angles as two separate equations.

- **Radius:** Since r is a real number, we obtain $r^3 = 2$, which implies $r = \sqrt[3]{2}$.
- **Angle:** The angles need to be equivalent but not necessarily equal. If they differ by a multiple of 2π, that is fine! Thus, we have $3\theta = 0 + 2\pi k$ for any integer k. Dividing both sides by 3, we have

  $$\theta = \frac{0 + 2\pi k}{3} = \ldots, \frac{-4\pi}{3}, \frac{-2\pi}{3}, 0, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{6\pi}{3}, \ldots$$

  However, if we use more values of θ beyond just $0, \frac{2\pi}{3}, \frac{4\pi}{3}$, the solutions will repeat since cosine and sine have period 2π. Thus, we use just those three angles.

Putting together our r and θ values, we have our three roots:

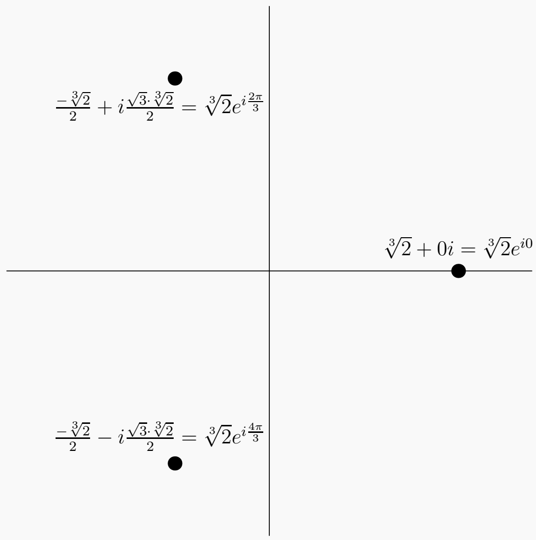
$$z = \sqrt[3]{2}e^{i0}, \sqrt[3]{2}e^{i\frac{2\pi}{3}}, \sqrt[3]{2}e^{i\frac{4\pi}{3}}$$

Thus we have our roots in complex polar form. We use Euler's Identity to turn these back into complex cartesian form:

$$z = \sqrt[3]{2}\cos(0) + i\sqrt[3]{2}\sin(0),\ \sqrt[3]{2}\cos(\frac{2\pi}{3}) + i\sqrt[3]{2}\sin(\frac{2\pi}{3}),\ \sqrt[3]{2}\cos(\frac{4\pi}{3}) + i\sqrt[3]{2}\sin(\frac{4\pi}{3})$$

At last, we use the unit circle to evaluate these and plot in the complex plane.

$$z = \sqrt[3]{2}, -\frac{\sqrt[3]{2}}{2} + i\frac{\sqrt[3]{2}\sqrt{3}}{2}, -\frac{\sqrt[3]{2}}{2} - i\frac{\sqrt[3]{2}\sqrt{3}}{2}$$



Exercise 8.2.1.2. Checking Once Again ☕☕

Cube each of the answers from the previous problem. Verify in each case you get 2!

It turns out to be of particular importance to find roots of 1. Define the n^{th} *roots of unity* to be the solutions to the equation:

$$z^n = 1$$

Lets play around and see if we can find some neat properties!

Exercise 8.2.1.3. Roots of Unity ☕☕☕☕

- Find all square roots of unity. Write your answers in both cartesian and polar complex form, and plot them in the complex plane. (the case where $n = 2$)

- Find all third roots of unity.

- Find all fourth roots of unity.

- Find all fifth roots of unity.

- Find all sixth roots of unity.

- Fill out the following table:

n	Σ_n	Π_n
2		
3		
4		
5		
6		

where Σ_n represents the sum of all n^{th} roots of unity and Π_n represents the product of all n^{th} roots of unity. (**Hint:** It's easier to add in cartesian, and easier to multiply in polar.)

- Based on your above data gathered, conjecture a formula for both Σ_n and Π_n. Prove your conjecture is correct. (**Hint:** Consider the roots of the polynomial $z^n - 1$ and how that polynomial would factor based on those roots. Then consider the degree zero and degree $n-1$ coefficients.)

Using the same techniques we can answer the following question, "what is the square root of i?" Keep in mind there are technically two square roots of i, the two solutions to the equation $z^2 = i$.

Exercise 8.2.1.4. Square Roots of i

- Find the square roots of i. Write your answers in complex cartesian form.

- Square your answers back out (in complex cartesian form) and verify that you do in fact get

i when you square them.

Exercise 8.2.1.5. Cubed Roots of i ☕☕☕

- Find all cubed roots of i. That is, find all complex numbers z such that $z^3 = i$. Write your answers in $a + bi$ form.

- Take the cube of each of your roots to verify that you do in fact get i as the third power.

8.2.2 Proving Trig Identities

Remember how there are 47,000 useful but impossible to remember trigonometric identities? No? Well, that shows how hard they are to remember. Believe it or not, most of them can be constructed very quickly and easily from Euler's Identity!

Example 8.2.2.1. The Sine and Cosine Double-Angle Identities

To construct the sine and cosine double-angle formulas, we can manipulate the expression $e^{2\theta}$. We proceed with the following chain of equality:

$$\begin{aligned}\cos(2\theta) + i\sin(2\theta) &= e^{i\cdot 2\theta} \\ &= \left(e^{i\theta}\right)^2 \\ &= (\cos(\theta) + i\sin(\theta))^2 \\ &= \cos^2(\theta) + 2\cos(\theta)i\sin(\theta) + i^2\sin^2(\theta) \\ &= \left(\cos^2(\theta) - \sin^2(\theta)\right) + i\,(2\sin(\theta)\cos(\theta))\end{aligned}$$

We now equate real parts to obtain the cosine double-angle identity:

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

Similarly, we equate imaginary parts to obtain the sine double-angle identity:

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

Exercise 8.2.2.2. Annotate! ☕

Write a short justification alongside each line of computation above.

Exercise 8.2.2.3. Angle-Sum Identities ☕☕

- Expand the expression $e^{i(A+B)}$ into real and imaginary parts using Euler's Identity.

- Expand the expression $e^{iA}e^{iB}$ into real and imaginary parts using Euler's Identity twice, once per factor. Multiply out the resulting terms into an expression of the format

$$f(A,B) + ig(A,B)$$

where f is the function corresponding the real part of that expression, and g corresponds to the imaginary part.

- Equate real and imaginary parts to produce the angle sum identities for cos and sin, respectively! (**Hint:** We're using the fact that two complex numbers $a + bi$ and $c + di$ are equal if and only if $a = c$ and $b = d$.)

8.2.3 Natural Logarithm of a Complex Number

We now show how to compute the natural logarithm of a complex number. As usual, polar form will be critical.

- Given a complex number z, we first write z in polar form $z = re^{i\theta}$, where r is a positive real number and $\theta \in [-\pi/2, 3\pi/2)$.
- Split apart using the property of logarithms and cancel the log with the exponential:

$$\ln(z) = \ln\left(re^{i\theta}\right) = \ln(r) + ln\left(e^{i\theta}\right) = \ln(r) + i\theta$$

Note that in principle there is no reason we had to pick our angle θ in that particular interval. One can construct a perfectly well-defined logarithm from choosing a different domain for θ. This is similar to the construction of the inverse trig functions, where one must restrict the domain in some manner, so we tend to just choose a default interval to restrict to and stick with it.

Try the above method to compute each of the following logarithms. Write each in the standard complex cartesian form $a + bi$:

Exercise 8.2.3.1. Complex Logarithms ☕☕

- $\ln(2)$
- $\ln(-2)$
- $\ln(i)$
- $\ln(1+i)$
- $\ln(3-4i)$

8.2.4 Complex Exponentials

Recall our trick for dealing with strange bases:

$$w_1{}^{w_2} = e^{\ln(w_1{}^{w_2})} = e^{w_2 \ln(w_1)}$$

This provides the advantage of moving us back to the familiar base e from the unfamiliar base w_1. This will make a complex exponential base manageable!

Exercise 8.2.4.1. Complex Exponentials ☕☕

1. Use the above trick to compute the derivative of the function 2^x.

2. Use the above trick to compute $(1+i)^i$.

3. Use the above trick to compute i^{1+i}.

4. Use the above trick to compute $(1+i)^{1+i}$.

8.2.5 Partial Fractions via Complex Numbers

If we are using complex numbers, there are no more irreducible quadratics! This gives us an interesting alternate way to perform PFD, since all polynomials will fully factor into linear factors.

Exercise 8.2.5.1. PFD over the Complex Numbers ☕☕☕

1. Find an antiderivative of $\frac{4-2x^2}{x^3+4x}$ via a partial fraction decomposition over the real numbers.

2. Find an antiderivative of $\frac{4-2x^2}{x^3+4x}$ via a partial fraction decomposition over the complex numbers.

3. Verify your answers are compatible.

Chapter 9

Introduction to Probability: Integrals and Series in Action

In probability theory, it turns out that many fundamental questions cannot be answered without the power series tools we developed here in Calculus 2. In particular, in the following exercises we will use the geometric series and the square of the geometric series.

Exercise 9.0.0.1. Recalling Some Important Power Series ☕

Write the power series centered at zero for these functions below:

$\frac{1}{1-x} =$

$\frac{1}{(1-x)^2} =$

Let's begin with a few definitions.

Definition 9.0.0.2. Probability

- The *probability* of an event A is the likelihood of the event happening, measured as a number between zero and one, where probability zero means A is impossible and probability one means A is guaranteed to happen. Everything inbetween is scaled proportionally, for example probability three-fourths means that in four trials, we would expect the event A to happen three times and not happen one time.
- A *discrete event* is an event where there is a finite or countable number of total possible outcomes for the event. For example, flipping a coin is a discrete event because there are only two possible outcomes. However, measuring the temperature at a particular time of day could be considered to not be a discrete event since your temperature could be viewed to potentially be any real number between absolute zero and infinity.

One way to compute the probability of a discrete event is to compute the number of ways the event itself can happen and then divide by the total number of possibilities.

Exercise 9.0.0.3. Discrete Event Probabilities ☕

- What is the probability of flipping a tails on one coin toss?

- What is the probability of rolling a number less than five if you roll a standard six sided die? Clearly indicate what are the number of successes and what is the number of total number of possibilities.

- What is the probability of picking a queen if you randomly draw one card out of a standard deck of 52 cards? Clearly indicate what are the number of successes and what is the number of total number of possibilities.

One reason for looking at probabilities of events is to answer the following question: "What is the expected outcome of the event?" Essentially we want to know what to expect before an event happens.

Definition 9.0.0.4. Expected Value

Define the *expected value* (also known as the *first moment*) of an event X as follows. Let D be the set of all possible outcomes of event X. Then the *expected value of* X, written $E(X)$ is

$$E(X) = \sum_{x \in D} x \cdot P(X = x)$$

That is to say, if we take each outcome times the probability of that outcome and sum them, we get the expected outcome. It is a weighted average across all outcomes, weighted by how likely each outcome is.

Example 9.0.0.5. Rolling a Die

For example, suppose we asked for the expected value of rolling one die. Each of the numbers has probability one-sixth of coming up. Thus the expected value of rolling a die is

$$\frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6 = 3.5$$

Exercise 9.0.0.6. Rolling Two Die ☕☕

What is the expected value of rolling two die? (**Hint:** Draw a 6x6 table to enumerate all possibilities where the columns correspond to the first die and the rows correspond to the second die.)

Most of the time it's harder to compute the number of successful events or the number of total events. The key trick in computing these is the following:

Theorem 9.0.0.7. Fundamental Principle of Counting

The *Fundamental Principle of Counting* says the following: if event A can occur n ways and event B can occur in m ways. then the number of ways events A and B can occur together is $n \cdot m$.

Note that this only holds if events A and B are *independent*. This means that for any outcome of A, any outcome of B can then occur, and vise-versa. That is, the outcome of event A in no way restricts what is possible for event B.

Exercise 9.0.0.8. Independent Events ☕☕

- Let event A be rolling the first of two die. Let event B be rolling the second of two die. Explain why events A and B are independent. In light of this, use the Fundamental Principle of Counting to compute the total number of possible outcomes.

- You roll two die. What is the probability the total is either a 7 or 11? (Note this is the probability you'll win on the first roll of a round of Craps.)

With more than two independent events, we can iterate the Fundamental Principle of Counting and simply multiply together the number of possibilities for all events.

Example 9.0.0.9. Four Coin Tosses

Suppose we flip a coin four times. Each flip has two potential outcomes, so there is a total of $2 \cdot 2 \cdot 2 \cdot 2 = 16$ total possible outcomes. We can explicitly list them to see our computation is correct, where T is tails and H is heads, and the string of four characters is the sequence of outcomes of the four flips.

$$\begin{array}{rl}
1 & TTTT \\
2 & TTTH \\
3 & TTHT \\
4 & TTHH \\
5 & THTT \\
6 & THTH \\
7 & THHT \\
8 & THHH \\
9 & HTTT \\
10 & HTTH \\
11 & HTHT \\
12 & HTHH \\
13 & HHTT \\
14 & HHTH \\
15 & HHHT \\
16 & HHHH
\end{array}$$

Use the above table to answer the following questions.

Exercise 9.0.0.10. Probabilities with Coins ☕☕

- You start flipping a coin over and over. What is the probability you get a heads on the very first flip?
- You start flipping a coin over and over. What is the probability your first heads doesn't appear until the second flip?
- You start flipping a coin over and over. What is the probability your first heads doesn't appear until the third flip?
- You start flipping a coin over and over. What is the probability your first heads doesn't appear until the fourth flip?

Let's now generalize this. Let events A and B be independent. Define a new event $A \cap B$ to mean that A and B both happen. By definition of probability and by the Fundamental Principle of Counting, we have

$$\begin{aligned}
\text{Probability of } A \cap B =& \frac{\text{number of ways for both events to be successful}}{\text{number of total possible outcomes for the two events}} \\
=& \frac{\text{number of ways for } A \text{ to be successful times number of ways for } B \text{ to successful}}{\text{number of ways for } A \text{ to happen times number of ways for } B \text{ to happen}} \\
=& \frac{\text{number of ways for } A \text{ to be successful}}{\text{number of ways for } A \text{ to happen}} \cdot \frac{\text{number of ways for } B \text{ to be successful}}{\text{number of ways for } B \text{ to happen}} \\
=& \text{Probability of } A \cdot \text{Probability of } B
\end{aligned}$$

This provides a more structured view of what was going on in the coin exercises. For example, in for the probability of our first heads on the fourth flip, we could also have computed our probability as follows:

$$\begin{aligned}
&\text{Probability of the first heads appearing on the fourth flip} \\
&\quad =\text{Probabilty of NOT getting heads on flip 1} \\
&\quad\quad \text{and NOT getting heads on flip 2} \\
&\quad\quad \text{and NOT getting heads on flip 3} \\
&\quad\quad \text{and getting heads on flip 4} \\
&\quad =\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \\
&\quad =\frac{1}{16}
\end{aligned}$$

Now let's look at a slightly more complicated situation where the answer can be obtained via the same reasoning.

Exercise 9.0.0.11. Expected Number of Days You Can Avoid a Ticket ☕☕☕

Suppose we are a CU student. Parking permits are grossly overpriced, so we decide to just risk it and park illegally in our favorite Permits-Only parking spot. From past experience, we know there is only a $p = 15\%$ chance of getting a ticket on any particular day; it's not heavily policed. Let X the number of days until the first parking ticket occurs. That is, $P(X = n)$ is the probability of us getting away with this for the first $n - 1$ days and then getting our first parking ticket on day n. Fill out the following table:

n	$P(X = n)$
1	
2	
3	
4	
5	
6	
7	
8	

Notice that the list of probability values are entries in a geometric sequence! Thus a random variable such as the one above is called a *geometric random variable* and the corresponding distribution of probabilities is called the *geometric distribution*.

Exercise 9.0.0.12. Row n of the Table ☕☕☕

- What is the general formula for $P(X = n)$ in terms of n?

- If all is fair, we should be able to add up all of the above probabilities and get 1, since 1 is total probability of anything at all happening. Compute

$$\sum_{n=1}^{\infty} P(X = n)$$

  Do you get one as the sum?

- How many days can we expect to go before we get a ticket? That is to say, what is the expected value of X? Recall the formula for expected value:

$$E(X) = \sum_{n=1}^{\infty} n \cdot P(X = n)$$

Alright so before we call it good... no probability discussion is ever complete without a gambling example!

Exercise 9.0.0.13. Roulette ☕☕

A roulette wheel has 38 spots on it. Suppose “28” is our favorite number, so we keep playing it over and over again. How many spins of the wheel should we expect before it hits our number?

Selected Answers and Hints

Exercise 2.0.0.1. Saying that F is an antiderivative of f is equivalent to saying the derivative of F is f. That is, $F'(x) = f(x)$. The Fundamental Theorem of Calculus states that after antidifferentiating the integrand, one can plug the bounds into the antiderivative and take their difference in order to calculate the integral. Because $F'(x) = f(x)$ by the definition of an antiderivative, a good way to check that your antiderivative F is correct is to take its derivative F'. You should get the original function, f.

Exercise 2.1.1.1. $\int f'(g(x)) \cdot g'(x)\,\mathrm{d}x = \int (f(g(x)))'\,\mathrm{d}x = f(g(x)) + C$

Exercise 2.1.1.6. Use the substitutions $u = x^2 + x + 8, \ln(x)$, and $-x^2$. In the last case, the $\mathrm{d}u$ term has nothing to cancel the x with!

Exercise 2.1.2.2. To have four intervals in the Riemann sum, Δx would be 1 while Δu would be 2. Thus, the width of each rectangle is getting doubled, since to convert between u and x we use the formula $u = 2x + 1$. The "plus one" merely slides all the rectangles one unit to the right, but it does not stretch their width at all, so it does not affect their area. Thus, the slope of the graph of $u = 2x + 1$ is the only thing that mattered regarding our conversion between x and u. That is to say, the quantity $\mathrm{d}u/\mathrm{d}x$ gives us the scaling factor.

Exercise 2.1.2.3. The definite integral evaluates to roughly 0.95. The horizontal scaling factor at each x-coordinate should correspond to the derivative $\mathrm{d}u/\mathrm{d}x$ at each point.

Exercise 2.2.1.3. By factoring out the quantity $(x+1)^{3/2}$, both answers can be brought into the form $(x+1)^{3/2}\left(\frac{2}{5}x - \frac{4}{15}\right) + C$.

Exercise 2.2.2.2. Use the substitution $u = 1 - x^2$.

Exercise 2.2.2.3. The antiderivative is $x\ln(x) - x + C$.

Exercise 2.2.3.5. The antiderivative is $\frac{1}{2}\left(\sec(x)\tan(x) + \ln|\sec(x) + \tan(x)|\right) + C$.

Exercise 2.3.0.1. Use the substitution $u = \sqrt{x}$ to transform the first integral into $\int 2u\cos(u)\,\mathrm{d}u$.

Exercise 2.3.0.2. Choosing $u = \ln(x)$ will make the logarithm disappear upon differentiation. The opposite choice will not clean up the log.

Exercise 2.4.0.2. The antiderivative is $\frac{1}{3}\sin^3(x) + C$.

Exercise 2.4.1.2. Since seven is odd, when we pulled out one factor of sine, we ended up with the sixth power of sine remaining. Since six is even, we were able to express it as a power of a perfect square of sine, which in turn let us rewrite as cosines using the Pythagorean identity.

Exercise 2.4.1.3. The first antiderivative is $-\frac{1}{3}\cos^3(x)+\frac{2}{5}\cos^5(x)-\frac{1}{7}\cos^7(x)+C$. For the second, rewrite as $(1-\sin^2(x))^4\cos(x)$ and proceed by letting $u=\sin(x)$.

Exercise 2.4.1.4. Often when trying to show that two antiderivatives are compatible, it is easiest to verify that their difference is a constant.

Exercise 2.4.1.4. The substitution $u=\sin(x)$ is much cleaner since the other will involve having to expand a binomial to the fifth power. The antiderivative is $\frac{1}{12}\sin^{12}(x)-\frac{1}{14}\sin^{14}(x)+C$.

Exercise 2.4.2.1. The exponent on sine is zero, which is indeed even. Thus both exponents are even in this case.

Exercise 2.4.2.3. When all like terms are combined and the one-eighth is distributed, the result is $\frac{5}{16}x+\frac{1}{4}\sin(2x)-\frac{1}{48}\sin^3(2x)+\frac{3}{64}\sin(4x)+C$.

Exercise 2.4.2.4. The antiderivative to $\cos^6(x)$ came out to

$$\frac{5}{16}x+\frac{1}{4}\sin(2x)-\frac{1}{48}\sin^3(2x)+\frac{3}{64}\sin(4x)+C$$

Before we differentiate, first bash everything back down to an "x" in the argument using double angle identities. This produces

$$\frac{5}{16}x+\frac{1}{2}\sin(x)\cos(x)-\frac{1}{6}\sin^3(x)\cos^3(x)+\frac{3}{16}\sin(x)\cos^3(x)-\frac{3}{16}\sin^3(x)\cos(x)+C$$

Factor out a sine and use the Pythagorean Identity to get everything else in terms of cosine. This produces

$$\frac{5}{16}x+\sin(x)\left(\frac{5}{16}\cos(x)+\frac{5}{24}\cos^3(x)+\frac{1}{6}\cos^5(x)\right)+C$$

Then we differentiate and obtain

$$\frac{5}{16}+\cos(x)\left(\frac{5}{16}\cos(x)+\frac{5}{24}\cos^3(x)+\frac{1}{6}\cos^5(x)\right)-\sin^2(x)\left(\frac{5}{16}+\frac{5}{8}\cos^2(x)+\frac{5}{6}\cos^4(x)\right)$$

to which we apply the Pythagorean Identity $\sin^2(x)=1-\cos^2(x)$ to produce

$$\frac{5}{16}+\cos(x)\left(\frac{5}{16}\cos(x)+\frac{5}{24}\cos^3(x)+\frac{1}{6}\cos^5(x)\right)-\left(1-\cos^2(x)\right)\left(\frac{5}{16}+\frac{5}{8}\cos^2(x)+\frac{5}{6}\cos^4(x)\right)$$

This will simplify to $\cos^6(x)$ once you expand and combine like terms.

Exercise 2.4.2.5. For the first, apply the identity $\sin^2(3x)=\frac{1-\cos(6x)}{2}$ and proceed. For the second, notice that $\sin^4(x)$ can be rewritten as $\left(\sin^2(x)\right)^2$, after which the half-angle identity can be applied.

Exercise 2.5.1.3. First apply all the product and chain rules to reach the expression

$$\frac{3}{\sqrt{1-\frac{x^2}{4}}}+4\sqrt{1-\frac{x^2}{4}}+\frac{-x^2}{\sqrt{1-\frac{x^2}{4}}}+\sqrt{1-\frac{x^2}{4}}\left(1-\frac{3}{2}x^2\right)+\frac{-x}{4\sqrt{1-\frac{x^2}{4}}}\left(x-\frac{x^3}{2}\right)$$

Put all terms over the common denominator $\sqrt{4-x^2}$ and combine like terms in the numerator. Notice the numerator becomes $\left(4-x^2\right)^2$ and then reduce for the win!

Exercise 2.5.1.4. The antiderivative is $2^{18}\left(\frac{\left(1-x^2/16\right)^{9/2}}{9}-\frac{\left(1-x^2/16\right)^{7/2}}{7}\right)+C$

Exercise 2.5.2.3. The antiderivative is $\frac{x\sqrt{x^2-4}}{2} - 2\ln|x+\sqrt{x^2-4}| + C$.

Exercise 2.6.1.5. Using properties of logarithms, both answers should be able to be put in the form $\ln\left|\sqrt{\frac{x-1}{x+1}}\right| + C$

Exercise 2.6.3.1. •The function $\frac{1}{x^2-9x+20}$ has $\ln\left|\frac{x-5}{x-4}\right| + C$ as its antiderivative. •The factorization $x^4-9 = \left(x^2+3\right)\left(x-\sqrt{3}\right)\left(x+\sqrt{3}\right)$ will produce the following setup:

$$\frac{1}{x^4-9} = \frac{Ax+B}{x^2+3} + \frac{C}{x-\sqrt{3}} + \frac{D}{x+\sqrt{3}}$$

in which you can then solve for the coefficients and antidifferentiate. •The function $\frac{x^4}{x^2+1}$ has an irreducible quadratic for a denominator. However, the degree of the numerator is not smaller than the degree of the denominator. Thus, polynomial long division is the only step of PFD that is required in this case. •The antiderivative of $\frac{2}{x^5+2x3+x}$ is

$$2\ln|x| - \ln\left|x^2+1\right| + \frac{1}{x^2+1}$$

•The PFD will produce

$$\frac{x-2}{x^3+x^2+3x-5} = \frac{-\frac{1}{8}}{x-1} + \frac{\frac{1}{8}x+\frac{11}{8}}{x^2+2x+5}$$

While the first term is easy to integrate, the second is quite tricky! To hack through it, split it as follows:

$$\frac{\frac{1}{8}x+\frac{11}{8}}{x^2+2x+5} = \frac{\frac{1}{8}x+\frac{1}{8}}{x^2+2x+5} + \frac{\frac{10}{8}}{x^2+2x+5}$$

The first fraction can then be integrated via u-sub, while the second can be done via trig sub after completing the square on the denominator.

Exercise 2.6.3.2. For $\frac{1}{x^4-9x^2}$, keep in mind that x^2 is not an irreducible quadratic factor but rather a repeated linear factor. The PFD and integration will produce

$$\frac{1}{9x} + \frac{1}{54}\ln\left|\frac{x-3}{x+3}\right| + C$$

Exercise 3.1.0.4. The limits are 0, $1/\pi$, and -1.

Exercise 3.1.1.3. The limits are 0, -2, and e.

Exercise 3.1.1.4. The results are 1, 0, and 1.

Exercise 3.1.2.2. Their ratio converges to 3 (both numerically in the table, and analytically as evaluated by LHR). Since this is a nonzero constant, the two functions have the same growth order.

Exercise 3.1.2.3. In the first and third, the ratio between f and g seems to grow without bound, so f has larger growth order. In the second, the ratio of f to g seems to always be right around 2. Thus, they have the same growth order.

Exercise 3.2.1.8. The integrals evaluate to $2\sqrt{2}, \infty, \infty$, and ∞.

Exercise 3.2.2.4. •The area under xe^{-x^2} from zero to ∞ is $\frac{1}{2}$. •Splitting into two integrals at

$x = 0$ produces one of area one-half and one of area negative one-half, so the total integral is zero. •After applying IBP with $u = x$ and $\mathtt{d}v = xe^{-x^2}\,\mathtt{d}x$, one obtains $\frac{\sqrt{\pi}}{2}$ as the area under the curve. •The area under $\frac{1}{x\ln(x)}$ from 2 to ∞ is infinite. •The area under $\frac{1}{x(\ln(x))^2}$ from 2 to ∞ is $\frac{1}{\ln(2)}$. •An improper integral is defined using a limit, and here the limit does not exist, as the area keeps going up and down by the same amount forever.

Exercise 3.3.2.1. •The curves $y = x^3 + x^2 - x - 1$ and $y = x^3 - x^2 - x + 1$ intersect on the x axis at -1 and 1 and have area 8/3 between them. •The area inside the unit circle but above the line $y = 1/2$ is $\pi/3 - \sqrt{3}/4$. •Notice graphically that the curves intersect at $x = \pm\pi/4$. The area between curves is $\pi/4 - \ln(2)$.

Exercise 3.4.1.2. The exact arc length is $\frac{2\sqrt{5}+\ln\left|2+\sqrt{5}\right|}{4}$.

Exercise 3.4.3.1. The length of the graph of the natural logarithm from (1,0) to (e,1) is

$$\sqrt{e^2+1} + \frac{1}{2}\ln\left|\frac{\sqrt{e^2+1}-1}{\sqrt{e^2+1}+1}\right| - \sqrt{2} + \frac{1}{2}\ln\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|$$

which is roughly 2.003497. Also, notice that the natural exponential function is just the inverse of the natural logarithm; think about what this means regarding arc length!

Exercise 3.4.4.1. The two-frusta approximation is

$$\frac{\pi}{8}\left(\sqrt{5} + 3\sqrt{13}\right) \approx 5.126$$

The exact value of the surface area is

$$\frac{\pi}{6}\left(5\sqrt{5} - 1\right) \approx 5.3304$$

which is just slightly larger, as one would expect.

Exercise 3.4.6.1. The volume of the torus is

$$V = 2\pi^2 R r^2$$

and the surface area is

$$SA = 4\pi^2 R r$$

Exercise 3.5.2.1. The diagonals have the equations

$$y = \frac{b}{a}x \text{ and } y = \frac{b-2c}{a}x + c$$

with intersection point $(a/2, b/2)$, which is also the center of mass of the region.

Exercise 3.5.3.2. The coordinates of the vertices are (0,0), (0,c), and (a, b). Two of the medians are

$$y = \frac{b+c}{a}x \text{ and } y = \frac{2b-c}{2a}x + \frac{c}{2}$$

and their intersection point (and center of mass of the triangle) is $\left(\frac{a}{3}, \frac{b+c}{3}\right)$.

Exercise 3.5.4.1. The center of mass is $\left(0, \frac{4r}{3\pi}\right)$.

Exercise 4.4.1.3. $\bullet \frac{1}{n+1}$ $\bullet n+1$ $\bullet (n+2)(n+1)$ $\bullet (2n+2)(2n+1)$

Exercise 4.4.3.1. In the context of computing a limit to infinity, it is fine to replace $n!$ by $\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. Setting up limits of ratios and testing growth order with LHR and good old algebra will then verify that the order goes $n^2, e^n, n!, n^n$.

Exercise 4.5.0.3. Think about what happens if the common difference d is zero and if the common ratio r is 1.

Exercise 4.7.0.3. In most cases, it is easiest to just expand the sums on both sides and see what the terms look like.

Exercise 4.8.0.5. The totals are 500500, 1501500, 214214, and 245.

Exercise 4.9.0.5. \$5 billion $\cdot \frac{1-0.8^{13}}{1-0.8} \approx$ \$23.6 billion

Exercise 4.9.0.6. Try to notice how the summations relate to the Fibonacci numbers themselves!

Exercise 4.11.1.1. The sequence is $a_n = \frac{1}{2^{n+1}}$. Since this is a geometric sequence, the finite geometric series formula can be applied to then find the sequence of partial sums A_N.

Exercise 4.11.1.2. It ends up one-third of a meter forward from where it started.

Exercise 4.11.1.5. Yes, the series is geometric with initial term $\frac{3^5}{2^{11}}$ and common ratio 3/4. The infinite series totals to $\frac{3^5}{2^9}$.

Exercise 4.11.1.7. The partial sums are $A_N = 2(N+1)$. The infinite series is the limit of A_N as N goes to infinity, which here is clearly again infinity. Thus, the infinite series diverges.

Exercise 4.11.1.8. The partial sums are

$$A_N = \frac{N+1}{N+3}$$

for an infinite sum of 1.

Exercise 5.1.0.4. Written in sigma notation, the power series is $\sin(x) = \Sigma_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)!}x^{2n+1}$.

Exercise 5.1.0.5. Written in sigma notation, the power series is $e^x = \Sigma_{n=0}^{\infty} \frac{1}{n!}x^n$.

Exercise 5.1.0.6. The power series is $\frac{1}{1-x} = \Sigma_{n=0}^{\infty} x^n$. It is a geometric series with initial term 1 and common ratio x.

Exercise 5.1.0.8. When we try to plug in $x = 0$ to find a_0, we get $\ln(0)$ which is not a real number.

Exercise 5.1.0.9. The power series centered at one for the natural log is $\ln(x) = \Sigma_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$.

Exercise 5.4.1.1. If we substitute $x-1$ for x in the power series for sine, we get $\sin(x-1) = \Sigma_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)!}(x-1)^{2n+1}$. Likewise, substituting $2x$ for x in the power series for sine produces $\sin(2x) = \Sigma_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)!}(2x)^{2n+1} = \Sigma_{n=0}^{\infty}(-1)^n \frac{2^{2n+1}}{(2n+1)!}x^{2n+1}$.

Exercise 5.4.2.1. The power series $\frac{1}{x^2-x-12} = \Sigma_{n=0}^{\infty}\left(\frac{-1}{21\cdot(-3)^n} - \frac{1}{28\cdot 4^n}\right)x^n$ has IOC (-3,3). The power series $\frac{1}{x} = \Sigma_{n=0}^{\infty}\frac{(-1)^n}{5^{n+1}}(x-5)^n$ has IOC (0,10). It turns out these two examples generalize; for rational functions, the IOC will always just be the interval that goes from the center of the series outwards until it bumps into the nearest vertical asymptote!

Exercise 5.4.4.1. Antidifferentiate the geometric series to sneak up on $\ln(1-x)$.

Exercise 5.4.4.2. Each method should lead to

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$$

Exercise 5.6.0.2. If $n = 1$, we have the following degree one power series centered at $a = 4$:

$$f(x) = \sqrt{x} \approx 2 + \frac{1}{4}(x-4)$$

Since $n = 1$, we need the second derivative. We compute $|f''(x)| = \frac{1}{4x^{3/2}}$, which on the interval [4,4.1] has its maximum at $x = 4$. Thus, $M = 1/32$, which provides an error bound of

$$\frac{\frac{1}{32}\cdot|4-4.1|^2}{2!} = \frac{1}{6400}$$

Thus the error is definitely less than one thousandth, but not necessarily less than one ten thousandth. So for the approximation

$$\sqrt{4.1} \approx 2 + \frac{1}{4}(4.1-4) = 2.025$$

we can guarantee three digits past the decimal are correct but not necessarily the fourth. That is, 2.025 is certainly the correct decimal expansion for $\sqrt{4.1}$ rounded to the thousandths place. However, the digit 0 we implicitly have in the ten-thousandths place may or may not be correct. This process can be repeated for the other n values of two and three.

Exercise 6.3.0.4. The arc length is

$$\frac{6\sqrt{146} + \ln\left(\sqrt{73} + 6\sqrt{2}\right)}{6} \approx 12.55$$

Exercise 6.3.0.5. The arc length is $\sqrt{2}\left(e^{2\pi} - 1\right)$.

Exercise 6.5.2.7. Yes, it is in fact a circle with cartesian center $(0, 1/2)$ and radius $1/2$. This can be verified by demonstrating the polar equation converts to the cartesian equation

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$$

Exercise 6.6.0.2. The derivative is a constant; thus the graph is a straight line!

Exercise 6.7.0.4. The area between the curves is $\frac{\pi}{8}$.

Exercise 6.7.0.5. The area inside the inner loop of $r(\theta) = \frac{1}{2} + \cos(\theta)$ is $\frac{\pi}{4} - \frac{3\sqrt{3}}{8}$.

Exercise 7.2.0.4. Any solution to $\frac{dy}{dx} = xy + x$ can be written as $y = Ce^{\frac{x^2}{2}} - 1$ for some real number C. The second DE with initial condition has the solution

$$\frac{1}{2}e^{-y}\left(\sin(y) - \cos(y)\right) = -e^{-x}\left(3 + 2x + x^2\right) + \frac{5}{2}$$

Exercise 8.2.3.1. $\bullet\ln(2) = \ln(2) + 0i$ $\bullet\ln(-2) = \ln(2) + \pi i$ $\bullet\ln(i) = 0 + i\frac{\pi}{2}$ $\bullet\ln(1+i) = \ln\left(\sqrt{2}\right) + i\frac{\pi}{4}$ $\bullet\ln(3-4i) = \ln(5) + i\arctan\left(-\frac{3}{4}\right)$

Exercise 8.2.4.1. The number $(1+i)^{1+i}$ can be written in complex cartesian form as

$$\left(e^{\ln\left(\sqrt{2}\right)-\frac{\pi}{4}}\cos\left(\ln\left(\sqrt{2}\right) + \frac{\pi}{4}\right)\right) + i\left(e^{\ln\left(\sqrt{2}\right)-\frac{\pi}{4}}\sin\left(\ln\left(\sqrt{2}\right) + \frac{\pi}{4}\right)\right)$$

Exercise 8.2.5.1. The PFD over the complex numbers is

$$\frac{4-2x^2}{x^3+4x} = \frac{1}{x} - \frac{\frac{3}{2}}{x+2i} - \frac{\frac{3}{2}}{x-2i}$$

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