Calculus II

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Contents

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We hope you have fun with this resource and find it helpful! It is a beautiful subject, and we tried to honor that with a beautiful text. The text is still in its infancy, and we welcome any and all feedback you could give us. Thank you in advance for any comments, complaints, suggestions, and questions.

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Chapter 1

Overview

A Three-Part Course

The topics of Calculus II fall into three parts that each have an appropriate place in the story of the calculus sequence.

- **Part I: Integration.** The first part of the course ties loose ends from Calculus I. The ending of Calculus I showed that antiderivatives can be used to evaluate integrals via the Fundamental Theorem of Calculus. However, by the end of Calculus I, only the very simplest antiderivatives can actually be computed. Part one expands the student's knowledge of **techniques of antidifferentiation**. These techniques are subsequently put to use computing **length, area, volume, and center of mass**.
- **Part II: Sequences and Series.** This is the topic that makes up the body of Calculus II. **Sequences and series** embody the beauty of mathematics; from simple beginnings (a sequence is just a list... a series is just adding up a list of numbers...) it quickly leads to incredible structure, surprises, complexity, and open problems. **Power series** redefine commonly used transcendental functions (functions that are not computed using algebra, e.g. cosine). If you've ever wondered **what your calculator does when you press the cosine button**, this is where you find out! (**Hint:** It does not have a circle of radius one spinning around with a team of elves that measure x coordinates.)
- **Part III: Coming Attractions.** By the end of Calculus II, the student is ready for a *lot* of other classes. The end of Calculus II thus ends with a sampler platter of topics that show the vast knowledge base built upon the foundation laid in Calculus II. Here the text takes a bite out of **Differential Equations**, serves some polar and parametric coordinates as a palate cleanser before **Calculus III**, and tastes some **Complex Analysis** to aid in digestion of Differential Equations. For dessert, it serves a scoop of **Probability** with both discrete and continuous colored sprinkles.

How to Use This Book

This book is meant to facilitate *Active Learning* for students, instructors, and learning assistants. Active Learning is the process by which the student participates directly in the learning process by reading, writing, and interacting with peers. This contrasts the traditional model where the student passively listens to lecture while taking notes. This book is designed as a self-guided step-by-step exploration of the concepts. The text incorporates theory and examples together in order to lead the student to discovering new results while still being able to relate back to familiar topics of mathematics. Much of this text can be done independently by the student for class preparation. During class sessions, the instructor and/or learning assistants may find it advantageous to encourage group work while being available to assist students, and work with students on a one-on-one or small group basis. This is highly desirable as extensive research has shown that active learning improves student success and retention. (For example, see `www.pnas.org/content/111/23/8410` for Scott Freeman's metaanalysis of 225 studies supporting this claim.)

What is Different about this Book

If you leaf through the text, you'll quickly notice two major structural differences from many traditional calculus books:

1. The exercises are very intermingled with the readings. Gone is the traditional separation into "section" versus "exercises".
2. Whitespace was included for the student to write and work through exercises. Parts of pages have indeed been intentionally left blank.

A consequence of this structure is that the readings and exercise are closely linked. It is intended for the student to do the readings and exercises concurrently.

Ok... Why?

The goal of this structure is to help the student simulate the process by which a mathematician reads mathematics. When a mathematician reads a paper or book, he/she always has a pen in hand and is constantly working out little examples alongside and scribbling incomprehensible notes in bad handwriting. It takes a *long* time and a lot of experience to know how to come up with good questions to ask oneself or to know what examples to work out in order to to help oneself absorb the subject. Hence, the exercises sprinkled throughout the readings are meant to mimic the margin scribbles or side work a mathematician engages in during the act of reading mathematics.

The Legend of Coffee

A potential hazard of this self-guided approach is that while most examples are meant to be simple exercises to help with absorption of the topics, there are some examples that students may find quite difficult. To prevent students from spinning their wheels in frustration, we have labeled the difficulty of all exercises using coffee cups as follows:

Coffee Cup Legend

Symbol	Number of Cups	Description of Difficulty
☕	A One-Cup Problem	Easy warm-up suitable for class prep.
☕☕	A Two-Cup Problem	Slightly harder, solid groupwork exercise.
☕☕☕	A Three-Cup Problem	Substantial problem requiring significant effort.
☕☕☕☕	A Four-Cup Problem	Difficult problem requiring effort and creativity!

Glossary of Symbols

In Precalculus and Calculus I, there is a wide range of how much notation from Set Theory gets used. To get everyone on the same page, here is a short list of some notation we will use in this text.

Sets and Elements

Often in mathematics, we construct collections of objects called *sets*.

- If an object x is in a set A, we say x is an *element* of A and write $x \in A$.
- If an object x is not in a set A, we say x is not an element of A and write $x \notin A$.

Any particular object is either an element of a set or it is not. We do not allow for an object to be partially contained in a set, nor do we allow for an object to appear multiple times in a set. Often we use curly braces around a comma-separated list to indicate what the elements are.

Example 1.0.0.1. A Prime Example

Suppose P is the set of all prime numbers. We write

$$P = \{2, 3, 5, 7, 11, 13, 17, \ldots\}$$

For example, $2 \in P$ and $65,537 \in P$, but $4 \notin P$.

Some Famous Sets of Numbers

The following are fundamental sets of numbers used in Calculus 2.

- **Natural Numbers:** The set $\mathbb{N}$ of natural numbers is the set of all positive whole numbers, along with zero. That is,

  $$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \ldots\}$$

  Note that in many other sources, zero is not included in the natural numbers. Both are widely used; be aware the choice on this convention will change throughout your mathematical travels!

- **Integers:** The set of integers $\mathbb{Z}$ is the set of all whole numbers, whether they are positive, negative, or zero. That is,

  $$\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

- **Rational Numbers:** The set of rational numbers $\mathbb{Q}$ is the set of all numbers expressible as a fraction whose numerator and denominator are both integers.

- **Real Numbers:** The set of real numbers $\mathbb{R}$ is the set of all numbers expressible as a decimal.

- **Complex Numbers:** The set $\mathbb{C}$ of complex numbers is the set of all numbers formed as a real number (called the real part) plus a real number times i (called the imaginary part), where i is a symbol such that $i^2 = -1$.

Set-Builder Notation

The most common notation used to construct sets is *set-builder notation*, in which one specifies a name for the elements being considered and then some property $P(x)$ that is the membership test for an object x to be an element of the set. Specifically,

$$A = \{x \in B : S(x)\}$$

means that an object x chosen from B is an element of the set A if and only if the claim $S(x)$ is true about x. Sometimes the “ $\in B''$ gets dropped if it is clear from context what set the elements are being chosen from. The set-builder notation above gets read as “the set of all x in B such that $S(x)$”. One can think of this as running through all elements of B and throwing away any that do not meet the condition described by S.

Example 1.0.0.2. Interval Notation

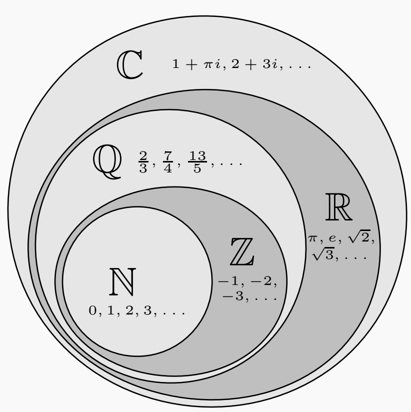
Interval notation can be expressed in set-builder notation as follows:

- $(a, b) = \{x \in \mathbb{R} : a < x < b\}$
- $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$
- $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$
- $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$

Example 1.0.0.3. Rational, Real, and Complex in Set-Builder Notation

Set-builder notation is often used to express the sets of rational, real, and complex numbers as follows:

- $\mathbb{Q} = \left\{\frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0\right\}$
- $\mathbb{R} = \{0.a_0a_1a_2a_3a_4 \ldots \times 10^n : n \in \mathbb{N}, a_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ where $i \in \mathbb{N}\}$ Note this is essentially scientific notation; the concatenation of the a_i's represents the digits in a base-ten decimal expansion.
- $\mathbb{C} = \{a + bi : a \in \mathbb{R}, b \in \mathbb{R}\}$



Part I

Integration

Chapter 2

How to Find an Antiderivative

The Fundamental Theorem of Calculus says that an integral (defined as the area under a curve) can be easily evaluated via antiderivative. However, it turns out to be very difficult and sometimes impossible to find an antiderivative! In this chapter, we give several commonly used methods for antidifferentiation.

Exercise 2.0.0.1. What is an Antiderivative Again? ☕

- Complete the definition of antiderivative. That is, if $f(x)$ is a function, then we say $F(x)$ is an antiderivative of $f(x)$ if and only if...

A Solution:

$$F'(x) = f(x)$$

- How do you use antiderivatives to evaluate definite integrals? Describe in a short sentence below.

A Solution: After finding the antiderivative, you evaluate it at the end and the start of the range over which you want the definite integral, then subtract the latter from the former. That is,

$$\int_a^b f(x)\,\mathtt{d}x = F(b) - F(a)$$

- Once you found an antiderivative, what could you do to check that it is correct? (Besides just computing it again!)

A Solution: Take the derivative of the antiderivative and make sure you get the original function back.

Saying that F is an antiderivative of f is equivalent to saying the derivative of F is f. That is, $F'(x) = f(x)$. The Fundamental Theorem of Calculus states that after antidifferentiating the integrand, one can plug the bounds into the antiderivative and take their difference in order to calculate the integral. Because $F'(x) = f(x)$ by the definition of an antiderivative, a good way to check that your antiderivative F is correct is to take its derivative F'. You should get the original function, f.

2.1 The Method of u-Substitution

2.1.1 Undoing the Chain Rule

The technique of u-substitution (affectionately known as "u-sub" from here on) can be seen as the reverse of the chain rule for antiderivatives.

Exercise 2.1.1.1. What Was the Chain Rule Again? ☕

- First, write down the chain rule.

$$(f(g(x)))' =$$

A Solution:

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

- Take the antiderivative of both sides of that equation.

$$\int \qquad\qquad \mathtt{d}x = f(g(x)) + C$$

A Solution:

$$\int f'(g(x)) \cdot g'(x)\, \mathtt{d}x = \int (f(g(x)))'\, \mathtt{d}x = f(g(x)) + C$$

In practice, we often make the substitution $u = g(x)$ to condense the notation. This will take a nastier integral with respect to x and replace it by a hopefully friendlier integral with respect to u. This process of transforming from x to u involves the following three steps:

1. **Choose u:** Pick u to be equal to some expression involving x. Frequently, it is helpful to pick u to be some "inner function" in a composition of functions that appears in the integrand. However, there is a *lot* of freedom regarding what substitution you make. Some choices of u will be helpful, and others will not be! It is important to be brave and just try some.

2. **Differentiate u:** Once you have a formula for u, differentiate with respect to x to get a formula for $\frac{\mathtt{d}u}{\mathtt{d}x}$. This will tell us what the conversion factor is between x units and u units.

3. **Solve for $\mathtt{d}x$:** Use your derivative to solve for $\mathtt{d}x$. Substitute that expression for the $\mathtt{d}x$ in the integral to replace it with $\mathtt{d}u$.

For the sake of having this process in a nice little formula box, here is the above paragraph rewritten concisely and precisely.

u-Substitution

$$\int f'(g(x))\, g'(x)\,\mathtt{d}x = \int f'(u)\,\mathtt{d}u = f(u) + C = f(g(x)) + C$$

Example 2.1.1.2. An Example of Integration via u-sub

To evaluate $\int x\cos\left(x^2\right)\mathtt{d}x$, we identify $u = x^2$ as a plausible choice based on our recollection of chain rule. This gives the following change of variables:

Three Steps of u-Substitution

Choice of u	Differentiate u	Solve for $\mathtt{d}x$
$u = x^2$	$\frac{\mathtt{d}u}{\mathtt{d}x} = 2x$	$\mathtt{d}x = \frac{1}{2x}\,\mathtt{d}u$

We now replace x^2 by u and replace $\mathtt{d}x$ by $\frac{1}{2x}\,\mathtt{d}u$ in our integral.

$$\int x\cos\left(x^2\right)\mathtt{d}x = \int x\cdot\cos(u)\frac{1}{2x}\,\mathtt{d}u = \frac{1}{2}\int \cos(u)\,\mathtt{d}u = \frac{1}{2}\sin(u) + C = \frac{1}{2}\sin\left(x^2\right) + C$$

Exercise 2.1.1.3. Checking Our Work ☕

As a follow up to the previous example, differentiate the answer to verify that you end up with the original integrand!

$$\frac{\mathtt{d}}{\mathtt{d}x}\left(\frac{1}{2}\sin\left(x^2\right) + C\right) =$$

A Solution:

$$\frac{\mathtt{d}}{\mathtt{d}x}\left(\frac{1}{2}\sin\left(x^2\right) + C\right) = \frac{1}{2}\cdot\frac{\mathtt{d}}{\mathtt{d}x}\sin\left(x^2\right) + \frac{\mathtt{d}}{\mathtt{d}x}C = \frac{1}{2}\cos\left(x^2\right)\cdot 2x + 0 = x\cos\left(x^2\right)$$

Example 2.1.1.4. A Trickier u-sub

Suppose we wish to evaluate the following integral:

$$\int \frac{\sqrt[3]{x}}{\sqrt[3]{x}+1}\, \mathtt{d}x$$

One possible approach is to let u be the denominator. The denominator can be thought of as the "inner function" inside a reciprocal function and thus often makes a good choice for u.

Three Steps of u-Substitution		
Choice of u	Differentiate u	Solve for $\mathtt{d}x$
$u = \sqrt[3]{x}+1$	$\frac{\mathtt{d}u}{\mathtt{d}x} = \frac{1}{3}x^{-2/3}$	$\mathtt{d}x = 3x^{2/3}\,\mathtt{d}u$

We now perform the substitutions on the denominator and the $\mathtt{d}x$.

$$\int \frac{\sqrt[3]{x}}{\sqrt[3]{x}+1}\, \mathtt{d}x = \int \frac{\sqrt[3]{x}}{u} 3x^{2/3}\, \mathtt{d}u = 3\int \frac{x}{u}\, \mathtt{d}u$$

At the moment, it seems like things are going very poorly! We hoped that x in the numerator would nicely cancel out, like it did back in the more civilized age of Exercise 2.1.1.2. To fix this, we solve for x in the equation $u = \sqrt[3]{x}+1$ to obtain $x = (u-1)^3$. We now substitute that expression for x in the integral.

$$\begin{aligned} 3\int \frac{x}{u}\, \mathtt{d}u &= 3\int \frac{(u-1)^3}{u}\, \mathtt{d}u \\ &= 3\int \frac{u^3 - 3u^2 + 3u - 1}{u}\, \mathtt{d}u \\ &= 3\int u^2 - 3u + 3 - \frac{1}{u}\, \mathtt{d}u \\ &= u^3 - \frac{9}{2}u^2 + 9u - 3\ln|u| + C \\ &= \left(\sqrt[3]{x}+1\right)^3 - \frac{9}{2}\left(\sqrt[3]{x}+1\right)^2 + 9\left(\sqrt[3]{x}+1\right) - 3\ln\left|\sqrt[3]{x}+1\right| + C \\ &= x - \frac{3}{2}\sqrt[3]{x}^2 + 3\sqrt[3]{x} - 3\ln\left|\sqrt[3]{x}+1\right| + C \end{aligned}$$

Exercise 2.1.1.5. Missing Constants ☕

In the above example, all of the constant terms disappeared on the final step! Was that ok?

Exercise 2.1.1.6. Practice with u-sub ☕☕

- Evaluate $\int \frac{6x+3}{x^2+x+8}\,\mathrm{d}x$.

A Solution:

$$\int \frac{6x+3}{x^2+x+8}\,\mathrm{d}x = 3\int \frac{2x+1}{x^2+x+8}\,\mathrm{d}x$$
$$\text{Let } u = x^2+x+8$$
$$\text{Then } \mathrm{d}u = 2x+1\,\mathrm{d}x$$
$$3\int \frac{\mathrm{d}u}{u} = 3\ln u + C = 3\ln\left(x^2+x+8\right) + C$$

- Evaluate $\int \frac{(\ln(x))^2}{x}\,\mathrm{d}x$.

A Solution:

$$\text{Let } u = \ln(x)$$
$$\text{Then } \mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$$
$$\int u^2\,\mathrm{d}u = \frac{u^3}{3} + C = \frac{1}{3}\left(\ln(x)\right)^3 + C$$

- Evaluate $\int xe^{-x^2}\,\mathrm{d}x$.

A Solution:

$$\text{Let } u = -x^2$$
$$\text{Then } \mathrm{d}u = -2x\,\mathrm{d}x \text{ and } \mathrm{d}x = -\frac{1}{2x}\,\mathrm{d}u$$
$$\int xe^{-x^2}\left(-\frac{1}{2x}\,\mathrm{d}u\right) = -\frac{1}{2}\int e^u\,\mathrm{d}u$$
$$-\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C$$

- Consider the integral
$$\int e^{\left(x^2\right)}\,\mathrm{d}x$$
Explain in words why the substitution $u = x^2$ will not work in this case. Where do you get stuck?

A Solution:

$$\text{Let } u = x^2$$
$$\text{Then } \mathrm{d}u = 2x\,\mathrm{d}x \text{ and } \mathrm{d}x = \frac{1}{2x}\,\mathrm{d}u$$

$$\int e^{x^2}\left(\frac{1}{2x}\,\mathrm{d}u\right) = \frac{1}{2}\int \frac{e^u}{x}\,\mathrm{d}u$$

Here we are stuck because there is no way to cancel x and we can't integrate with two variables.

Exercise 2.1.1.7. Create an Integral! ☕☕☕

Come up with your own integral that can be evaluated by u-sub. Find a partner and trade! See if you can evaluate each other's integrals with u-sub, or explain to your partner why theirs cannot be evaluated using u-sub.

2.1.2 Change of Coordinates and u-Substitution

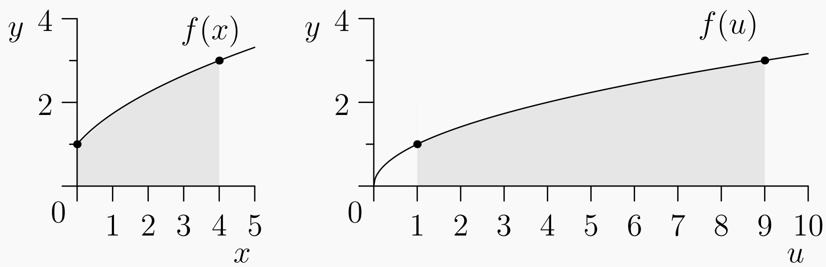
The method of u-substitution is actually a special case of a more general notion, *change of coordinates*. This will be studied more thoroughly and in more generality in Calculus 3.

Example 2.1.2.1. How u-sub Affects Area

Suppose we wish to compute the following integral:

$$\int_{x=0}^{x=4} \sqrt{2x+1}\,\mathrm{d}x$$

We interpret this as the area under the curve $f(x) = \sqrt{2x+1}$ from $x = 0$ to $x = 4$ as drawn below. We apply the substitution $u = 2x+1$. This transforms a region in the xy-plane to a region in the uy-plane.



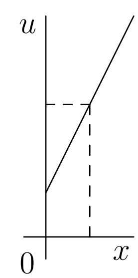
Notice the heights are exactly the same, but the widths have changed by a factor of 2. Thus to equate the x integral to the u integral, we need to divide the u integral by 2 in order to fix the fact that we doubled all the widths! In particular,

$$\int_{x=0}^{x=4} \sqrt{2x+1}\,\mathrm{d}x = \frac{1}{2}\int_{u=1}^{u=9} \sqrt{u}\,\mathrm{d}u.$$

Note that this geometric interpretation corresponds perfectly to what happens in the Riemann sum definition of an integral. For a fixed number of rectangles, the Δx in a Riemann sum of the area under $\sqrt{2x+1}$ would be exactly half of the Δu in a corresponding Riemann sum of the area under $\sqrt{u}$.

Exercise 2.1.2.2. Relationship between Δx and Δu

To illustrate the final claim of the example, draw an evenly-spaced four-rectangle Riemann Sum in each of the two shaded regions above.



Graph of $u = 2x+1$, constant derivative 2.

- What is your value of Δx?

A Solution:

$$\Delta x = 1$$

- What is your value of Δu?

A Solution:

$$\Delta u = 2$$

- Explain where the u bounds of 1 and 9 come from.

A Solution: Since $u = 2x+1$, when $x = 0$, $u = 1$, and when $x = 4$, $u = 9$. The "+1" shifts u 1 unit to the right, just as it would in a graph transformation.

- In the conversion $u = 2x+1$, the multiplication by 2 had all the effects described above. But, why did the "plus one" not affect area? (**Hint:** What would the u bounds have been, and what would the region have looked like, if the "plus one" was not there?)

A Solution: Sliding the graph has no effect on its area. If the substitution had been $u = 2x$, then the u bounds would have been $0-8$, but the graph would also have started at -1, so the shaded region would be the same in relation to the curve.

- In the figure at right, what aspect of that graph corresponds to the scaling factor between x and u?

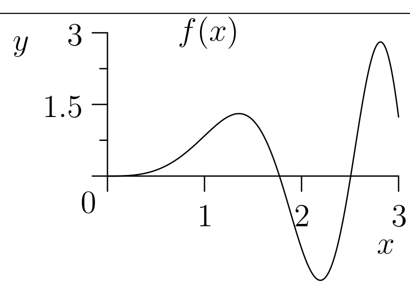
A Solution: The scaling factor is represented by the slope $\frac{du}{dx}$ of the line in the graph at right.

The moral to the story is that the scaling factor between Δx and Δu is in fact the derivative of u with respect to x. In order to translate from x to u in our integral, we must divide by $\frac{du}{dx}$. In this case, that was just the constant 2. What is fascinating is that this approach of dividing off by the scaling factor still works, even when the scaling factor is not just a constant. Hence if we use $u = x^2$, we must divide by $\frac{du}{dx} = 2x$, as we do in many of the above examples. It is still a scaling factor, but one that changes depending on the input!

Exercise 2.1.2.3. Looking Graphically at a u-sub

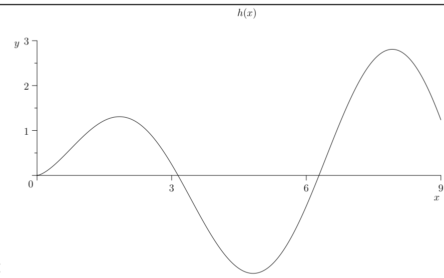
We now look at a case where the scaling factor depends on x.

- Plot a graph of the function $f(x) = x\sin(x^2)$ on the domain $[0, 3]$. (You may use a graphing utility to assist you.)



A Solution:

- Plot a graph of the function $h(u) = \sqrt{u}\sin(u)$ on the domain $[0, 9]$. (You may use a graphing utility to assist you.)



A Solution:

- Can you see the horizontal scaling factor change at different points of the graph? How stretched out does it seem to be at $x = 1$? How stretched out does it seem to be at $x = 3$?

A Solution: At $x = 1$, the scaling factor is about 1. At $x = 3$ the scaling factor is about 9.

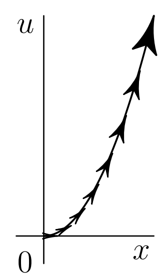
- Show algebraically that $u = x^2$ is the substitution that turns that function $f(x)$ into the corresponding function $h(u)$.

A Solution:

$$\text{Let } u = x^2$$

$$\text{Then } x = \sqrt{u}$$

$$\text{Substituting: } x\sin x^2 = \sqrt{u}\sin u$$



Graph of $u = x^2$, changing derivative of $2x$.

- Evaluate the integral $\int_{x=0}^{x=3} x\sin(x^2)\,\mathtt{d}x$.

A Solution:

$$\text{Let } u = x^2\text{, then } \frac{\mathtt{d}u}{\mathtt{d}x} = 2x\text{, and } \mathtt{d}x = \frac{\mathtt{d}u}{2x}$$

Substituting into the original function:

$$\int_{x=0}^{x=3} x\sin(u)\frac{\mathtt{d}u}{2x} = \frac{1}{2}\int_{u=0}^{u=9}\sin(u)\,\mathtt{d}u = \frac{1}{2}\left[-\cos(u)\right]_0^9 = \frac{1}{2}\left(-\cos 9 + 1\right) \approx 0.955$$

- In the figure at right, what aspect of that graph corresponds to the scaling factor between x and u?

A Solution: The slope at each point, $\frac{\mathtt{d}u}{\mathtt{d}x} = 2x$, is the scaling factor.

2.1.3 Antiderivatives of the Six Trig Functions

In Calculus I, we found the derivatives of all six trig functions. List those below.

Exercise 2.1.3.1. Recalling the Derivatives of the Six Trig Functions

Write the derivative of each of the following trig functions:

- $\frac{\mathrm{d}}{\mathrm{d}x}(\sin(x)) =$
- $\frac{\mathrm{d}}{\mathrm{d}x}(\cos(x)) =$
- $\frac{\mathrm{d}}{\mathrm{d}x}(\tan(x)) =$
- $\frac{\mathrm{d}}{\mathrm{d}x}(\cot(x)) =$
- $\frac{\mathrm{d}}{\mathrm{d}x}(\sec(x)) =$
- $\frac{\mathrm{d}}{\mathrm{d}x}(\csc(x)) =$

A Solution:

- $\cos(x)$
- $-\sin(x)$
- $\sec^2(x)$
- $-\csc^2(x)$
- $\sec(x)\tan(x)$
- $-\csc(x)\cot(x)$

From these, we easily obtain the antiderivatives of sine and cosine.

Exercise 2.1.3.2. Integrals of Sine and Cosine

Use the derivatives above to compute the following antiderivatives.

- $\int \sin(x)\,\mathrm{d}x =$
- $\int \cos(x)\,\mathrm{d}x =$

A Solution:

- $\int \sin(x)\,\mathrm{d}x = -\cos(x) + C$
- $\int \cos(x)\,\mathrm{d}x = \sin(x) + C$

For tangent and cotangent, we need u-sub.

Example 2.1.3.3. Antiderivative of Tangent

We compute the antiderivative of tangent by rewriting as $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and then using the substitution $u = \cos(x)$. Differentiating both sides produces $\mathtt{d}x = \frac{\mathtt{d}u}{-\sin(x)}$. We now apply these substitutions:

$$\begin{aligned}\int \tan(x)\,\mathtt{d}x &= \int \frac{\sin(x)}{\cos(x)}\,\mathtt{d}x \\ &= \int \frac{\sin(x)}{u}\frac{\mathtt{d}u}{-\sin(x)} \\ &= -\int \frac{1}{u}\,\mathtt{d}u \\ &= -\ln|u| + C \\ &= -\ln|\cos(x)| + C\end{aligned}$$

The method used to antidifferentiate tangent can be adapted to also antidiffernticate cotangent.

Exercise 2.1.3.4. Integral of Cotangent ☕☕

Find the antiderivative of cotangent.

$$\int \cot(x)\,\mathtt{d}x =$$

A Solution:

$$\text{Let } u = \sin(x)\text{, then } \mathtt{d}x = \frac{\mathtt{d}u}{\cos x}$$

$$\begin{aligned}\int \cot(x)\,\mathtt{d}x &= \int \frac{\cos(x)}{\sin(x)}\,\mathtt{d}x \\ &= \int \frac{\cos(x)}{u}\frac{\mathtt{d}u}{\cos(x)} \\ &= \int \frac{1}{u}\,\mathtt{d}u \\ &= \ln|u| + C \\ &= \ln|\sin(x)| + C\end{aligned}$$

The antiderivative of secant is much trickier! The process is not intuitive and requires a rabbit out of a hat.

Example 2.1.3.5. Integral of Secant

Since multiplication by 1 does not change the integrand, we are free to multiply by 1 whenever it is helpful. Here, it turns out to be helpful to multiply by $\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}$. This is the rabbit.

$$\begin{aligned}
\int \sec(x)\,\mathtt{d}x &= \int \sec(x)\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\,\mathtt{d}x \\
&= \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\,\mathtt{d}x \\
&= \int \frac{\sec^2(x)+\sec(x)\tan(x)}{u}\,\frac{1}{\sec(x)\tan(x)+\sec^2(x)}\,\mathtt{d}u \\
&= \int \frac{1}{u}\,\mathtt{d}u \\
&= \ln|u| + C \\
&= \ln|\sec(x)+\tan(x)| + C
\end{aligned}$$

The above method can be adapted to antidifferentiate cosecant.

Exercise 2.1.3.6. Integral of Cosecant ☕☕

Find the antiderivative of cosecant.

$$\int \csc(x)\,\mathtt{d}x =$$

A Solution:

$$\begin{aligned}
\int \csc(x)\,\mathtt{d}x &= \int \csc(x)\frac{\csc(x)+\cot(x)}{\csc(x)+\cot(x)}\,\mathtt{d}x \\
&= \int \frac{\csc^2(x)+\csc(x)\cot(x)}{\csc(x)+\cot(x)}\,\mathtt{d}x \\
&= \int \frac{\csc^2(x)+\csc(x)\cot(x)}{u}\,\frac{1}{-\csc(x)\cot(x)-\csc^2(x)}\,\mathtt{d}u \\
&= -\int \frac{1}{u}\,\mathtt{d}u \\
&= -\ln|u| + C \\
&= -\ln|\csc(x)+\cot(x)| + C
\end{aligned}$$

2.2 Integration by Parts

Integration by parts (IBP) is the Product Rule spun around backwards to become a rule for antiderivatives rather than derivatives.

Exercise 2.2.0.1. Reversing the Product Rule

Fill in the blanks in the following construction of integration by parts:

- Recall the Product Rule for derivatives.

$$(f(x)g(x))' =$$

A Solution:

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

- Take an antiderivative of both sides.

$$= \int (f'(x)g(x))\,\mathtt{d}x + \int (f(x)g'(x))\,\mathtt{d}x$$

A Solution:

$$f(x)g(x) = \int (f'(x)g(x))\,\mathtt{d}x + \int (f(x)g'(x))\,\mathtt{d}x$$

- Rewrite the equation by subtracting the term $\int (f'(x)g(x))\,\mathtt{d}x$ from both sides.

$$=$$

A Solution:

$$f(x)g(x) - \int (f'(x)g(x))\,\mathtt{d}x = \int (f(x)g'(x))\,\mathtt{d}x$$

- To condense the notation, it is customary to make the substitutions $u = f(x)$ and $v = g(x)$. Thus, we say $\frac{\mathtt{d}u}{\mathtt{d}x} = f'(x)$ and similarly $\frac{\mathtt{d}v}{\mathtt{d}x} = g'(x)$. Multiply the $\mathtt{d}x$ to the right-hand side in both of those equations, we obtain

$$\mathtt{d}u =$$

and

$$\mathtt{d}v =$$

A Solution: $\mathtt{d}u = f'(x)\,\mathtt{d}x$ and $\mathtt{d}v = g'(x)\,\mathtt{d}x$

- Use these substitutions to replace all instances of x, f, and g by u and v and conclude the IBP formula.

A Solution:

$$uv - \int v\,\mathtt{d}u = \int u\,\mathtt{d}v$$

Just for sake of having it in its own box, here it is again!

Integration by Parts Formula
$\int u\,\mathtt{d}v = uv - \int v\,\mathtt{d}u$

We typically use this to integrate a product of functions in the case that u-substitution does not work. You can identify one factor of your integrand as u, the remaining factor as $\mathtt{d}v$, and plug into the IBP formula. There are three main types:

1. A product with one factor that becomes much simpler upon differentiation
2. A not-quite-a product that we turn into a product
3. An integrand that reappears after applying IBP

We illustrate each of these methods with an example.

2.2.1 A Product with One Factor That Becomes Much Simpler Upon Differentiation

We let u be whichever factor becomes simpler when it is differentiated. The other factor by default must then be set equal to $\mathtt{d}v$.

Example 2.2.1.1. Integrating a Product

Suppose we wish to find an antiderivative for the function $x \cdot \cos(x)$. We can either choose $u = x$ or $u = \cos(x)$. Since $u = x$ has lovely little constant function 1 as its derivative, whereas $u = \cos(x)$ would produce just another trig function as its derivative, we conclude $u = x$ is the better choice.

Choice of u and $\mathtt{d}v$	
$u = x$	$v = \sin(x)$
$\mathtt{d}u = \mathtt{d}x$	$\mathtt{d}v = \cos(x)\,\mathtt{d}x$

We are now ready to calculate the antiderivative via IBP:

$$\int x \cdot \cos(x)\,\mathtt{d}x = \int u\,\mathtt{d}v = uv - \int v\,\mathtt{d}u = x \cdot \sin(x) - \int \sin(x)\,\mathtt{d}x = x \cdot \sin(x) + \cos(x) + C$$

Exercise 2.2.1.2. Checking Our Work ☕

Take the derivative of our result, $x\sin(x) + \cos(x) + C$, to verify that it is in fact the correct antiderivative!

A Solution:

$$\frac{\mathtt{d}}{\mathtt{d}x}\left(x\sin(x) + \cos(x) + C\right) = x\cos(x) + \sin(x) - \sin(x) = x\cos(x)$$

Exercise 2.2.1.3. An Integral via both u-sub and IBP ☕☕☕

Consider the integral

$$\int x\sqrt{x+1}\,\mathtt{d}x$$

- Evaluate the integral using the u-sub $u = x + 1$.

  A Solution: Letting $u = x + 1$, then $\mathtt{d}u = \mathtt{d}x$.

$$\begin{aligned}\int (u-1)\sqrt{u}\,\mathtt{d}x &= \int u^{\frac{3}{2}} - u^{\frac{1}{2}}\,\mathtt{d}u \\ &= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} \\ &= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}}\end{aligned}$$

- Evaluate the integral using IBP, choosing $u = x$ and $\mathtt{d}v = \sqrt{x+1}\,\mathtt{d}x$.

A Solution: Letting $u = x$ and $\mathtt{d}v = \sqrt{x+1}$, then $\mathtt{d}u = \mathtt{d}x$ and $v = \frac{2}{3}(x+1)^{\frac{3}{2}}$.

$$\begin{aligned}
\int u\,\mathtt{d}v &= uv - \int v\,\mathtt{d}u \\
&= x\frac{2}{3}(x+1)^{\frac{3}{2}} - \frac{2}{3}\int (x+1)^{\frac{3}{2}}\,\mathtt{d}x \\
&= \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{2}{3}\left(\frac{5}{2}(x+1)^{\frac{5}{2}}\right) \\
&= \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{4}{15}(x+1)^{\frac{5}{2}}
\end{aligned}$$

- Your answers will appear very different! Is one incorrect? Or are they compatible?

A Solution: Starting with the first solution from the u-sub:

$$\begin{aligned}
\frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} &= (x+1)^{\frac{3}{2}}\left(\frac{2}{5}(x+1) - \frac{2}{3}\right) \\
&= (x+1)^{\frac{3}{2}}\left(\frac{2}{5}x + \frac{2}{5} - \frac{2}{3}\right) \\
&= (x+1)^{\frac{3}{2}}\left(\frac{2}{5}x - \frac{4}{15}\right)
\end{aligned}$$

Starting with the IBP solution:

$$\begin{aligned}
\frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{4}{15}(x+1)^{\frac{5}{2}} &= (x+1)^{\frac{3}{2}}\left(\frac{2}{3}x - \frac{4}{15}(x+1)\right) \\
&= (x+1)^{\frac{3}{2}}\left(\frac{2}{3}x - \frac{4}{15}x - \frac{4}{15}\right) \\
&= (x+1)^{\frac{3}{2}}\left(\frac{2}{5}x - \frac{4}{15}\right)
\end{aligned}$$

2.2.2 A Not-Quite-a Product That We Turn into a Product

Often, an integrand that does not appear to be a product can be rewritten as product in a helpful way. This often includes **rewriting the integrand as the integrand times one**. We let u be the entire integrand, leaving $\mathtt{d}v$ to just be the invisible 1 times $\mathtt{d}x$.

Example 2.2.2.1. Multiplying by 1 in an IBP

Suppose we wish to find an antiderivative for the function $\arccos(x)$. We identify $u = \arccos(x)$ which leaves $\mathtt{d}v = 1 \cdot \mathtt{d}x$. Thus we make the following declarations:

Choice of u and $\mathtt{d}v$	
$u = \arccos(x)$	$v = x$
$\mathtt{d}u = -\frac{1}{\sqrt{1-x^2}}\,\mathtt{d}x$	$\mathtt{d}v = 1 \cdot \mathtt{d}x$

We are now ready to calculate the antiderivative via IBP:

$$\int \arccos(x) \cdot 1 \cdot \mathtt{d}x = \int u\,\mathtt{d}v = uv - \int v\,\mathtt{d}u = x \cdot \arccos(x) - \int x\left(-\frac{1}{\sqrt{1-x^2}}\right)\mathtt{d}x$$

$$= x\arccos(x) + \int \frac{x}{\sqrt{1-x^2}}\mathtt{dx} = x\arccos(x) - \sqrt{1-x^2} + C$$

Exercise 2.2.2.2. Filling in the Details ☕☕

Notice that the very last step of the above example was in fact a u-substitution! Show the details of how that antiderivative was carried out.

$$\int \frac{x}{\sqrt{1-x^2}}\,\mathtt{d}x =$$

A Solution: Let $u = 1 - x^2$. Then $\mathtt{d}u = -2x\,\mathtt{d}x$.

$$\int \frac{x}{\sqrt{1-x^2}}\,\mathtt{d}x = -\frac{1}{2}\int \frac{\mathtt{d}u}{\sqrt{u}} \tag{2.1}$$

$$= -\frac{1}{2}\int u^{-\frac{1}{2}}\,\mathtt{d}u \tag{2.2}$$

$$= -\frac{1}{2} \cdot 2u^{\frac{1}{2}} + C \tag{2.3}$$

$$= -(1-x^2)^{\frac{1}{2}} + C \tag{2.4}$$

$$= -\sqrt{1-x^2} + C \tag{2.5}$$

$$\tag{2.6}$$

Exercise 2.2.2.3. The Antiderivative of the Natural Logarithm ☕☕

Apply the same technique to find an antiderivative for the function $\ln(x)$.

A Solution: Let $u = \ln(x)$ and $\mathtt{d}v = 1\,\mathtt{d}x$. Then $\mathtt{d}u = \frac{1}{x}\,\mathtt{d}x$ and $v = x$.

$$\int \ln(x)\,\mathtt{d}x = x\ln(x) - \int x\frac{1}{x}\,\mathtt{d}x \tag{2.7}$$
$$= x\ln(x) - \int \mathtt{d}x \tag{2.8}$$
$$= x\ln(x) - x + C \tag{2.9}$$
$$\tag{2.10}$$

2.2.3 An Integrand that Reappears After Applying IBP

Sometimes, we can get the original expression to come back after applying integration by parts one or more times. Once this occurs, you can **give some name to the integral (we will use I)** and solve for it as you would solve any equation in algebra!

Example 2.2.3.1. An Integrand that Reappears After IBP

Suppose we wish to find an antiderivative for the function $e^{2x}\cos(x)$. Call I the desired antiderivative. That is:

$$I = \int e^{2x}\cos(x)\,\mathtt{d}x$$

We now wish to apply IBP, so we make the following declarations:

Choice of u and $\mathtt{d}v$	
$u = e^{2x}$	$v = \sin(x)$
$\mathtt{d}u = 2e^{2x}\,\mathtt{d}x$	$\mathtt{d}v = \cos(x)\,\mathtt{d}x$

We are now ready to calculate the antiderivative via IBP:

$$\begin{aligned} I &= \int u\,\mathtt{d}v \\ &= uv - \int v\,\mathtt{d}u \\ &= e^{2x}\sin(x) - \int \sin(x)\cdot 2e^{2x}\,\mathtt{d}x \\ &= e^{2x}\sin(x) - 2\int e^{2x}\sin(x)\,\mathtt{d}x \end{aligned}$$

We notice now that the new integral is again a product of functions (and does not appear to be doable via u-sub) so we apply IBP once again with the following declarations (using new u and v):

Choice of u and $\mathtt{d}v$	
$u = e^{2x}$	$v = -\cos(x)$
$\mathtt{d}u = 2e^{2x}\,\mathtt{d}x$	$\mathtt{d}v = \sin(x)\,\mathtt{d}x$

We now proceed with the previous expression, using the new IBP setup and notice that the original integral I reappears:

$$\begin{aligned} I &= e^{2x}\sin(x) - 2\int u\,\mathtt{d}v \\ &= e^{2x}\sin(x) - 2\left(uv - \int v\,\mathtt{d}u\right) && (\bowtie) \\ &= e^{2x}\sin(x) - 2\left(e^{2x}(-\cos(x)) - \int(-\cos(x))2e^{2x}\,\mathtt{d}x\right) && (\bowtie) \\ &= e^{2x}\sin(x) + 2e^{2x}\cos(x) - 4\int e^{2x}\cos(x)\,\mathtt{d}x \\ &= e^{2x}\sin(x) + 2e^{2x}\cos(x) - 4I \end{aligned}$$

At first glimpse this seems troubling; we have reduced the problem we are trying to solve to solving the exact same problem that we are trying to solve! Yet upon further inspection, it becomes clear that this is in fact an equation involving I, and thus we can solve for it! Proceeding:

$$\begin{aligned} I &= e^{2x}\sin(x) + 2e^{2x}\cos(x) - 4I \\ 5I &= e^{2x}\sin(x) + 2e^{2x}\cos(x) \\ I &= \frac{e^{2x}\sin(x) + 2e^{2x}\cos(x)}{5} \end{aligned}$$

and we are done, concluding that

$$\int e^{2x}\cos(x)\,\mathtt{d}x = \frac{e^{2x}\sin(x) + 2e^{2x}\cos(x)}{5} + C$$

Exercise 2.2.3.2. Carefulness ☕

In the example above, there are two lines labeled with bowties ($\bowtie$). Explain briefly in a sentence or two why those giant parentheses are present. What would go wrong if those parentheses were not there?

A Solution: When IBP is used to evaluate an antiderivative, it is important to distribute the integral's coefficient to both parts of the new expression. If the parentheses weren't there, the coefficients would only be applied to the first expression in each IBP expansion.

Example 2.2.3.3. Another Reappearing IBP Integral

Suppose we wish to find an antiderivative for the function $\tan(x)\sec^2(x)$. We identify $\mathtt{d}v = \sec^2(x)\,\mathtt{d}x$ as having a nice clean antiderivative, which leaves $u = \tan(x)$ by default. Thus we make the following declarations:

Choice of u and $\mathtt{d}v$	
$u = \tan(x)$	$v = \tan(x)$
$\mathtt{d}u = \sec^2(x)\,\mathtt{d}x$	$\mathtt{d}v = \sec^2(x)\,\mathtt{d}x$

We are now ready to calculate the antiderivative via IBP:

$$\int \tan(x)\sec^2(x)\,\mathtt{d}x = \tan(x)\tan(x) - \int \tan(x)\sec^2(x)\,\mathtt{d}x$$

We notice that the original integral has reappeared! We give it the name I and solve. The equation becomes $I = \tan^2(x) - I$, which implies that $2I = \tan^2(x)$. Dividing by two produces the following result:

$$\int \tan(x)\sec^2(x)\,\mathtt{d}x = \frac{1}{2}\tan^2(x) + C$$

Exercise 2.2.3.4. Alternate Solutions ☕☕

Find the antiderivative of $\tan(x)\sec^2(x)$ yet again but by two different methods! In particular, try...

- ...a u-sub with $u = \tan(x)$.

- ...an IBP with $u = \sec(x)$ and $\mathtt{d}v = \tan(x)\sec(x)\,\mathtt{d}x$.

Confirm that your answers match the result of Exercise 2.2.3.3.

Exercise 2.2.3.5. A Tricky but Important One: Secant Cubed ☕☕☕

Find an antiderivative for the function $\sec^3(x)$. (**Hint:** Split the cube as $\sec^3(x) = \sec^2(x)\sec(x)$. Also, the Pythagorean Identity $\tan^2(x) = \sec^2(x) - 1$ will be useful.)

A Solution: Let $I = \int \sec^3(x)\,\mathtt{d}x$ then split the integral:

$$\int \sec^3 x\,\mathtt{d}x = \int \sec x \sec^2 x\,\mathtt{d}x$$

Then, let $u = \sec x$ and $\mathtt{d}v = \sec^2 x$. Then $\mathtt{d}u = \sec x \tan x \, \mathtt{d}x$ and $v = \tan x$.

$$\begin{aligned}
I = \int \sec x \sec^2 x \, \mathtt{d}x &= \sec x \tan x - \int \tan x \cdot \sec x \tan x \, \mathtt{d}x \\
&= \sec x \tan x - \int \tan^2 x \sec x \, \mathtt{d}x \\
&= \sec x \tan x - \int \left(sec^2 x - 1\right) \sec x \, \mathtt{d}x \\
&= \sec x \tan x - \int \sec^3 x - \sec x \, \mathtt{d}x \\
&= \sec x \tan x - \left(\int \sec^3 x \, \mathtt{d}x - \int \sec x \, \mathtt{d}x \right) \\
&= \sec x \tan x - \int \sec^3 x \, \mathtt{d}x + \ln|\sec x + \tan x| + C \\
I &= \sec x \tan x - I + \ln|\sec x + \tan x| + C \\
2I &= \sec x \tan x + \ln|\sec x + \tan x| + C \\
I &= \frac{\sec x \tan x + \ln|\sec x + \tan x|}{2} + C
\end{aligned}$$

2.3 Mixed Practice with Substitution/IBP

Sometimes it is not obvious which technique to use in solving a particular problem. One must often use more than one technique of integration in combination.

Exercise 2.3.0.1. Practice on u-sub and/or IBP ☕☕☕

- Find an antiderivative for the function $\cos(\sqrt{x})$.

A Solution: Start with a u-sub. Let $u = \sqrt{x}$. Then $\mathtt{d}u = \frac{1}{2\sqrt{x}} \mathtt{d}x$ and $\mathtt{d}x = 2\sqrt{x} \, \mathtt{d}u$

$$\begin{aligned}
\int \cos \sqrt{x} \, \mathtt{d}x = \int \cos u \cdot 2u \, \mathtt{d}u &= 2 \int u \cos u \, \mathtt{d}u \\
&= 2\left(u \sin u + \cos u + C\right), \text{ Example 2.2.1.1} \\
&= 2\sqrt{x} \sin\left(\sqrt{x}\right) + 2\cos\left(\sqrt{x}\right) + C
\end{aligned}$$

- Evaluate $\int e^{\sqrt{2x}} \, \mathtt{d}x$.

A Solution: Start with a u-sub. Let $u = \sqrt{2x}$. Then $\mathtt{d}u = \frac{1}{\sqrt{2x}} \mathtt{d}x$ and $\mathtt{d}x = \sqrt{2x} \, \mathtt{d}u$

$$\int e^{\sqrt{2x}} \, \mathtt{d}x = \int e^u \cdot \sqrt{2x} \, \mathtt{d}u = \int u e^u \, \mathtt{d}u$$

Then, proceed with an IBP, using $u = u$, $\mathtt{d}v = e^u\,\mathtt{d}u$, $\mathtt{d}u = \mathtt{d}u$, and $v = e^u$.

$$\begin{aligned}\int ue^u\,\mathtt{d}u &= ue^u - \int e^u\,\mathtt{d}u\\ &= ue^u - e^u + C\\ &= e^u(u-1) + C\\ &= e^{\sqrt{2x}}\left(\sqrt{2x}-1\right) + C\end{aligned}$$

- Evaluate $\int \arcsin(5x)\,\mathtt{d}x$.

A Solution: Start with a u-sub. Let $u = 5x$, $\mathtt{d}u = 5\,\mathtt{d}x$:

$$\int \arcsin(5x)\,\mathtt{d}x = \frac{1}{5}\int \arcsin(u)\,\mathtt{d}u$$

Then, use IBP (we'll use p, q in place of u, v to avoid confusion): Let $p = \arcsin(u)$ and $\mathtt{d}q = \mathtt{d}u$, then $\mathtt{d}p = \frac{1}{\sqrt{1-u^2}}\,\mathtt{d}u$ and $q = u$:

$$\frac{1}{5}\int \arcsin(u)\,\mathtt{d}u = \frac{1}{5}\left(u\arcsin(u) - \int \frac{u}{\sqrt{1-u^2}}\,\mathtt{d}u\right)$$

Then, we apply another u-sub (we will use v to avoid confusion), with $v = 1 - u^2$ and $\mathtt{d}v = -2\,\mathtt{d}u$, so that $\mathtt{d}u = -\frac{\mathtt{d}v}{2u}$

$$\begin{aligned}\frac{1}{5}\left(u\arcsin(u) - \int \frac{u}{\sqrt{1-u^2}}\,\mathtt{d}u\right) &= \frac{1}{5}\left(u\arcsin(u) - \int \frac{u}{\sqrt{v}} \cdot -\frac{\mathtt{d}v}{2u}\right)\\ &= \frac{1}{5}\left(u\arcsin(u) + \frac{1}{2}\int v^{-\frac{1}{2}}\,\mathtt{d}v\right)\end{aligned}$$

Proceeding:

$$\begin{aligned}\frac{1}{5}\left(u\arcsin(u) + \frac{1}{2}\int v^{-\frac{1}{2}}\,\mathtt{d}v\right) &= \frac{1}{5}\left(u\arcsin(u) + \frac{1}{2}\cdot 2v^{\frac{1}{2}} + C\right)\\ &= \frac{1}{5}\left(u\arcsin(u) + v^{\frac{1}{2}} + C\right)\end{aligned}$$

Now we back substitute:

$$\begin{aligned}\frac{1}{5}\left(u\arcsin(u) + v^{\frac{1}{2}} + C\right) &= \frac{1}{5}\left(u\arcsin(u) + \sqrt{1-u^2} + C\right)\\ &= \frac{1}{5}\left(5x\arcsin(5x) + \sqrt{1-(5x)^2} + C\right)\\ &= x\arcsin(5x) + \frac{1}{5}\sqrt{1-(5x)^2} + C\end{aligned}$$

- Evaluate $\int e^{2x}\sin(2x)\,\mathtt{d}x$.

A Solution: u-sub: $u = 2x$, $\mathtt{d}u = 5\,\mathtt{d}x$:

$$\int e^{2x}\sin(2x)\,\mathtt{d}x = \frac{1}{2}\int e^{u}\sin(u)\,\mathtt{d}u$$

IBP with $p = \sin(u)$ and $\mathtt{d}q = e^{u}\,\mathtt{d}u$, then $\mathtt{d}p = \cos(u)\,\mathtt{d}u$ and $q = e^{u}$:

$$\frac{1}{2}\int e^{u}\sin(u)\,\mathtt{d}u = \frac{1}{2}\left(e^{u}\sin(u) - \int e^{u}\cos(u)\,\mathtt{d}u\right) \qquad (\bowtie)$$

Apply a second IBP with $p = \cos(u)$ and $\mathtt{d}q = e^{u}\,\mathtt{d}u$, then $\mathtt{d}p = -\sin(u)\,\mathtt{d}u$ and $q = e^{u}$

$$\frac{1}{2}\left(e^{u}\sin(u) - \int e^{u}\cos(u)\,\mathtt{d}u\right) = \frac{1}{2}\left(e^{u}\sin(u) - \left(e^{u}\cos(u) + \int e^{u}\sin(u)\,\mathtt{d}u\right)\right)$$

Notice that $\int e^{u}\sin(u)\,\mathtt{d}u$ is the LHS from $\bowtie$. So we can set $\int e^{u}\sin(u)\,\mathtt{d}u = I$ and equate the last step with $\bowtie$.

$$\begin{aligned}
\frac{1}{2}\int e^{u}\sin(u)\,\mathtt{d}u = \frac{1}{2}I &= \frac{1}{2}\left(e^{u}\sin(u) - \left(e^{u}\cos(u) + I\right)\right)\\
\frac{1}{2}I &= \frac{1}{2}e^{u}\sin(u) - \frac{1}{2}e^{u}\cos(u) - \frac{1}{2}I\\
I &= \frac{1}{2}e^{u}\sin(u) - \frac{1}{2}e^{u}\cos(u)\\
\int e^{u}\sin(u)\,\mathtt{d}u &= \frac{1}{2}e^{u}\sin(u) - \frac{1}{2}e^{u}\cos(u) + C\\
2\int e^{2x}\sin(2x)\,\mathtt{d}u &= \frac{1}{2}e^{2x}\left(\sin(2x) - \cos(2x)\right) + C, \text{ Remember } \mathtt{d}u = 2\,\mathtt{d}x\\
\int e^{2x}\sin(2x)\,\mathtt{d}2x &= \frac{1}{4}e^{2x}\left(\sin(2x) - \cos(2x)\right) + C
\end{aligned}$$

Exercise 2.3.0.2. Who is u vs Who is $\mathtt{d}v$?

Suppose we wish to find an antiderivative for the function $x^{2.5}\ln(x)$. There are two natural choices for u. We can let $u = x^{2.5}$ and $\mathtt{d}v = \ln(x)\,\mathtt{d}x$, or we can let can let $u = \ln(x)$ and $\mathtt{d}v = x^{2.5}\,\mathtt{d}x$.

- Apply just the first step of IBP with $u = x^{2.5}$ and $\mathtt{d}v = \ln(x)\,\mathtt{d}x$.

$$\int x^{2.5}\ln(x)\,\mathtt{d}x =$$

A Solution:

$$\int x^{2.5}\ln(x)\,\mathtt{d}x = x^{2.5}(x\ln(x)-x) - \int (x\ln(x)-x)\left(2.5x^{1.5}\right)\mathtt{d}x$$

- Apply just the first step of IBP with $u=\ln(x)$ and $\mathtt{d}v = x^{2.5}\,\mathtt{d}x$.

$$\int x^{2.5}\ln(x)\,\mathtt{d}x =$$

A Solution:

$$\int x^{2.5}\ln(x)\,\mathtt{d}x = \frac{x^{3.5}}{3.5}\ln(x) - \frac{1}{3.5}\int x^{2.5}\,\mathtt{d}x$$

- Write a short explanation regarding which choice of u will be easier to use to evaluate the integral and why.

A Solution: Choosing $u=\ln(x)$ will make the logarithm disappear upon differentiation, so all we have to evaluate is the integral of a power of x. The opposite choice will not clean up the log.

- Carry out the integral using whichever choice you decided was easier.

$$\int x^{2.5}\ln(x)\,\mathtt{d}x =$$

A Solution:

$$\frac{x^{3.5}}{3.5}\ln(x) - \frac{1}{3.5}\int x^{2.5}\,\mathtt{d}x = \frac{x^{3.5}}{3.5}\ln(x) - \frac{1}{3.5}\cdot\frac{1}{3.5}x^{3.5} = \frac{x^{3.5}}{3.5}\left(\ln(x) - \frac{1}{3.5}\right) + C$$

- Differentiate your answer to check that your antiderivative is correct.

A Solution:

$$\begin{aligned}\frac{\mathtt{d}}{\mathtt{d}x}\left(\frac{x^{3.5}}{3.5}\left(\ln(x)-\frac{1}{3.5}\right)+C\right) &= x^{2.5}\left(\ln(x)-\frac{1}{3.5}\right) + \frac{x^{3.5}}{3.5}\left(\frac{1}{x}\right)\\ &= x^{2.5}\ln(x) - \frac{x^{2.5}}{3.5} + \frac{x^{2.5}}{3.5}\\ &= x^{2.5}\ln(x)\end{aligned}$$

2.4 Integrating Products of Powers of Sine and Cosine

In this section, we give an algorithm to find an antiderivative of the form

$$\int \sin^n(x)\cos^m(x)\,\mathtt{d}x$$

for $n, m \in \mathbb{N}$.

Exercise 2.4.0.1. Knowledge is Power ☕

There are two exponents in the integrand above.

- What symbol above is the exponent of sine?
- What symbol above is the exponent of cosine?

Note that some sine-cosine integrals can be done by techniques you have already learned. For example, n or m is equal to 1, ordinary u-substitution will work just fine!

Exercise 2.4.0.2. u-sub with Sines and Cosines ☕☕

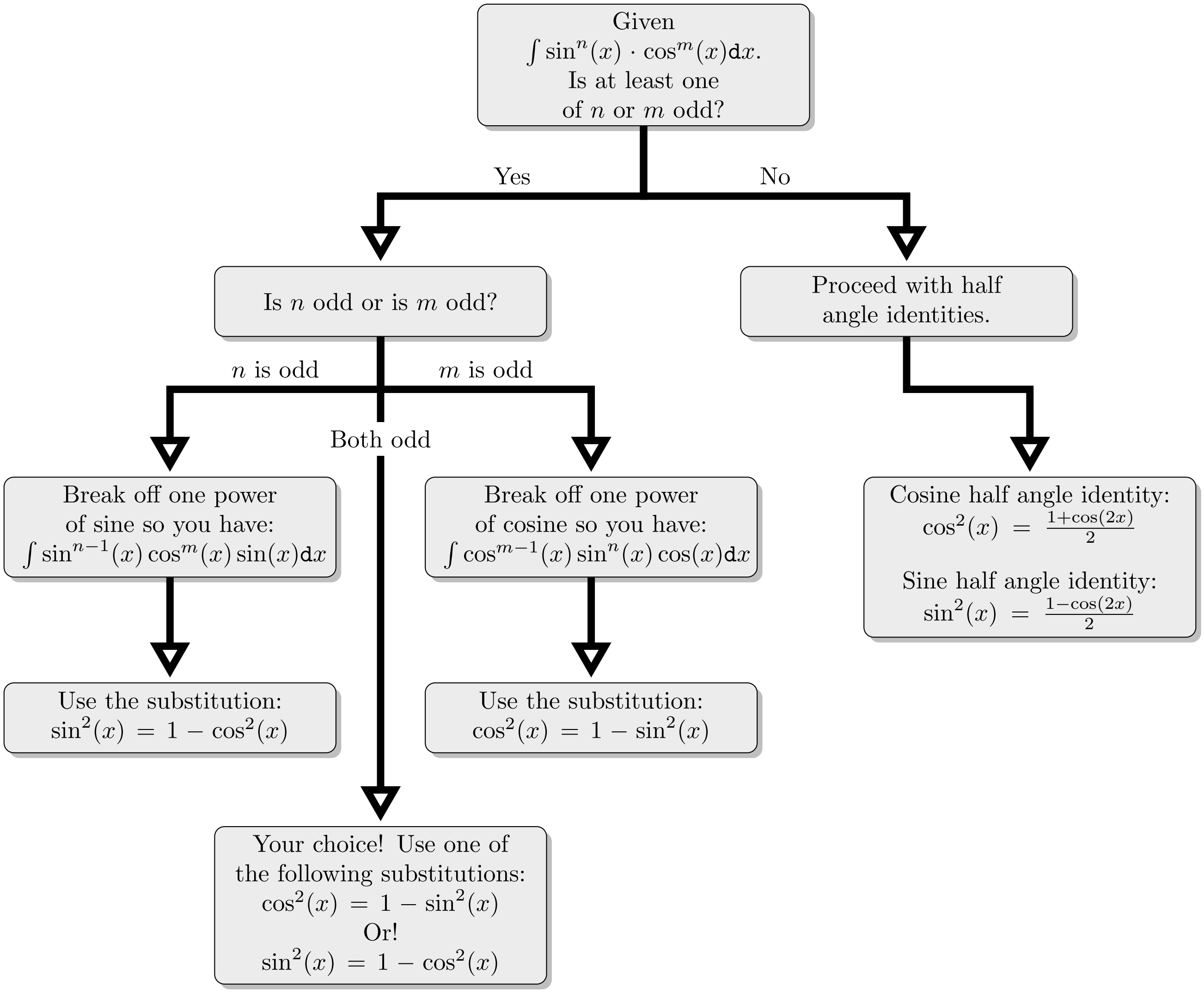
Evaluate the following integral using the substitution $u = \sin(x)$:

$$\int \sin^2(x)\cos(x)\,\mathtt{d}x$$

A Solution: Let $u = \sin(x)$, $\mathtt{d}u = \cos(x)$. Then:

$$\begin{aligned}\int \sin^2(x)\cos(x)\,\mathtt{d}x &= \int u^2\,\mathtt{d}u\\ &= \frac{1}{3}u^3 + C\\ &= \frac{1}{3}\sin^3(x) + C\end{aligned}$$

There are two types of integrals containing powers of sine and cosine. The first type is the case where we have at least one odd exponent; the second type is where both exponents are even. We show an overview of how to handle each case in the following awesome flow chart:



2.4.1 At Least One Odd Power

Recall the Pythagorean identity for sine and cosine (written in two useful forms here):

Pythagorean Theorem Slightly Rewritten	
$\cos^2(x) = 1 - \sin^2(x)$	$\sin^2(x) = 1 - \cos^2(x)$

If at least one exponent is odd, we pull one of those functions out for the "$\mathtt{d}u$" and perform u-sub. We then use the Pythagorean trig identity to rewrite sine and cosine in terms of each other as needed.

Example 2.4.1.1. Odd Power Case

Here we compute the integral

$$\int \sin^7(x) \cos^2(x) \, \mathtt{d}x$$

. In this case, we proceed using the substitution $u = \cos(x)$, so $\mathtt{d}x = \frac{1}{-\sin(x)} \, \mathtt{d}u$.

$$\begin{aligned}
\int \sin^7(x)\cos^2(x)\,\mathtt{d}x &= \int \sin^6(x)\cos^2(x)\sin(x)\,\mathtt{d}x \\
&= \int \left(\sin^2(x)\right)^3 \cos^2(x)\sin(x)\frac{1}{-\sin(x)}\,\mathtt{d}u \\
&= \int \left(1-\cos^2(x)\right)^3 \cos^2(x)(-1)\,\mathtt{d}u \\
&= -\int \left(1-u^2\right)^3 u^2\,\mathtt{d}u \\
&= -\int \left(1-3u^2+3u^4-u^6\right) u^2\,\mathtt{d}u \\
&= -\int \left(u^2-3u^4+3u^6-u^8\right)\mathtt{d}u \\
&= -\left(\frac{1}{3}u^3-\frac{3}{5}u^5+\frac{3}{7}u^7-\frac{1}{9}u^9\right)+C \\
&= -\frac{1}{3}\cos^3(x)+\frac{3}{5}\cos^5(x)-\frac{3}{7}\cos^7(x)+\frac{1}{9}\cos^9(x)+C
\end{aligned}$$

Exercise 2.4.1.2. Why Odd Mattered ☕

In Example 2.4.1.1, the exponent of sine (in this case, the number 7) being odd really mattered. If that 7 were replaced by an even number instead, why would this approach have failed? Answer in a few short sentences below.

A Solution: If the exponent of sine had been even, then we couldn't have used the Pythagorean identity to express it in terms of cosine, and still had an extra sine to use with the u-sub, which would have prevented us from expressing all the parts in terms of a single expression.

Exercise 2.4.1.3. Try a Few with Odd Exponents ☕☕

- Find an antiderivative for the function $\sin^5(x)\cos^2(x)$.

A Solution:

$$\begin{aligned}
\int \sin^5(x)\cos^2(x)\,\mathtt{d}x &= \int \sin^4(x)\cos^2(x)\sin(x)\,\mathtt{d}x \\
&= \int \left(\sin^2(x)\right)^2 \cos^2(x)\sin(x)\,\mathtt{d}x \\
&= \int \left(1-\cos^2(x)\right)^2 \cos^2(x)\sin(x)\,\mathtt{d}x
\end{aligned}$$

Let $u = \cos(x)$ so $\mathtt{d}u = -\sin(x)$ and $\mathtt{d}x = \frac{1}{-\sin(x)}\,\mathtt{d}u$

$$\begin{aligned}
\int \left(1-\cos^2(x)\right)^2 \cos^2(x)\sin(x)\,\mathtt{d}x &= \int \left(1-u^2\right)^2 u^2 \sin(x)\frac{1}{-\sin(x)}\,\mathtt{d}u \\
&= \int \left(1-u^2\right)^2 u^2(-1)\,\mathtt{d}u \\
&= -\int \left(1-2u^2+u^4\right) u^2\,\mathtt{d}u \\
&= -\int u^2-2u^4+u^6\,\mathtt{d}u \\
&= -\left(\frac{1}{3}u^3-\frac{2}{5}u^5+\frac{1}{7}u^7+C\right) \\
&= -\frac{1}{3}\cos^3(x)+\frac{2}{5}\cos^5(x)-\frac{1}{7}\cos^7(x)+C
\end{aligned}$$

- Evaluate $\int \cos^9(x)\,\mathtt{d}x$. (**Hint:** Pascal's Triangle will be extremely helpful!)

A Solution:

$$\begin{aligned}
\int \cos^9(x)\,\mathtt{d}x &= \int \cos^8(x)\cos(x)\,\mathtt{d}x \\
&= \int (1-\sin^2(x))^4\cos(x)\,\mathtt{d}x \\
&\text{Let } u=\sin(x) \\
&= \int (1-u^2)^4\,\mathtt{d}u \\
&= \int 1-4u^2+6u^2-4u^6+u^8\,\mathtt{d}u \\
&= u-\frac{4}{3}u^3+\frac{6}{3}u^3-\frac{4}{7}u^7+\frac{1}{9}u^9+C \\
&= \sin(x)-\frac{4}{3}\sin^3(x)+\frac{6}{3}\sin^3(x)-\frac{4}{7}\sin^7(x)+\frac{1}{9}\sin^9(x)+C
\end{aligned}$$

Exercise 2.4.1.4. Two Different Options ☕☕

- Consider $\int \cos(x)\sin^3(x)\,\mathtt{d}x$.
  - Compute this integral using $u = \cos(x)$.

A Solution:

$$\begin{aligned}
\int \cos(x)\sin^3(x)\,\mathtt{d}x &= \int \cos(x)\left(1-\cos^2(x)\right)\sin(x)\,\mathtt{d}x \\
&= -\int u\left(1-u^2\right)\mathtt{d}u \\
&= -\int u - u^3\,\mathtt{d}u \\
&= -\left(\frac{1}{2}u^2 - \frac{1}{4}u^4\right) + C \\
&= -\frac{1}{2}\cos^2(x) + \frac{1}{4}\cos^4(x) + C
\end{aligned}$$

– Compute this integral using $u = \sin(x)$.

A Solution:

$$\begin{aligned}
\int \cos(x)\sin^3(x)\,\mathtt{d}x &= \int u^3\,\mathtt{d}u \\
&= \frac{1}{4}\sin^4(x) + C
\end{aligned}$$

– Your two answers will appear very different! Show that they are in fact compatible.

A Solution:

$$\begin{aligned}
-\frac{1}{2}\cos^2(x) + \frac{1}{4}\cos^4(x) + C &= -\frac{1}{2}(1-\sin^2(x)) + \frac{1}{4}(1-\sin^2(x))^2 + C \\
&= -\frac{1}{2} + \frac{1}{2}\sin^2(x) + \frac{1}{4}(1 - 2\sin^2(x) + \sin^4(x)) + C \\
&= -\frac{1}{2} + \frac{1}{2}\sin^2(x) + \frac{1}{4} - \frac{1}{2}\sin^2(x) + \frac{1}{4}\sin^4(x) + C \\
&= -\frac{1}{4} + \frac{1}{4}\sin^4(x) + C \\
&-\frac{1}{4} \text{ is absorbed into } C \text{ leaving} \\
&= \frac{1}{4}\sin^4(x) + C
\end{aligned}$$

- Consider $\int \cos^3(x)\sin^{11}(x)\,\mathtt{d}x$.
  - Can you compute this integral using $u = \cos(x)$? Explain.

A Solution: Taking $u = \cos(x)$, we can express the integral as $\int \cos^3(x)\left(1-\cos^2(x)\right)^5 \cos(x)\,\mathtt{d}x = \int u^3\left(1-u^2\right)^5 \mathtt{d}u$

– Can you compute this integral using $u = \sin(x)$? Explain.

A Solution: We can express the integral as $\left(1-sin^2(x)\right)\sin^1 1(x)\cos(x)\,\mathtt{d}x = \int\left(1-u^2\right)\left(u^{11}\right)\mathtt{d}x$

– Which of the two above substitutions will be easier to use? Carry out the integration, using the easier of the two.

A Solution: The second option, $u = \sin(x)$ will be easier because it avoids expading a binomial to the 5th power. Let $u = \sin(x)$

$$\begin{aligned}\int \cos^3(x)\sin^{11}(x)\,\mathtt{d}x &= \left(1-u^2\right)u^{11}\,\mathtt{d}u\\ &= \int u^{11} - u^{13}\,\mathtt{d}u\\ &= \frac{1}{12}u^{12} - \frac{1}{14}u^{14} + C\\ &= \frac{1}{12}\sin^{12}(x) - \frac{1}{14}\sin^{14}(x) + C\end{aligned}$$

2.4.2 Both Even Powers

Recall the Half-Angle Identities!

Half-Angle Identities	
$\cos^2(x) = \frac{1+\cos(2x)}{2}$	$\sin^2(x) = \frac{1-\cos(2x)}{2}$

If the powers of sine and cosine are both even, we use the half-angle identities for both sine and cosine. This can get quite messy, but it works!

Exercise 2.4.2.1. Just Cosines without Sine ☕

Consider the following integral:

$$\int \cos^6(x)\,\mathtt{d}x$$

Here the exponent on cosine is the even number 6. What is the exponent of sine in that integrand? Is that an even number?

A Solution: The exponent on sine is zero, which is indeed even. Thus both exponents are even in this case.

Example 2.4.2.2. Carrying Out Antidifferentiation with the Half-Angle Identities

We now show how the half-angle identities help antidifferentiate the sixth power of cosine.

$$\begin{aligned}\int \cos^6(x)\,\mathtt{d}x &= \int \left(\cos^2(x)\right)^3 \mathtt{d}x \\ &= \int \left(\frac{1+\cos(2x)}{2}\right)^3 \mathtt{d}x \\ &= \frac{1}{8}\int 1 + 3\cos(2x) + 3\cos^2(2x) + \cos^3(2x)\,\mathtt{d}x \\ &= \frac{1}{8}\left(\int 1\,\mathtt{d}x + \int 3\cos(2x)\,\mathtt{d}x + \int 3\cos^2(2x)\,\mathtt{d}x + \int \cos^3(2x)\,\mathtt{d}x\right)\end{aligned}$$

Notice that we now have four integrals. The first is easy, the second is a u-substitution, and the third is another even power of cosine (where we again use the half-angle identity). Finally, the fourth is an odd power of cosine, so we can use the technique from Section 2.4.1.

Exercise 2.4.2.3. Finishing the Example ☕☕

Carry out each of these processes to compute the four integrals:

- $\int 1\,\mathtt{d}x$

A Solution:

$$\int 1\,\mathtt{d}x = x + C$$

- $\int 3\cos(2x)\,\mathtt{d}x$

A Solution: Let $u = 2x$. Then,

$$\int 3\cos 2x\,\mathtt{d}x = \frac{3}{2}\int \cos u\,\mathtt{d}u = \frac{3}{2}\sin 2x + C$$

- $\int 3\cos^2(2x)\,\mathtt{d}x$

A Solution:

$$\int 3\cos^2(2x)\,\mathtt{d}x = 3\int \frac{1+\cos 4x}{2}\,\mathtt{d}x = 3\left(\frac{1}{2}x + \frac{1}{8}\sin 4x + C\right) = \frac{3}{2}x + \frac{3}{8}\sin 4x + C$$

- $\int \cos^3(2x)\,\mathtt{d}x$

A Solution: Let $u = \sin 2x$ and $\mathtt{d}u = 2\cos 2x\,\mathtt{d}x$. Then,

$$\begin{aligned}\int \cos^3(2x)\,\mathtt{d}x &= \int \cos 2x\left(1-\sin^2 2x\right)\mathtt{d}x\\ &= \frac{1}{2}\int 1-u^2\,\mathtt{d}u\\ &= \frac{1}{2}u-\frac{1}{6}u^3+C\\ &= \frac{1}{2}\sin 2x-\frac{1}{6}\sin^3 2x+C\end{aligned}$$

Add your antiderivatives together and combine like terms to produce your final answer for the integral! Oh and remember that one-eighth.

$$\int \cos^6(x)\,\mathtt{d}x =$$

A Solution:

$$\begin{aligned}&\frac{1}{8}\left(x+\frac{3}{2}\sin 2x+\frac{3}{2}x+\frac{3}{8}\sin 4x+\frac{1}{2}\sin 2x-\frac{1}{6}\sin^3 2x+C\right)\\ &=\frac{1}{8}\left(\frac{5}{2}x+2\sin 2x+\frac{3}{8}\sin 4x-\frac{1}{6}\sin^3 2x+C\right)\\ &=\frac{5}{16}x+\frac{1}{4}\sin(2x)-\frac{1}{48}\sin^3(2x)+\frac{3}{64}\sin(4x)+C\end{aligned}$$

Exercise 2.4.2.4. Checking the Previous Example ☕☕☕

Differentiate your answer and verify you get the original integrand back.

A Solution:

$$\frac{\mathtt{d}}{\mathtt{d}x}\left(\frac{5}{16}x+\frac{1}{4}\sin(2x)-\frac{1}{48}\sin^3(2x)+\frac{3}{64}\sin(4x)+C\right)$$

Before we differentiate, first bash everything back down to an "x" in the argument using double angle identities. This produces

$$\frac{5}{16}x+\frac{1}{2}\sin(x)\cos(x)-\frac{1}{6}\sin^3(x)\cos^3(x)+\frac{3}{16}\sin(x)\cos^3(x)-\frac{3}{16}\sin^3(x)\cos(x)+C$$

Factor out a sine and use the Pythagorean Identity to get everything else in terms of cosine. This produces

$$\frac{5}{16}x + \sin(x)\left(\frac{5}{16}\cos(x) + \frac{5}{24}\cos^3(x) + \frac{1}{6}\cos^5(x)\right) + C$$

Then we differentiate and obtain

$$\frac{5}{16} + \cos(x)\left(\frac{5}{16}\cos(x) + \frac{5}{24}\cos^3(x) + \frac{1}{6}\cos^5(x)\right) - \sin^2(x)\left(\frac{5}{16} + \frac{5}{8}\cos^2(x) + \frac{5}{6}\cos^4(x)\right)$$

to which we apply the Pythagorean Identity $\sin^2(x) = 1 - \cos^2(x)$ to produce

$$\frac{5}{16} + \cos(x)\left(\frac{5}{16}\cos(x) + \frac{5}{24}\cos^3(x) + \frac{1}{6}\cos^5(x)\right) - \left(1 - \cos^2(x)\right)\left(\frac{5}{16} + \frac{5}{8}\cos^2(x) + \frac{5}{6}\cos^4(x)\right)$$

This will simplify to $\cos^6(x)$ once you expand and combine like terms.

Exercise 2.4.2.5. Practice with the Even Case ☕☕

- Find an antiderivative for the function $\sin^2(3x)$.

A Solution:

$$\begin{aligned}
\int \sin^2(3x) &= \int \frac{1-\cos(6x)}{2}\,\mathtt{d}x \\
&= \int \frac{1}{2} - \frac{1}{2}\cos(6x)\,\mathtt{d}x \\
\text{Let } u = 6x, \mathtt{d}u = 6\,\mathtt{d}x \quad & \\
&= \frac{1}{2}x - \frac{1}{2}\cdot\frac{1}{6}\int \cos u\,\mathtt{d}u \\
&= \frac{1}{2}x - \frac{1}{12}\sin u + C \\
&= \frac{1}{2}x - \frac{1}{12}\sin 6x + C
\end{aligned}$$

- Find an antiderivative for the function $\sin^4(x)$.

A Solution:

$$\begin{aligned}
\int \sin^4(x)\,\mathtt{d}x &= \int \left(\sin^2(x)\right)^2 \mathtt{d}x \\
&= \int \left(\frac{1-\cos(2x)}{2}\right)^2 \mathtt{d}x \\
&= \int \frac{1-2\cos(2x)+cos^2(2x)}{4}\,\mathtt{d}x \\
&= \int \frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{4}\cdot\frac{1+\cos(4x)}{2}\,\mathtt{d}x \\
&= \int \frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{8} + \frac{1}{8}\cos(4x)\,\mathtt{d}x \\
&= \int \frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)\,\mathtt{d}x \\
&= \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C
\end{aligned}$$

- Find an antiderivative for the function $\sin^2(x)\cos^2(x)$.

A Solution:

$$\begin{aligned}
\int \sin^2(x)\cos^2(x)\,\mathtt{d}x &= \int \sin^2(x)\left(1-\sin^2(x)\right)\mathtt{d}x \\
&= \int \sin^2(x) - \sin^4(x)\,\mathtt{d}x
\end{aligned}$$

From previous examples:

$$\begin{aligned}
\int \sin^2(x)\,\mathtt{d}x &= \frac{1}{2}x - \frac{1}{2}\sin x + C \\
\int \sin^4(x)\,\mathtt{d}x &= \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C \\
\int \sin^2(x) - \sin^4(x)\,\mathtt{d}x &= \frac{1}{2}x - \frac{1}{2}\sin x - \left(\frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x)\right) + C \\
&= \frac{1}{2}x - \frac{1}{2}\sin x - \frac{3}{8}x + \frac{1}{4}\sin(2x) - \frac{1}{32}\sin(4x) + C \\
&= \frac{1}{8}x - \frac{1}{2}\sin x + \frac{1}{4}\sin(2x) - \frac{1}{32}\sin(4x) + C
\end{aligned}$$

2.5 Trigonometric Substitution

Though in theory you could use any trigonometric function, the three commonly used trigonometric substitutions are sine, tangent, and secant. The substitutions are motivated by the Pythagorean Identities from trigonometry.

Exercise 2.5.0.1. Recalling the Pythagorean Identities ☕

- Start with the Pythagorean Identity for sine and cosine:

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

- Subtract $\sin^2(\theta)$ from both sides. Write the resulting equation below.

A Solution:

$$\cos^2(\theta) = 1 - \sin^2(\theta)$$

- Again, start with the Pythagorean Identity for sine and cosine. What would we have to divide both sides by in order to get the corresponding identity for tangent and secant (written below)?

$$1 + \tan^2(\theta) = \sec^2(\theta)$$

A Solution: Divide everything by $\cos^2(\theta)$

$$\frac{\cos^2(\theta)}{\cos^2(\theta)} + \frac{\sin^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$$

$$1 + \tan^2(\theta) = \sec^2(\theta)$$

- How would you then obtain the identity below?

$$\sec^2(\theta) - 1 = \tan^2(\theta)$$

A Solution: Subtract 1 from both sides.

You can take any of the identities above and multiply both sides by a^2 (where a represents an arbitrary positive real constant) to produce a more general identity. This is what results in the commonly used trigonometric substitutions for integrals, summarized in the table below.

Trigonometric Substitutions

If you see...	...make the substitution...	...because...
$a^2 - x^2$	$x = a \sin(\theta)$	$a^2 - a^2 \sin^2(\theta) = a^2 \cos^2(\theta)$
$a^2 + x^2$	$x = a \tan(\theta)$	$a^2 + a^2 \tan^2(\theta) = a^2 \sec^2(\theta)$
$x^2 - a^2$	$x = a \sec(\theta)$	$a^2 \sec^2(\theta) - a^2 = a^2 \tan^2(\theta)$

Exercise 2.5.0.2. Why Only Three Cases?

In the table above, we have cases for how to clean up expressions of the form $a^2 - x^2$, $a^2 + x^2$, and $x^2 - a^2$. Why is there not a fourth case for $x^2 + a^2$?

A Solution: The case $x^2 + a^2$ is equivalent to the case $a^2 + x^2$, so we can rearrange the expression and use the substitution $x = a\tan(\theta)$

2.5.1 Sine Substitution

When we see an expression of the form $a^2 - x^2$ in the integrand, we think of the identity $1 - \sin^2(\theta) = \cos^2(\theta)$. This motivates the following substitution:

Sine Substitution
$x = a \cdot \sin(\theta)$

The next example will require use of the Double-Angle Identities for sine and cosine. We recall these before we dive in!

Exercise 2.5.1.1. Recalling the Double-Angle Formulas

- The double-angle formula for sine is $\sin(2\theta) =$

  A Solution: $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

- The double-angle formula for cosine is $\cos(2\theta) =$

  A Solution: $\cos(2\theta) = \cos^2(\theta) - \cos^2(\theta)$

- What do you get if you apply the sine double-angle identity to $\sin(4\theta)$? Specifically, think of $\sin(4\theta)$ as $\sin(2 \cdot 2\theta)$.

A Solution: $\sin(4\theta) = 2\sin(2\theta)\cos(2\theta)$

We now put our sine substitution to use to evaluate an antiderivative!

Example 2.5.1.2. Using a Sine Substitution

Suppose we wish to evaluate

$$\int \left(4 - x^2\right)^{3/2} \mathtt{d}x$$

We use the substitution suggested above, specifically

$$x = 2 \cdot \sin(\theta).$$

We then differentiate both sides to find the conversion between the differentials and then multiply both sides by $\mathtt{d}\theta$:

$$\frac{\mathtt{d}x}{\mathtt{d}\theta} = 2 \cdot \cos(\theta)$$

We now use the above equations to substitute for x and $\mathtt{d}x$ in the integral:

$$\begin{aligned}
\int \left(4 - x^2\right)^{3/2} \mathtt{d}x &= \int \left(4 - (2 \cdot \sin(\theta))^2\right)^{3/2} 2 \cdot \cos(\theta)\mathtt{d}\theta \\
&= 2 \int \left(4 - 4 \cdot \sin^2(\theta)\right)^{3/2} \cos(\theta)\mathtt{d}\theta \\
&= 2 \int \left(4\left(1 - \sin^2(\theta)\right)\right)^{3/2} \cos(\theta)\mathtt{d}\theta \\
&= 2 \int \left(4\left(\cos^2(\theta)\right)\right)^{3/2} \cos(\theta)\mathtt{d}\theta \\
&= 2 \int (4)^{3/2} \left(\cos^2(\theta)\right)^{3/2} \cos(\theta)\mathtt{d}\theta \\
&= 16 \int \cos^3(\theta) \cdot \cos(\theta)\mathtt{d}\theta \\
&= 16 \int \cos^4(\theta)\mathtt{d}\theta
\end{aligned}$$

Recall the previous section where we learned how to antidifferentiate even powers of sine and cosine! Accordingly, we use the half-angle identities.

$$
\begin{aligned}
\int \left(4-x^2\right)^{3/2} \mathtt{d}x &= 16 \int \left(\cos^2(\theta)\right)^2 \mathtt{d}\theta \\
&= 16 \int \left(\frac{1+\cos(2\theta)}{2}\right)^2 \mathtt{d}\theta \\
&= 16 \int \frac{1+2\cdot\cos(2\theta)+\cos^2(2\theta)}{4} \mathtt{d}\theta \\
&= 4 \int 1+2\cdot\cos(2\theta)+\cos^2(2\theta) \mathtt{d}\theta \\
&= 4 \int 1+2\cdot\cos(2\theta)+\frac{1+\cos(4\theta)}{2} \mathtt{d}\theta \\
&= 4 \int \frac{3}{2}+2\cdot\cos(2\theta)+\frac{1}{2}\cos(4\theta) \mathtt{d}\theta \\
&= 4\left(\frac{3}{2}\theta+\sin(2\theta)+\frac{1}{8}\sin(4\theta)\right)+C \\
&= 6\theta+4\cdot\sin(2\theta)+\frac{1}{2}\sin(4\theta)+C
\end{aligned}
$$

We have successfully taken the antiderivative! However, it still remains to unwind the trigonometric substitution back in terms of x rather than θ. Our original substitution argument is θ, whereas currently we have 2θ and 4θ as arguments. In order to resolve this, we use the sine and cosine double angle formulas and the Pythagorean identity. Proceeding:

$$
\begin{aligned}
\int \left(4-x^2\right)^{3/2} \mathtt{d}x &= 6\theta+4\cdot\sin(2\theta)+\frac{1}{2}\sin(4\theta)+C \\
&= 6\theta+4\cdot 2\cdot\sin(\theta)\cos(\theta)+\sin(2\theta)\cos(2\theta)+C \\
&= 6\theta+4\cdot 2\cdot\sin(\theta)\cos(\theta)+2\cdot\sin(\theta)\cos(\theta)\left(\cos^2(\theta)-\sin^2(\theta)\right)+C \\
&= 6\theta+8\cdot\sin(\theta)\sqrt{1-\sin^2(\theta)}+2\cdot\sin(\theta)\sqrt{1-\sin^2(\theta)}\left(1-2\sin^2(\theta)\right)+C \\
&= 6\cdot\arcsin\left(\frac{x}{2}\right)+8\frac{x}{2}\sqrt{1-\frac{x^2}{4}}+2\frac{x}{2}\sqrt{1-\frac{x^2}{4}}\left(1-2\frac{x^2}{4}\right)+C \\
&= 6\cdot\arcsin\left(\frac{x}{2}\right)+4x\sqrt{1-\frac{x^2}{4}}+\sqrt{1-\frac{x^2}{4}}\left(x-\frac{x^3}{2}\right)+C
\end{aligned}
$$

Exercise 2.5.1.3. Checking Our Work ☕☕☕

Verify the result of the previous example by differentiating!

A Solution: First apply all the product and chain rules to reach the expression

$$
\frac{3}{\sqrt{1-\frac{x^2}{4}}}+4\sqrt{1-\frac{x^2}{4}}+\frac{-x^2}{\sqrt{1-\frac{x^2}{4}}}+\sqrt{1-\frac{x^2}{4}}\left(1-\frac{3}{2}x^2\right)+\frac{-x}{4\sqrt{1-\frac{x^2}{4}}}\left(x-\frac{x^3}{2}\right)
$$

Put all terms over the common denominator $\sqrt{4-x^2}$ and combine like terms in the numerator.

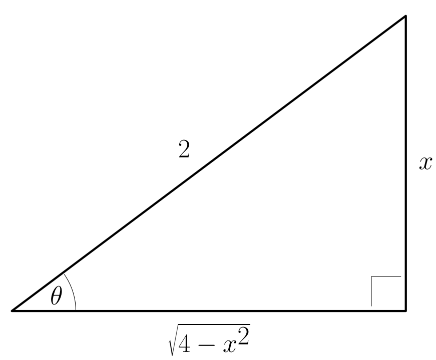
$$
\begin{aligned}
&= \frac{3\cdot 2}{\sqrt{4-x^2}} + \frac{2\cdot 4\left(1-\frac{x^2}{4}\right)-2x^2}{\sqrt{4-x^2}} + \frac{4(1-\frac{x^2}{4})\left(1-\frac{3}{2}x^2\right)-x\left(x-\frac{x^3}{2}\right)}{4\sqrt{1-\frac{x^2}{4}}}\\
&= \frac{6}{\sqrt{4-x^2}} + \frac{8-2x^2-2x^2}{\sqrt{4-x^2}} + \frac{4-6x^2-x^2+\frac{3}{2}x^4-x^2+\frac{x^4}{2}}{4\sqrt{1-\frac{x^2}{4}}}\\
&= \frac{6}{\sqrt{4-x^2}} + \frac{8-4x^2}{\sqrt{4-x^2}} + \frac{2-4x^2+x^4}{\sqrt{4-x^2}}\\
&= \frac{16-8x^2+x^4}{\sqrt{4-x^2}}\\
&= \frac{\left(4-x^2\right)^2}{\sqrt{4-x^2}}\\
&= \left(4-x^2\right)^{\frac{3}{2}}
\end{aligned}
$$

An Alternate Approach

In the above example, we made it back from θ to x by just bashing it to bits with trig identities. Sometimes a cleaner approach can be to use a little geometry. Since we had the substitution $x = 2\sin(\theta)$, we can divide both sides by 2 to obtain the following:

$$\sin(\theta) = \frac{x}{2}$$

Since sine is the ratio of the opposite side to the hypotenuse in a right triangle, we can label the opposite side as x and the hypotenuse as 2.



This makes it easier to know what to substitute for other trig functions of θ. For example, cosine is the ratio of the adjacent side length to the hypotenuse, so we have:

$$\cos(\theta) = \frac{\sqrt{4-x^2}}{2}$$

Exercise 2.5.1.4. Try One on your own! ☕☕☕

Evaluate the following antiderivative:

$$\int x^3 \left(16 - x^2\right)^{5/2} \,\mathtt{d}x$$

(**Hint:** Recall our methods for integrating powers of sines and cosines!)

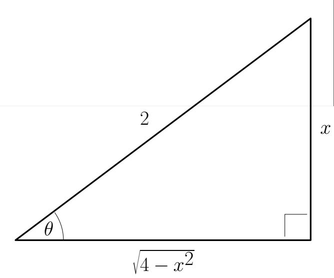
A Solution: We have an expression of the form $a^2 - x^2$, so we can use a sine substitution and let $x = 4\sin(\theta), \mathtt{d}x = 4\cos(\theta)\,\mathtt{d}\theta$.

$$\begin{aligned}
\int x^3 \left(16 - x^2\right)^{\frac{5}{2}} \mathtt{d}x &= \int \left(4\sin(\theta)\right)^3 \left(16 - 16\sin^2(\theta)\right)^{\frac{5}{2}} 4\cos(\theta)\,\mathtt{d}\theta \\
&= \int 4^3 \sin^3(\theta) \left(16\left(1 - \sin^2(\theta)\right)\right)^{\frac{5}{2}} 4\cos(\theta)\,\mathtt{d}\theta \\
&= \int 4^3 \sin^3(\theta) \cdot 4^5 \left(\cos^2(\theta)\right)^{\frac{5}{2}} \cdot 4\cos(\theta)\,\mathtt{d}\theta \\
&= 4^9 \int \sin^3(\theta) \cdot \cos^5(\theta) \cdot \cos(\theta)\,\mathtt{d}\theta \\
&= 4^9 \int \sin^3(\theta) \cos^6(\theta)\,\mathtt{d}\theta
\end{aligned}$$

Now we can use our tools for integrating powers of sine and cosine, with $u = \cos(\theta)$ and $\mathtt{d}u = -\sin(\theta)$.

$$\begin{aligned}
4^9 \int \sin^3(\theta)\cos^6(\theta)\,\mathtt{d}\theta &= 4^9 \int \left(\sin^2(\theta)\right)\left(\sin(\theta)\right)\cos^6(\theta)\,\mathtt{d}\theta \\
&= 4^9 \int \left(1 - \cos^2(\theta)\right)\left(\sin(\theta)\right)\cos^6(\theta)\,\mathtt{d}\theta \\
&= -4^9 \int \left(1 - u^2\right) u^6 \,\mathtt{d}u \\
&= -4^9 \int u^6 - u^8 \,\mathtt{d}u \\
&= -4^9 \left(\frac{1}{7}u^7 - \frac{1}{8}u^8\right) + C \\
&= -4^9 \left(\frac{1}{7}\cos^7(\theta) - \frac{1}{9}\cos^9(\theta)\right) + C
\end{aligned}$$

To unwind in terms of x, we can use the definition of sine and cosine as side ratios of a triange. If $x = 4\sin(\theta)$, we can draw the triangle: From this, you can see that $\cos(\theta) = \frac{\sqrt{4^2-x^2}}{4}$. We can substitute that



in for the $\cos(\theta)$'s in the answer:

$$\begin{aligned}
-4^9\left(\frac{1}{7}\cos^7(\theta)-\frac{1}{9}\cos^9(\theta)\right)+C &= -4^9\left(\frac{1}{7}\left(\frac{\sqrt{4^2-x^2}}{4}\right)^7-\frac{1}{9}\left(\frac{\sqrt{4^2-x^2}}{4}\right)^9\right)+C\\
&= -4^9\left(\frac{1}{7}\left(\frac{4\sqrt{1-\frac{x^2}{16}}}{4}\right)^7-\frac{1}{9}\left(\frac{4\sqrt{1-\frac{x^2}{16}}}{4}\right)^9\right)+C\\
&= -4^9\left(\frac{1}{7}\left(\sqrt{1-\frac{x^2}{16}}\right)^7-\frac{1}{9}\left(\sqrt{1-\frac{x^2}{16}}\right)^9\right)+C\\
&= 2^{18}\left(\frac{\left(1-\frac{x^2}{16}\right)^{9/2}}{9}-\frac{\left(1-\frac{x^2}{16}\right)^{7/2}}{7}\right)+C
\end{aligned}$$

And that's it.

2.5.2 Secant Substitution

When we see an expression of the form x^2-a^2 in the integrand, we think of the identity $\sec^2(\theta)-1=\tan^2(\theta)$, so we use the following substitution:

Secant Substitution

$x=a\cdot\sec(\theta)$

Example 2.5.2.1. A Secant Substitution

Suppose we wish to evaluate the following integral:

$$\int\frac{1}{x^4-9x^2}\,\mathtt{d}x$$

Since $x^4-9x^2=x^2\left(x^2-9\right)$, we use the following substitution:

$$\begin{aligned}
x&=3\sec(\theta)\\
\mathtt{d}x&=3\sec(\theta)\tan(\theta)\,\mathtt{d}\theta
\end{aligned}$$

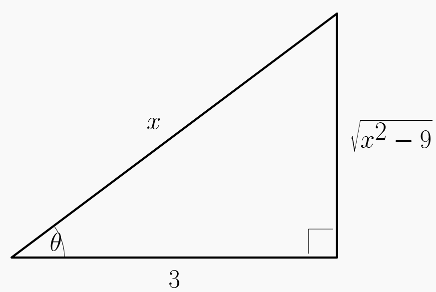
We now apply these substitutions to rewrite the integral in terms of θ.

$$\begin{aligned}
\int \frac{1}{x^4 - 9x^2} \, \mathtt{d}x &= \int \frac{1}{x^2 \left(x^2 - 9\right)} \, \mathtt{d}x \\
&= \int \frac{3 \sec(\theta) \tan(\theta)}{9 \sec^2(\theta) \left(9 \sec^2(\theta) - 9\right)} \, \mathtt{d}\theta \\
&= \int \frac{3 \sec(\theta) \tan(\theta)}{81 \sec^2(\theta) \tan^2(\theta)} \, \mathtt{d}\theta \\
&= \frac{1}{27} \int \frac{1}{\sec(\theta) \tan(\theta)} \, \mathtt{d}\theta \\
&= \frac{1}{27} \int \frac{\cos^2(\theta)}{\sin(\theta)} \, \mathtt{d}\theta \\
&= \frac{1}{27} \int \frac{1 - \sin^2(\theta)}{\sin(\theta)} \, \mathtt{d}\theta \\
&= \frac{1}{27} \int \frac{1}{\sin(\theta)} \, \mathtt{d}\theta - \frac{1}{27} \int \frac{\sin^2(\theta)}{\sin(\theta)} \, \mathtt{d}\theta \\
&= \frac{1}{27} \int \csc(\theta) \, \mathtt{d}\theta - \frac{1}{27} \int \sin(\theta) \, \mathtt{d}\theta \\
&= -\frac{1}{27} \ln\left|\csc(\theta) + \cot(\theta)\right| + \frac{1}{27} \cos(\theta) + C
\end{aligned}$$

Here we have successfully taken the antiderivative, and now need to just get back to x from θ. We draw a triangle and label the sides according to our substitution. In particular,

$$\sec(\theta) = \frac{x}{3} = \frac{\text{hypotenuse}}{\text{adjacent}}$$

so we can let the hypotenuse be x and the adjacent side be 3.



This enables us to compute the other trig functions using this triangle.

Exercise 2.5.2.2. Getting from θ back to x ☕

Complete the above example by using the triangle to find the values of the other trig functions.

$$\begin{aligned}
\cos(\theta) &= \\
\cot(\theta) &= \\
\csc(\theta) &=
\end{aligned}$$

Then plug these expressions back into our antiderivative to get a final answer in terms of x rather than θ. Make these substitutions and then simplify to verify the final answer shown below. Show your work below.

$$\begin{aligned}\int \frac{1}{x^4-9x^2}\,\mathtt{d}x &= -\frac{1}{27}\ln\left|\csc(\theta)+\cot(\theta)\right| + \frac{1}{27}\cos(\theta)+C \\ &= \\ &= \\ &= \frac{1}{9x}-\frac{1}{27}\ln\left|\frac{x+3}{\sqrt{x^2-9}}\right|+C\end{aligned}$$

Exercise 2.5.2.3. Yes You Can! Take the Cant Out of Secant! ☕☕☕

Evaluate the following antiderivative:

$$\int \sqrt{x^2-4}\,\mathtt{d}x.$$

A Solution: We have an expression of the form x^2-a^2, so we can use a secant substitution and let $x=2\sec(\theta), \mathtt{d}x=2\sec(\theta)\tan(\theta)\,\mathtt{d}\theta$.

$$\begin{aligned}\int \sqrt{x^2-4}\,\mathtt{d}x &= \int \sqrt{4\sec^2(\theta)-4}\cdot 2\sec(\theta)\tan(\theta)\,\mathtt{d}\theta \\ &= \int \sqrt{4\sec^2(\theta)-4}\cdot 2\sec(\theta)\tan(\theta)\,\mathtt{d}\theta \\ &= \int 2\sqrt{\sec^2(\theta)-1}\cdot 2\sec(\theta)\tan(\theta)\,\mathtt{d}\theta \\ &= 4\int \sqrt{\tan^2(\theta)}\cdot\sec(\theta)\tan(\theta)\,\mathtt{d}\theta \\ &= 4\int \sec(\theta)\tan^2(\theta)\,\mathtt{d}\theta\end{aligned}$$

Using the Pythagorean Identity, we can split the integral:

$$\begin{aligned}4\int \sec(\theta)\tan^2(\theta)\,\mathtt{d}\theta &= 4\int \sec(\theta)\left(\sec^2(\theta)-1\right)\mathtt{d}\theta \\ &= 4\int \sec^3(\theta)-\sec(\theta)\,\mathtt{d}\theta \\ &= 4\left[\int \sec^3(\theta)\,\mathtt{d}\theta - \int \sec(\theta)\,\mathtt{d}\theta\right]\end{aligned}$$

From previous exercises, we can evaluate these as:

$$4\left[\int \sec^3(\theta)\,\mathtt{d}\theta - \int \sec(\theta)\,\mathtt{d}\theta\right] = 4\left[\frac{\sec(\theta)\tan(\theta)+\ln\left|\sec(\theta)+\tan(\theta)\right|}{2} - \ln\left|\sec(\theta)+\tan(\theta)\right|\right]$$

To unwind, note that $x=2\sec(\theta)$ so that $\sec(\theta)=\frac{x}{2}$, which gives us enough information to draw a right triangle (try it) and find the side ratios, giving us $\tan(\theta) = \frac{\sqrt{x^2-4}}{2}$.

$$\begin{aligned}
&4\left[\frac{\sec(\theta)\tan(\theta)+\ln|\sec(\theta)+\tan(\theta)|}{2} - \ln|\sec(\theta)+\tan(\theta)|\right] + C \\
&= 4\left[\frac{\frac{x}{2}\cdot\frac{\sqrt{x^2-4}}{2} + \ln\left|\frac{x}{2}+\frac{\sqrt{x^2-4}}{2}\right|}{2} - \ln\left|\frac{x}{2}+\frac{\sqrt{x^2-4}}{2}\right|\right] + C \\
&= 4\left[\frac{\frac{x\sqrt{x^2-4}}{4} + \ln\left|\frac{x+\sqrt{x^2-4}}{2}\right|}{2} - \ln\left|\frac{x+\sqrt{x^2-4}}{2}\right|\right] + C \\
&= \frac{x\sqrt{x^2-4}}{2} + 2\ln\left|\frac{x+\sqrt{x^2-4}}{2}\right| - 4\ln\left|\frac{x+\sqrt{x^2-4}}{2}\right| + C \\
&= \frac{x\sqrt{x^2-4}}{2} - 2\ln\left|\frac{x+\sqrt{x^2-4}}{2}\right| + C
\end{aligned}$$

We can clean this up a bit by absorbing part of the logarithm into the constant:

$$\begin{aligned}
\frac{x\sqrt{x^2-4}}{2} - 2\ln\left|\frac{x+\sqrt{x^2-4}}{2}\right| + C &= \frac{x\sqrt{x^2-4}}{2} - \left(2\ln\left|x+\sqrt{x^2-4}\right| - 2\ln|2|\right) + C \\
&= \frac{x\sqrt{x^2-4}}{2} - 2\ln\left|x+\sqrt{x^2-4}\right| + C
\end{aligned}$$

2.5.3 Tangent Substitution

When we see an expression of the form a^2+x^2 or x^2+a^2 (which are the same) in the integrand, we think of the identity $\tan^2(\theta)+1 = \sec^2(\theta)$, so we use the following substitution:

Tangent Substitution

$$x = a\cdot\tan(\theta)$$

Exercise 2.5.3.1. Revisiting an Old Friend ☕☕

- Recall the derivative of arctangent:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arctan(x)) =$$

A Solution:

$$\frac{1}{1+x^2}$$

- We should be able to reverse the above by taking the antiderivative of the right-hand side. Perform this antiderivative using the substitution $x = \tan(\theta)$:

$$\int \frac{1}{1+x^2}\,\mathtt{d}x =$$

A Solution: Let $x = \tan(\theta), \mathtt{d}x = \sec^2(\theta)$.

$$\begin{aligned}\int \frac{1}{1+x^2}\,\mathtt{d}x &= \int \frac{1}{1+tan^2(\theta)}\sec^2(\theta)\,\mathtt{d}\theta \\ &= \int \frac{\sec^2(\theta)}{\sec^2(\theta)}\,\mathtt{d}\theta \\ &= \int 1\,\mathtt{d}\theta \\ &= \theta + C \\ &= \arctan(x) + C\end{aligned}$$

2.5.4 Preprocessing with Algebra or u-sub

Often we need to do a little algebra and/or u-sub to get the integrand into a form where we can then perform trig sub.

Exercise 2.5.4.1. A Bit of Algebra to Help Us ☕☕☕

- Explain why the two following expressions are equal:

$$(4x^2+1)^2 = 16\left(x^2 + \left(\frac{1}{2}\right)^2\right)^2$$

A Solution:

$$\begin{aligned}(4x^2+1)^2 &= \left(4\left(x^2+\frac{1}{4}\right)\right)^2 \\ &= (4)^2\left(x^2 + \left(\frac{1}{2}\right)^2\right)^2 \\ &= 16\left(x^2 + \left(\frac{1}{2}\right)^2\right)^2\end{aligned}$$

- Use the equality above along with a tangent substitution to evaluate the following antiderivative:

$$\int \frac{1}{(4x^2+1)^2}\,\mathtt{d}x$$

A Solution:

$$\int \frac{1}{(4x^2+1)^2}\,\mathtt{d}x = \int \frac{1}{16\left(x^2+\left(\frac{1}{2}\right)^2\right)^2}\,\mathtt{d}x$$

Let $x = \frac{1}{2}\tan(\theta), \mathtt{d}x = \frac{1}{2}\sec^2(\theta)$.

$$= \int \frac{1}{16\left(\frac{1}{4}\tan^2(\theta)+\left(\frac{1}{2}\right)^2\right)^2}\cdot\frac{1}{2}\sec^2(\theta)\,\mathtt{d}\theta$$

$$= \frac{1}{32}\int \frac{\sec^2(\theta)}{\left(\frac{1}{4}\tan^2(\theta)+\left(\frac{1}{2}\right)^2\right)^2}\,\mathtt{d}\theta$$

$$= \frac{1}{32}\int \frac{\sec^2(\theta)}{\left(\frac{1}{4}\sec^2(\theta)\right)^2}\,\mathtt{d}\theta$$

$$= \frac{1}{32}\int \frac{\sec^2(\theta)}{\frac{1}{16}\sec^4(\theta)}\,\mathtt{d}\theta$$

$$= \frac{1}{2}\int \frac{1}{\sec^2(\theta)}\,\mathtt{d}\theta$$

$$= \frac{1}{2}\int \cos^2(\theta)\,\mathtt{d}\theta$$

$$= \frac{1}{2}\int \frac{1+\cos(2\theta)}{2}\,\mathtt{d}\theta$$

$$= \frac{1}{2}\int \frac{1}{2}+\frac{1}{2}\cos(2\theta)\,\mathtt{d}\theta$$

$$= \frac{1}{2}\left(\frac{1}{2}\theta+\frac{1}{4}\sin(2\theta)+C\right)$$

$$= \frac{1}{4}\theta+\frac{1}{8}\sin(2\theta)+C$$

$$= \frac{1}{4}\theta+\frac{1}{4}\sin(\theta)\cos(\theta)+C$$

Since, $\tan(\theta) = 2x$, we construct a triangle to obtain $\cos(\theta)$ and $\sin(\theta)$.

$$\sin(\theta)) = \frac{2x}{\sqrt{4x^2-1}}, \cos(\theta)) = \frac{1}{\sqrt{4x^2-1}}$$

$$= \frac{1}{4}\arctan(2x)+\frac{1}{4}\frac{2x}{\sqrt{4x^2-1}}\cdot\frac{1}{\sqrt{4x^2-1}}+C$$

$$= \frac{1}{4}\arctan(2x)+\frac{x}{8x^2-2}+C$$

A trick from algebra that is often used with trigonometric substitution is completing the square. You might need to complete the square to get it into a form where a trig sub will work.

Example 2.5.4.2. Completing the Square

Suppose we wish to find an antiderivative for the function $\left(x^2+x-1\right)^{-2}$. We begin by completing the square on the quadratic polynomial:

$$\begin{aligned} x^2+x-1 &= x^2+x+\frac{1}{4}-\frac{1}{4}-1 \\ &= \left(x+\frac{1}{2}\right)^2-\frac{5}{4} \end{aligned}$$

We now use the substitution that this quadratic motivates. Namely, we pick $a=\frac{\sqrt{5}}{2}$ since we want its square to be five-fourths. Where x used to go in the problems above, we now have an $x+\frac{1}{2}$. Thus our substitution is $x+\frac{1}{2}=\frac{\sqrt{5}}{2}\sec(\theta)$, or more explicitly:

$$x=\frac{\sqrt{5}}{2}\sec(\theta)-\frac{1}{2}$$

Taking the derivative of both sides with respect to θ shows that

$$\mathtt{d}x=\frac{\sqrt{5}}{2}\sec(\theta)\tan(\theta)\mathtt{d}\theta$$

Exercise 2.5.4.3. Completing the Example ☕☕☕

Use the substitutions suggested in the example above to find the antiderivative.

$$\int\frac{1}{(x^2+x-1)^2}\mathtt{d}x=$$

A Solution: With $x+\frac{1}{2}=\frac{\sqrt{5}}{2}\sec(\theta)$, and $\mathtt{d}x=\frac{\sqrt{5}}{2}\sec(\theta)\tan(\theta)\,\mathtt{d}\theta$, we have:

$$\begin{aligned}
\int \frac{1}{(x^2+x-1)^2}\,\mathtt{d}x = \int \frac{1}{(x+\frac{1}{2})^2-\frac{5}{4}}\,\mathtt{d}x &= \int \frac{\frac{\sqrt{5}}{2}\sec(\theta)\tan(\theta)}{\left(\frac{\sqrt{5}}{2}\sec(\theta)\right)^2-\frac{5}{4}}\,\mathtt{d}\theta \\
&= \frac{\sqrt{5}}{2}\int \frac{\sec(\theta)\tan(\theta)}{\frac{5}{4}\sec^2(\theta)-\frac{5}{4}}\,\mathtt{d}\theta \\
&= \frac{2\sqrt{5}}{5}\int \frac{\sec(\theta)\tan(\theta)}{\tan^2(\theta)}\,\mathtt{d}\theta \\
&= \frac{2\sqrt{5}}{5}\int \frac{\sec(\theta)}{\tan(\theta)}\,\mathtt{d}\theta \\
&= \frac{2\sqrt{5}}{5}\int \csc(\theta)\,\mathtt{d}\theta \\
&= -\frac{2}{\sqrt{5}}\ln|\csc(\theta)+\cot(\theta)|+C
\end{aligned}$$

Using the initial substitution $\sec(\theta)=\frac{2}{\sqrt{5}}\left(x+\frac{1}{2}\right)$, we can draw a triangle and derive the other sides to get $\csc\theta=\frac{2x+1}{2\sqrt{x^2+x-1}}$ and $\cot\theta=\frac{\sqrt{5}}{2\sqrt{x^2+x-1}}$

$$\begin{aligned}
-\frac{2}{\sqrt{5}}\ln|\csc(\theta)+\cot(\theta)|+C &= -\frac{2}{\sqrt{5}}\ln\left|\frac{2x+1}{2\sqrt{x^2+x-1}}+\frac{\sqrt{5}}{2\sqrt{x^2+x-1}}\right|+C \\
&= \frac{-2\ln\left|\frac{2x+1+\sqrt{5}}{2\sqrt{x^2+x-1}}\right|}{\sqrt{5}}+C \\
&= \frac{-2\ln\left|2x+1+\sqrt{5}\right|+2\ln\left|2\sqrt{x^2+x-1}\right|}{\sqrt{5}}+C \\
&= \frac{-2\ln\left|2x+1+\sqrt{5}\right|+\ln\left(\left|2\sqrt{x^2+x-1}\right|\right)^2}{\sqrt{5}}+C \\
&= \frac{-2\ln\left|2x+1+\sqrt{5}\right|+\ln\left|4x^2+4x-4\right|}{\sqrt{5}}+C \\
&= \frac{-2\ln\left|2x+1+\sqrt{5}\right|+\ln\left|4(x+\frac{1+\sqrt{5}}{2})(x+\frac{1-\sqrt{5}}{2})\right|}{\sqrt{5}}+C \\
&= \frac{-2\ln\left|2x+1+\sqrt{5}\right|+\ln\left|4\cdot\frac{1}{2}(2x+1+\sqrt{5})\cdot\frac{1}{2}(2x+1-\sqrt{5})\right|}{\sqrt{5}}+C \\
&= \frac{-2\ln\left|2x+1+\sqrt{5}\right|+\ln\left|(2x+1+\sqrt{5})(2x+1-\sqrt{5})\right|}{\sqrt{5}}+C \\
&= \frac{-2\ln\left|2x+1+\sqrt{5}\right|+\ln\left|(2x+1+\sqrt{5})\right|+\ln\left|(2x+1-\sqrt{5})\right|}{\sqrt{5}}+C \\
&= \frac{\ln\left|(2x+1-\sqrt{5})\right|-\ln\left|2x+1+\sqrt{5}\right|}{\sqrt{5}}+C
\end{aligned}$$

Exercise 2.5.4.4. Try One On Your Own

Evaluate the following antiderivative:

$$\int \frac{x}{\sqrt{2x^2-4x-7}}\,\mathtt{d}x$$

A Solution: Completing the square first, we get:

$$\int \frac{x}{\sqrt{2x^2-4x-7}}\,\mathtt{d}x = \int \frac{x}{\sqrt{2\left(x^2-2x+1\right)-2-7}}\,\mathtt{d}x = \int \frac{x}{\sqrt{2\left(x-1\right)^2-9}}\,\mathtt{d}x$$

We want an expression of the form $a^2\sec^2\theta - a^2$, so if we let $(x-1) = \frac{3}{\sqrt{2}}\sec\theta$, we get the desired result.

$$\begin{aligned}
\int \frac{x}{\sqrt{2\left(x-1\right)^2-9}}\,\mathtt{d}x &= \int \frac{\frac{3}{\sqrt{2}}\sec(\theta)+1}{\sqrt{2\left(\frac{3}{\sqrt{2}}\sec\theta\right)^2-9}} \cdot \frac{3}{\sqrt{2}}\sec(\theta)\tan(\theta)\,\mathtt{d}x \\
&= \int \frac{\frac{9}{2}\sec(\theta)+\frac{3}{\sqrt{2}}}{\sqrt{2\left(\frac{9}{2}\sec^2(\theta)\right)-9}} \cdot \sec(\theta)\tan(\theta)\,\mathtt{d}x \\
&= \int \frac{\frac{9}{2}\sec(\theta)+\frac{3}{\sqrt{2}}}{\sqrt{9\sec^2(\theta)-9}} \cdot \sec(\theta)\tan(\theta)\,\mathtt{d}x \\
&= \int \frac{\frac{9}{2}\sec(\theta)+\frac{3}{\sqrt{2}}}{\sqrt{9\tan^2(\theta)}} \cdot \sec(\theta)\tan(\theta)\,\mathtt{d}x \\
&= \int \frac{\frac{9}{2}\sec(\theta)+\frac{3}{\sqrt{2}}}{3\tan^{(}\theta)} \cdot \sec(\theta)\tan(\theta)\,\mathtt{d}x \\
&= \int \frac{3}{2}\sec^2(\theta)+\frac{1}{\sqrt{2}}\sec(\theta)\,\mathtt{d}x \\
&= \frac{3}{2}\tan(\theta)+\frac{1}{\sqrt{2}}\ln\left|\sec(\theta)+\tan(\theta)\right|+C
\end{aligned}$$

Using our initial substitution, we have $\sec(\theta) = \frac{\sqrt{2}}{3}(x-1)$. Drawing a triangle with hypotenuse $\sqrt{2}(x-1)$ and bottom leg 3, we can determine $\tan(\theta) = \frac{\sqrt{2x^2-4x-7}}{3}$. Plugging these in for the trig functions gives us:

$$\begin{aligned}
\frac{3}{2}\tan(\theta)+\frac{1}{\sqrt{2}}\ln\left|\sec(\theta)+\tan(\theta)\right|+C &= \frac{3}{2}\frac{\sqrt{2x^2-4x-7}}{3}+\frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2}}{3}(x-1)+\frac{\sqrt{2x^2-4x-7}}{3}\right|+C \\
&= \frac{\sqrt{2x^2-4x-7}}{2}+\frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2}x-\sqrt{2}+\sqrt{2x^2-4x-7}}{3}\right|+C
\end{aligned}$$

We can clean this result up a bit by noticing that we can split the logarithm on the right hand side and that $\ln 3$ is a constant that we can roll into C.

$$\begin{aligned}
&= \frac{\sqrt{2x^2-4x-7}}{2} + \frac{1}{\sqrt{2}} \ln\left|\frac{\sqrt{2}x-\sqrt{2}+\sqrt{2x^2-4x-7}}{3}\right| + C \\
&= \frac{\sqrt{2x^2-4x-7}}{2} + \frac{1}{\sqrt{2}} \left(\ln\left|\sqrt{2}x-\sqrt{2}+\sqrt{2x^2-4x-7}\right| - \ln|3|\right) + C \\
&= \frac{\sqrt{2x^2-4x-7}}{2} + \frac{1}{\sqrt{2}} \ln\left|\sqrt{2}x-\sqrt{2}+\sqrt{2x^2-4x-7}\right| + C
\end{aligned}$$

Furthermore, we can multiply and divide everything in the logarithm by $\sqrt{2}$ as well, and pull out $\ln\sqrt{2}$ as another constant.

$$\begin{aligned}
&= \frac{\sqrt{2x^2-4x-7}}{2} + \frac{1}{\sqrt{2}} \ln\left|\sqrt{2}x-\sqrt{2}+\sqrt{2x^2-4x-7}\right| + C \\
&= \frac{\sqrt{2x^2-4x-7}}{2} + \frac{1}{\sqrt{2}} \ln\left|\frac{\sqrt{2}\left(\sqrt{2}x-\sqrt{2}+\sqrt{2x^2-4x-7}\right)}{\sqrt{2}}\right| + C \\
&= \frac{\sqrt{2x^2-4x-7}}{2} + \frac{1}{\sqrt{2}} \left(\ln\left|2x-2+\sqrt{4x^2-8x-14}\right| - \ln|\sqrt{2}|\right) + C \\
&= \frac{\sqrt{2x^2-4x-7}}{2} + \frac{1}{\sqrt{2}} \ln\left|2x-2+\sqrt{4x^2-8x-14}\right| + C
\end{aligned}$$

Lastly, we can rationalize the denominator and factor out $\frac{1}{2}$.

$$\begin{aligned}
&= \frac{\sqrt{2x^2-4x-7}}{2} + \frac{1}{\sqrt{2}} \ln\left|2x-2+\sqrt{4x^2-8x-14}\right| + C \\
&= \frac{\sqrt{2x^2-4x-7}}{2} + \frac{\sqrt{2}}{2} \ln\left|2x-2+\sqrt{4x^2-8x-14}\right| + C \\
&= \frac{1}{2}\left(\sqrt{2x^2-4x-7} + \sqrt{2}\ln\left|2x-2+\sqrt{4x^2-8x-14}\right|\right) + C
\end{aligned}$$

. Thus, our final expression:

$$\int \frac{x}{\sqrt{2x^2-4x-7}}\,\mathtt{d}x = \frac{1}{2}\left(\sqrt{2x^2-4x-7} + \sqrt{2}\ln\left|2x-2+\sqrt{4x^2-8x-14}\right|\right) + C$$

2.6 Partial Fraction Decomposition

In this section, we will combine the techniques of all previous sections and learn how to antidifferentiate rational functions!

Exercise 2.6.0.1. What is a Rational Function Again? ☕

What is the definition of a *rational function*?

A Solution: A rational function is a function that is expressed as a ratio, with some power of x in the denominator. Think a polynomial divided by a polynomial.

A partial fraction decomposition (PFD) is a way to decompose a rational function (a polynomial divided by a polynomial) as a sum of simpler rational functions. This is purely an algebraic trick that fundamentally does not involve calculus. It is useful in many contexts! Here we apply it to (of course) finding antiderivatives. Typically, a given rational function is too challenging to antidifferentiate as is. Once we break it up into smaller pieces via PFD, it becomes manageable.

The fundamental idea is simple. If we have a fraction that has more than one factor in the denominator, we can rewrite it as a sum of fractions whose denominators have the original denominator as their least common multiple.

Exercise 2.6.0.2. Trying This with Integers Before We Go to Polynomials ☕

Consider the fraction $\frac{1}{6}$. We notice that the denominator, six, is equal to two times three. Thus, we attempt to write one-sixth as a sum of fractions whose denominators are two and three.
Find integers A and B such that:

$$\frac{1}{6} = \frac{A}{2} + \frac{B}{3}$$

Check your answer by adding the fractions on the right hand side back together to verify you get one-sixth.

A Solution:

$$3A + 2B = 1$$

$$A = \frac{1 - 2B}{3}$$

Let $B = 2$. Then $A = \frac{1-4}{3} = -1$.

$$-\frac{1}{2} + \frac{2}{3} = \frac{-3 + 4}{3 \cdot 2} = \frac{1}{6}$$

2.6.1 Warming Up with a Small Example

Partial fraction decomposition is the same idea, except we are working with polynomials rather than just integers.

Example 2.6.1.1. Our First Decomposition!

To decompose the fraction $\frac{1}{x^2-1}$, we first factor the denominator into $x^2 - 1 = (x-1)(x+1)$. Thus, we look for an expression of the form

$$\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$$

for some numbers A and B. To find such A and B, we multiply both sides by $x^2 - 1$ to produce the polynomial equation

$$1 = A(x+1) + B(x-1)$$

Since we want the expressions to be equal for all values of x, we pick convenient values of x to plug in to solve for A and B.

- Set $x = 1$:
$$1 = A \cdot (2) + B \cdot (0) \implies A = \frac{1}{2}$$

- Set $x = -1$:
$$1 = A \cdot (0) + B \cdot (-2) \implies B = -\frac{1}{2}$$

At last, we have obtained the partial fraction decomposition!

$$\frac{1}{x^2-1} = \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}$$

Exercise 2.6.1.2. Checking Our Work ☕

Take the right-hand side of the above equation and add the two fractions together by finding a common denominator. Verify that their sum is the original rational function $\frac{1}{x^2-1}$.

A Solution:

$$\begin{aligned}\frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1} &= \frac{\frac{1}{2}(x+1) - \frac{1}{2}(x-1)}{(x-1)(x+1)} \\ &= \frac{\frac{x}{2} + \frac{1}{2} - \left(\frac{x}{2} - \frac{1}{2}\right)}{x^2-1} \\ &= \frac{\frac{x}{2} + \frac{1}{2} - \frac{x}{2} + \frac{1}{2}}{x^2-1} \\ &= \frac{1}{x^2-1}\end{aligned}$$

Example 2.6.1.3. Finding the Same PFD by Expanding and Equating Coefficients

We repeat the above example but demonstrate an alternate way to find our coefficients. Recall the equation

$$1 = A(x+1) + B(x-1)$$

In the previous example, we proceeded by plugging in numerical values for x. Instead, we could fully multiply out the polynomials and combine like terms. This produces

$$1 = (A+B)x + (A-B)$$

We can pad the left-hand side with a degree one term with coefficient zero to put both sides in the form "number times x plus number".

$$0x + 1 = (A+B)x + (A-B)$$

Now we can construct a system of two equations in two unknowns by equating one coefficient at a time. Specifically, we build it as:

Degree zero coefficient of LHS = Degree zero coefficient of RHS	$\Longrightarrow$	$1 = A - B$
Degree one coefficient of LHS = Degree one coefficient of RHS	$\Longrightarrow$	$0 = A + B$

The resulting linear system in two equations and two unknowns can then be solved via any applicable method (substitution, elimination, matrices, etc).

Exercise 2.6.1.4. Solve the System ☕

Solve the linear system of two equations and two unknowns in the example above. Verify you obtain the same values for A and B that we found in Example 2.6.1.1.

Exercise 2.6.1.5. Using a PFD to Find an Antiderivative ☕☕

- Find an antiderivative of $\frac{1}{x^2-1}$ by antidifferentiating

$$\frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}$$

A Solution:

$$\begin{aligned}
\int \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}\,\mathtt{d}x &= \int \frac{\frac{1}{2}}{x-1}\,\mathtt{d}x - \int \frac{\frac{1}{2}}{x+1}\,\mathtt{d}x \\
&= \frac{1}{2}\int \frac{1}{x-1}\,\mathtt{d}x - \frac{1}{2}\int \frac{1}{x+1}\,\mathtt{d}x \\
&= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C \\
&= \frac{1}{2}\left(\ln|x-1| - \ln|x+1|\right) + C \\
&= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C \\
&= \ln\left|\sqrt{\frac{x-1}{x+1}}\right| + C
\end{aligned}$$

- Verify the answer is the same as what you would get if you had taken the antiderivative of $\frac{1}{x^2-1}$ using the trigonometric substitution $x = \sec(\theta)$.

A Solution: With $x = \sec(\theta)$ and $\mathtt{d}x = \sec(\theta)\tan(\theta)\,\mathtt{d}\theta$ we have:

$$\begin{aligned}
\int \frac{1}{x^2-1}\,\mathtt{d}x &= \int \frac{1}{\sec^2(\theta)-1}\sec(\theta)\tan(\theta)\,\mathtt{d}\theta \\
&= \int \frac{1}{\tan^2(\theta)}\sec(\theta)\tan(\theta)\,\mathtt{d}\theta \\
&= \int \frac{\sec(\theta)}{\tan(\theta)}\,\mathtt{d}\theta \\
&= \int \frac{1}{\cos(\theta)}\cdot\frac{\cos(\theta)}{\sin(\theta)}\,\mathtt{d}\theta \\
&= \int \frac{1}{\sin(\theta)}\,\mathtt{d}\theta \\
&= \int \csc(\theta)\,\mathtt{d}\theta \\
&= -\ln|\csc(\theta)+\cot(\theta)| + C
\end{aligned}$$

Constructing a triangle with $x = \sec(\theta)$, we can determine $\csc(\theta) = \frac{x}{\sqrt{x^2-1}}$ and $\cot(\theta) =$

$\frac{1}{\sqrt{x^2-1}}$. So we have:

$$\begin{aligned}
-\ln|\csc(\theta)+\cot(\theta)|+C &= -\ln\left|\frac{x}{\sqrt{x^2-1}}+\frac{1}{\sqrt{x^2-1}}\right|+C \\
&= -\ln\left|\frac{x+1}{(\sqrt{x-1})(\sqrt{x+1})}\right|+C \\
&= -\ln\left|\frac{\sqrt{x+1}}{\sqrt{x-1}}\right|+C \\
&= \ln\left|\left(\sqrt{\frac{x+1}{x-1}}\right)^{-1}\right|+C \\
&= \ln\left|\sqrt{\frac{x-1}{x+1}}\right|+C
\end{aligned}$$

And we have the same result.

It turns out there are three strange things that can happen when finding a PFD, namely:

1. The degree of the numerator is greater than or equal to the degree of the denominator.
2. The denominator has one or more irreducible quadratic factors (where irreducible quadratic means a degree two polynomial that has no real roots).
3. The denominator has one or more repeated factors.

Each has a particular workaround. Below, we describe these methods and show a corresponding hideous example that demonstrates all of these steps.

Exercise 2.6.1.6. Reminding Ourselves of Some Language ☕

- What exactly does *irreducible quadratic* mean?

A Solution: A quadratic that can't be factored into real roots.

- Give an example of a quadratic polynomial that is irreducible.

A Solution: x^2+x+1

- Give an example of a quadratic polynomial that is not irreducible.

A Solution: x^2+2x+1

- Is the polynomial x^2 an irreducible quadratic? Explain why or why not.

A Solution: No. It has real roots $x = 0$.

2.6.2 The General Method of PFD

The process for performing a partial fraction decomposition of $\frac{p(x)}{q(x)}$ is as follows:

1. **Polynomial Long Division:** If the degree of $p(x)$ is not strictly smaller than the degree of $q(x)$, start by performing polynomial long division to split the fraction into a quotient and remainder. In the remainder term, the numerator will now have degree less than the denominator.
2. **Factor Denominator:** Factor the denominator into a product of powers of linear and irreducible quadratic polynomials.
3. **Set Up Terms in the Summation:**
   (a) **Linear Factors:** If the denominator is divisible by $(x-r)^n$ for some real number r and positive natural number n, we build terms that look like
   $$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_n}{(x-r)^n}$$
   where the A_i represent unknown real constants. That is, you use all consecutive powers of a linear factor as denominators and have arbitrary constants as numerators.
   (b) **Irreducible Quadratic Factors:** Let b and c be real numbers and suppose x^2+bx+c is an irreducible quadratic. If the denominator is divisible by $(x^2+bx+c)^n$ for some positive natural number n, we build terms that look like
   $$\frac{A_1x+B_1}{x^2+bx+c} + \frac{A_2x+B_2}{(x^2+bx+c)^2} + \frac{A_3x+B_3}{(x^2+bx+c)^3} + \cdots + \frac{A_nx+B_n}{(x^2+bx+c)^n}$$
   where the A_i and B_i represent unknown real constants. That is, you use all consecutive powers of a linear factor as denominators and have arbitrary constants as numerators.
4. **Clear Denominators:** Multiply each side of your equation by the denominator $q(x)$ to clear all fractions.
5. **Solve for Unknowns:** Solve for the unknown constants by plugging in convenient values of x (since we want the expression to be true for all values of x). The roots of $q(x)$ are always good choices for x values, but other friendly numbers like zero or one are also often helpful.
6. **Plug Values Back into the Previously Unknown Numerators:** Plug your constants back in to conclude the equality of your original rational expression with its PFD.

Example 2.6.2.1. An Epic PFD

We now find the partial fraction decomposition of the rational function

$$r(x) = \frac{x^7 + 8x^6 + 25x^5 + 52x^4 + 79x^3 + 13x^2 - 61x + 81}{x^6 + 9x^5 + 28x^4 + 36x^3 + 27x^2 + 27x}$$

This rational function has quotient $x - 1$ and remainder $6x^5 + 44x^4 + 88x^3 + 13x^2 - 34x + 81$ upon long division. So, for our first step in the decomposition we have

$$r(x) = x - 1 + \frac{6x^5 + 44x^4 + 88x^3 + 13x^2 - 34x + 81}{x^6 + 9x^5 + 28x^4 + 36x^3 + 27x^2 + 27x}$$

We now ignore the quotient and work on breaking up the fractional piece. The denominator is divisible by x, so we factor that out. Next, we use the Rational Root Theorem to form a list of possible roots and divide off the corresponding factors as we find them. Working out all the algebra, we conclude the denominator factors as

$$x^6 + 9x^5 + 28x^4 + 36x^3 + 27x^2 + 27x = x(x + 3)^3(x^2 + 1)$$

In this particular setting, x, $x + 3$, $(x + 3)^2$, and $(x + 3)^3$ are the relevant powers of linear factors. The factor $x^2 + 1$ is the only irreducible quadratic. (**Note:** $(x+3)^2$ is not an irreducible quadratic term; it is a common mistake to consider it so. It is a power of a linear term and should be treated as such.) We now set up our sum.

$$\frac{6x^5 + 44x^4 + 88x^3 + 13x^2 - 34x + 81}{x(x + 3)^3(x^2 + 1)} = \frac{A}{x} + \frac{B}{x + 3} + \frac{C}{(x + 3)^2} + \frac{D}{(x + 3)^3} + \frac{Ex + F}{x^2 + 1}$$

Since fractions are a pain, we get rid of them! Multiplying both sides by $x(x + 3)^3(x^2 + 1)$, our equation becomes

$$6x^5 + 44x^4 + 88x^3 + 13x^2 - 34x + 81$$
$$= A(x+3)^3(x^2+1) + Bx(x+3)^2(x^2+1) + Cx(x+3)(x^2+1) + Dx(x^2+1) + (Ex+F)x(x+3)^3$$

We now solve for our unknown coefficients. It is highly convenient to set $x = 0$. This produces the equation $81 = A(3)^3$ which implies $A = 3$. Similarly, we set $x = -3$. This produces the equation

$$6(-3)^5 + 44(-3)^4 + 88(-3)^3 + 13(-3)^2 - 34(-3) + 81 = D(-3)((-3)^2 + 1)$$

which simplifies to $30 = D(-30)$ which implies $D = -1$. We have now run out of the most convenient values to choose for x, namely the roots of the denominator. At this point, we unfortunately need to do something messy! We can either plug in less than optimal values of x, for example $x = 1$, then $x = -1$, then $x = 2$, etc, and solve the resulting simultaneous system of equations that results. Or, we can multiply out the polynomials and equate coefficients one degree at a time (the method of Example 2.6.1.3). Carrying out either of these methods will produce

$$B = 2, C = 1, E = 1, F = -5$$

At last, we plug the values for the constants A, B, C, D, E, and F back into the original decomposition (with quotient). Our final PFD is

$$\frac{x^7 + 8x^6 + 25x^5 + 52x^4 + 79x^3 + 13x^2 - 61x + 81}{x^6 + 9x^5 + 28x^4 + 36x^3 + 27x^2 + 27x} = x - 1 + \frac{3}{x} + \frac{2}{x + 3} + \frac{1}{(x + 3)^2} - \frac{1}{(x + 3)^3} + \frac{x - 5}{x^2 + 1}$$

Exercise 2.6.2.2. Identifying the Steps of PFD ☕

In the ridiculous example above, label each of the six steps of partial fraction decomposition. Where exactly does each step occur?

Exercise 2.6.2.3. Which Type of Numerator Goes Where? ☕

In the above example, notice that the factor $(x+3)^2$ corresponded to a term of the form

$$\frac{C}{(x+3)^2}$$

and not a term of the form

$$\frac{Cx+D}{(x+3)^2}.$$

Why was this the case?

Well, that's the process of partial fraction decomposition! Why are we doing it in a calculus course? Because a generic rational function is really hard to integrate, but the partial fraction decomposition is made up of simpler terms that are much easier to integrate. Let's find the antiderivative of that beast above!

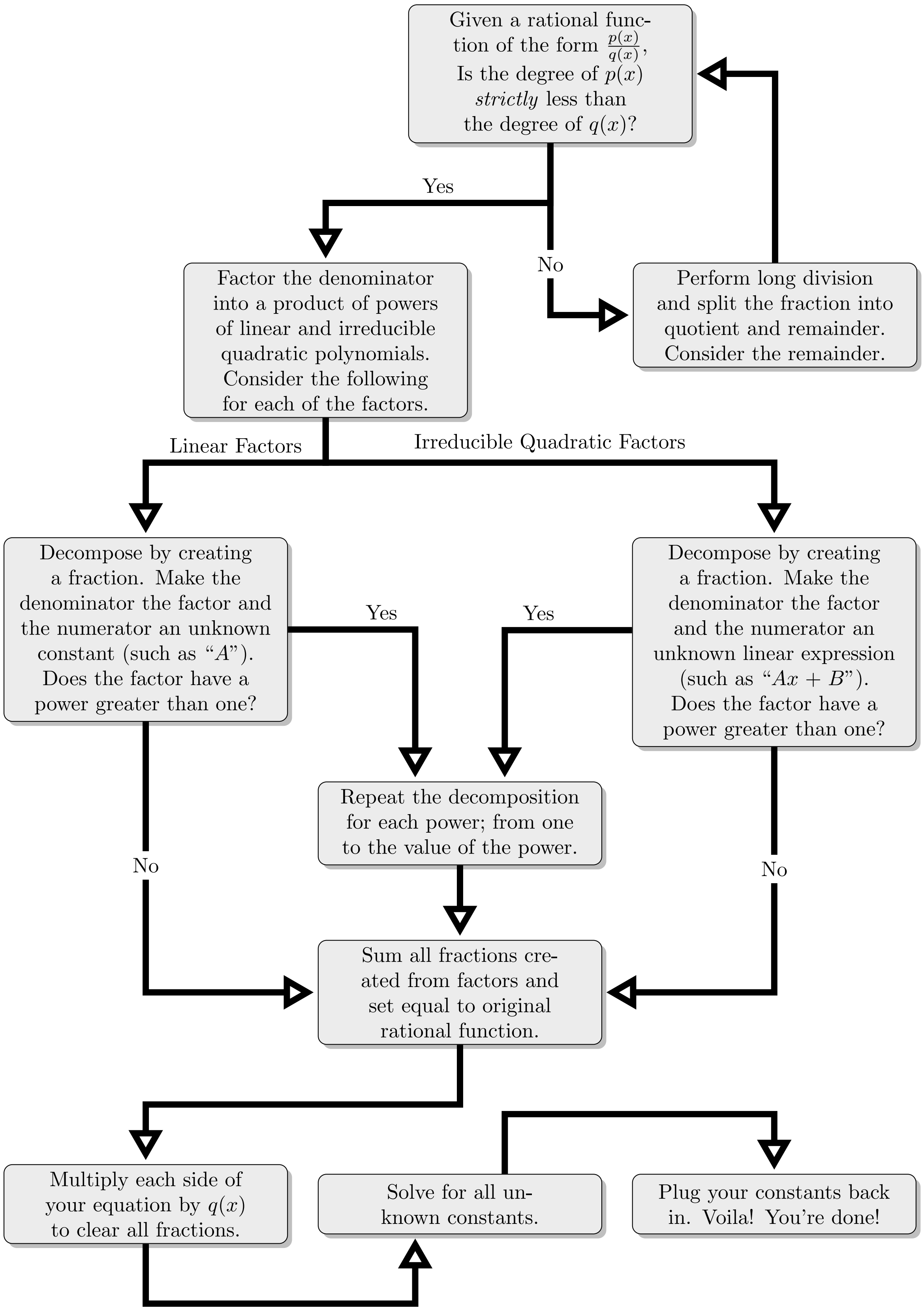
Example 2.6.2.4. Return of the Son of *Using a PFD to Find an Antiderivative*

We apply our PFD to compute the following antiderivative:

$$\begin{aligned}
&\int \left(\frac{x^7+8x^6+25x^5+52x^4+79x^3+13x^2-61x+81}{x^6+9x^5+28x^4+36x^3+27x^2+27x}\right) \mathtt{d}x \\
&= \int \left(x-1+\frac{3}{x}+\frac{2}{x+3}+\frac{1}{(x+3)^2}-\frac{1}{(x+3)^3}+\frac{x-5}{x^2+1}\right) \mathtt{d}x \\
&= \frac{x^2}{2}-x+3\ln(x)+2\ln(x+3)+-\frac{1}{x+3}+\frac{1}{2(x+3)^2}+\int \frac{x}{x^2+1}\,\mathtt{d}x+\int \frac{-5}{x^2+1}\,\mathtt{d}x \\
&= \frac{x^2}{2}-x+3\ln(x)+2\ln(x+3)+-\frac{1}{x+3}+\frac{1}{2(x+3)^2}+\frac{1}{2}\ln(x^2+1)-5\arctan(x)
\end{aligned}$$

Oh, and um, plus C.

2.6.3 Sweet PFD Flow Chart



Exercise 2.6.3.1. Now you cry! I mean, try! ☕☕☕

Find the following antiderivatives. Keep in mind that not every step of PFD will necessarily occur in every problem!

- $\displaystyle\int \frac{1}{x^2-9x+20}\,\mathtt{d}x$

A Solution: Let's start by breaking up the fraction. We can factor $x^2-9x+20$ into $(x-5)(x-4)$.

$$\frac{1}{x^2-9x+20} = \frac{A}{x-5} + \frac{B}{x-4}$$

Multiplying by the common denominator gives

$$1 = A(x-4) + B(x-5)$$

First let $x = 5$:

$$1 = A(5-4) + B(5-5) = A$$

Then let $x = 4$:

$$1 = A(4-4) + B(4-5) = -B$$

$$A = 1, B = -1$$

Now we can proceed with our integral:

$$\begin{aligned}\int \frac{1}{x-5} - \frac{1}{x-4}\,\mathtt{d}x &= \ln|x-5| - \ln|x-4| + C\\ &= \ln\left|\frac{x-5}{x-4}\right| + C\end{aligned}$$

- $\displaystyle\int \frac{1}{x^4-9}\,\mathtt{d}x$

A Solution: x^4-9 factors to $(x^2+3)(x^2-3)$. (x^2-3) further factors to $(x-\sqrt{3})(x+\sqrt{3})$. So our partial fraction decomposition is:

$$\frac{1}{x^4-9} = \frac{Ax+B}{x^2+3} + \frac{C}{x-\sqrt{3}} + \frac{D}{x+\sqrt{3}}$$

Multiplying both sides by the common denominator gives:

$$1 = (Ax+B)(x^2-3) + C(x^2+3)(x+\sqrt{3}) + D(x^2+3)(x-\sqrt{3})$$

We can let $x = \sqrt{3}$:

$$1 = (Ax+B)(3-3) + C(3+3)(\sqrt{3}+\sqrt{3}) + D(3+3)(\sqrt{3}-\sqrt{3})$$

$$1 = C(6)(2\sqrt{3}) = 12\sqrt{3}C$$

So $C = \frac{1}{12\sqrt{3}}$. Let $x = -\sqrt{3}$:

$$1 = (Ax+B)(3-3) + C(3+3)(-\sqrt{3}+\sqrt{3}) + D(3+3)(-\sqrt{3}-\sqrt{3})$$

$$1 = D(6)(-2\sqrt{3}) = -12\sqrt{3}D$$

So $D = -\frac{1}{12\sqrt{3}}$. To get A and B, substitute everything we have and multiply out the polynomial:

$$1 = (Ax+B)(x^2-3) + \frac{1}{12\sqrt{3}}(x^2+3)(x+\sqrt{3}) + \frac{-1}{12\sqrt{3}}(x^2+3)(x-\sqrt{3})$$

$$1 = Ax^3 - 3Ax + Bx^2 - 3B + \frac{1}{12\sqrt{3}}(x^3 + x^2\sqrt{3} + 3x + 3\sqrt{3}) - \frac{1}{12\sqrt{3}}(x^3 - x^2\sqrt{3} + 3x - 3\sqrt{3})$$

$$1 = Ax^3 - 3Ax + Bx^2 - 3B + \frac{1}{12\sqrt{3}}(x^3 + x^2\sqrt{3} + 3x + 3\sqrt{3}) - \frac{1}{12\sqrt{3}}(x^3 - x^2\sqrt{3} + 3x - 3\sqrt{3})$$

$$1 = Ax^3 - 3Ax + Bx^2 - 3B + \frac{x^2}{12} + \frac{1}{4} + \frac{x^2}{12} + \frac{1}{4}$$

$$1 = Ax^3 - 3Ax + Bx^2 - 3B + \frac{x^2}{6} + \frac{1}{2}$$

Before proceeding, note that A is the only coefficient with an x^3 term on either side. Since there is no x^3 on the LHS, and none on the RHS to cancel out Ax^3, we know $A = 0$. We can substitute that in to simplify further:

$$1 = Bx^2 - 3B + \frac{x^2}{6} + \frac{1}{2}$$

Now, note that the x^2 term on the LHS is 0, so $Bx^2 + \frac{1}{6}x^2 = 0$, which tells us $B = -\frac{1}{6}$. We can check that by looking at the constant term, which is 1. $-3B + \frac{1}{2} = 1$. Using $B = -\frac{1}{6} : -3\frac{-1}{6} + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1$. Plug all the terms back into the original PFD:

$$\begin{aligned}\int \frac{-\frac{1}{6}}{x^2+3} + \frac{\frac{1}{12\sqrt{3}}}{x-\sqrt{3}} + \frac{-\frac{1}{12\sqrt{3}}}{x+\sqrt{3}}\,\mathtt{d}x &= -\frac{1}{6}\int \frac{1}{x^2+3}\,\mathtt{d}x + \frac{1}{12\sqrt{3}}\int \frac{1}{x-\sqrt{3}}\,\mathtt{d}x - \frac{1}{12\sqrt{3}}\int \frac{1}{x+\sqrt{3}}\,\mathtt{d}x \\ &= -\frac{1}{12}\ln|x^2+3| + \frac{1}{12\sqrt{3}}\ln|x-\sqrt{3}| - \frac{1}{12\sqrt{3}}\ln|x+\sqrt{3}| + C \\ &= -\frac{1}{12}\ln|x^2+3| + \frac{1}{12\sqrt{3}}\ln\left|\frac{x-\sqrt{3}}{x+\sqrt{3}}\right| + C\end{aligned}$$

- $\displaystyle\int \frac{x^4}{x^2+1}\,\mathtt{d}x$

A Solution: We use polynomial long division, so that

$$\frac{x^4}{x^2+1} = x^2 - 1 + \frac{1}{x^2+1}$$

Then, the antiderivative is fairly straightfowarad:

$$\begin{aligned}\int \frac{x^4}{x^2+1}\,\mathtt{d}x &= \int x^2 - 1 + \frac{1}{x^2+1}\,\mathtt{d}x \\ &= \frac{1}{3}x^3 - x + \frac{1}{2}\ln|x^2+1| + C\end{aligned}$$

- $\displaystyle\int \frac{2}{x^5+2x^3+x}\,\mathtt{d}x$

A Solution: We start by factoring the fraction:

$$\frac{2}{x^5+2x^3+x} = \frac{1}{(x)(x^4+2x^2+1)} = \frac{1}{(x)(x^2+1)^2}$$

Our PFD breaks apart into:

$$2\left(\frac{1}{x^5+2x^3+x}\right) = 2\left(\frac{A}{x} + \frac{Bx+c}{x^2+1} + \frac{Dx+E}{(x^2+2)^2}\right)$$

From here, the simplest approach is to multiply both sides by the least common denominator and compare terms of like power, and then solve the resulting system of equations.

$$1 = A(x^2+1)^2 + (Bx+C)(x)(x^2+1) + (Dx+E)(x)$$

We have an easy win by first setting $x = 0 : 1 = A$.

$$1 = x^4 + 2x^2 + 1 + Bx^4 + Bx^2 + Cx^3 + Cx + Dx^2 + Ex$$

Collecting like terms and comparing to the LHS, we have:

$$\begin{aligned}(B+1)x^4 &= 0x^4\\ (C)x^3 &= 0x^3\\ (2+B+D)x^2 &= 0x^2\\ (E)x &= 0x\end{aligned}$$

Solving these systems of equations, we get $A = 1, B = -1, C = 0, D = -1, E = 0$, and our PFD becomes:

$$2\left(\frac{1}{x^5+2x^3+x}\right) = 2\left(\frac{1}{x} + \frac{-1x}{x^2+1} + \frac{-1x}{(x^2+2)^2}\right)$$

Solving our antiderivative now is much easier:

$$\begin{aligned}\int 2\left(\frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2}\right)\mathtt{d}x &= 2\int \frac{1}{x} - \frac{x}{x^2+1} - \frac{x}{(x^2+1)^2}\,\mathtt{d}x\\ &= 2\left(\ln|x| - \frac{1}{2}\ln|x^2+1| + \frac{1}{2}\frac{1}{x^2+1} + C\right)\\ &= 2\ln|x| - \ln|x^2+1| + \frac{1}{x^2+1} + C\end{aligned}$$

- $\displaystyle\int \frac{x-2}{x^3+x^2+3x-5}\,\mathtt{d}x$

A Solution: Our PFD is:

$$\frac{x-2}{x^3+x^2+3x-5} = \frac{x-2}{(x-1)(x^2+2x+5)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+2x+5}$$

Then we can solve for A, B and C:

$$\begin{aligned} x-1 &= A(x^2+2x+5)+(Bx+C)(x-1) \\ &= Ax^2+2Ax+5A+Bx^2-Bx+Cx-C \end{aligned}$$

Grouping by like power terms:

$$\begin{aligned} (A+B)x^2 &= 0x^2 \\ (2A-B+C)x &= x \\ (5A-C) &= -1 \end{aligned}$$

Exercise 2.6.3.2. Revisiting an Old Friend ☕☕☕

Recall Example 2.5.2.1, where we found the antiderivative of

$$\frac{1}{x^4-9x^2}$$

via trig sub. Find this antiderivative again but via PFD! Verify your answer is compatible with what trig sub produced.

A Solution: The fraction breaks up into $\frac{1}{x^2(x-3)(x+3)}$, so we construct the partial fractions:

$$\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-3}+\frac{D}{x+3}$$

. Clearing the denominator produces:

$$A(x)(x-3)(x+3)+B(x-3)(x+3)+C(x^2)(x+3)+D(x^2)(x-3)=1$$

. We can clear out the fractions by settings x equal to the roots of the denominator:

$$\begin{aligned} x=0 &: -9B=1 \implies B=-\frac{1}{9} \\ x=3 &: C(9)(6)=1 \implies C=\frac{1}{54} \\ x=-3 &: D(9)(-6)=1 \implies D=-\frac{1}{54} \end{aligned}$$

Since A has an x^3 power and nothing else does, we know $A=0$. So our integral is

$$\int \frac{1}{9x^2}+\frac{1}{54(x-3)}-\frac{1}{54(x+3)}\,\mathtt{d}x$$

Solving this produces

$$\begin{aligned} \frac{1}{9x}+\frac{1}{54}\ln|x-3|-\frac{1}{54}\ln|x+3| &= \frac{1}{9x}+\frac{1}{54}\left(\ln|x-3|-\ln|x+3|\right) \\ &= \frac{1}{9x}+\frac{1}{54}\ln\left|\frac{x-3}{x+3}\right| \end{aligned}$$

2.7 Chapter Summary

In this chapter, we tackled a very difficult question, namely

Given a function $f(x)$, how does one find an antiderivative?

Though there are many functions out there that do not have a closed form antiderivative, we explored **five** techniques that can get you there in a great many cases! Here are brief descriptions of the five:

1. **U-substitution:** Try to clean up an integral by making a substitution of the form $u = g(x)$. Often $g(x)$ is chosen to be the inner function in some function composition appearing in the integrand.

2. **Integration by Parts:** This is the product rule for antiderivatives. We identify two factors in the integrand and call one u while the other is called $\mathtt{d}v$. We then apply the IBP formula:

$$\int u \,\mathtt{d}v = uv - \int v \,\mathtt{d}u.$$

   In general, one tries to pick u to be something that is cleaner when differentiated and $\mathtt{d}v$ to be something we can antidifferentiate.

3. **Products of sines and cosines:** Any expression of the form

$$\int \sin^n(x)\cos^m(x)\,\mathtt{d}x$$

   for $n, m \in \mathbb{N}$ can be integrated by using the appropriate trig identities based on the parity of n and m.

4. **Trigonometric Substitution:** If you see quadratic polynomials in your integrand, you can likely clean things up with a trigonometric substitution. In particular,

If you see...	...make the substitution...	...because...
$a^2 - x^2$	$x = a\sin(\theta)$	$a^2 - a^2\sin^2(\theta) = a^2\cos^2(\theta)$
$a^2 + x^2$	$x = a\tan(\theta)$	$a^2 + a^2\tan^2(\theta) = a^2\sec^2(\theta)$
$x^2 - a^2$	$x = a\sec(\theta)$	$a^2\sec^2(\theta) - a^2 = a^2\tan^2(\theta)$

5. **Partial Fraction Decomposition:** This is the general method by which we can integrate any expression of the form

$$\int \frac{p(x)}{q(x)}\,\mathtt{d}x$$

   where $p(x)$ and $q(x)$ are polynomials.

Don't forget that you can check your work on any antiderivative by differentiating your answer. The result should be the original integrand!

2.8 Mixed Practice

2.8.1 Warm Ups

These are good problems for reinforcing the vocabulary and foundational concepts of this chapter.

Exercise 2.8.1.1. ☕

Find the antiderivative of $\frac{1}{1+x}$ using the substitution $u = 1 + x$.

A Solution: $\int \frac{1}{1+x}\,\mathtt{d}x$
Let $u = 1 + x$ then $\mathtt{d}u = \mathtt{d}x$
so $\int \frac{1}{1+x}\,\mathtt{d}x = \int \frac{1}{u}\,\mathtt{d}u = \ln(u) + C = \ln(1 + x) + C$

Exercise 2.8.1.2. ☕☕

Find the antiderivative

$$\int \frac{\sqrt{x}}{\sqrt{x}+1}\,\mathtt{d}x$$

using the substitution $u = \sqrt{x} + 1$.

A Solution: $\int \frac{\sqrt{x}}{\sqrt{x}+1} dx$
Let $u = \sqrt{x} + 1$ then $du = 1/2x^{-1/2}dx$ so $2x^{1/2}du = dx$
so $\int \frac{\sqrt{x}}{\sqrt{x}+1} dx = \int \frac{\sqrt{x}}{u}(2x^{1/2}du) = 2\int \frac{x}{u} du$ but since $u = \sqrt{x} + 1 \Rightarrow x = (u-1)^2$ we have $2\int \frac{x}{u} du = 2\int \frac{(u-1)^2}{u} du = 2\int \frac{u^2-2u+1}{u} du = 2\int (u - 2 + \frac{1}{u}) du = 2(u^2/2 - 2u + \ln|u|) + C$
$= (\sqrt{x}+1)^2 - 4(\sqrt{x}+1) + 2\ln|\sqrt{x}+1| + C = x - 2\sqrt{x} + 2\ln|\sqrt{x}+1| + C$

Exercise 2.8.1.3. ☕☕

Compute the exact value of the following definite integral:

$$\int_{x=1}^{x=\sqrt{3}} \frac{1}{\sqrt{x^2+1}}\,\mathtt{d}x.$$

A Solution: Use $x = \tan\theta$ then $\mathtt{d}x = \sec^2\theta\,\mathtt{d}\theta$. Also, $x^2 = \tan^2\theta$ and $\tan^2\theta + 1 = \sec^2\theta$. So

$$\int_1^{\sqrt{3}} \frac{1}{\sqrt{x^2+1}}\,\mathtt{d}x = \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{3}} \frac{1}{\sqrt{\tan^2\theta+1}} \cdot \sec^2\theta\,\mathtt{d}\theta = \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{3}} \frac{1}{\sec\theta} \cdot \sec^2\theta\,\mathtt{d}\theta = \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{3}} \sec\theta\,\mathtt{d}\theta$$

$$= \ln|\sec\theta + \tan\theta|\Big|_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{3}} = \ln|\sec(\pi/3)+\tan(\pi/3)| - \ln|\sec(\pi/4)+\tan(\pi/4)| = \ln\left|\frac{2+\sqrt{3}}{\sqrt{2}+1}\right|$$

Exercise 2.8.1.4. ☕☕

Calculate the antiderivative:

$$\int \frac{1}{x^4 - x^2}\,\mathtt{d}x$$

via partial fraction decomposition.

A Solution: First let $\frac{1}{x^4-x^2} = \frac{1}{x^2(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x} + \frac{D}{x^2}$ then we have $1 = Ax^2(x+1) + Bx^2(x-1) + Cx(x+1)(x-1) + D(x+1)(x-1)$
Let $x = 0$ then $1 = D(1)(-1) = -D$ so $D = -1$
Let $x = 1$ then $1 = A(2)$ so $A = \frac{1}{2}$
Let $x = -1$ then $1 = B(-2)$ so $B = -\frac{1}{2}$
We can find C using the degree 3 coefficients of the equation
$1 = Ax^2(x+1) + Bx^2(x-1) + Cx(x+1)(x-1) + D(x+1)(x-1)$
so we have $0 = A + B + C \Rightarrow -A - B = C \Rightarrow -\frac{1}{2} + \frac{1}{2} = C \Rightarrow C = 0$
So we have

$$\int \frac{1}{x^4-x^2}dx = \int \frac{\frac{1}{2}}{x-1} + \frac{-\frac{1}{2}}{x+1} + \frac{0}{x} + \frac{-1}{x^2}dx = \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + \frac{1}{x} + C$$

2.8.2 Sample Test Problems

Exercise 2.8.2.1. ☕☕

Consider $\int \cos^{13} x \sin^5 x\,\mathtt{d}x$.

- Can you compute this integral using $u = \cos x$? Explain.

  A Solution: Yes, use one factor of $\sin x$ for the $\mathtt{d}u$. Specifically, $u = \cos x$ implies $\mathtt{d}u = -\sin x\,\mathtt{d}x$ and use $\sin^2 x = 1 - \cos^2 x$. We now calculate the integral as $\int \cos^{13} x \sin^5 x dx = \int \cos^{13} x \sin^4 x \sin x\,\mathtt{d}x = -\int u^{13}(1-u^2)^2\,\mathtt{d}u = -\int u^{13} - 2u^{15} + u^{17}\,\mathtt{d}u = -\frac{u^{14}}{14} + \frac{2u^{16}}{16} + \frac{u^{18}}{18} = -\frac{\cos^{18} x}{18} + \frac{\cos^{16} x}{8} - \frac{\cos^{14} x}{14} + C.$

- Can you compute this integral using $u = \sin(x)$? Explain.

A Solution:

- Which of the two above substitutions will be easier to use? Carry out the integration, using the easier of the two.

A Solution:

Exercise 2.8.2.2. ☕☕☕

Evaluate the integral
$$\int \csc^3(x)\, \mathtt{d}x$$
via IBP.

A Solution: Use $u = \csc(x)$ and $\mathtt{d}v = \csc^2(x)\, \mathtt{d}x$.

Exercise 2.8.2.3. ☕☕☕

Consider the following antiderivative:
$$\int \frac{1}{x^2 - 16}\, \mathtt{d}x$$

- Compute the above antiderivative via a partial fraction decomposition.

A Solution: $\frac{1}{x^2-16} = \frac{A}{x+4} + \frac{B}{x-4} \Rightarrow 1 = A(x-4) + B(x+4)$
Set $x = -4 \to 1 = A \cdot (-8) \to A = -\frac{1}{8}$
and set $x = 4 \to 1 = B \cdot 8 \to B = \frac{1}{8}$ So we have
$$\int \frac{1}{x^2-16}dx = -\frac{1}{8}\int \frac{1}{x+4}dx + \frac{1}{8}\int \frac{1}{x-4}dx = -\frac{1}{8}\ln|x+4| + \frac{1}{8}\ln|x-4| = \frac{1}{8}\ln\left|\frac{x-4}{x+4}\right| + C$$

- Compute the above antiderivative via trigonometric substitution.

A Solution: Use $x = 4\sec\theta$ then $dx = 4\sec\theta\tan\theta \quad d\theta$ also $x^2 = 16\sec^2\theta$ and $16\sec^2\theta - 16 = 16\tan^2\theta$ So
$$\int \frac{1}{x^2-16}dx = \int \frac{1}{16\sec^2\theta - 16} 4\sec\theta\tan\theta d\theta = \int \frac{1}{16\tan^2\theta} 4\sec\theta\tan\theta d\theta = \int \frac{\sec\theta}{4\tan\theta} d\theta$$

$$= \frac{1}{4}\int \csc\theta d\theta = -\frac{1}{4}\ln|\csc\theta + \cot\theta| + C = -\frac{1}{4}\ln|\frac{x}{\sqrt{x^2-16}} + \frac{4}{\sqrt{x^2-16}}| + C = -\frac{1}{4}\ln\left|\frac{x+4}{\sqrt{x^2-16}}\right| + C$$

because $x = 4\sec\theta \Rightarrow \cos\theta = \frac{4}{x} \Rightarrow \cos^2\theta = \frac{16}{x^2} = 1 - \sin^2\theta \Rightarrow \sin^2\theta = 1 - \frac{16}{x^2} = \frac{x^2-16}{x^2} \Rightarrow \sin\theta = \frac{\sqrt{x^2-16}}{x} \Rightarrow \csc\theta = \frac{x}{\sqrt{x^2-16}}$ also since we defined $x = 4\sec\theta$ then $16\sec^2\theta - 16 = x^2 - 16 = 16\tan^2\theta \Rightarrow \sqrt{x^2-16} = 4\tan\theta \Rightarrow \cot\theta = \frac{4}{\sqrt{x^2-16}}$

- Your answers may appear very different! Verify that they are in fact equivalent.

A Solution: Start with the answer from the previous part and use properties of the natural logarithm as follows:
$-\frac{1}{4}\ln\left|\frac{x+4}{\sqrt{x^2-16}}\right| = -\frac{1}{4}\ln|x+4| + \frac{1}{4}\ln|\sqrt{x^2-16}|$
$= -\frac{1}{4}\ln|x+4| + \frac{1}{4}\frac{1}{2}\ln|x^2-16|$
$= -\frac{1}{4}\ln|x+4| + \frac{1}{8}\ln|x-4| + \frac{1}{8}\ln|x+4|$
$= -\frac{1}{8}\ln|x+4| + \frac{1}{8}\ln|x-4|$
$= \frac{1}{8}\ln\left|\frac{x-4}{x+4}\right|$.

Exercise 2.8.2.4. ☕☕☕☕

- Perform a Partial Fraction Decomposition on the following rational function:

$$\frac{x^3}{x^3 - 3x^2 + 4}$$

A Solution: Note the powers of the numerator and denominator are the same, PFD requires the numerator to be less than the denominator. So start with long division

$$\begin{array}{r|l} & \quad 1 \\ \hline x^3 - 3x^2 + 4 & \quad x^3 \\ & -x^3 + 3x^2 - 4 \\ \hline & \quad 3x^2 - 4 \end{array}$$

so we have $\frac{x^3}{x^3-3x^2+4} = 1 + \frac{3x^2-4}{x^3-3x^2+4}$ Now we need to factor $x^3 - 3x^2 + 4$ we can try multiple options with synthetic division and the rational zero theorem.
A good guess is -1 since $(-1)^3 - 3(-1)^2 + 4 = 0$ so a factor is $(x+1)$ use long division to

factor

$$
\begin{array}{r|l}
 & x^2 - 4x + 4 \\
\hline
x+1 & x^3 - 3x^2 \qquad + 4 \\
 & -x^3 - x^2 \\
\hline
 & -4x^2 \\
 & 4x^2 + 4x \\
\hline
 & 4x + 4 \\
 & -4x - 4 \\
\hline
 & 0
\end{array}
$$

now we have $\frac{x^3}{x^3-3x^2+4} = 1 + \frac{3x^2-4}{(x+1)(x^2-4x+4)} = 1 + \frac{3x^2-4}{(x+1)(x-2)^2}$ use PFD on $\frac{3x^2-4}{(x+1)(x-2)^2}$ and we have
$\frac{3x^2-4}{(x+1)(x-2)^2} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} \Rightarrow 3x^2 - 4 = A(x-2)^2 + B(x+1)(x-2) + C(x+1)$
Let $x = -1$ then $3 - 4 = -1 = A(-3)^2 = 9A \Rightarrow A = -\frac{1}{9}$
Let $x = 2$ then $3(2)^2 - 4 = 8 = C(3) \Rightarrow C = \frac{8}{3}$
Use degree 2 coefficients to get $3 = A + B \Rightarrow 3 = -\frac{1}{9} + B \Rightarrow 3 + \frac{1}{9} = \frac{28}{9} = B$
We now have $\frac{x^3}{x^3-3x^2+4} = 1 + \frac{-\frac{1}{9}}{x+1} + \frac{\frac{28}{9}}{x-2} + \frac{\frac{8}{3}}{(x-2)^2}$

- Use your work from the previous part to evaluate the following antiderivative:

$$\int \frac{x^3}{x^3 - 3x^2 + 4}\, dx$$

Exercise 2.8.2.5.

Evaluate the following antiderivative using Integration by Parts:

$$\int \sec^5 x\, dx.$$

Hint: The two integrals from Subsections 2.1.3 and 2.2.3 listed below may be helpful!

$$\int \sec x\, dx = \ln|\sec x + \tan x| + C$$

$$\int \sec^3 x\, dx = \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C$$

A Solution: Let $u = \sec^3 x$ then $du = 3\sec^2 x \sec x \tan x dx$
Let $dv = \sec^2 x dx$ then $v = \tan x$
$\int \sec^5 x dx = \sec^3 x \tan x - 3\int \tan^2 x \sec^3 x dx = \sec^3 x \tan x - 3\int (\sec^2 x - 1)\sec^3 x dx$
$= \sec^3 x \tan x - 3\int (\sec^5 x - \sec^3 x) dx = \sec^3 x \tan x - 3\int \sec^5 x dx + \int \sec^3 x dx$
$= \sec^3 x \tan x + \frac{3}{2}(\sec x \tan x + \ln|\sec x + \tan x|) - 3\int \sec^5 x dx$ but then we have
$4\int \sec^5 x dx = \sec^3 x \tan x + \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|)$
so $\int \sec^5 x dx = \frac{\sec^3 x \tan x + \frac{3}{2}(\sec x \tan x + \ln|\sec x + \tan x|)}{4} = \frac{1}{4}\sec^3 x \tan x + \frac{3}{8}\sec x \tan x +$

$\frac{3}{8} \ln|\sec x + \tan x| + C$

Part II

Coming Attractions

Chapter 3

Introduction to Calculus III: Parametric and Polar

We begin by briefly thinking about the word *dimension*.

Exercise 3.0.0.1. Dimension ☕

One intuitive notion of dimension comes from the idea of how you would assign units to measure it. If an object has length, you would call it one-dimensional. If an object has area, it is called two-dimensional. If it has volume, it is called three-dimensional. State the dimension of each of the following objects:

- $\{x \in \mathbb{R} : x < 2\}$
- $\{(x, y) \in \mathbb{R}^2 : x < 2\}$
- The closed interval $[2, 3]$
- The circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$
- The disc $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$

In Calculus III, you will redo all of the key concepts of Calculus I and II but in three (or more) dimensions. Often the difficultly of higher-dimensional calculus is notational more than anything! In three or more dimensions, it becomes messier to write down the same concepts. To make this cleaner, we develop better languages for points and curves beyond our standard coordinate system.

3.1 Parametric Curves

Many of the objects we study, like circles or graphs of functions, are one-dimensional objects even though we usually view them as embedded in a two-dimensional plane. Thus, we can represent both x and y

(the two dimensions) in terms of the same parameter t.

Definition 3.1.0.1. Parametric Curve

Let $x(t)$ and $y(t)$ be functions of t and let $D \subset \mathbb{R}$. The corresponding *parametric curve* is the set of points

$$\{(x(t), y(t)) : t \in D\}.$$

Typically, D is an interval or union of intervals. We can graph most curves by just selecting t values from the domain D and plotting the corresponding points.

Exercise 3.1.0.2. A Warm-up Parametric Curve

Consider the parametric curve

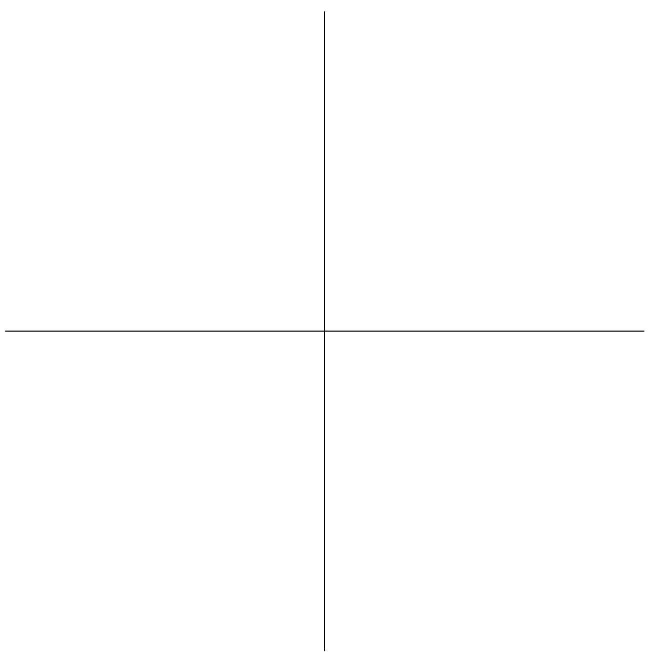
$$\{(2t, 3t+1) : t \in [-1, 3]\}.$$

That is, $x(t) = 2t$, $y(t) = 3t + 1$, and $-1 \leq t \leq 3$.

- Use the above formulas for $x(t)$ and $y(t)$ and the following t-values selected from $D = [-1, 3]$ to fill out the following table:

t	-1	0	1	2	3
$x(t)$					
$y(t)$					

- Plot those five points on the axes below. What type of shape does it appear to be?



- Solve the equation $x = 2t$ for t. Substitute this expression for t into the equation $y = 3t + 1$.

What does this new equation tell you about the parametric curve?

Here is an example of a parametric curve used in Trigonometry (though not called so at the time).

Exercise 3.1.0.3. The Unit Circle ☕☕

- Explain why the parametric curve

$$C_1 = \{(\cos(t), \sin(t)) : t \in [0, 2\pi]\}.$$

  is the familiar unit circle from trigonometry.

- Consider the curve

$$C_2 = \{(\sin(t), \cos(t)) : t \in [0, 2\pi]\}.$$

  How are the curves C_1 and C_2 similar? How are they different?

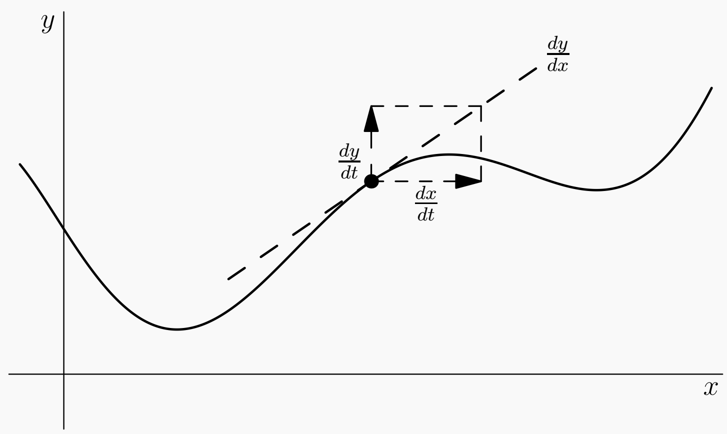
3.2 Derivatives of Parametric Curves: Slopes of Tangent Lines

To compute the derivative of a parametric curve, we recall that the slope of a line is the change in y-coordinate divided by the change in x-coordinate. In the context of parametric curves, these can be computed as rates of change with respect to the parameter t.

Definition 3.2.0.1. Parametric Derivatives

Let $(x(t), y(t))$ be a parametric curve. Then the slope of the tangent line can be computed as

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$



Notice that the above formula is just a slightly rearranged version of the chain rule. In particular, if we consider a portion of the graph of $(x(t), y(t))$ that passes the Vertical Line Test, then we can consider y as a function of x, which x in turn is a function of t. So if we wanted to ask how y changes with respect to t, we would have to take the rate of change of y with respect to x and multiply it by the rate of change of x with respect to t (by the chain rule). Expressing this Chain Rule in symbols instead of words, we have

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

Exercise 3.2.0.2. Understanding the Definition ☕

How would you get from the chain rule application shown above to our definition of parametric derivatives?

Example 3.2.0.3. Parametric Derivatives on a Parabola

Consider the parametric curve given by

$$\left\{ \left(t^2, t\right) : t \in [0, \infty) \right\}.$$

To find the slope of a tangent line to this parabola, we can use the parametric derivative formula as follows:

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \\ &= \frac{\frac{d}{dt}(t)}{\frac{d}{dt}(t^2)} \\ &= \frac{1}{2t}.\end{aligned}$$

Alternately, we could convert the curve to a cartesian equation and differentiate with respect to y. Proceeding, we notice this curve is contained in the graph of $y = \sqrt{x}$, since the formulas $x = t^2$ and $y = t$ satisfy that relationship. Thus, we can differentiate y with the power rule.

$$\begin{aligned}\frac{dy}{dx} &= \left(\sqrt{x}\right)' \\ &= \frac{1}{2\sqrt{x}}\end{aligned}$$

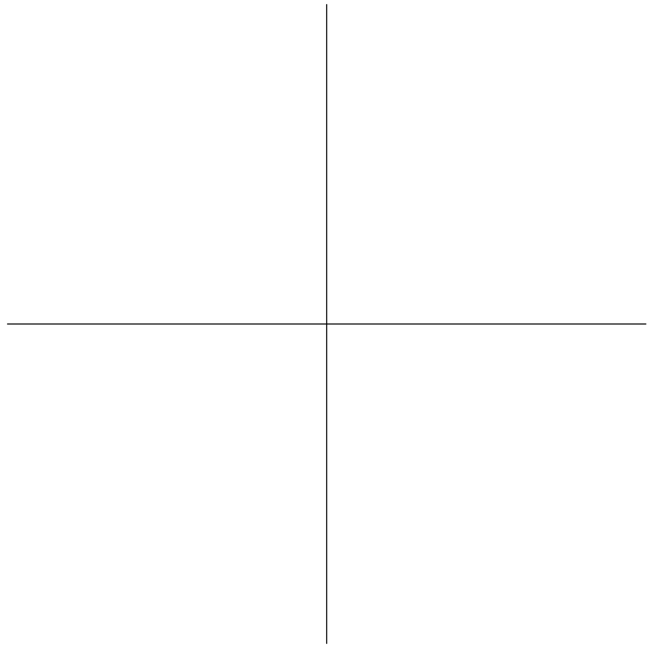
Exercise 3.2.0.4. Equivalence of the Results ☕☕

In the above example, we have two distinct expressions for $\frac{dy}{dx}$. Explain why they are in fact equivalent.

Exercise 3.2.0.5. The Tangent Line to an Ellipse ☕☕

- Plot the parametric curve given by

$$\{(2\cos(t), \sin(t)) : t \in [0, 2\pi]\}.$$



- Find the point on the graph located at $t = \pi/4$, and find the slope of the tangent line at that point using the parametric derivative formula. Sketch the tangent line on your graph above.

- Verify the above curve is in fact the ellipse given by $\frac{x^2}{4} + y^2 = 1$.

- Use implicit differentiation on the equation $\frac{x^2}{4} + y^2 = 1$ to find $\frac{dy}{dx}$ at that same point and verify your answers match!

Exercise 3.2.0.6. Finding a Parameterization ☕☕☕

Find a parameterization of the path that consists of two full clockwise laps around the ellipse given by

$$\frac{(x-3)^2}{4} + (y-3)^2 = 1$$

starting from the point (3,2).

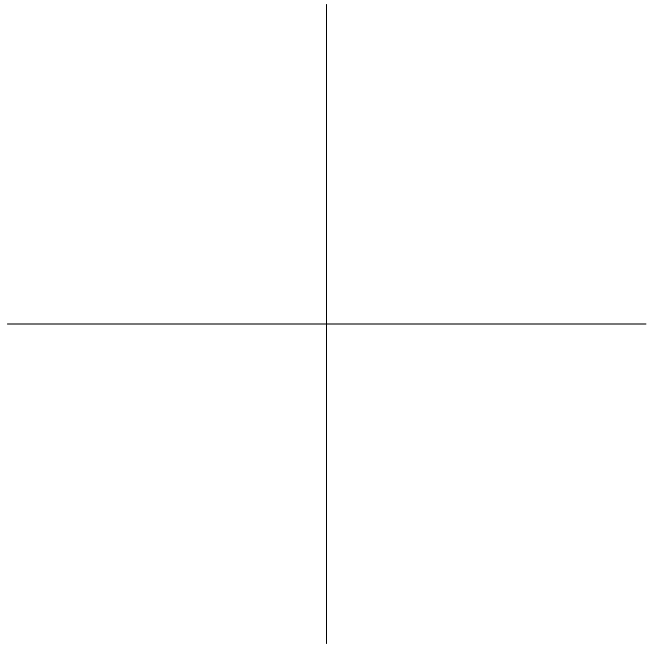
Exercise 3.2.0.7. A Hyperbola ☕☕

Consider the parametric curve given by the following:

$$x(t) = e^t - e^{-t},$$
$$y(t) = e^t + e^{-t},$$
$$t \in [0, \infty).$$

- Show that the above curve is contained in the hyperbola $y^2 - x^2 = 4$.

- Graph the parametric curve.



- Find dy/dx using the parametric formula for derivatives. Take the limit as t approaches infinity and interpret on your graph.

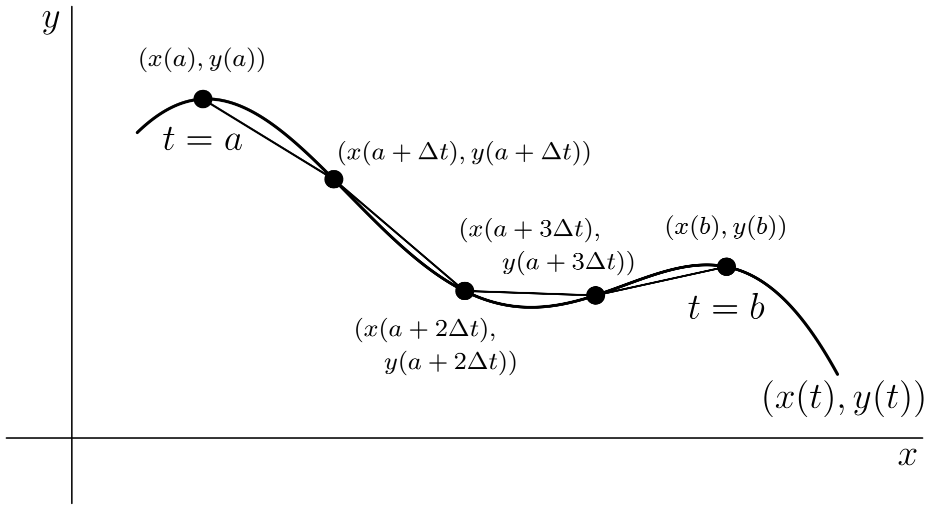
3.3 Integrals of Parametric Curves: Arc Length

The length of a parametric curve is given by the following formula.

Theorem 3.3.0.1. Parametric Arc Length

Let a parametric curve C be given by $(x(t), y(t))$ for $a \leq t \leq b$. Then the arc length is computed via

$$\int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, \mathtt{d}t.$$



The construction here is nearly identical to the construction of the arc length of the graph of a function in Section **??**. We select points corresponding to t-values along the curve, compute the sum of the lengths of the line segments connecting them, and take the limit as the number of line segments goes to infinity.

Exercise 3.3.0.2. Fill in the Blanks! Derivation of the Arc Length Formula

Let $t_0, t_1, t_2, \ldots, t_n$ be equally spaced points in the interval $[a, b]$. That is, $t_0 = a$, $t_n = b$, and for each $i \in \{0, 1, 2, \ldots, n-1\}$, $\Delta t =$ ____________________.
With this setup, if we want the length of a line segment connecting points $(x(t_{i+1}), y(t_{i+1}))$ and $(x(t_i), y(t_i))$, we would use the Pythagorean Theorem to obtain

$$\sqrt{\left(\qquad\qquad\qquad\qquad \right)}$$

as the length.

$$\begin{aligned}
L &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \sqrt{\qquad\qquad\qquad\qquad} \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} \sqrt{\qquad\qquad\qquad\qquad} \sqrt{(\Delta t)^2} \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} \sqrt{\qquad\qquad\qquad\qquad} \Delta t \\
&= \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, \mathsf{d}t
\end{aligned}$$

As usual, when first trying out a new tool, it is best to use it in a case where you already know the answer.

Exercise 3.3.0.3. Checking the Circumference of a Circle ☕☕

Consider the parametrization

$$x(t) = r\cos(t)$$
$$y(t) = r\sin(t)$$

for $t \in [0, 2\pi]$.

- Explain why this is a parameterization of a circle of radius r.

- Use the parametric arc length formula to compute the length of the curve. Compare it to your known formula for the circumference of a circle. Does the answer make sense?

Exercise 3.3.0.4. A Familiar Conic in Disguise ☕☕☕

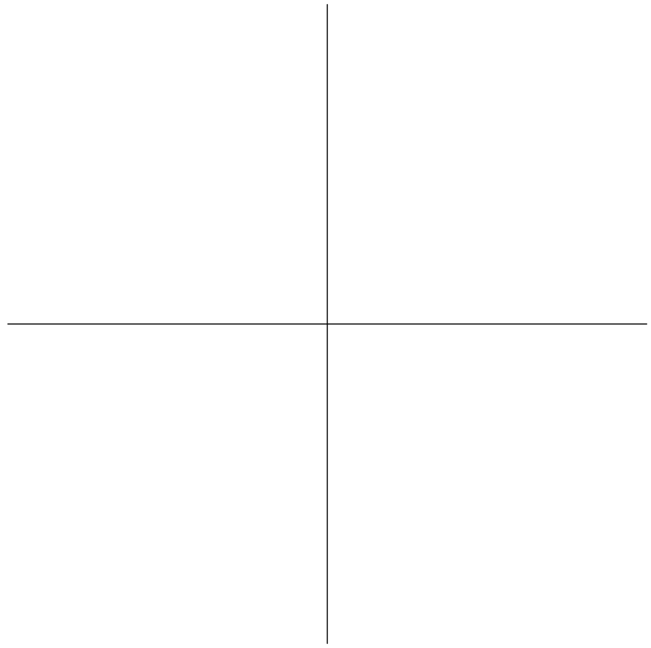
Consider the parametrization

$$x(t) = \sqrt{|t|}$$
$$y(t) = 3t - 1$$

for $t \in [-2, 2]$.

- Convert this to a cartesian equation. What kind of shape is it?

- Sketch the curve. Indicate any vertical or horizontal tangent lines and where they occur.



- Use the parametric arc length formula to compute the length of the curve. Does the answer make sense?

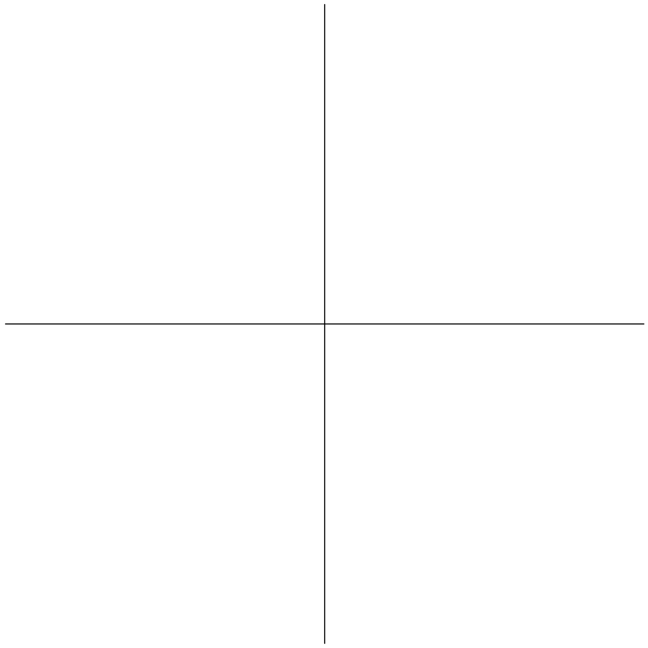
Ok, time to finally play with a curve that is not just a conic.

Exercise 3.3.0.5. Analyzing a Stranger Curve

- Sketch the graph of the following parametric curve C:

$$C = \left\{ \left(e^t \cos(t), e^t \sin(t) \right) : 0 \le t \le 2\pi \right\}.$$

Include labels of points on the graph at $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.



- Where does the above graph have vertical tangent lines? Where does the above graph have horizontal tangent lines? Mark them on your graph.

- What is the length of C?

3.4 Hyperbolic Sine and Cosine

You may have seen in a previous course (or if not, then here they are!) the definitions of the *hyperbolic sine* and *hyperbolic cosine* functions. They are typically defined as follows:

- $\cosh(t) = \frac{e^t+e^{-t}}{2}$
- $\sinh(t) = \frac{e^t-e^{-t}}{2}$

This of course prompts the question: "why do these e things get called sine or cosine?" We answer this question below.

Exercise 3.4.0.1. Power Series for Hyperbolic Sine and Cosine

- Find a power series for cosh by using what we know about the series for the exponential function. How does the resulting series relate to cosine?

- Find a power series for sinh by using what we know about the series for the exponential function. How does the resulting series relate to sine?

And of course there are more questions prompted here: "Why do these e things get called hyperbolic? What do these hyperbolic functions have to do with hyperbolas?"

Exercise 3.4.0.2. Parametric Curve Generated by Hyperbolic Sine and Cosine

Consider the parametric curve

$$\{(\cosh(t), \sinh(t)) : t \in \mathbb{R}\}.$$

- Verify this parametric curve satisfies the cartesian equation for a hyperbola given by

$$x^2 - y^2 = 1$$

by plugging the exponential definitions for our hyperbolic trig functions in for x and y.

- Again, verify this parametric curve satisfies the cartesian equation for the same hyperbola given by

$$x^2 - y^2 = 1$$

by plugging the power series formulas for our hyperbolic trig functions in for x and y.

3.5 Polar Coordinates

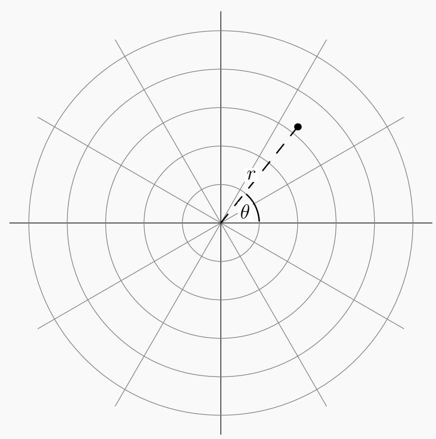
We use the *(horizontal,vertical)* coordinate system so much that it is easy to think that it is somehow inherent to a plane. However, a plane is just a geometric object; a coordinate system is an arbitrary system of labels that we slap on after the fact. Here we explore a different commonly used coordinate system, *polar coordinates*.

3.5.1 Points in Polar Coordinates

Assume we chose an origin in the plane, a direction that we call the positive x-axis, and some point along that ray that marks off unit distance. We now define coordinates for the rest of the plane based on these choices.

Definition 3.5.1.1. Plotting Points in Polar Coordinates

The point (θ, r) is the point located at an angle θ radians counterclockwise from the positive x-axis, a distance of r units from the origin.



Notice the angles are measured in the same manner as on the unit circle in trigonometry. The difference here is we allow any real number r as radius, rather than only radius one. We do allow r to be a negative number, in which case we travel “backwards” along the ray given by θ.

Example 3.5.1.2. Polar Coordinates are not Unique!

Be warned that any given point will have many different representations in polar coordinates. For example, consider the cartesian point (1,-1). In polar coordinates, we have many ways to represent this point. We can think of the angle as $\theta = -\pi/4$ and the radius as $r = \sqrt{2}$. We can also think of the angle as $\theta = 7\pi/4$ and the radius as $r = \sqrt{2}$. Yet another valid way to reach that same point is to use angle $\theta = 3\pi/4$ and the radius $r = -\sqrt{2}$. Thus, in polar coordinates we have that

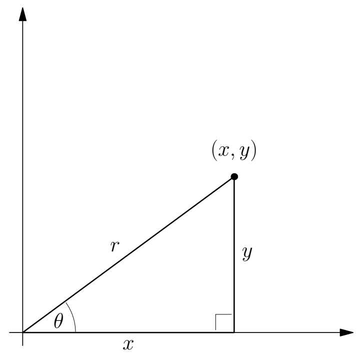
$$\left(-\pi/4, \sqrt{2}\right) = \left(7\pi/4, \sqrt{2}\right) = \left(3\pi/4, -\sqrt{2}\right)$$

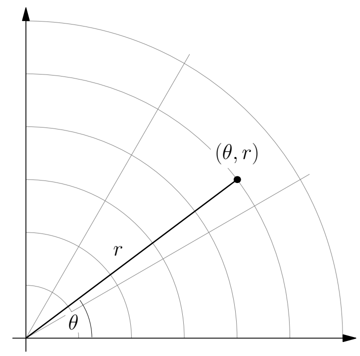
all represent the same point.

We now see how right-triangle trigonometry allows us to convert between polar coordinates and cartesian coordinates.

Exercise 3.5.1.3. Converting Between Polar and Cartesian ☕☕☕

- See the diagram below, with a point in QI labeled with both cartesian and polar measurements. For each of the conversion formulas listed below, write a short sentence afterwards explaining how it comes from the diagram.





  - $x^2 + y^2 = r^2$
  - $x = r\cos(\theta)$
  - $y = r\sin(\theta)$
  - $\tan(\theta) = \frac{y}{x}$
  - $\theta = \arctan\left(\frac{y}{x}\right)$

- If the point of interest were in a different quadrant, do the above formulas still hold? Do any of them require adjustment? Explain.

Exercise 3.5.1.4. Plotting in Polar ☕☕

- Plot the polar point $(5\pi/4{,}4)$. What are its cartesian coordinates?

- Consider the cartesian point (2,0). What are *all* possible ways of writing that point in polar coordinates?

Exercise 3.5.1.5. Do Any Points Have the Same Name? ☕☕☕

Do any points happen to have the same label in both polar and cartesian coordinates? Find all points that do, and explain why there are no more!

It is worth noting why "polar coordinates" are called what they are called. Cartesian coordinates look like a grid of horizontal and vertical lines. This is a great approximation of what latitude and longitude lines look like if you are standing at a random point on earth and think of your surroundings as approximated by a plane. But, if you are standing at the north or south pole, the latitude and longitude lines do not in any way look like a grid!

Exercise 3.5.1.6. Justifying the Name ☕☕

What do the lattitude and longitutde lines look like if you are standing at the north or south pole? Draw a small graph below.

Exercise 3.5.1.7. The Idea of Coordinate Systems ☕☕☕☕

Create another coordinate system for the plane that is not cartesian and is not polar! Describe your system of labeling all the points!

3.5.2 Graphs of Equations and Functions in Polar Coordinates

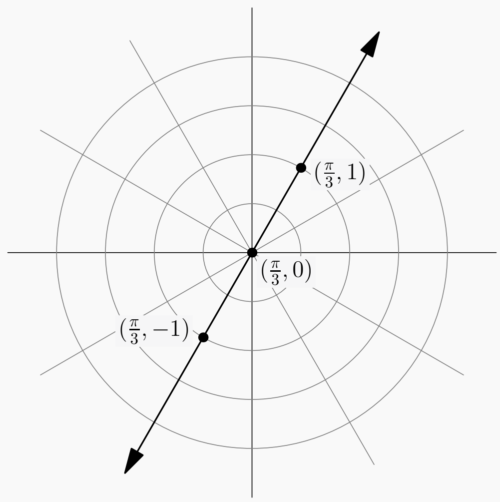
An equation in polar coordinates is an equality between expressions involving r and θ. We often wish to view the solutions visually by plotting all points (θ, r) in the plane that make the equations true (just as one would in cartesian).

Example 3.5.2.1. Graphing a Polar Equation

Suppose we wish to graph the equation

$$\theta = \pi/3$$

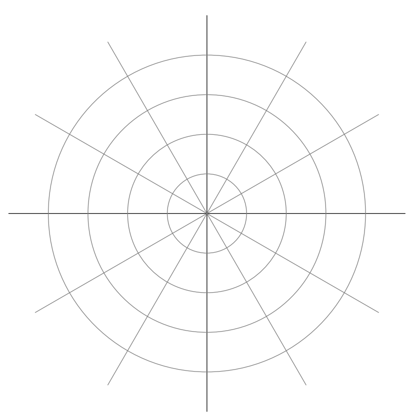
A point satisfies that equation if and only if the angle is $\pi/3$. The radius r is free to be any real number -positive, negative, or zero. For example, points that satisfy the equation include $(\pi/3, 1), (\pi/3, 0)$, and $(\pi/3, -1)$. Thus the graph is a line through the origin at 60° to the positive x-axis.



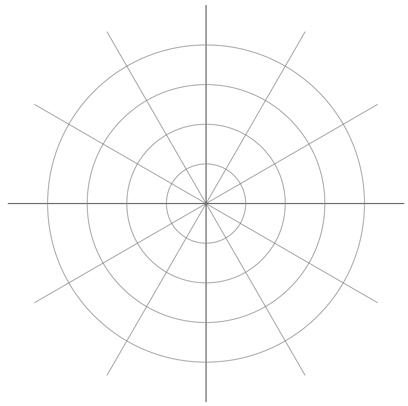
Exercise 3.5.2.2. Graphing Equations ☕☕

Graph the following equations.

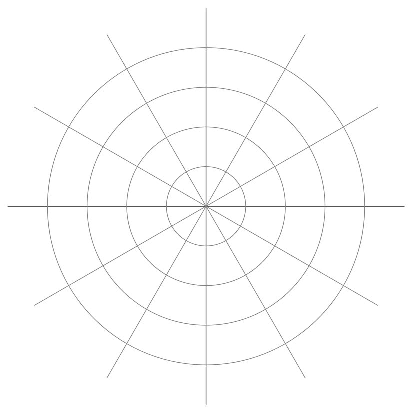
- $r = 2$



- $r = -2$



- $\theta^2 = \pi^2/4$



If the equation can be solved for r, we can consider r as a function of the independent variable θ. To graph a function, we simply make an input-output table of θ values and corresponding $r(\theta)$ values and plot the corresponding points $(\theta, r(\theta))$.

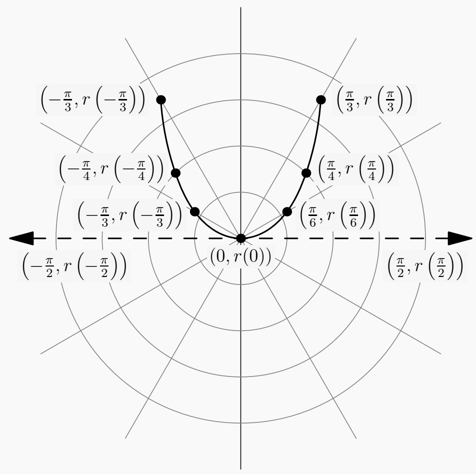
Example 3.5.2.3. Graphing a Polar Function

Plot the polar function

$$r(\theta) = \tan(\theta)$$

over the domain $-\pi/2 < \theta < \pi/2$. We select input values for θ that are clean unit circle values to plot.

θ	$-\pi/2$	$-\pi/3$	$-\pi/4$	$-\pi/6$	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$r(\theta)$	DNE	$-\sqrt{3}$	-1	$-\sqrt{3}/3$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	DNE



Exercise 3.5.2.4. Analyzing the Graph ☕☕

Does the graph appear to have any asymptotes? If so, where?

Looking at the graph prompts the question "can we find a Cartesian equation that describes the same set of points"? Here we use the formulas from Exercise 3.5.3 to rewrite all instances of r and θ in terms of x and y. Also, perhaps the cartesian equation can confirm our asymptote suspicions above!

Example 3.5.2.5. Converting to Cartesian

Let's find a cartesian equation for the graph of $r(\theta) = \tan(\theta)$ from the previous example. Since we do not have a particularly clean conversion formula for r itself but rather for r^2, it can be helpful to either multiply both sides by r or square both sides. In this case, squaring both sides will be cleaner so we take that path. Proceeding:

$$\begin{aligned}
r &= \tan(\theta) \\
r^2 &= (\tan(\theta))^2 \\
x^2 + y^2 &= \left(\frac{y}{x}\right)^2 \\
x^2 + y^2 &= \left(\frac{y}{x}\right)^2 \\
x^4 + x^2y^2 &= y^2 \\
x^4 + \left(x^2 - 1\right) y^2 &= 0 \\
y^2 &= \frac{x^4}{1 - x^2} \\
y &= \pm\frac{x^2}{\sqrt{1 - x^2}}
\end{aligned}$$

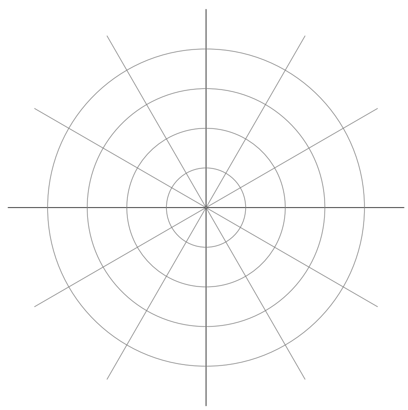
Exercise 3.5.2.6. Analyzing the Graph, Round II ☕☕

Does the cartesian formula tell you anything further about the apparent asymptotes on the graph?

The next exercise shows why converting a polar graph to cartesian coordinates can help analyze the geometry of the graph.

Exercise 3.5.2.7. Graphing a Function and Converting ☕☕☕

- Graph the function $r(\theta) = \sin(\theta)$. Does it look like a circle?



- Is it a circle? If so, what is the center and radius? Convert the equation to cartesian coordinates to confirm!

3.6 Derivatives in Polar Coordinates

Suppose we have the graph of a polar function $r(\theta)$, and we would like to find the slope of the tangent line at a point. We can consider this graph to be a parameterized curve by treating $t = \theta$ as the parameter. Specifically, the parameterization is given by

$$x(t) = r(t)\cos(t)$$
$$y(t) = r(t)\sin(t).$$

Exercise 3.6.0.1. Deriving the Derivative ☕☕

Use the formula for the derivative of a parametric curve to find the formula for the derivative of a polar graph.

Exercise 3.6.0.2. Using the Formula ☕☕

Use the polar derivative formula above to find the slope of the graph of $r(\theta) = \sec(\theta)$. What does this let you conclude about that graph?

3.7 Area in Polar Coordinates

To compute area in polar coordinates, we essentially repeat the process of taking a Riemann sum. Rather than using rectangles however, we use sectors of circles.

Exercise 3.7.0.1. Area of a Single Sector ☕☕

- What is the area of an entire circle with radius r? Draw the circle.

- Within your circle, draw a sector of that circle with angle θ. What proportion of the area of the entire circle does that sector occupy?

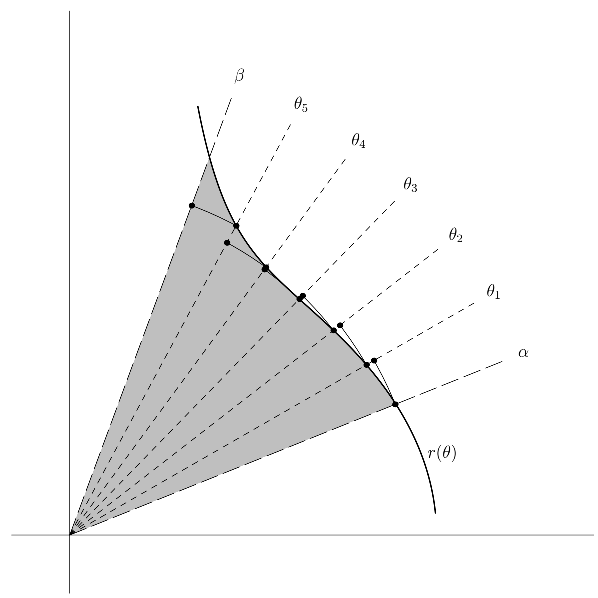
- Explain why the area of that sector is $A = \frac{1}{2}r^2\theta$.

We now repeat the process of taking a Riemann sum using sectors of circles. In particular, say we wish to find the area under the graph of $r(\theta)$ between two rays specified by angles $\theta = \alpha$ and $\theta = \beta$.

Let $\theta_0, \theta_1, \theta_2, \ldots, \theta_n$ be equally spaced angles from α to β. That is, $\theta_0 = \alpha$, $\theta_n = b$, and for each $i \in \{0, 1, 2, \ldots, n-1\}$, $\Delta\theta = \theta_{i+1} - \theta_i = \frac{\beta-\alpha}{n}$.

$$
\begin{aligned}
A &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{2} r^2(\theta_i)\,\Delta\theta \\
&= \frac{1}{2} \lim_{n\to\infty} \sum_{i=0}^{n-1} r^2(\theta_i)\,\Delta\theta \\
&= \frac{1}{2} \int_{\theta=\alpha}^{\theta=\beta} r^2(\theta)\,\mathtt{d}\theta
\end{aligned}
$$

Thus, we have the formula for polar area!



Theorem 3.7.0.2. Polar Area

The area under a polar curve $r(\theta)$ between $\theta = \alpha$ and $\theta = \beta$ is

$$A = \frac{1}{2} \int_{\theta=\alpha}^{\theta=\beta} r^2(\theta)\,\mathtt{d}\theta$$

Sometimes a helpful way to remember the above formula is to write it as

$$A = \int_{\theta=\alpha}^{\theta=\beta} \pi r^2(\theta) \frac{\mathtt{d}\theta}{2\pi}.$$

This way you can think of the integrand as the area of a circle being multiplied by what ratio of 2π radians the change in θ is occupying. Canceling the two π's and pulling the $\frac{1}{2}$ outside of the integral lands you back at the Polar Area formula.

Exercise 3.7.0.3. Looking for Patterns ☕☕☕

Fill out the table! Carry out the instructions listed below for each given value of n.

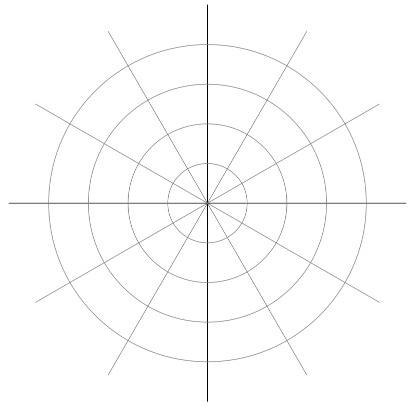
- Plot the graph of $r(\theta)$.
- Find the area inside just one "petal".
- What patterns in n do you see? What can you say about the percent of the unit circle that lies inside rather than outside the graph?

n	Graph of $r(\theta) = \sin(n\theta)$	Area of One Petal
2		
3		
4		
5		
n		

n	Graph of $r(\theta) = \cos(n\theta)$	Area of One Petal
2		
3		
4		
5		
n		

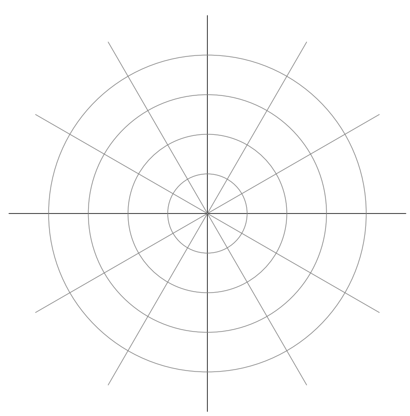
Exercise 3.7.0.4. Area Bounded by Two Polar Curves ☕☕☕

Plot both $r_1(\theta) = \frac{1}{2}\sec(\theta)$ and $r_2(\theta) = \cos(\theta)$ on the same set of axes. Find the area of the region that is to the right of r_1 but inside r_2.



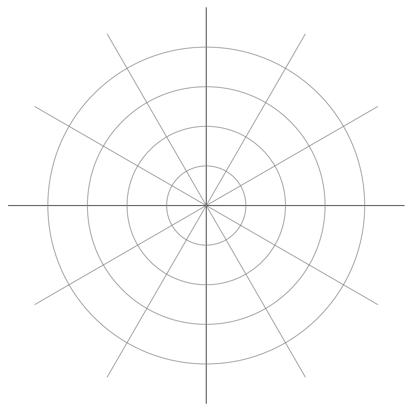
Exercise 3.7.0.5. Mixed Practice with Polar Curves ☕☕☕

- – Sketch the graph of $r(\theta) = 2\cos(2\theta)$.



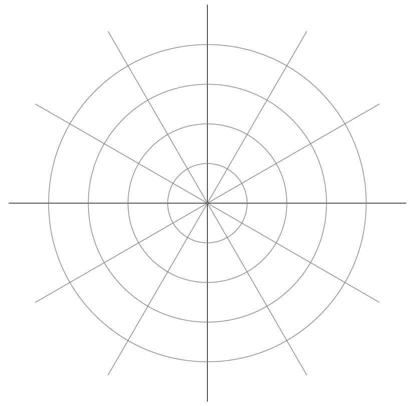
  - Convert the above curve to cartesian. That is, find a polynomial equation in x and y whose solution set describes the same set of points. (**Hint:** Begin by applying the cosine double-angle identity!)

- – Sketch the graph of $r(\theta) = \frac{1}{2} + \cos(\theta)$.



  - Find the area enclosed by the inner loop of the graph.

- – Plot the graphs of both $r_1(\theta) = 1 + \cos(\theta)$ and $r_2(\theta) = 1 - \cos(\theta)$ on the same axes.



– Shade the region contained inside both curves. Find its area.

3.8 Microphone Design

A microphone is a device that picks up sound (variations in air pressure) and produces an electrical signal. For any microphone, sound engineers want what is called the *polar pattern*, a graph indicating all locations from which sound is picked up with equal intensity. Microphones that are physically designed differently will have different polar patterns.

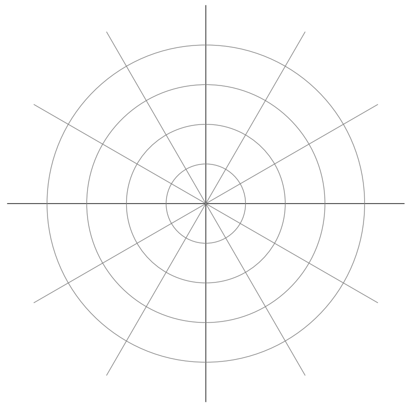
The key element to a microphone is some mechanical device that the waves of air pressure can compress. There are two basic types of devices:

3.8.1 Diaphragm

A spherical diaphragm responds equally to changes in air pressure from any side. Thus given a sound of a particular volume, the response in the microphone sensitivity is proportional to the distance from the diaphragm. A microphone with such a diaphragm is called an *omnidirectional microphone* and is represented by the polar pattern $r(\theta) = 1$, since the microphone has equal sensitivity to all points on a circle.

Exercise 3.8.1.1. Diaphragm Microphones

Plot the polar pattern for a diaphragm microphone function below.

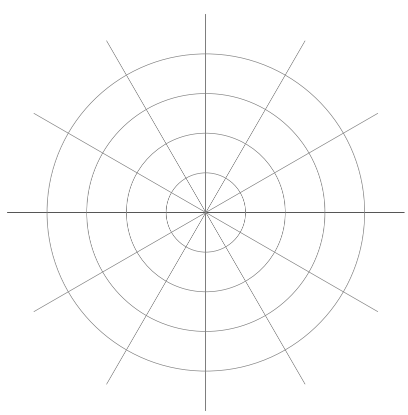


3.8.2 Ribbon

The other main type of device is a ribbon that floats in a magnetic field. Since it is a horizontal ribbon, it picks up changes in air pressure proportion to the sine of the angle to the source. (Imagine for example in physics a force pushing on a wall at an angle... the force that goes into the wall is not equal to the magnitude of the whole force but rather the magnitude times sine of the angle.) A microphone equipped with such a device is called a *ribbon microphone* or a *figure eight microphone* and has polar pattern given by $r(\theta) = |\sin(\theta)|$. Here we are taking absolute values because we are just denoting sensitivity, not the wave itself.

Exercise 3.8.2.1. Ribbon Microphones

Plot the polar pattern for a ribbon microphone function below.



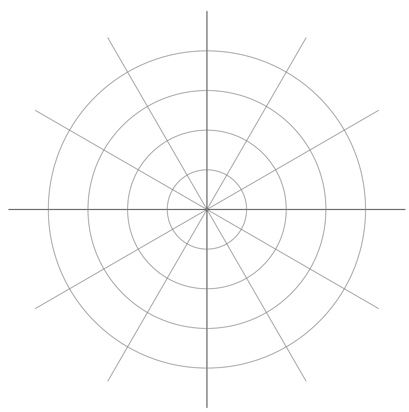
3.8.3 Cardioid

There are many situations where one of the above microphones is perfect for the purpose at hand. However, when a band is playing live music on a stage, the above two microphones do not work. The basic setup is the following: if a singer sings into the microphone, the main speakers are pointed towards the audience and not towards the singer. Thus, it is necessary to have monitors (smaller speakers pointing the opposite direction) so that the singer can hear herself. However, if the microphone picks up the sound coming out of the monitor, it's going to be again reproducing the same sound it just heard. The waves combine amplitude again and again, and this leads to that horrible high-pitched screeching noise known as feedback.

The solution to this is to design a microphone that picks up sound from one side but not from the other. The ingenious way engineers figured out how to do this was to simply make a microphone with *both* a ribbon and a diaphragm inside! The waves produced add to each other to make a single signal. Thus the sine of the ribbon will combine with the diaphragm's signal on one side, but cancel it out on the other! The polar pattern is given by the function $r(\theta) = 1 + \sin(\theta)$ (adding the waves together). Such a mic is called a *cardioid microphone* and is the standard mic for onstage live sound. The Shure 57 and Shure 58 are cardioid microphones and have been the standard mic used onstage for about 40 years now!

Exercise 3.8.3.1. Cardiod Microphones ☕☕

- Plot the polar pattern corresponding to the above described cardioid microphone.



- Find the area of the region where sounds are at least as sensitive as they are on the boundary of that cardioid. (That is, find the area enclosed by the above polar curve.)

3.9 Chapter Summary

Here we introduced two new languages for describing curves in the plane, parametric and polar.

1. **Parametric:** A **parametric curve** is the set of points $(x(t), y(t))$ for some specified domain of t values.

   (a) **Graphing:** Pick a helpful spread of t values and plot the resulting points $(x(t), y(t))$ to get some idea of the shape. Often **converting to cartesian** by eliminating t and finding a direct relationship between x and y can be helpful.

   (b) **Derivatives:** The **slope of the tangent line to a parametric curve** can be found by $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\,\mathrm{d}t}{\mathrm{d}x/\,\mathrm{d}t}$.

   (c) **Integrals:** The **length of a parametric curve** can be found by integrating the distance formula. If the parameter domain is the closed interval $D = [a, b]$, then the length is

$$\int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,\mathrm{d}t.$$

2. **Polar:** The system of **polar coordinates** describes the plane as (θ, r) where θ is the counterclockwise angle from the positive x axis and r is the signed distance from the origin.

   (a) **Graphing:** Given a polar function $r(\theta)$, pick a helpful spread of θ values and plot the resulting points $(\theta, r(\theta))$ to get some idea of the shape. Often **converting to cartesian** can be helpful. To convert, use the relationships

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \\ r^2 &= x^2 + y^2 \end{aligned}$$

   or any other helpful relationship that follows from the triangle with angle θ, adjacent side x, opposite side y, and hypothenuse r.

   (b) **Derivatives:** To find the **slope of the tangent line to a polar graph**, convert to parametric by letting $t = \theta$. Specifically, set

$$\begin{aligned} x(t) &= r(t)\cos(t) \\ y(t) &= r(t)\sin(t) \end{aligned}$$

   and then use the formula for a parametric derivative.

   (c) **Integrals:** To find **area under a polar graph**, perform a Riemann sum with sectors of circles (rather than rectangles as we did initially). The π cancels to give us our polar area formula as seen below.

$$\begin{aligned} A &= \int_{\theta=\alpha}^{\theta=\beta} \underbrace{\pi r^2(\theta)}_{\text{Area of a circle}} \underbrace{\frac{\mathrm{d}\theta}{2\pi}}_{\text{proportion of full circle's radians}} \\ &= \frac{1}{2}\int_{\theta=\alpha}^{\theta=\beta} r^2(\theta)\,\mathrm{d}\theta \end{aligned}$$

3.10 Mixed Practice

3.10.1 Warm Ups

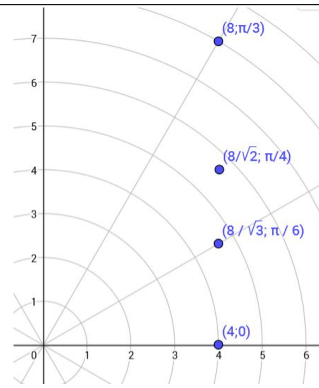
These are good problems for reinforcing the vocabulary and foundational concepts of this chapter.

Exercise 3.10.1.1. ☕☕

a.) Graph the polar function $r(\theta) = 4\sec(\theta)$ by picking a spread of θ values and making an input-output table.

A Solution:

θ	0	$\pi/6$	$\pi/4$	$\pi/3$
$r(\theta)$	4	$8/\sqrt{3}$	$8/\sqrt{2}$	8



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b.) Convert to cartesian coordinates to show that the graph is in fact just a line!

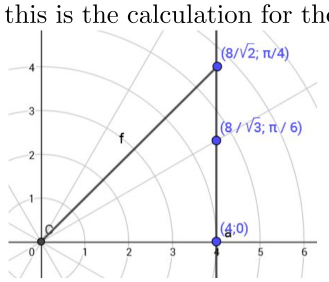
A Solution: $r = 4\sec\theta \Longrightarrow r\cos\theta = 4 \Longrightarrow x = 4$ is a vertical line.

---

c.) Sketch the polar region whose area corresponds to the following integral:

$$A = \int_{\theta=0}^{\theta=\pi/4} \frac{1}{2}\left(4\sec(\theta)\right)^2 \,\mathrm{d}\theta$$

A Solution: Since this is the calculation for the area under the curve $4\sec\theta$ in polar form, we have the region:



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d.) What shape is the region sketched in c)? Find the area by basic geometry.

A Solution: It is an isosceles triangle with hypotenuse $4\sqrt{2}$ and sides 4 so $A = \frac{1}{2}b \cdot h = \frac{1}{2}4 \cdot 4 = 8$ so we get

e.) Find the area by computing the integral. Verify your answers match.

A Solution: $A = \int_0^{\pi/4} \frac{1}{2}(4\sec\theta)^2 \, d\theta = \int_0^{\pi/4} \frac{1}{2}(16\sec^2\theta) \, d\theta = 8\tan\theta\Big|_0^{\pi/4} = 8(\tan\pi/4 - \tan 0) = 8 \cdot 1 - 8 \cdot 0 = 8$ They are the same.

Exercise 3.10.1.2. ☕☕

Consider the parameterization

$$\{(4t-1, 6t) : t \in [0, 2]\}$$

That is, $x(t) = 4t - 1$ and $y(t) = 6t$ as t roams from 0 to 2.

a.) Describe the shape of this parametric curve.

A Solution: It is a line segment that lies on the line $\frac{x-1}{4} = t = \frac{y}{6} \leftrightarrow y = \frac{3}{2}x - \frac{3}{2}$ between $(-1, 0)$ and $(7, 12)$

b.) What is the slope of the tangent line to the parametric curve at $t = 1$?

A Solution: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6}{4} = \frac{3}{2}$ which is the slope of the line.

c.) Use the parametric arc length formula to compute the length of the parameterized curve from part a).

A Solution: $L = \int_{t=0}^{t=2} \sqrt{(x'(t))^2 + (y'(t))^2} \, dt = \int_0^2 \sqrt{4^2 + 6^2} \, dt = \int_0^2 \sqrt{52} \, dt = 2\sqrt{13}t\Big|_0^2 = 4\sqrt{13}$

3.10.2 Sample Test Problems

Exercise 3.10.2.1.

State the power series definitions for hyperbolic sine and cosine:

- $\cosh(t) =$
- $\sinh(t) =$

A Solution: $\cosh t = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \cdots$
$\sinh t = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots$

---

a.) Use the power series for hyperbolic sine to compute its derivative.

A Solution: $\frac{d \ \sinh(t)}{dt} = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots = \cosh(t)$

b.) Use the power series for hyperbolic cosine to compute its derivative.

A Solution: $\frac{d \ \cosh(t)}{dt} = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots = \sinh(t)$

c.) Consider the parametric curve

$$\{(\cosh(t), \sinh(t)) : t \in \mathbb{R}\}$$

Verify out to degree six that this parametric curve also satisfies the cartesian equation for the same hyperbola given by:

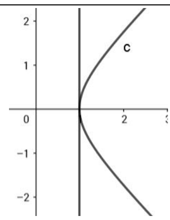
$$x^2 - y^2 = 1$$

by plugging the power series formulas for our hyperbolic trig functions in for x and y.

A Solution: $\cosh^2 t - \sinh^2 t = \left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \cdots\right)^2 - \left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right)^2 = \left(1 + \frac{t^2}{2!} + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^4}{4!} + \frac{t^4}{2!2!} + \frac{t^6}{6!} + \frac{t^6}{6!} + \frac{t^6}{4!2!} + \frac{t^6}{4!2!} + \cdots\right) - \left(t^2 + \frac{t^4}{3!} + \frac{t^4}{3!} + \frac{t^6}{3!3!} + \frac{t^6}{5!} + \frac{t^6}{5!} + \cdots\right) = 1 - \left(\frac{2}{2!} - 1\right) t^2 + \left(\frac{2}{4!} + \frac{1}{2!2!} - \frac{2}{3!}\right) t^4 + \left(\frac{2}{6!} + \frac{2}{4!2!} - \frac{1}{3!3!} - \frac{2}{5!}\right) t^6 + \cdots = 1 + (1-1)t^2 + \left(\frac{1}{12} + \frac{1}{4} - \frac{1}{3}\right) t^4 + \left(\frac{1}{6\cdot5\cdot4\cdot3} + \frac{1}{4\cdot3\cdot2} - \frac{1}{36} - \frac{1}{5\cdot4\cdot3}\right)^6 + \cdots = 1 + 0t^2 + \left(\frac{1+3-4}{12}\right) t^4 + \left(\frac{1}{6\cdot5\cdot4\cdot3} + \frac{15}{6\cdot5\cdot4\cdot3} - \frac{10}{6\cdot5\cdot4\cdot3} - \frac{6}{6\cdot5\cdot4\cdot3}\right) t^6 \cdots = 1 + 0t^2 + 0t^4 + 0t^6 + \cdots = 1$

d.) Find the slope of the tangent line at $t = 0$ to the parametric curve using your derivatives computed above.

A Solution: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cosh(t)}{\sinh(t)}$ Evaluate at $t = 0$ to get $\frac{\cosh(0)}{\sinh(0)} = \frac{1+\frac{0^2}{2!}+\frac{0^4}{4!}+\frac{0^6}{6!}+\cdots}{0+\frac{0^3}{3!}+\frac{0^5}{5!}+\cdots} = \frac{1}{0}$ which is a vertical line e.) Graph the parametric curve along with the tangent line you found in the previous part.

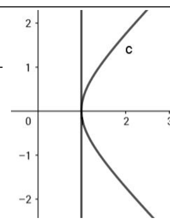
A Solution: 

Exercise 3.10.2.2.

a.) Graph the polar function $r(\theta) = \sin(2\theta)$.

A Solution:

θ	$r(\theta)$
0	1
$\pi/4$	0
$\pi/2$	-1
$3\pi/4$	0
π	1
$5\pi/4$	0
$3\pi/2$	-1
$7\pi/4$	0
2π	1



b.) Find the area enclosed by one loop of that function.

A Solution: $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}(\cos 2\theta)^2 \, d\theta = \frac{1}{2}\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \cos^2(2\theta) \, d\theta = \frac{1}{2}\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{1+\cos(4\theta)}{2} \, d\theta =$
$\frac{1}{4}\left(\theta + \frac{\sin(4\theta)}{4}\right)\Bigg|_{\frac{-\pi}{4}}^{\frac{\pi}{4}} = \frac{1}{4}\left(\frac{\pi}{4} + \frac{\sin(\pi)}{4}\right) - \frac{1}{4}\left(\frac{-\pi}{4} + \frac{\sin(-\pi)}{4}\right) = \frac{\pi}{16} + \frac{\pi}{16} = \frac{\pi}{8}$

Chapter 4

Introduction to Differential Equations

In this course, we got really good at two things: finding antiderivatives and using power series. It is no accident that the study of differential equations relies primarily on those two techniques! Here we show just two methods for solving differential equations: separation of variables, based on antidifferentiation, and power series solutions, based on power series (really!).

4.1 What is a Differential Equation?

Definition 4.1.0.1. Differential Equation

A *differential equation* (DE) is an equation involving a variable (say y) that stands for some unknown function, and also involving one or more derivatives of y. The *solution* to a differential equation is the set of all functions y that make the equation true.

We begin with a nice bridge troll riddle. We ask "What functions are equal to their own derivative?".

Example 4.1.0.2. Functions Equal to Their Own Derivative

To state this question in the language of differential equations, we say that we wish to solve the DE

$$y' = y.$$

Exercise 4.1.0.3. Guess and Check ☕

Can you think of any functions that are equal to their own derivative? Do you think you have all of them, or are some likely still out there?

As you can see, guess and check is not a good method for solving even the simplest of differential equations. We now take a more structured approach.

4.2 Separable Equations

Definition 4.2.0.1. Separable

Let x be the independent variable and let y represent an unknown function of x. A differential equation is *separable* if and only if it can be written in the form

$$\frac{dy}{dx} = F(x)G(y)$$

for some functions F and G.

Our method for solving a separable differential equation is as follows:

1. Write right-hand side of the differential equation in factored form, one function of x times one function of y.

2. Separate variables by multiplying both sides by $\frac{1}{G(y)}\,\mathtt{d}x$.

3. Antidifferentiate both sides.

4. Solve for y, if possible. (If not, we at least have an implicit solution.)

We try out this method on the previous example.

Example 4.2.0.2. Separation of Variables

Notice the differential equation

$$y' = y$$

is separable because it can be rewritten as

$$\frac{\mathtt{d}y}{\mathtt{d}x} = (1)(y).$$

That is, our factored form uses the functions $F(x) = 1$ and $G(y) = y$. We now perform separation of variables and antidifferentiate both sides.

$$\begin{aligned}
\frac{1}{y}\,\mathtt{d}y &= 1\,\mathtt{d}x \\
\int \frac{1}{y}\,\mathtt{d}y &= \int 1\,\mathtt{d}x \\
\ln|y| &= x + C \\
|y| &= e^{x+C} \\
y &= \pm e^{C} e^{x} \\
y &= Ce^{x}
\end{aligned}$$

Notice that on the last line for simplicity, we clean up the constant $\pm e^C$ by just calling it C.

Exercise 4.2.0.3. Analyzing the Example ☕

- Why were we able to just put a $+C$ on one side when we integrated? What would have happened if we put it on both sides?
- When we renamed $\pm e^C$ as C, we technically introduced a new solution. The expression $\pm e^C$ is incapable of being equal to zero, but C can be. Verify that the $C = 0$ solution is valid to include as a solution to the differential equation.

Exercise 4.2.0.4. More Complicated DEs ☕☕☕

- Solve the following differential equation via separation of variables:

$$\frac{dy}{dx} = xy + x$$

- Solve the following Initial Value Problem via separation of variables:

$$\frac{dy}{dx} = e^{y-x}\sec(y)(1+x^2)$$

  Note that you will not be able to obtain an explicit formula for y in terms of x but rather an implicit solution. Use the initial condition $y(0) = 0$ to solve for C.

4.3 Power Series Solutions

Power series provide a very effective method for solving differential equations. The steps are simple:

- Set the unknown function y equal to an unknown power series.
- Plug the power series in for all occurrences of y. Expand and combine like terms.
- Equate coefficients one degree at a time (much like we do when solving for unknowns in a PFD).
- Solve for the coefficients a_0, a_1, $a_2, \ldots$ one at a time in terms of a_0.
- Plug those coefficients back into the power series expansion for y to obtain a power series solution.
- Try to recognize that power series as a known function or variant thereof.

The process is thus very mechanical, but sometimes working through the details becomes a bit messy. We repeat the previous example with this new method.

Example 4.3.0.1. Revisiting Our First DE

Here we solve

$$y' = y$$

using power series. First, let y be written as a generic unknown power series as follows:

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

Plug this expression into the differential equation and expand.

$$\begin{aligned}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots\right)' &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \\ a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots\end{aligned}$$

We now equate coefficients one degree at a time, and solve for every coefficient in terms of a_0.

$$\begin{array}{lrcl} \text{Degree 0:} & a_1 = a_0 & \Longrightarrow & a_1 = a_0 \\ \text{Degree 1:} & 2a_2 = a_1 & \Longrightarrow & a_2 = \frac{1}{2} a_0 \\ \text{Degree 2:} & 3a_3 = a_2 & \Longrightarrow & a_3 = \frac{1}{3!} a_0 \\ \text{Degree 3:} & 4a_4 = a_3 & \Longrightarrow & a_4 = \frac{1}{4!} a_0 \\ \vdots & \vdots & & \vdots \\ \text{Degree } n-1\text{:} & n a_n = a_{n-1} & \Longrightarrow & a_n = \frac{1}{n!} a_0 \end{array}$$

We can now plug all coefficients back into our expression for y and simplify until we obtain a closed form for y.

$$\begin{aligned} y(x) &= a_0 + a_0 x + \frac{1}{2!} a_0 x^2 + \frac{1}{3!} a_0 x^3 + \frac{1}{4!} a_0 x^4 + \cdots \\ &= a_0 \left(1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \cdots\right) \\ &= a_0 e^x \end{aligned}$$

Notice we have obtained the same solution as via separation of variables! Clearly, the power series solution was way more work. The reason it is so valuable though is that there are many DEs which are not separable but for which the power series method works just fine.

Exercise 4.3.0.2. Comparing the Methods ☕☕☕

- - Show the differential equation $\frac{dy}{dx} = yx$ is separable and use this to separate variables and solve the differential equation.
  - Solve the same differential equation via power series. Confirm you get the same answer.
- - Explain why the differential equation $\frac{dy}{dx} = yx + x + 1$ is not separable.

- Solve the same differential equation via power series.

- Check your answer is correct by plugging it back into the original DE.

- Consider the DE given by

$$y(0) = 1$$
$$y'(0) = 0$$
$$y'' = -y.$$

Solve this DE via power series (use the initial conditions to solve for a_0 and a_1).

4.4 Modeling with Differential Equations

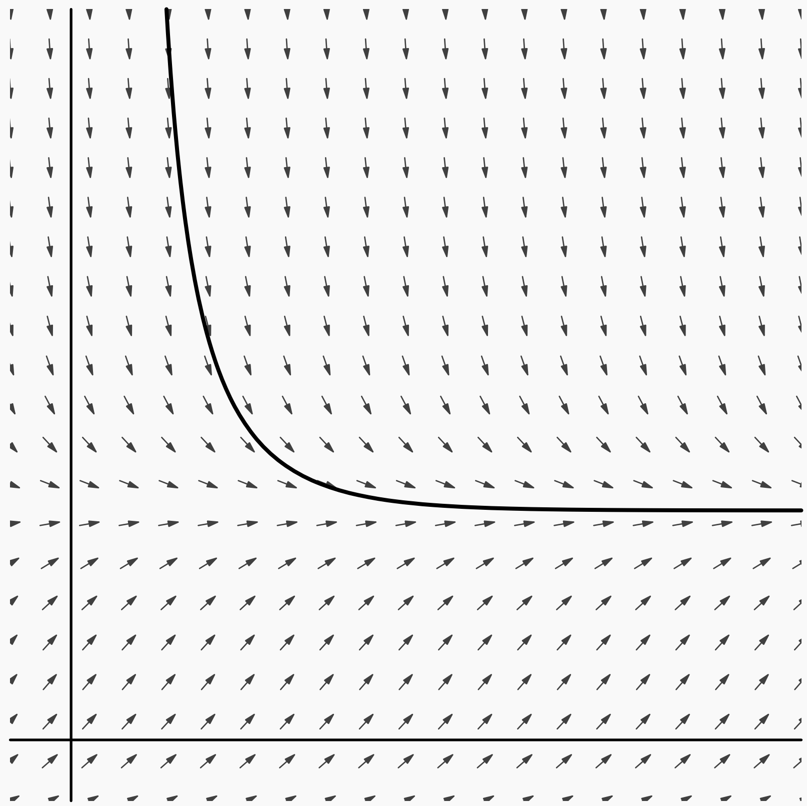
Differential equations are used extensively in applied mathematics and the sciences to describe models, which are then solved using mathematics to find explicit formulas for the quantities of interest.

Example 4.4.0.1. Newton's Law of Cooling

Newton's Law of Cooling states that the rate of change of the temperature of a small object in a room is proportional to the difference between room temperature and the temperature of the object. If A is the constant that represents the ambient temperature (room temperature), $T(t)$ represents the temperature of the room at time t, and k is the constant of proportionality, then this situation can be modeled by

$$\frac{dT}{dt} = k(T - A).$$

Here we introduce the idea of a *slope field*, a grid of small dashes that indicate the slope $\frac{\mathrm{d}T}{\mathrm{d}t}$ at every point (t, T) in the plane. Here we draw a slope field that governs solution curves to this model and show one sample solution curve.



Exercise 4.4.0.2. Newton's Law of Cooling ☕☕

- Label the above diagram. What variables do the axes correspond to? Can you find where the horizontal line $T = A$ is located?

- In this model, would it make sense that the proportionality constant k is positive or negative? Why?

- Solve the differential equation by separation of variables.

- Solve the differential equation by power series.

Note that those solutions give explicit formulas for the solution curves above, which is significantly more useful than just thinking of it intuitively as "following the arrows".

Exercise 4.4.0.3. Malthusian Population Model

A simple intuitive population model can be stated as follows:

If there are more individuals in a population, there will be more babies produced.

Here is a slightly more technical restatement of the same idea:

The rate of growth of a population is proportional to the size of the population.

- Let $P(t)$ be the size of the population at time t. Rewrite the above growth principal in the language of differential equations.

- Solve your differential equation using power series.

- Solve your differential equation using separation of variables and confirm that your answers match.

- Use your formula to find the limit of $P(t)$ as t approaches infinity.

- Under what real life conditions might this model be realistic? Under what conditions might this model be unrealistic?

As you probably noticed, the above model is slightly ridiculous for large time values since it would claim that eventually any species would fill up the entire visible universe with bodies. So let's adjust it to fix that unrealistic assumption. Here's an upgrade:

Exercise 4.4.0.4. Logistic Population Model ☕☕☕

If there are more individuals in a population, there will be more babies produced, but then it slows down as it approaches some sort of maximum possible population (a limit perhaps based on food supply, available habitat, etc).

Here is a slightly more technical restatement of the same idea:

The rate of growth of a population is jointly proportional to both the size of the population and the distance from some maximum possible population.

- Let $P(t)$ be the size of the population at time t and let M for maximum be a constant that the population cannot exceed. Rewrite the above growth principal in the language of differential equations.

- Solve your differential equation using power series out to a degree two approximation (this will be much too difficult to solve the whole thing using power series!).

- Solve your differential equation using separation of variables and confirm that your answers match out to the degree two approximation.

- Use your formula to find the limit of $P(t)$ as t approaches infinity.

- Suppose you started with population $P(0) = 2M$. What would your model predict would happen to the population?

4.5 Chapter Summary

A **differential equation** is an equation involving an unknown function $y(x)$ and one or more of its derivatives. The goal is to solve for an infinite family of functions $y(x)$ that satisfy the equation, or to find just a single function that solves the equation if a suitable **initial condition** is provided. There are many methods for solving DEs, but we focused on two in particular.

1. **Separation of variables:** To solve via separation of variables, we follow the following steps:

   - Write right-hand side of the differential equation in factored form, producing a DE in the form $\frac{dy}{dx} = F(x)G(y)$ (if possible).
   - Separate variables by multiplying both sides by $\frac{1}{G(y)}\,\mathtt{d}x$.
   - Antidifferentiate both sides.
   - Solve for y, if possible. (If not, we at least have an implicit solution.)

   This method is usually quite straightforward to carry out. The drawback is that most Differential Equations are not separable, meaning that it is impossible to write it as $\frac{dy}{dx} = F(x)G(y)$. Thus, the method usually fails as soon as it gets started.

2. **Power series solutions:** This method is far more robust as it does not depend on the DE being separable. It applies to more DEs, but be warned it is typically far messier! The steps are as follows:

   - Set the unknown function y equal to an unknown power series:

   $$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots .$$

   - Plug the power series into the DE for all occurrences of y. Expand and combine like terms.
   - Equate coefficients one degree at a time. This will create an infinite system of equations of the following form:

   $$\begin{aligned}
   \text{Left-hand side degree zero coefficient} &= \text{Right-hand side degree zero coefficient}\\
   \text{Left-hand side degree one coefficient} &= \text{Right-hand side degree one coefficient}\\
   \text{Left-hand side degree two coefficient} &= \text{Right-hand side degree two coefficient}\\
   \text{Left-hand side degree three coefficient} &= \text{Right-hand side degree three coefficient}\\
   &\vdots
   \end{aligned}$$

   - Solve for the coefficients a_0, a_1, $a_2, \ldots$ one at a time in terms of a_0. (If you have an initial condition, a_0 will just be a number. Otherwise leave everything in terms of a_0.)
   - Plug those coefficients back into the power series expansion for y to obtain a power series solution.
   - Try to recognize that power series as a known function or variant thereof.

   One can visualize solutions to DEs via a **slope field**, a grid of arrows that shows the value of $\frac{\mathtt{d}y}{\mathtt{d}x}$ at each point as the slope of the arrow. The solutions to the differential equation are functions that essentially follow the directions given by the arrows, starting at some initial condition.

4.6 Mixed Practice

4.6.1 Warm Ups

These are good problems for reinforcing the vocabulary and foundational concepts of this chapter.

Exercise 4.6.1.1. ☕

Find all functions that equal their own second derivative. That is to say, use power series to solve the following differential equation:

$$y'' = y$$

A Solution: Let $y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ then $y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots$ and $y'' = 2a_2 + 3 \cdot 2a_3x + 4 \cdot 3a_4x^2 + 5 \cdot 4a_5x^3 \cdots$ Equate coefficients one degree at a time:
$a_0 = 2a_2 \to a_2 = \frac{a_0}{2}$
$a_1 = 3 \cdot 2a_3 \to a_3 = \frac{a_1}{3!}$
$a_2 = 4 \cdot 3a_4 \to a_4 = \frac{a_0}{4!}$
$a_3 = 5 \cdot 4a_5 \to a_5 = \frac{a_1}{5!}$
So we have if n is odd then $a_n = \frac{a_1}{n!}$ and if n is even, we have $a_n = \frac{a_0}{n!}$
Put it all together to get $y = a_0 + a_1x + \frac{a_0}{2}x^2 + \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \cdots$ notice if you group the even and odd degrees you get $y = (a_0 + \frac{a_0}{2}x^2 + \frac{a_0}{4!}x^4 + \cdots) + (a_1x + \frac{a_1}{3!}x^3 + \frac{a_1}{5!} + \cdots) = a_0\cosh(x) + a_1\sinh(x)$
Thus, linear combinations of hyperbolic sine and hyperbolic cosine functions are the only functions that equal their own second derivatives.

4.6.2 Sample Test Problems

Exercise 4.6.2.1. ☕☕☕

a.) Find the set of all solutions to the following differential equation using power series. Do not leave your answer as a power series but rather turn it back into a closed explicit formula using familiar functions.

$$\frac{dy}{dx} = y - x - 2$$

A Solution: Let $y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ a power series. Then $y' = \frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + \cdots$ So we have $a_1 + 2a_2x + 3a_3x^2 + \cdots = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots - x - 2 = (a_0 - 2) + (a_1 - 1)x + a_2x^2 + a_3x^3 + \cdots$ Which means $a_1 = a_0 - 2$
$2a_2 = (a_1 - 1) = (a_0 - 2 - 1) = a_0 - 3$ so $a_2 = \frac{a_0 - 3}{2}$
$3a_3 = a_2 = a_2 = \frac{a_0-3}{2}$ So $a_3 = \frac{a_0-3}{3!}$
$4a_4 = a_3 = \frac{a_0-3}{3!}$ so $a_4 = \frac{a_0-3}{4!}$ etc
so we have $y = a_0 + (a_0 - 2)x + \frac{a_0-3}{2}x^2 + \frac{a_0-3}{3!}x^3 + \frac{a_0-3}{4!}x^4 + \cdots = a_0 + (a_0 - 2)x + (a_0 -$

$3)\frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!}x^4 + \cdots = a_0 + (a_0 - 2)x + (a_0 - 3)\sum_{x=0}^{\infty} \frac{x^n}{n!} - (a_0 - 3) - (a_0 - 3)x$ The part $-(a_0 - 3) - (a_0 - 3)x$ is necessary to exclude the first 2 terms of $(a_0 - 3)\sum_{x=0}^{\infty} \frac{x^n}{n!} = (a_0 - 3)e^x$ So we actually have $y = a_0 - (a_0 - 3) + (a_0 - 2)x - (a_0 - 3)x + (a_0 - 3)e^x = 3 + x + (a_0 - 3)e^x$

---

b.) Plug your answer back into the DE to verify it is correct.

A Solution: If $y = 3 + x + (a_0 - 3)e^x$ then $\frac{dy}{dx} = 1 + (a_0 - 3)e^x$ but $y - x - 2 = 3 + x + (a_0 - 3)e^x - x - 2 = 1 + (a_0 - 3)e^x$ So they match.

Exercise 4.6.2.2. ☕☕☕

a.) Explain why one cannot use separation of variables to solve the differential equation

$$\frac{dy}{dx} = 2y + x$$

A Solution: The right-hand side $2y+x$ does not factor into a function of y times a function of x, so the equation is not separable.

---

b.) Solve the above differential equation using power series. Recognize your answer as a known function!

A Solution: Let $y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ then $\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots$ so we have $a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots = 2(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots) + x = 2a_0 + (2a_1 + 1)x + 2a_2x^2 + 2a_3x^3 + 2a_4x^4 + \cdots$ set corresponding coefficients equal to each other to get:
$a_1 = 2a_0$
$2a_2 = 2a_1 + 1 = 4a_0 + 1 \to a_2 = \frac{4a_0+1}{2}$
$3a_3 = 2a_2 = 8a_0 + 2 \to a_3 = \frac{8a_0+2}{3!}$
$4a_4 = 2a_3 = \frac{16a_0+4}{3!} \to a_4 = \frac{16a_0+4}{4}$
$a_n = \frac{2^n a_0 + 2^{n-2}}{n!}$ So $y = a_0 + 2a_0x + \frac{2^2a_0+2^{2-2}}{2!}x^2 + \frac{2^3a_0+2^{3-2}}{3!}x^3 + \cdots = a_0 + 2a_0x + \sum_{n=2}^{\infty} \frac{2^n a_0+2^{n-2}}{n!}$
Note that $\sum_{n=2}^{\infty} \frac{2^n a_0+2^{n-2}}{n!} = (a_0 + 2^{-2}) \sum_{n=2}^{\infty} \frac{(2x)^n}{n!} = (a_0 + 2^{-2}) \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} - (a_0 + 2^{-2}) - (a_0 + 2^{-2})(2x) = (a_0 + 2^{-2})e^{2x} - (a_0 + 2^{-2}) - (a_0 + 2^{-2})(2x)$ So that we now have $y = a_0 + 2a_0x + (a_0 + 2^{-2})e^{2x} - (a_0 + 2^{-2}) - (a_0 + 2^{-2})(2x) = -\frac{1}{4} - \frac{1}{2}x + (a_0 + 2^{-2})e^{2x}$ Finally we have $y = -\frac{1}{4} - \frac{1}{2} + Ce^{2x}$

Chapter 5

Introduction to Complex Numbers

The extension from the real numbers to the complex numbers has far-reaching affects. In this chapter, we give a brief introduction to complex numbers and then show how they interact with almost every topic in the course!

5.1 Complex Numbers

The complex numbers arise out of the fact that the simple little equation $x^2 + 1 = 0$ has no solution over the reals. Thus, we create the number i to represent a root of that polynomial. That is, $i^2 + 1 = 0$.

Definition 5.1.0.1. Complex Numbers

The set of *complex numbers* is the set of all numbers that can be written in the form $a + bi$ for real numbers a and b.

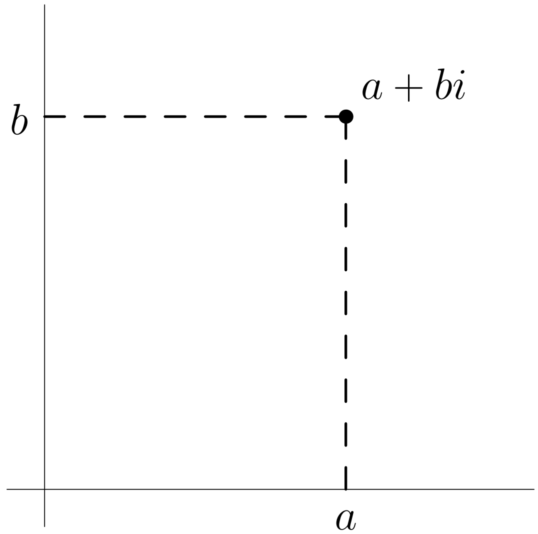
We perform arithmetic in the complex numbers using the usual rules of arithmetic and algebra along with the extra identity $i^2 = -1$.

Exercise 5.1.0.2. Containment of the Reals ☕

- Is 3 a complex number? Can you write 3 in the form $a + bi$ for real numbers a and b?

- Does the set of complex numbers contain all real numbers?

We can visualize complex numbers in the complex plane, where a (the *real part*) is the horizontal component and b (the *imaginary part*) is the vertical.



5.2 Euler's Identity and Consequences

Look again at the power series for the exponential function, sine, and cosine:

$$\begin{aligned} e^x &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 + \cdots \\ \cos(x) &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots \\ \sin(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots \end{aligned}$$

You have to wonder if there is some way to add together sine and cosine to get the exponential function! Sure the signs are off, but otherwise things seem so right. Sine has all the odd factorial denominators, cosine has all the even factorial denominators, and the exponential function has all of them! It turns out that i is exactly the constant we need to fix those minus signs!

Exercise 5.2.0.1. Proof of Euler's Identity

- Write out a power series for $e^{i\theta}$.

- Write out a power series for $\cos(\theta) + i\sin(\theta)$.

- Verify the two are equal!

The fact that there is any relationship whatsoever between sine, cosine, and e is very surprising when you think of how differently those quantities are defined! We again state this incredible theorem, Euler's Identity!

Theorem 5.2.0.2. Euler's Identity

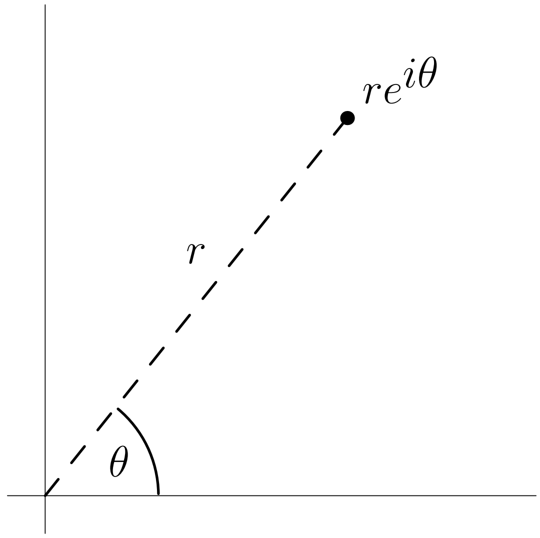
For any real number θ,

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

If we multiply both sides by a real number r, we then obtain

$$re^{i\theta} = r\cos(\theta) + ir\sin(\theta).$$

We notice that the horizontal component, $r\cos(\theta)$, is in fact the conversion for x into polar coordinates. Likewise, $r\sin(\theta)$ is the conversion for y into polar coordinates. This means that the complex number $re^{i\theta}$ is in fact the point located at angle θ and radius r in the complex plane.



5.2.1 Complex Roots

One interesting fact about the complex numbers is that the number of n^{th} roots of *every* real number is exactly n. So every number has two square roots, three cubed roots, and so on. We use $re^{i\theta}$ form to find these complex roots.

Example 5.2.1.1. The Cubed Roots of Two

To find all cubed roots of two, we solve the equation

$$z^3 = 2.$$

We begin by putting both z and 2 in complex polar form. We write $z = re^{i\theta}$ and $2 = 2e^{i0}$. We plug these into the equation, expand the powers.

$$\begin{aligned} z^3 &= 2 \\ \left(re^{i\theta}\right)^3 &= 2e^{i0} \\ r^3 e^{i3\theta} &= 2e^{i0} \end{aligned}$$

We now equate the radius and the angles as two separate equations.

- **Radius:** Since r is a real number, we obtain $r^3 = 2$, which implies $r = \sqrt[3]{2}$.
- **Angle:** The angles need to be equivalent but not necessarily equal. If they differ by a multiple of 2π, that is fine! Thus, we have $3\theta = 0 + 2\pi k$ for any integer k. Dividing both sides by 3, we have

  $$\theta = \frac{0 + 2\pi k}{3} = \ldots, \frac{-4\pi}{3}, \frac{-2\pi}{3}, 0, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{6\pi}{3}, \ldots.$$

  However, if we use more values of θ beyond just $0, \frac{2\pi}{3}, \frac{4\pi}{3}$, the solutions will repeat since cosine and sine have period 2π. Thus, we use just those three angles.

Putting together our r and θ values, we have the following three roots:

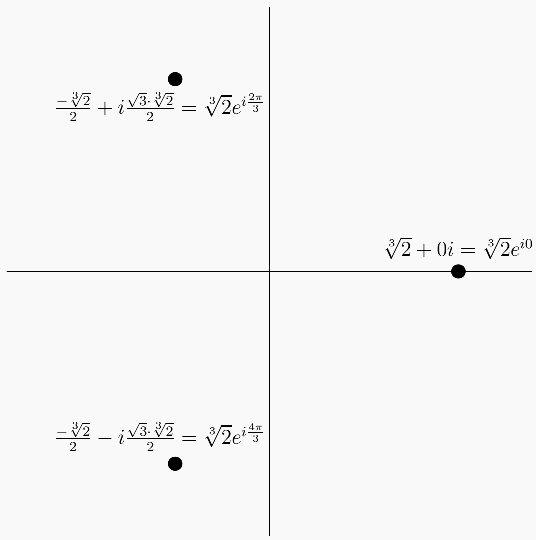
$$z = \sqrt[3]{2}e^{i0}, \sqrt[3]{2}e^{i\frac{2\pi}{3}}, \sqrt[3]{2}e^{i\frac{4\pi}{3}}.$$

Thus, we have our roots in complex polar form. We use Euler's Identity to turn these back into complex cartesian form as follows:

$$z = \sqrt[3]{2}\cos(0) + i\sqrt[3]{2}\sin(0),\ \sqrt[3]{2}\cos(\frac{2\pi}{3}) + i\sqrt[3]{2}\sin(\frac{2\pi}{3}),\ \sqrt[3]{2}\cos(\frac{4\pi}{3}) + i\sqrt[3]{2}\sin(\frac{4\pi}{3}).$$

At last, we use the unit circle to evaluate these and plot in the complex plane.

$$z = \sqrt[3]{2}, -\frac{\sqrt[3]{2}}{2} + i\frac{\sqrt[3]{2}\sqrt{3}}{2}, -\frac{\sqrt[3]{2}}{2} - i\frac{\sqrt[3]{2}\sqrt{3}}{2}$$



Exercise 5.2.1.2. Checking Once Again ☕☕

Cube each of the answers from the previous problem. Verify in each case you get 2!

It turns out to be of particular importance to find roots of 1. Define the n^{th} *roots of unity* to be the solutions to the equation

$$z^n = 1.$$

Lets play around and see if we can find some neat properties!

Exercise 5.2.1.3. Roots of Unity ☕☕☕☕

- Find all square roots of unity. Write your answers in both cartesian and polar complex form, and plot them in the complex plane. (the case where $n = 2$)

- Find all third roots of unity.

- Find all fourth roots of unity.

- Find all fifth roots of unity.

- Find all sixth roots of unity.

- Fill out the following table:

n	Σ_n	Π_n
2		
3		
4		
5		
6		

where Σ_n represents the sum of all n^{th} roots of unity and Π_n represents the product of all n^{th} roots of unity. (**Hint:** It's easier to add in cartesian, and easier to multiply in polar.)

- Based on your above data gathered, conjecture a formula for both Σ_n and Π_n. Prove your conjecture is correct. (**Hint:** Consider the roots of the polynomial $z^n - 1$ and how that polynomial would factor based on those roots. Then consider the degree zero and degree $n-1$ coefficients.)

Using the same techniques we can answer the following question, "what is the square root of i?" Keep in mind there are technically two square roots of i, the two solutions to the equation $z^2 = i$.

Exercise 5.2.1.4. Square Roots of i ☕☕

- Find the square roots of i. Write your answers in complex cartesian form.

- Square your answers back out (in complex cartesian form) and verify that you do in fact get

i when you square them.

Exercise 5.2.1.5. Cubed Roots of i ☕☕☕

- Find all cubed roots of i. That is, find all complex numbers z such that $z^3 = i$. Write your answers in $a + bi$ form.

- Take the cube of each of your roots to verify that you do in fact get i as the third power.

5.2.2 Proving Trig Identities

Remember how there are 47,000 useful but impossible to remember trigonometric identities? No? Well, that shows how hard they are to remember. Believe it or not, most of them can be constructed very quickly and easily from Euler's Identity!

Example 5.2.2.1. The Sine and Cosine Double-Angle Identities

To construct the sine and cosine double-angle formulas, we can manipulate the expression $e^{2\theta}$. We proceed with the following chain of equality:

$$\begin{aligned}\cos(2\theta) + i\sin(2\theta) &= e^{i\cdot 2\theta} \\ &= \left(e^{i\theta}\right)^2 \\ &= (\cos(\theta) + i\sin(\theta))^2 \\ &= \cos^2(\theta) + 2\cos(\theta) i \sin(\theta) + i^2\sin^2(\theta) \\ &= \left(\cos^2(\theta) - \sin^2(\theta)\right) + i\left(2\sin(\theta)\cos(\theta)\right).\end{aligned}$$

We now equate real parts to obtain the cosine double-angle identity

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta).$$

Similarly, we equate imaginary parts to obtain the sine double-angle identity

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta).$$

Exercise 5.2.2.2. Annotate! ☕

Write a short justification alongside each line of computation above.

Exercise 5.2.2.3. Angle-Sum Identities ☕☕

- Expand the expression $e^{i(A+B)}$ into real and imaginary parts using Euler's Identity.

- Expand the expression $e^{iA}e^{iB}$ into real and imaginary parts using Euler's Identity twice, once per factor. Multiply out the resulting terms into an expression of the format

$$f(A,B) + ig(A,B)$$

where f is the function corresponding the real part of that expression, and g corresponds to the imaginary part.

- Equate real and imaginary parts to produce the angle sum identities for cos and sin, respectively! (**Hint:** We're using the fact that two complex numbers $a+bi$ and $c+di$ are equal if and only if $a=c$ and $b=d$.)

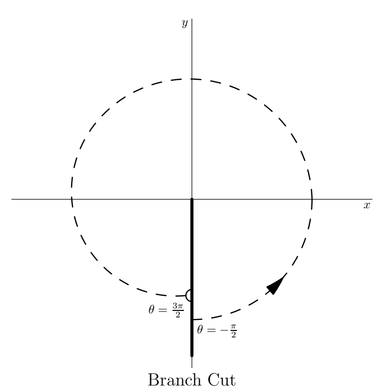
5.2.3 Natural Logarithm of a Complex Number

We now show how to compute the natural logarithm of a complex number. As usual, polar form will be critical.

- Given a complex number z, we first write z in polar form $z = re^{i\theta}$, where r is a positive real number and $\theta \in [-\pi/2, 3\pi/2)$. This choice of interval for θ is often called a *branch cut* and is essentially a domain restriction for the exponential function (since it fails to be one-to-one over the complex numbers).

- Split apart using the property of logarithms and cancel the log with the exponential as follows:

$$\ln(z) = \ln\left(re^{i\theta}\right) = \ln(r) + \ln\left(e^{i\theta}\right) = \ln(r) + i\theta.$$



Example 5.2.3.1. Natural Log of i

Here we compute the mysterious quantity $\ln(i)$. We begin by rewriting i as

$$i = 1 \cdot e^{i\pi/2}.$$

Notice that we chose the angle $\theta = \pi/2$ to be within the branch cut specified above. From here, we split using log properties as follows:

$$\begin{aligned} \ln(i) &= \ln\left(1 \cdot e^{i\pi/2}\right) \\ &= \ln(1) + \ln\left(e^{i\pi/2}\right) \\ &= 0 + i\frac{\pi}{2}. \end{aligned}$$

Thus, $\ln(i) = \frac{\pi}{2}i$.

Note that in principle there is no reason we had to pick our angle θ in that particular interval. One can construct a perfectly well-defined logarithm from choosing a different domain for θ. This is similar to the construction of the inverse trig functions, where one must restrict the domain in some manner, so we tend to just choose a default interval to restrict to and stick with it.

Exercise 5.2.3.2. Complex Logarithms ☕☕

Try the above method to compute each of the following logarithms. Write each in the standard complex cartesian form $a + bi$.

- $\ln(2)$
- $\ln(-2)$
- $\ln(1+i)$
- $\ln(3-4i)$

5.2.4 Complex Exponentials

Recall our trick for dealing with strange bases:

$$ {w_1}^{w_2} = e^{\mathtt{ln}({w_1}^{w_2})} = e^{w_2 \mathtt{ln}(w_1)}. $$

This provides the advantage of moving us back to the familiar base e from the unfamiliar base w_1. This will make a complex exponential base manageable!

Example 5.2.4.1. Computation of Hammurabi

Here we perform two exponentials; we have an i to an i, and a 2 to the 2^{th}. Using the above trick and the value of $\ln(i)$ computed in Example 5.2.3.1, we find

$$ \begin{aligned} i^i 2^2 &= 4i^i \\ &= 4e^{\ln\left(i^i\right)} \\ &= 4e^{i\ln(i)} \\ &= 4e^{i\frac{\pi}{2}i} \\ &= 4e^{-\frac{\pi}{2}}. \end{aligned} $$

Notice there was no need to decompose further using Euler's Identity here; the end result of i raised to the i power is in fact a real number!

Exercise 5.2.4.2. Complex Exponentials ☕☕

- Use the above trick to compute $(1+i)^i$.

- Use the above trick to compute i^{1+i}.

- Use the above trick to compute $(1+i)^{1+i}$.

5.3 Partial Fractions via Complex Numbers

If we are using complex numbers, there are no more irreducible quadratics! This gives us an interesting alternate way to perform PFD, since all polynomials will fully factor into linear factors.

Exercise 5.3.0.1. PFD over the Complex Numbers ☕☕☕

- Find an antiderivative of $\frac{4-2x^2}{x^3+4x}$ via a partial fraction decomposition over the real numbers.

- Find an antiderivative of $\frac{4-2x^2}{x^3+4x}$ via a partial fraction decomposition over the complex numbers.

- Verify your answers are compatible.

Since Euler's Identity relates the exponential function to trigonometric functions, it is plausible that there would be analagous relationships out there between logarithmic and inverse trigonometric functions over the complex numbers! At a very crude level, one can think of just taking an inverse function of both sides of Euler's Identity. It turns out that PFD over the complex numbers is the right tool to make this formal!

Exercise 5.3.0.2. Inverse Tangent and Natural Log

Recall that by trigonometric substitution, we have

$$\int \frac{1}{x^2+1}\,\mathtt{d}x = \arctan(x) + C.$$

Compute the same antiderivative but using a PFD over the complex numbers. In particular, carry out the following steps:

- Factor x^2+1 over the complex numbers.

- Find A and B in the decomposition

$$\frac{1}{x^2+1} = \frac{A}{x+i} + \frac{B}{x-i}.$$

- Find the antiderivative, simply integrating terms of the form $\frac{c_1}{x+c_2}$ as $c_1 \ln(x+c_2)$ for complex constants c_1 and c_2, just as you would for real constants.

- Conclude that

$$\arctan(x) = \frac{1}{2i}\ln\left(\frac{x-i}{x+i}\right) + C$$

for some constant C.

- Solve for C by letting $x \to \infty$ on both sides to get a relationship between inverse tangent and the natural logarithm! (It is valid for positive x.)

5.4 Chapter Summary

The set of **complex numbers** is the set of all numbers expressible as $a + bi$ for real numbers a and b. In the **complex plane**, we plot a as the horizontal and b as the vertical. Allowing complex numbers in our calculus adventures relates many seemingly unrelated objects!

1. **Euler's Identity and consequences:** By setting $x = i\theta$ in our power series for the exponential function, we obtain **Euler's Identity**. This is the relationship

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

which is usually then multiplied by r to obtain

$$re^{i\theta} = \underbrace{r\cos(\theta)}_{x \text{ in polar coords}} + i \underbrace{r\sin(\theta)}_{y \text{ in polar coords}}.$$

This means that in the complex plane we have $re^{i\theta}$ as the point at angle θ and radius r. This relationship has many consequences, including the following:

   (a) **Proving trig identities:** Properties of exponentials can be turned into trig identities using Euler's Identity.

   (b) **Calculating roots of complex numbers:** To find the n^{th} roots of a complex number $a+bi$, notice that this is the same as solving the equation $z^n = a + bi$. Rewrite everything in polar form, distribute the n power, and equate radius and angles to find the roots.

   (c) **Calculating logarithms of complex numbers:** To compute $\ln(a + bi)$, write $a+bi$ in polar form with an angle chosen in the branch cut $-\pi/2 \leq \theta < 3\pi/2$. From there, use properties of logs to simplify the expression.

   (d) **Calculating exponentials with a complex base:** Rewrite as "e to the ln" of the expression and then use the method for complex logarithms described above.

2. **Revisiting PFD with complex numbers:** With complex numbers, there is no need for the irreducible quadratic case in a PFD. Instead, we can completely factor the denominator of any rational function into a product of powers of linear factors.

5.5 Mixed Practice

5.5.1 Warm Ups

These are good problems for reinforcing the vocabulary and foundational concepts of this chapter.

Exercise 5.5.1.1. ☕☕

a.) Find all square roots of $-i$. Write your answers in complex cartesian form.

A Solution: If $z^2 = e^{i(-\pi/2)}$ and $z = re^{i\theta}$ then $r^2 e^{i2\theta} = e^{i(-\pi/2}$ so $r = 1$ and $2\theta = \frac{-\pi}{2} + 2\pi k$ Thus $\theta = \frac{-\pi}{4} + \pi k = \frac{-\pi}{4}, \frac{3\pi}{4}$ So $z = e^{-i(\pi/4)}, e^{i(3\pi/4)} = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

b.) Take your answers and square them to verify their square is $-i$ as you claim above.

A Solution: $\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)^2 = \frac{2}{4} - 2i\frac{2}{4} + i^2\frac{2}{4} = \frac{1}{2} - \frac{1}{2} - i = -i$ and $\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)^2 = \frac{2}{4} - 2i\frac{2}{4} + i^2\frac{2}{4} = \frac{1}{2} - \frac{1}{2} - i = -i$

5.5.2 Sample Test Problems

Exercise 5.5.2.1. ☕☕☕

Compute the following complex numbers in standard $a + bi$ form for $a, b \in \mathbb{R}$. List all values if there are multiple answers.

a.) i^{2015}

A Solution: $i^{2015} = i^{2012+3} = i^{2012}i^3 = 1 \cdot i^3 = -i$

b.) $i^{(2i)}$

A Solution: $i^{(2i)} = e^{\ln i^{(2i)}} = e^{2i \ln i}$ Note that $i = \cos(\pi/2) + i\sin(\pi/2) = e^{i\pi/2}$ So $e^{2i\ln i} = e^{2i \ln e^{i\pi/2}} = e^{2i \cdot i\pi/2} = e^{-\pi}$

c.) $1 + i\pi - \frac{\pi^2}{2!} - \frac{i\pi^3}{3!} + \frac{\pi^4}{4!} + \frac{i\pi^5}{5!} - \frac{\pi^6}{6!} - \frac{i\pi^7}{7!} + \frac{\pi^8}{8!} + \cdots$

A Solution: $1 + i\pi - \frac{\pi^2}{2!} - \frac{i\pi^3}{3!} + \frac{\pi^4}{4!} + \frac{i\pi^5}{5!} - \frac{\pi^6}{6!} - \frac{i\pi^7}{7!} + \frac{\pi^8}{8!} + \cdots = 1 + i\pi + \frac{(i\pi)^2}{2!} + \frac{(i\pi)^3}{3!} +$

$\frac{(i\pi)^4}{4!} + \frac{(i\pi)^5}{5!} - \frac{(i\pi)^6}{6!} - \frac{(i\pi)^7}{7!} + \frac{(i\pi)^8}{8!} + \cdots = e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$

---

d.) $\sqrt[3]{i}$

A Solution: $\left(r(e^{i\theta})\right)^3 = r^3 e^{i3\theta} = i = e^{i\pi/2}$
$\Rightarrow r = 1$ and $3\theta = \frac{\pi}{2} + 2\pi k$
$\Rightarrow \theta = \frac{\pi}{6} + \frac{2\pi}{3}k \Rightarrow \frac{\pi}{6}, \frac{\pi}{6} + \frac{2\pi}{3}, \frac{\pi}{6} + \frac{4\pi}{3}$
$\Rightarrow \sqrt[3]{i} = e^{i\pi/6}, e^{i5\pi/6}, e^{i3\pi/2} = \frac{\sqrt{3}}{2} + \frac{i}{2}, -\frac{\sqrt{3}}{2} + \frac{i}{2}, -i$

---

e.) $\ln\left(4\sqrt{3} + i\right)$

A Solution: First convert $4\sqrt{3} + i$ to polar using a triangle where $4\sqrt{3}$ is the horizontal side and 1 is the vertical side. Then $r = \sqrt{(4\sqrt{3})^2 + 1^2} = \sqrt{16 \cdot 3 + 1} = \sqrt{49} = 7$ also, $\theta = \arctan\left(\frac{1}{4\sqrt{3}}\right) = \arctan\left(\frac{\sqrt{3}}{12}\right)$ Thus $ln(4\sqrt{3}+i) = \ln\left(7e^{i\arctan\sqrt{3}/12}\right) = \ln 7 + i\arctan\sqrt{3}/12$

Exercise 5.5.2.2. ☕☕

a.) State and prove Euler's Identity using power series.

A Solution: $\cos(\theta) + i\sin(\theta) = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!}\cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!}\cdots\right) = \left(1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!}\cdots\right) = \left(1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!}\cdots\right) = e^{i\theta}$

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b.) Multiply both sides of Euler's Identity by r. Explain how this formula relates to our conversion between polar and cartesian coordinates.

A Solution: $r\cos(\theta) + ri\sin(\theta) = re^{i\theta}$ This shows that if $x + iy$ is the **horizontal + i vertical** representation of a complex number, then $x = r\cos(\theta)$ and $y = ri\sin(\theta)$ so that r and θ represent a radius and an angle.

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c.) Prove the sine double-angle identity using Euler's Identity.

A Solution: Prove $\sin(2\theta) = 2\cos(\theta)\sin(\theta)$
Start with $\cos(2\theta) + i\sin(2\theta) = e^{i(2\theta)} = (e^{i\theta})^2 = (\cos(\theta) + i\sin(\theta))^2 = \cos^2(\theta) + 2i\sin(\theta)\cos(\theta) + i^2\sin^2(\theta) = (\cos^2(\theta) - \sin^2(\theta)) + i(2\sin(\theta)\cos(\theta))$ So $\cos(2\theta) = \cos^2(\theta) -$

$\sin^2(\theta)$ and $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

Selected Answers and Hints

Exercise 2.1.1.1. $\int f'(g(x)) \cdot g'(x)\, \mathtt{d}x = \int (f(g(x)))'\, \mathtt{d}x = f(g(x)) + C$

Exercise 2.1.1.6. Use the substitutions $u = x^2 + x + 8, \ln(x)$, and $-x^2$. In the last case, the $\mathtt{d}u$ term has nothing to cancel the x with!

Exercise 2.1.2.2. To have four intervals in the Riemann sum, Δx would be 1 while Δu would be 2. Thus, the width of each rectangle is getting doubled, since to convert between u and x we use the formula $u = 2x + 1$. The "plus one" merely slides all the rectangles one unit to the right, but it does not stretch their width at all, so it does not affect their area. Thus, the slope of the graph of $u = 2x + 1$ is the only thing that mattered regarding our conversion between x and u. That is to say, the quantity $\mathtt{d}u/\,\mathtt{d}x$ gives us the scaling factor.

Exercise 2.1.2.3. The definite integral evaluates to roughly 0.95. The horizontal scaling factor at each x-coordinate should correspond to the derivative $\mathtt{d}u/\,\mathtt{d}x$ at each point.

Exercise 2.2.1.3. By factoring out the quantity $(x + 1)^{3/2}$, both answers can be brought into the form $(x + 1)^{3/2}\left(\frac{2}{5}x - \frac{4}{15}\right) + C$.

Exercise 2.2.2.2. Use the substitution $u = 1 - x^2$.

Exercise 2.2.2.3. The antiderivative is $x \ln(x) - x + C$.

Exercise 2.2.3.5. The antiderivative is $\frac{1}{2}(\sec(x)\tan(x) + \ln|\sec(x) + \tan(x)|) + C$.

Exercise 2.3.0.1. Use the substitution $u = \sqrt{x}$ to transform the first integral into $\int 2u \cos(u)\, \mathtt{d}u$.

Exercise 2.3.0.2. Choosing $u = \ln(x)$ will make the logarithm disappear upon differentiation. The opposite choice will not clean up the log.

Exercise 2.4.0.2. The antiderivative is $\frac{1}{3}\sin^3(x) + C$.

Exercise 2.4.1.2. Since seven is odd, when we pulled out one factor of sine, we ended up with the sixth power of sine remaining. Since six is even, we were able to express it as a power of a perfect square of sine, which in turn let us rewrite as cosines using the Pythagorean identity.

Exercise 2.4.1.3. The first antiderivative is $-\frac{1}{3}\cos^3(x) + \frac{2}{5}\cos^5(x) - \frac{1}{7}\cos^7(x) + C$. For the second, rewrite as $(1 - \sin^2(x))^4 \cos(x)$ and proceed by letting $u = \sin(x)$.

Exercise 2.4.1.4. Often when trying to show that two antiderivatives are compatible, it is easiest to verify that their difference is a constant.

Exercise 2.4.1.4. The substitution $u = \sin(x)$ is much cleaner since the other will involve having to expand a binomial to the fifth power. The antiderivative is $\frac{1}{12}\sin^{12}(x) - \frac{1}{14}\sin^{14}(x) + C$.

Exercise 2.4.2.1. The exponent on sine is zero, which is indeed even. Thus both exponents are even in this case.

Exercise 2.4.2.3. When all like terms are combined and the one-eighth is distributed, the result is $\frac{5}{16}x + \frac{1}{4}\sin(2x) - \frac{1}{48}\sin^3(2x) + \frac{3}{64}\sin(4x) + C$.

Exercise 2.4.2.4. The antiderivative to $\cos^6(x)$ came out to

$$\frac{5}{16}x + \frac{1}{4}\sin(2x) - \frac{1}{48}\sin^3(2x) + \frac{3}{64}\sin(4x) + C$$

Before we differentiate, first bash everything back down to an "x" in the argument using double angle identities. This produces

$$\frac{5}{16}x + \frac{1}{2}\sin(x)\cos(x) - \frac{1}{6}\sin^3(x)\cos^3(x) + \frac{3}{16}\sin(x)\cos^3(x) - \frac{3}{16}\sin^3(x)\cos(x) + C$$

Factor out a sine and use the Pythagorean Identity to get everything else in terms of cosine. This produces

$$\frac{5}{16}x + \sin(x)\left(\frac{5}{16}\cos(x) + \frac{5}{24}\cos^3(x) + \frac{1}{6}\cos^5(x)\right) + C$$

Then we differentiate and obtain

$$\frac{5}{16} + \cos(x)\left(\frac{5}{16}\cos(x) + \frac{5}{24}\cos^3(x) + \frac{1}{6}\cos^5(x)\right) - \sin^2(x)\left(\frac{5}{16} + \frac{5}{8}\cos^2(x) + \frac{5}{6}\cos^4(x)\right)$$

to which we apply the Pythagorean Identity $\sin^2(x) = 1 - \cos^2(x)$ to produce

$$\frac{5}{16} + \cos(x)\left(\frac{5}{16}\cos(x) + \frac{5}{24}\cos^3(x) + \frac{1}{6}\cos^5(x)\right) - \left(1 - \cos^2(x)\right)\left(\frac{5}{16} + \frac{5}{8}\cos^2(x) + \frac{5}{6}\cos^4(x)\right)$$

This will simplify to $\cos^6(x)$ once you expand and combine like terms.

Exercise 2.4.2.5. For the first, apply the identity $\sin^2(3x) = \frac{1-\cos(6x)}{2}$ and proceed. For the second, notice that $\sin^4(x)$ can be rewritten as $\left(\sin^2(x)\right)^2$, after which the half-angle identity can be applied.

Exercise 2.5.1.3. First apply all the product and chain rules to reach the expression

$$\frac{3}{\sqrt{1-\frac{x^2}{4}}} + 4\sqrt{1-\frac{x^2}{4}} + \frac{-x^2}{\sqrt{1-\frac{x^2}{4}}} + \sqrt{1-\frac{x^2}{4}}\left(1 - \frac{3}{2}x^2\right) + \frac{-x}{4\sqrt{1-\frac{x^2}{4}}}\left(x - \frac{x^3}{2}\right)$$

Put all terms over the common denominator $\sqrt{4-x^2}$ and combine like terms in the numerator. Notice the numerator becomes $\left(4-x^2\right)^2$ and then reduce for the win!

Exercise 2.5.1.4. The antiderivative is $2^{18}\left(\frac{\left(1-x^2/16\right)^{9/2}}{9} - \frac{\left(1-x^2/16\right)^{7/2}}{7}\right) + C$

Exercise 2.5.2.3. Exercise 2.2.3.5 will be helpful! The antiderivative is $\frac{x\sqrt{x^2-4}}{2} - 2\ln|x + \sqrt{x^2-4}| + C$.

Exercise 2.5.4.3. The antiderivative is $-\frac{1}{5}\frac{2x+1}{x^2+x-1} + \frac{4\sqrt{5}}{25}\ln\left(\frac{2x+1+\sqrt{5}}{2\sqrt{x^2+x-1}}\right) + C$. Note that one can expand using properties of logarithms and then rename C as $C - \frac{4\sqrt{5}}{25}\ln(2)$ since it is anyhow just an arbitrary constant. Thus, we can slightly clean up the answer to become $-\frac{1}{5}\frac{2x+1}{x^2+x-1}+\frac{4\sqrt{5}}{25}\ln\left(2x+1+\sqrt{5}\right)-\frac{2\sqrt{5}}{25}\ln\left(x^2+x-1\right)+C$.

Exercise 2.6.1.5. Using properties of logarithms, both answers should be able to be put in the form $\ln\left|\sqrt{\frac{x-1}{x+1}}\right|+C$

Exercise 2.6.3.1. •The function $\frac{1}{x^2-9x+20}$ has $\ln\left|\frac{x-5}{x-4}\right|+C$ as its antiderivative. •The factorization $x^4-9=\left(x^2+3\right)\left(x-\sqrt{3}\right)\left(x+\sqrt{3}\right)$ will produce the following setup:

$$\frac{1}{x^4-9}=\frac{Ax+B}{x^2+3}+\frac{C}{x-\sqrt{3}}+\frac{D}{x+\sqrt{3}}$$

in which you can then solve for the coefficients and antidifferentiate. •The function $\frac{x^4}{x^2+1}$ has an irreducible quadratic for a denominator. However, the degree of the numerator is not smaller than the degree of the denominator. Thus, polynomial long division is the only step of PFD that is required in this case. •The antiderivative of $\frac{2}{x^5+2x3+x}$ is

$$2\ln|x|-\ln\left|x^2+1\right|+\frac{1}{x^2+1}$$

•The PFD will produce

$$\frac{x-2}{x^3+x^2+3x-5}=\frac{-\frac{1}{8}}{x-1}+\frac{\frac{1}{8}x+\frac{11}{8}}{x^2+2x+5}$$

While the first term is easy to integrate, the second is quite tricky! To hack through it, split it as follows:

$$\frac{\frac{1}{8}x+\frac{11}{8}}{x^2+2x+5}=\frac{\frac{1}{8}x+\frac{1}{8}}{x^2+2x+5}+\frac{\frac{10}{8}}{x^2+2x+5}$$

The first fraction can then be integrated via u-sub, while the second can be done via trig sub after completing the square on the denominator.

Exercise 2.6.3.2. For $\frac{1}{x^4-9x^2}$, keep in mind that x^2 is not an irreducible quadratic factor but rather a repeated linear factor. The PFD and integration will produce

$$\frac{1}{9x}+\frac{1}{54}\ln\left|\frac{x-3}{x+3}\right|+C$$

Exercise 2.8.1.1. $\ln(1+x)+C$

Exercise 2.8.1.2. $x+-2\sqrt{x}+2\ln|\sqrt{x}+1|+C$

Exercise 2.8.1.3. $\ln\left|\frac{2+\sqrt{3}}{\sqrt{2}+1}\right|$

Exercise 2.8.1.4. $\frac{1}{2}\ln|x-1|-\frac{1}{2}\ln|x+1|+\frac{1}{x}+C$

Exercise 2.8.2.1. $-\frac{\cos^{18}x}{18}+\frac{\cos^{16}x}{8}-\frac{\cos^{14}x}{14}+C$

Exercise 2.8.2.2. The antiderivative is $-\frac{1}{2}\left(\csc(x)\cot(x)+\ln|\csc(x)+\cot(x)|\right)+C$

Exercise 2.8.2.3. $\frac{1}{8}\ln\left|\frac{x-4}{x+4}\right|+C$

Exercise 2.8.2.4. •$\frac{x^3}{x^3-3x^2+4}=1+\frac{-\frac{1}{9}}{x+1}+\frac{\frac{28}{9}}{x-2}+\frac{\frac{8}{3}}{(x-2)^2}$

• $\int 1 + \frac{-\frac{1}{9}}{x+1} + \frac{\frac{28}{9}}{x-2} + \frac{\frac{8}{3}}{(x-2)^2}\,\mathsf{d}x = x - \frac{1}{9}\ln|x+1| + \frac{28}{9}\ln|x-2| - \frac{8}{3}\frac{1}{(x-2)} + C$

Exercise 2.8.2.5. $\frac{1}{4}\sec^3 x\tan x + \frac{3}{8}\sec x\tan x + \frac{3}{8}\ln|\sec x + \tan x| + C$

Exercise 3.1.0.2. This parametric curve is the line $y = \frac{3}{2}x + 1$.

Exercise 3.1.0.3. The two curves are the same points in the plane. Both start at the point (1,0) at time $t = 0$, but C_1 then proceeds counter-clockwise while C_2 proceeds clockwise.

Exercise 3.3.0.4. The arc length is

$$\frac{6\sqrt{146} + \ln\left(\sqrt{73} + 6\sqrt{2}\right)}{6} \approx 12.55.$$

Also, to handle the absolute value, just find the arc length on the interval [0,2] where you can ignore the absolute value and then apply symmetry.

Exercise 3.3.0.5. The arc length is $\sqrt{2}\left(e^{2\pi} - 1\right)$.

Exercise 3.5.2.7. Yes, it is in fact a circle with cartesian center $(0, 1/2)$ and radius $1/2$. This can be verified by demonstrating the polar equation converts to the cartesian equation

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2.$$

Exercise 3.6.0.2. The derivative is a constant; thus the graph is a straight line!

Exercise 3.7.0.4. The area between the curves is $\frac{\pi}{8}$.

Exercise 3.7.0.5. The area inside the inner loop of $r(\theta) = \frac{1}{2} + \cos(\theta)$ is $\frac{\pi}{4} - \frac{3\sqrt{3}}{8}$.

Exercise 3.10.1.1. a. $r(0) = 4$, $r(\pi/6) = 8/\sqrt{3}$, $r(\pi/4) = 8/\sqrt{2} = 4\sqrt{2}$, $r(\pi/3) = 8$ b. $r = 4\sec\theta \implies r\cos\theta = 4 \implies x = 4$ is a vertical line. c. d. It is an isosceles triangle with hypotenuse $4\sqrt{2}$ and sides 4 $A = 8$ e. $A = 8$ They are the same.

Exercise 3.10.1.2. a.) It is a line segment that lies on the line $\frac{x-1}{4} = t = \frac{y}{6} \leftrightarrow y = \frac{3}{2}x - \frac{3}{2}$ between $(-1, 0)$ and $(7, 12)$
b.) $\frac{3}{2}$
c.) $4\sqrt{13}$

Exercise 3.10.2.1. $\cosh t = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \cdots$
$\sinh t = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots$
a. $\frac{d\ \sinh(t)}{dt} = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots = \cosh(t)$
b. $\frac{d\ \cosh(t)}{dt} = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots = \sinh(t)$
c. $\cosh^2 t - \sinh^2 t = \left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \cdots\right)^2 - \left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right)^2 = 1$
d. $\frac{1}{0}$ which is a vertical line

Exercise 3.10.2.2. b.) $\frac{\pi}{8}$

Exercise 4.2.0.4. Any solution to $\frac{dy}{dx} = xy + x$ can be written as $y = Ce^{\frac{x^2}{2}} - 1$ for some real number C. The second DE with initial condition has the solution

$$\frac{1}{2}e^{-y}\left(\sin(y) - \cos(y)\right) = -e^{-x}\left(3 + 2x + x^2\right) + \frac{5}{2}.$$

Exercise 4.6.1.1. Linear combinations of hyperbolic sine and hyperbolic cosine functions are the only functions that equal their own second derivatives.

Exercise 4.6.2.1. a. $y = 3 + x + (a_0 - 3)e^x$ b. If $y = 3 + x + (a_0 - 3)e^x$ then $\frac{dy}{dx} = 1 + (a_0 - 3)e^x$ but $y - x - 2 = 3 + x + (a_0 - 3)e^x - x - 2 = 1 + (a_0 - 3)e^x$ So they match.

Exercise 4.6.2.2. a.) The right-hand side $2y + x$ does not factor into a function of y times a function of x so there can be no separation of variables.
b.) $y = -\frac{1}{4} - \frac{1}{2}x + Ce^{2x}$

Exercise 5.2.3.2. •$\ln(2) = \ln(2) + 0i$ •$\ln(-2) = \ln(2) + \pi i$ •$\ln(1 + i) = \ln\left(\sqrt{2}\right) + i\frac{\pi}{4}$ •$\ln(3 - 4i) = \ln(5) + i\arctan\left(-\frac{3}{4}\right)$

Exercise 5.2.4.2. The number $(1 + i)^{1+i}$ can be written in complex cartesian form as

$$\left(e^{\ln\left(\sqrt{2}\right) - \frac{\pi}{4}}\cos\left(\ln\left(\sqrt{2}\right) + \frac{\pi}{4}\right)\right) + i\left(e^{\ln\left(\sqrt{2}\right) - \frac{\pi}{4}}\sin\left(\ln\left(\sqrt{2}\right) + \frac{\pi}{4}\right)\right).$$

Exercise 5.3.0.1. The PFD over the complex numbers is

$$\frac{4 - 2x^2}{x^3 + 4x} = \frac{1}{x} - \frac{\frac{3}{2}}{x + 2i} - \frac{\frac{3}{2}}{x - 2i}.$$

Exercise 5.3.0.2. In taking the limit, the logarithmic term will approach zero. Thus $C = \pi/2$.

Exercise 5.5.1.1. $z = e^{i(-\pi/4)}, e^{i(3\pi/4)} = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

Exercise 5.5.2.1. a.) $-i$
b.) $e^{-\pi}$
c.) -1
d.) $\frac{\sqrt{3}}{2} + \frac{i}{2}, -\frac{\sqrt{3}}{2} + \frac{i}{2}, -i$
e.) $\ln 7 + i\arctan\sqrt{3}/12$

Exercise 5.5.2.2. b. $r\cos(\theta) + ri\sin(\theta) = re^{i\theta}$ This shows that r and θ represent a radius and an angle.

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