Calculus II

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Contents

Acknowledgements

Thank you so much for choosing this text and a big thanks to all who made this possible!

Kenneth would like to thank Front Range Community College for their relentless dedication to excellence, inclusivity, and student success. To help Ken help our students, they supported this project with an Innovation Grant. In particular, thanks to FRCC president Andy Dorsey for believing in the project and Christy Gomez for her vision, structure, and pragmatism. He would like to thank his family Ken, Gina, Maria, Bryan, and Keenan for the (even in a family that loves combinatorics) countless mathematical adventures throughout the years. He is also grateful for the training he received from generous mentors while he was just a lowly graduate student at Colorado State University – especially from Alexander Hulpke and Tim Penttila. Lastly, thanks to his wife Faith for all the love and support throughout the project. She learned to make coffee just so she could help, despite thinking it's the most disgusting liquid in the visible universe.

Jenna would like to thank the fantastic faculty at Front Range Community College – Boulder County Campus and University of Colorado Boulder for their training and dedication to helping her become a better mathematician. She would also like to thank the Monks family who has done so much to help her mathematically and encouraged her to continue to grow. Finally, thanks to her family for brewing the coffee and putting up with her hermit tendencies.

Thanks to Joe Darschewski, FRCC-BCC's excellent Calculus instructor, for his being the first instructor to use the text and offer an enormous amount of helpful feedback. He also typeset the new Mixed Practice sections, which will be very helpful to the students!

We hope you have fun with this resource and find it helpful! It is a beautiful subject, and we tried to honor that with a beautiful text. The text is still in its infancy, and we welcome any and all feedback you could give us. Thank you in advance for any comments, complaints, suggestions, and questions.

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Chapter 1

Overview

A Three-Part Course

The topics of Calculus II fall into three parts that each have an appropriate place in the story of the calculus sequence.

- **Part I: Integration.** The first part of the course ties loose ends from Calculus I. The ending of Calculus I showed that antiderivatives can be used to evaluate integrals via the Fundamental Theorem of Calculus. However, by the end of Calculus I, only the very simplest antiderivatives can actually be computed. Part one expands the student's knowledge of **techniques of antidifferentiation**. These techniques are subsequently put to use computing **length, area, volume, and center of mass**.
- **Part II: Sequences and Series.** This is the topic that makes up the body of Calculus II. **Sequences and series** embody the beauty of mathematics; from simple beginnings (a sequence is just a list... a series is just adding up a list of numbers...) it quickly leads to incredible structure, surprises, complexity, and open problems. **Power series** redefine commonly used transcendental functions (functions that are not computed using algebra, e.g. cosine). If you've ever wondered **what your calculator does when you press the cosine button**, this is where you find out! (**Hint:** It does not have a circle of radius one spinning around with a team of elves that measure x coordinates.)
- **Part III: Coming Attractions.** By the end of Calculus II, the student is ready for a *lot* of other classes. The end of Calculus II thus ends with a sampler platter of topics that show the vast knowledge base built upon the foundation laid in Calculus II. Here the text takes a bite out of **Differential Equations**, serves some polar and parametric coordinates as a palate cleanser before **Calculus III**, and tastes some **Complex Analysis** to aid in digestion of Differential Equations. For dessert, it serves a scoop of **Probability** with both discrete and continuous colored sprinkles.

How to Use This Book

This book is meant to facilitate *Active Learning* for students, instructors, and learning assistants. Active Learning is the process by which the student participates directly in the learning process by reading, writing, and interacting with peers. This contrasts the traditional model where the student passively listens to lecture while taking notes. This book is designed as a self-guided step-by-step exploration of the concepts. The text incorporates theory and examples together in order to lead the student to discovering new results while still being able to relate back to familiar topics of mathematics. Much of this text can be done independently by the student for class preparation. During class sessions, the instructor and/or learning assistants may find it advantageous to encourage group work while being available to assist students, and work with students on a one-on-one or small group basis. This is highly desirable as extensive research has shown that active learning improves student success and retention. (For example, see `www.pnas.org/content/111/23/8410` for Scott Freeman's metaanalysis of 225 studies supporting this claim.)

What is Different about this Book

If you leaf through the text, you'll quickly notice two major structural differences from many traditional calculus books:

1. The exercises are very intermingled with the readings. Gone is the traditional separation into "section" versus "exercises".
2. Whitespace was included for the student to write and work through exercises. Parts of pages have indeed been intentionally left blank.

A consequence of this structure is that the readings and exercise are closely linked. It is intended for the student to do the readings and exercises concurrently.

Ok... Why?

The goal of this structure is to help the student simulate the process by which a mathematician reads mathematics. When a mathematician reads a paper or book, he/she always has a pen in hand and is constantly working out little examples alongside and scribbling incomprehensible notes in bad handwriting. It takes a *long* time and a lot of experience to know how to come up with good questions to ask oneself or to know what examples to work out in order to to help oneself absorb the subject. Hence, the exercises sprinkled throughout the readings are meant to mimic the margin scribbles or side work a mathematician engages in during the act of reading mathematics.

The Legend of Coffee

A potential hazard of this self-guided approach is that while most examples are meant to be simple exercises to help with absorption of the topics, there are some examples that students may find quite difficult. To prevent students from spinning their wheels in frustration, we have labeled the difficulty of all exercises using coffee cups as follows:

Coffee Cup Legend

Symbol	Number of Cups	Description of Difficulty
☕	A One-Cup Problem	Easy warm-up suitable for class prep.
☕☕	A Two-Cup Problem	Slightly harder, solid groupwork exercise.
☕☕☕	A Three-Cup Problem	Substantial problem requiring significant effort.
☕☕☕☕	A Four-Cup Problem	Difficult problem requiring effort and creativity!

Glossary of Symbols

In Precalculus and Calculus I, there is a wide range of how much notation from Set Theory gets used. To get everyone on the same page, here is a short list of some notation we will use in this text.

Sets and Elements

Often in mathematics, we construct collections of objects called *sets*.

- If an object x is in a set A, we say x is an *element* of A and write $x \in A$.
- If an object x is not in a set A, we say x is not an element of A and write $x \notin A$.

Any particular object is either an element of a set or it is not. We do not allow for an object to be partially contained in a set, nor do we allow for an object to appear multiple times in a set. Often we use curly braces around a comma-separated list to indicate what the elements are.

Example 1.0.0.1. A Prime Example

Suppose P is the set of all prime numbers. We write

$$P = \{2, 3, 5, 7, 11, 13, 17, \ldots\}$$

For example, $2 \in P$ and $65,537 \in P$, but $4 \notin P$.

Some Famous Sets of Numbers

The following are fundamental sets of numbers used in Calculus 2.

- **Natural Numbers:** The set $\mathbb{N}$ of natural numbers is the set of all positive whole numbers, along with zero. That is,

  $$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \ldots\}$$

  Note that in many other sources, zero is not included in the natural numbers. Both are widely used; be aware the choice on this convention will change throughout your mathematical travels!

- **Integers:** The set of integers $\mathbb{Z}$ is the set of all whole numbers, whether they are positive, negative, or zero. That is,

  $$\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

- **Rational Numbers:** The set of rational numbers $\mathbb{Q}$ is the set of all numbers expressible as a fraction whose numerator and denominator are both integers.

- **Real Numbers:** The set of real numbers $\mathbb{R}$ is the set of all numbers expressible as a decimal.

- **Complex Numbers:** The set $\mathbb{C}$ of complex numbers is the set of all numbers formed as a real number (called the real part) plus a real number times i (called the imaginary part), where i is a symbol such that $i^2 = -1$.

Set-Builder Notation

The most common notation used to construct sets is *set-builder notation*, in which one specifies a name for the elements being considered and then some property $P(x)$ that is the membership test for an object x to be an element of the set. Specifically,

$$A = \{x \in B : S(x)\}$$

means that an object x chosen from B is an element of the set A if and only if the claim $S(x)$ is true about x. Sometimes the "$\in B''$ gets dropped if it is clear from context what set the elements are being chosen from. The set-builder notation above gets read as "the set of all x in B such that $S(x)$". One can think of this as running through all elements of B and throwing away any that do not meet the condition described by S.

Example 1.0.0.2. Interval Notation

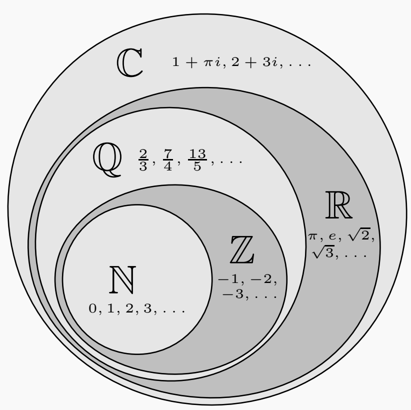
Interval notation can be expressed in set-builder notation as follows:

- $(a, b) = \{x \in \mathbb{R} : a < x < b\}$
- $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$
- $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$
- $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$

Example 1.0.0.3. Rational, Real, and Complex in Set-Builder Notation

Set-builder notation is often used to express the sets of rational, real, and complex numbers as follows:

- $\mathbb{Q} = \left\{\frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0\right\}$
- $\mathbb{R} = \{0.a_0a_1a_2a_3a_4 \ldots \times 10^n : n \in \mathbb{N}, a_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ where $i \in \mathbb{N}\}$ Note this is essentially scientific notation; the concatenation of the a_i's represents the digits in a base-ten decimal expansion.
- $\mathbb{C} = \{a + bi : a \in \mathbb{R}, b \in \mathbb{R}\}$



Part I

Integration

Part II

Sequences and Series

Chapter 2

Sequences and Series: Commas and Plus Signs Run Amok

2.1 Definition of Sequences

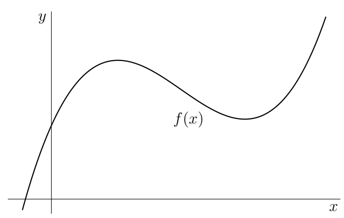
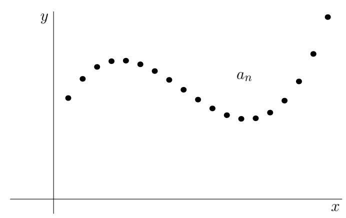
Definition 2.1.0.1. Sequence

A *sequence* is a function whose domain is a subset of $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, the set of natural numbers.

Typically for $n \in \mathbb{N}$, we write a_n as the output corresponding to the input n. The output could technically be an object of any type, but in this course we usually use real numbers or complex numbers as our outputs. Technically the sequence itself is the map $n \mapsto a_n$, though since this is a bit cumbersome to write, we often write just a_n to refer to the entire sequence, similar to how we write $f(x)$ for a function on the real numbers. When the outputs are real numbers, we can graph sequences as a collection of points of the form (input,output) just as we would for functions on the real numbers.

2.1.1 Sequences vs Functions on the Real Numbers

The chart below compares sequences (functions whose domain is the natural numbers) and functions whose domain is the real numbers.

Trait or Notation	**Function on Reals**	**Sequence**
Default Independent Variable	x	n
Default Formula Notation	$f(x)$	a_n
Domain D	Subset of $\mathbb{R}$	Subset of $\mathbb{N}$
Graph	$\{(x, f(x)) : x \in D\}$ 	$\{(n, a_n) : n \in D\}$ 

Less formally, a sequence is simply a list of objects. The correspondence between sequences as maps and sequences as lists of objects is that the k^{th} object in the list is the output corresponding to $k-1$ under the map. That is, the map $n \mapsto a_n$ corresponds to the list $a_0, a_1, a_2, a_3, \ldots$.

Example 2.1.1.1. The Sequence of Even Natural Numbers

Consider the list of nonnegative even numbers: $0, 2, 4, 6, 8, 10, \ldots$. We can view this as the map $n \mapsto 2n$, which we can see as a function on the naturals with the following inputs and outputs:

$$\begin{gathered} 0 \mapsto 0 \\ 1 \mapsto 2 \\ 2 \mapsto 4 \\ 3 \mapsto 6 \\ 4 \mapsto 8 \\ \vdots \end{gathered}$$

2.2 Notation for Sequences

There are two main ways that we define sequences: as explicit formulas and as recursive formulas. The next two subsections describe these methods.

2.2.1 Explicit Formulas

Often times we define a sequence via what is called an *explicit formula* or a *closed formula*. This is a formula given in terms of n that shows explicitly how to compute the output corresponding to an input n in finitely many steps expressed in our usual language of algebraic and transcendental functions.

Example 2.2.1.1. The Sequence of Even Natural Numbers: Explicit Formula

The sequence of even natural numbers defined above has

$$a_n = 2n$$

as its explicit formula.

Exercise 2.2.1.2. Practice with Explicit Formulas ☕☕

Find an explicit formula for the sequence of...

- ...odd natural numbers.

$$1, 3, 5, 7, \ldots$$

A Solution: $a_n = 2n + 1, n \in \{0, 1, 2, \ldots\}$

- ...even integers starting at -4 and counting upwards, two at a time.

$$-4, -2, 0, 2, \ldots$$

A Solution: $-4 + 2n, n \in \{0, 1, 2, \ldots\}$

- ...all multiples of 5, starting from 20 and counting downwards.

$$20, 15, 10, 5, 0, -5, \ldots$$

A Solution: $a_n = 20 - 5n, n \in \{0, 1, 2, \ldots\}$

- ...all natural numbers that are one more than a multiple of 3.

$$1, 4, 7, 10, \ldots$$

A Solution: $a_n = 3n + 1, n \in \{0, 1, 2, \ldots\}$

- ...consecutive powers of 2, starting from 1.

$$1, 2, 4, 8, \ldots$$

A Solution: $a_n = 2^n, n \in \{0, 1, 2, \ldots\}$

- ...terms that alternate forever between positive and negative one.

$$1, -1, 1, -1, \ldots$$

A Solution: $a_n = (-1)^n, n \in \{0, 1, 2, \ldots\}$

- ...fractions whose numerators are all even natural numbers starting from zero and whose denominators are all odd natural numbers starting from 1.

$$\frac{0}{1}, \frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \ldots$$

A Solution: $a_n = \frac{2n}{2n+1}, n \in \{0, 1, 2, \ldots\}$

2.2.2 Recursive Formulas

Recursion is a beautiful, powerful, and useful concept. It is the idea of defining a structure in terms of smaller instances of that same type of structure. In the case of sequences, we want to define a later term a_n as a formula given in terms of a_k for some k values strictly less than n. This definition of later terms built out of previous terms is called the *recursion*. Additionally, a recursive definition requires the definition of some initial term or terms to get the process rolling. These early terms are called the *base cases* or *initial terms*.

Returning to our favorite little example once again, we ask how we can find a recursive formula for the sequence of even numbers. Notice how the later terms relate to the earlier terms; each term is exactly two more than the previous term. We build a recursive formula out of this observation.

Example 2.2.2.1. The Sequence of Even Natural Numbers: Recursive Formula

$$\begin{aligned} a_0 &= 0 \\ a_n &= 2 + a_{n-1} \text{ for } n \geq 1 \end{aligned}$$

Exercise 2.2.2.2. Absorbing the Language ☕

In the recursive formula above, which expression is the base case? Which part is the recursion?

A Solution: $a_0 = 0$ is the base case and $a_n = 2 + a_{n-1}$ is the recursion.

Exercise 2.2.2.3. Practice with Recursive Formulas ☕☕

Find a recursive formula for the sequence of...

- ...odd natural numbers.

$$1, 3, 5, 7, \ldots$$

A Solution:

$$a_0 = 1$$
$$a_n = a_{n-1} + 2$$

- ...even integers starting at -4 and counting upwards, two at a time.

$$-4, -2, 0, 2, \ldots$$

A Solution:

$$a_0 = -4$$
$$a_n = a_{n-1} + 2$$

- ...all multiples of 5, starting from 20 and counting downwards.

$$20, 15, 10, 5, 0, -5, \ldots$$

A Solution:

$$a_0 = 20$$
$$a_n = a_{n-1} - 5$$

- ...all natural numbers that are one more than a multiple of 3.

$$1, 4, 7, 10, \ldots$$

A Solution:

$$a_0 = 1$$
$$a_n = a_{n-1} + 3$$

- ...consecutive powers of 2, starting from 1.

$$1, 2, 4, 8, \ldots$$

A Solution:

$$a_0 = 1$$
$$a_n = a_{n-1} \cdot 2$$

- ...terms that alternate forever between positive and negative one.

$$1, -1, 1, -1, \ldots$$

A Solution:

$$a_0 = 1$$
$$a_n = a_{n-1} \cdot (-1)$$

- ...fractions whose numerators are all even natural numbers starting from zero and whose denominators are all odd natural numbers starting from 1.

$$\frac{0}{1}, \frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \ldots$$

A Solution:

$$a_0 = \frac{0}{1}, a_n = \frac{1}{n^2 + n} + a_{n-1}.$$

This is obtained by calculating the difference $a_n - a_{n-1}$ and then rearranging the equatin to solve for a_n

.

2.3 Factorials

Some sequences have no simple explicit formula and are most easily thought of recursively. The sequence of factorials is a famous example of this type.

2.3.1 Recursive Formula for Factorials

Example 2.3.1.1. Factorials, Defined Recursively

Consider the following recursively defined sequence:

$$\begin{aligned} a_0 &= 1 \\ a_n &= n \cdot a_{n-1} \text{ for } n \geq 1 \end{aligned}$$

We can unwind this recursion a bit to obtain a more accessible expression for factorials. Observe the following calculations based on the base case and recursion given above:

$$\begin{aligned} a_0 &= 1 = 1 \\ a_1 &= 1 \cdot a_0 = 1 \\ a_2 &= 2 \cdot a_1 = 2 \cdot 1 = 2 \\ a_3 &= 3 \cdot a_2 = 3 \cdot 2 \cdot 1 = 6 \\ a_4 &= 4 \cdot a_3 = 4 \cdot 3 \cdot 2 \cdot 1 = 24 \\ a_5 &= 5 \cdot a_4 = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \end{aligned}$$

This sequence comes up so frequently that we give it its own symbol, the exclamation point! Since the factorial of n always amounts to the product of all natural numbers greater than or equal to 1 but less than or equal to n, we write the following:

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$$

Note that almost any expression involving a shady "$\cdots$" is truthfully a recursion in disguise!

Exercise 2.3.1.2. Why is the Factorial of Zero Equal to One? ☕

Looking carefully at the above definition, you will notice that

$$0! = 1$$

It is a common mistake to compute 0! as 0 instead. Here is one way to see why it should in fact be 1.

- If you compute 2^2, how many numbers are you multiplying together?
- If you compute 2^1, how many numbers are you multiplying together?
- If you compute 2^0, how many numbers are you multiplying together?

- Right, zero numbers are being multiplied together. A product like this is called an *empty product* and is always defined to be one, since that is the multiplicative identity.
- If you compute 3! as the product of all natural numbers greater than or equal to 1 and but less than or equal to 3, how many numbers are you multiplying together?
- If you compute 2! as the product of all natural numbers greater than or equal to 1 and but less than or equal to 2, how many numbers are you multiplying together?
- If you compute 1! as the product of all natural numbers greater than or equal to 1 and but less than or equal to 1, how many numbers are you multiplying together?
- If you compute 0! as the product of all natural numbers greater than or equal to 1 and but less than or equal to 0, how many numbers are you multiplying together?
- Since 0! is also an empty product (much like 2^0), what should we define it to be?

The following type of simplification will occur frequently throughout the our adventures in infinite series and power series. They all follow directly from the recursive definition of factorials.

Example 2.3.1.3. Simplifying Factorials

Let n represent a positive natural number. Consider the expression $\frac{(n+2)!}{n!}$. The numerator represents the product of all natural numbers between $n+2$ and 1, inclusive. The denominator represents the product of all natural numbers between n and 1, inclusive. We expand out these products and then cancel whatever factors they have in common.

$$\begin{aligned}\frac{(n+2)!}{n!} &= \frac{(n+2)(n+1)(n)(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{(n)(n-1)(n-2)\cdots 3\cdot 2\cdot 1}\\ &= (n+2)(n+1)\end{aligned}$$

Thus, that ratio of factorials cleans up to just a polynomial!

Exercise 2.3.1.4. Simplifying Factorials ☕☕

Let n be a natural number greater than or equal to 1. Reduce the following fractions! (Are they fractorials?)

- $\frac{n!}{(n+1)!}$

  A Solution:

  $$\frac{1}{n+1}$$

- $\frac{(n+1)!}{n!}$

  A Solution:

  $$n+1$$

- $\frac{(n+2)!}{n!}$

A Solution:

$$(n+1)(n+1)$$

- $\frac{(2n+2)!}{(2n)!}$

A Solution:

$$(2n+2)(2n+1)$$

2.3.2 Explicit Formula for Factorials: The Gamma Function

The gamma function, denoted Γ, is the most common smooth interpolation of the factorial function on the positive integers. The gamma function is defined using improper integrals.

Definition 2.3.2.1. The Gamma Function

For $n \in \mathbb{R}$, define

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} \, \mathtt{d}x$$

That is, $\Gamma(n)$ is the area under the graph of $x^{n-1}e^{-x}$ in the first quadrant.

Exercise 2.3.2.2. Computing Values of Gamma

- To see the manner in which the gamma function provides a continuous analog of the factorial function, fill out the values in the following table:

n	$n!$	$\Gamma(n)$
1		
2		
3		
4		

A Solution:

n	$n!$	$\Gamma(n)$
1	1	1
2	2	1
3	6	2
4	24	6

Computing the values of the gamma function will require quite a bit of work. You don't have to show all details of the integrals above, but make sure you are comfortable doing such manipulations by hand.

- Given the table above, conjecture an explicit formula for the factorials. It won't be a nice little algebraic formula, but express it in terms of an improper integral. Write the conjecture below.

$$n! = \int_0^\infty \qquad\qquad\qquad\qquad \mathtt{d}x$$

A Solution:

$$n! = \int_0^\infty x^n e^{-x}\,\mathtt{d}x$$

Observe that you were be able to use the smaller instances of the gamma function to help you compute the larger instances! That is, when you apply integration by parts to compute $\Gamma(n)$, it will produce an expression that involves the integral you computed for $\Gamma(n-1)$.

It turns out this relationship is exactly what shows that the Γ function will *always* match the values of the factorial function. For factorial, we have:

$$n! = n \cdot (n-1)!$$

Exercise 2.3.2.3. Gamma Recursion ☕☕☕

What is the corresponding relationship for the Gamma function? Specifically, how does $\Gamma(n)$ relate to $\Gamma(n-1)$? Write your answer below.

A Solution: We can compute this by looking at $\Gamma(n+1)$ since it's a little easier to work with, and computing it with integration by parts, with $u = x^n, \mathtt{d}u = nx^{n-1}\,\mathtt{d}x, v = e^{-x}, \mathtt{d}v = -e^{-x}$:

$$\begin{aligned}\Gamma(n+1) &= \int_0^\infty x^n e^{-x}\,\mathtt{d}x \\ &= -x^n e^{-x}\Big]_0^\infty + n\int x^{n-1}e^{-x}\,\mathtt{d}x \\ &= 0 + n\Gamma(n)\end{aligned}$$

So $\Gamma(n+1) = n\Gamma(n)$ and:

$$\Gamma(n) = (n-1)\Gamma(n-1).$$

2.3.3 An Approximation for the Factorials

As mentioned above, there does not exist a simple algebraic explicit formula for the factorial function. However **Stirling's Formula** gives a very nice explicit asymptotic formula.

Stirling's Formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

That is to say, as n approaches infinity, $n!$ approaches $\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. We don't have the tools to fully prove Stirling's Formula in this course, though it will be occasionally helpful to have a rough measure of the growth order of a factorial function.

2.4 Arithmetic and Geometric Sequences

Here we provide the definitions for two particularly famous families of sequences, arithmetic and geometric.

Definition 2.4.0.1. Arithmetic Sequence

A sequence a_n is called *arithmetic* if and only if there exists some real constant d such that $a_{n+1} - a_n = d$ for all natural numbers n. In such a sequence, the number d is called the *common difference*.

Definition 2.4.0.2. Geometric Sequence

A sequence a_n is called *geometric* if and only if there exists some real constant r such that $a_{n+1}/a_n = r$ for all natural numbers n. In such a sequence, the number r is called the *common ratio*.

Exercise 2.4.0.3. Playing with the Definition ☕☕

- Return to Exercise 2.2.1.2. Which of those sequences are arithmetic? For those that are, what is the common difference?

A Solution:

1. The sequence of odd natural numbers $1, 3, 5, 7, \ldots$ is arithmetic and has difference of 2.
2. The sequence of even integers $-4, -2, 0, 2, \ldots$ is arithmetic and has difference 2.
3. The sequence of multiples of 5 is arithmetic and has difference -5, since the later term is less than the previous term.
4. The sequence of natural numbers one more than a multiple of 3, given by $1, 4, 7, 10, \ldots$ is arithmetic and has difference of 3.

- Return again to Exercise 2.2.1.2. Which of those sequences are geometric? For those that are, what is the common ratio?

  A Solution:

  1. The sequence of consecutive powers of 2, give by $1, 2, 4, 8, \ldots$ is geometric, and has common ratio 2.
  2. The sequence of alternating ones, $1, -1, 1, -1, \ldots$ is geometric, with common ratio -1.

- Give an informal definition of an arithmetic sequence. (Think of what you would say if you had to explain what it was to a fifth grader).

  A Solution: An arithmetic sequence grows (or decreases) by the same amount at each step. The step size from one number to the next is always the same. You add each term to the same number to get the next term.

- Give an informal definition of a geometric sequence. (Think again of what you would say if you had to explain what it was to a fifth grader).

  A Solution: A geometric sequence grows (or shrinks) more and more at each step. You multiply each term by the same number to get the next term.

- Give an example of a sequence that is arithmetic but not geometric.

  A Solution: $1.1, 1.2, 1.3, 1.4, \ldots$.

- Give an example of a sequence that is geometric but not arithmetic.

  A Solution: $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \cdots$.

- Can a sequence simultaneously be both arithmetic and geometric? If it is possible, give an example of such a sequence. If it is not possible, explain why it is not possible.

  A Solution: The sequence $1, 1, 1, 1, \ldots$ is both geometric (common ratio is 1) and arithmetic (common difference 0).

Exercise 2.4.0.4. Converting Between Recursive and Explicit Definitions ☕☕

- Write a sentence that explains the difference between defining a sequence recursively vs defining a sequence explicitly.

A Solution: Defining a sequence recursively means defining each term in relation to the previous (or several previous) terms. To find a given term, you have to calculate all the ones before it.

Defining a sequence explicitly allows us to get any term on its own.

- Consider the following recursively-defined sequence:

$$a_0 = 5$$
$$a_n = 2 \cdot a_{n-1}$$

Write out the first five terms of this sequence. Can you find an explicit formula?

A Solution:

$$5, 10, 20, 40, 80, \ldots.$$

The explicit formula is given by $5 \cdot 2^n$.

- Consider the following explicitly-defined sequence:

$$a_n = 3n - 2$$

Write out the first five terms of this sequence. Can you find an recursive formula?

A Solution:

$$-2, 1, 4, 7, 10, 13, \ldots.$$

A recursive form is given by

$$a_0 = -2$$
$$a_{n+1} = a_n + 3.$$

2.4.1 Explicit Formulae for Arithmetic and Geometric Sequences

Example 2.4.1.1. Explicit Formula for Arithmetic Sequences

Since every arithmetic sequence starts with some initial term a_0 and then adds the same number d each time to get from term to term, we can say the explicit formula will always have the same form. In particular, we have the terms of the sequence as follows:

$$a_0, a_0 + d, a_0 + 2d, a_0 + 3d, \cdots .$$

Thus, a generic term of the sequence looks like

$$a_n = a_0 + dn$$

where a_0 is the initial term and d is the common difference.

Exercise 2.4.1.2. Explicit Formula for Geometric Sequences ☕☕

Repeat the process of the above example to demonstrate that every geometric sequence has explicit formula

$$a_n = a_0 r^n$$

where a_0 is the initial term and r is the common ratio.

A Solution: Since a geometric sequence starts with a_0 and then multiplies each term by the same common ratio, we get the sequence

$$a_0, a_0 \cdot r, a_0 \cdot r^2, a_0 \cdot r^3, \ldots .$$

Then some generic term is

$$a_n = a_0 r^n.$$

2.5 Absolute Convergence and Rearrangements

When we add up finitely many numbers, we take properties like commutativity and associativity for granted. We add up numbers in whatever order is most convenient. With infinite series, we cannot be quite so cavalier!

Example 2.5.0.1. Did You Know that Zero Equals One?

Here we use the fact that 0=-1+1.

$$\begin{aligned}
1 &= 1 + 0 + 0 + 0 + 0 + \cdots \\
&= 1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + (-1 + 1) + \cdots \\
&= 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - \cdots \\
&= (1 - 1) + (1 - 1) + (1 - 1) + (1 - 1) + (1 + -1) + \cdots \\
&= 0 + 0 + 0 + 0 + 0 + \cdots \\
&= 0
\end{aligned}$$

Exercise 2.5.0.2. What? ☕☕☕

Let us now correctly analyze the infinite sum

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - \cdots$$

- Consider the sequence $a_n = (-1)^n$. Use the Geometric Series Formula to find the corresponding sequence of partial sums A_N.

  A Solution:
  $$\frac{1 - (-1)^{N+1}}{2}$$

- What is the limit of the sequence of partial sums?

  A Solution: The limit does not exist, since it alternates between 1 and -1.

- Thus, what is the correct value of the infinite series $1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - \cdots$?

  A Solution: The sum is not defined.

It turns out that the key lies in the distinction between a series being convergent vs being *absolutely* *convergent*, a stronger type of convergence.

Definition 2.5.0.3. Absolute Convergence

An infinite series $\sum_{n=0}^{\infty} a_n$ is *absolutely convergent* if and only if $\sum_{n=0}^{\infty} |a_n|$ is convergent.

Absolute convergence is the idea that it wasn't just some sort of cancellation of positive and negative terms that let the partial sums stabilize. Rather, the magnitudes of the terms were going to zero quickly enough. To test this, we just take the term-by-term absolute value of the series and see if the resulting series still converges.

Example 2.5.0.4. An Absolutely Convergent Series

The infinite geometric series

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$$

is absolutely convergent because

$$|1| + \left|-\frac{1}{2}\right| + \left|\frac{1}{4}\right| + \left|-\frac{1}{8}\right| + \left|\frac{1}{16}\right| + \cdots = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 2.$$

The term=by-term absolute value of the series still converges, so the original series is declared absolutely convergent.

Contrast this concept with the following definition, a weaker form of convergence called conditional.

Definition 2.5.0.5. Conditional Convergence

An infinite series $\sum_{n=0}^{\infty} a_n$ is *conditionally convergent* if and only if it converges but $\sum_{n=0}^{\infty} |a_n|$ diverges.

Example 2.5.0.6. A Conditionally Convergent Series

The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots$$

is conditionally convergent because

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

diverges. In particular, it is the harmonic series which totals to ∞ (as we will see in Example 2.6.1.2).

Exercise 2.5.0.7. A Maybe Absolutely Convergent Series ☕☕

Is the infinite geometric series

$$0.1 - 0.02 + 0.004 - 0.0008 + 0.00016 - \cdots$$

absolutely convergent or conditionally convergent? Explain.

A Solution: If we take the absolute value of each of the terms and look consider that series, we get

$$\begin{aligned} &|0.1| + |-0.02| + |0.004| + |-0.0008| + |0.000016| + \cdots \\ &= 0.1 + 0.02 + 0.004 + 0.0008 + 0.000016 + \cdots . \end{aligned}$$

This has common ratio $\frac{2}{10}$ and a starting term of $\frac{1}{10}$, so we can use our Geometric Series formula to get

$$\sum_{n=0}^{N} a_n = \frac{1}{10} \cdot \frac{1 - \frac{2}{10}^{N+1}}{1 - \frac{2}{10}}.$$

Taking the limit of this

$$\lim_{N\to\infty} \frac{1}{10} \cdot \frac{1 - \frac{2}{10}^{N+1}}{\frac{8}{10}} = \frac{1}{10} \cdot \frac{10}{8} = \frac{1}{8}$$

.

It turns out that for absolutely convergent series, rearranging of terms and any sort of normal algebraic manipulation is fine. This is a theorem that is rather difficult to prove and will be saved for a later mathematical adventure. For this course, we will just use it!

2.6 Convergence Tests

It is often too difficult to determine the exact value of an infinite series (if it converges at all). Thus, we usually settle for the knowledge *that* a series converges (or diverges) as opposed to finding *what* number it converges to. By "settling" as such, we are not actually giving up too much. If we can guarantee that a series converges, it means it is safe to approximate it by just taking a partial sum with lots of terms.

Exercise 2.6.0.1. Why Convergence is So Critical ☕☕

Why would it not make sense to approximate a divergent infinite series using a partial sum with lots of terms?

Below, we detail eight commonly used tests for convergence.

1. No Hope Test
2. Geometric Series Test
3. Integral Test
4. p Test
5. Alternating Series Test
6. Limit Comparison Test
7. Ratio Test
8. Direct Comparison Test

2.6.1 No Hope Test and the Harmonic Series

The No Hope Test is sometimes also referred to as the Divergence Test or the n^{th} Term Test. Intuitively, this test says that if the terms of the sequence a_n do not go to zero, then their sum has no hope of converging. We give a more formal statement here.

Theorem 2.6.1.1. No Hope Test

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=0}^{\infty} a_n$ diverges.

It does *not* say the following:

$$\text{If } \lim_{n\to\infty} a_n = 0, \text{ then } \sum_{n=0}^{\infty} a_n \text{ converges.} \qquad (☹)$$

This is a fallacy and a *very* common mistake. If the terms of the sequence go to zero, then the series has some chance of converging, but it is no guarantee. Here is a classic counterexample, the harmonic series! The following divergence proof by Johann Bernoulli comes from the mid-seventeenth century.

Example 2.6.1.2. Divergence of the Harmonic Series

Here we show that

$$\sum_{n=0}^{\infty} \frac{1}{n} = \infty$$

We accomplish this by showing the partial sums will exceed any sum of one-half added to itself again and again. Proceeding:

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \cdots \\
&> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \cdots \\
&= 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \frac{8}{16} + \cdots \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots \\
&= \infty
\end{aligned}$$

Since the harmonic series is greater than a sum of one-half added to itself infinitely many times, it is infinite.

Exercise 2.6.1.3. Justifying the Steps

Annotate the above proof with a short comment justifying each line of equality or inequality.

Exercise 2.6.1.4. Revisiting a Convergent Series

In the previous section, we showed that

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1.$$

If you attempt to use Bernoulli's argument to show that it diverges, where does it break down? Why can't you just group terms together into batches that are at least size one-half?

Exercise 2.6.1.5. No Hope Test Backwards ☕

Explain why the Harmonic Series is a counterexample to the claim tagged with ☹.

Example 2.6.1.6. Where NHT Applies and Does Not Apply

- Consider the infinite series
$$\sum_{n=0}^{\infty} \arctan(n).$$
We notice that $\lim_{n\to\infty} \arctan(n) = \pi/2 \neq 0$. Thus, the series diverges by NHT, since we are essentially adding up infinitely many copies of $\pi/2$ to itself.

- Consider the infinite series
$$\sum_{n=0}^{\infty} \arctan\left(\frac{1}{n}\right).$$
Since $\lim_{n\to\infty} \arctan\left(\frac{1}{n}\right) = 0$, we cannot make any conclusion about this series via NHT. The summand approaches zero, so there is some hope of the series converging, but further work would be required to determine this.

Now try a few on your own!

Exercise 2.6.1.7. Practice with the No Hope Test ☕☕

- Use the No Hope Test to prove that the series $\sum_{n=1}^{\infty} \frac{\sqrt{2n^2+2}}{n}$ diverges.

- Use the No Hope Test to prove that the series $\sum_{n=1}^{\infty} \cos(1/n)$ diverges.

- What does the No Hope Test tell you about the convergence/divergence of the series $\sum_{n=1}^{\infty} \sin(1/n)$?

2.6.2 Geometric Series Test

This is essentially just a restatement of the main result of Subsection **??**, the Geometric Series Formula. Recall that a geometric series is a series of the form

$$\sum_{n=0}^{\infty} a \cdot r^n$$

for some real numbers a and r. That is to say, it is a series that has starting term a and common ratio r.

Theorem 2.6.2.1. Geometric Series Test

A geometric series converges if $|r| < 1$ (and in fact converges to the value $a/(1-r)$) and diverges otherwise.

Exercise 2.6.2.2. Practice with the Geometric Series ☕☕☕

- Explain why $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is not a geometric series (and thus this test would be inapplicable).
- Explain why the series $18 - 6 + 2 - \frac{2}{3} + \frac{2}{9} - \cdots$ converges absolutely.
- Give an example of a geometric series that converges conditionally, or explain why it is not possible to construct such a series.

Exercise 2.6.2.3. Why Not Arithmetic? ☕☕☕

Why is there *not* another result in this section called the "Arithmetic Series Test"? Why is only the geometric series getting to have all the fun?

2.6.3 Integral Test

Here we use integrals to test convergence of infinite series! Intuitively this test says the following:

An infinite series converges if and only if the corresponding improper integral converges.

Theorem 2.6.3.1. Integral Test

Let $a \in \mathbb{N}$ and $f(n)$ be a decreasing function on $[a, \infty)$. If $\int_{x=a}^{x=\infty} f(x)\,\mathtt{d}x$ converges, then $\sum_{n=a}^{\infty} f(n)$ converges as well. Conversely, if $\int_{x=a}^{x=\infty} f(x)\,\mathtt{d}x$ diverges, then $\sum_{n=a}^{\infty} f(n)$ diverges as well.

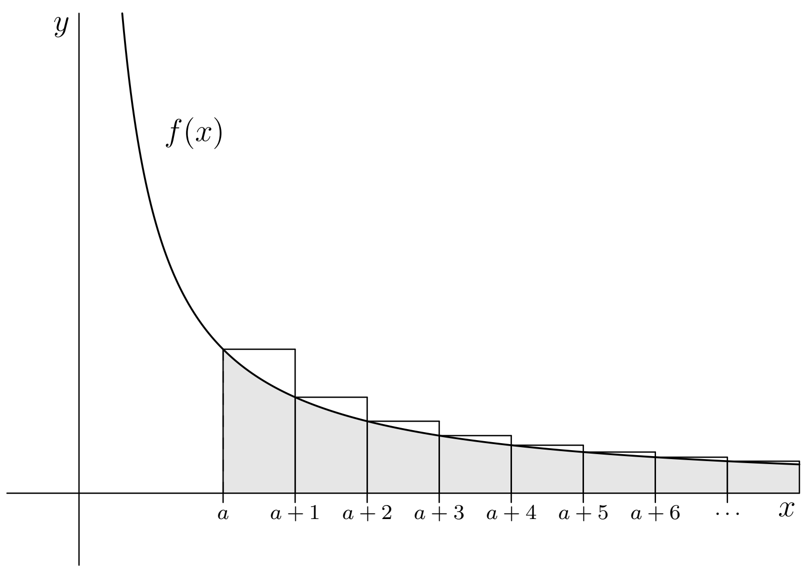
Here is a "proof by picture" to justify the Integral Test.

Exercise 2.6.3.2. Explaining the Integral Test ☕☕☕

Study the following diagrams and use them to determine why the series converges if and only if the series converges. Specifically:

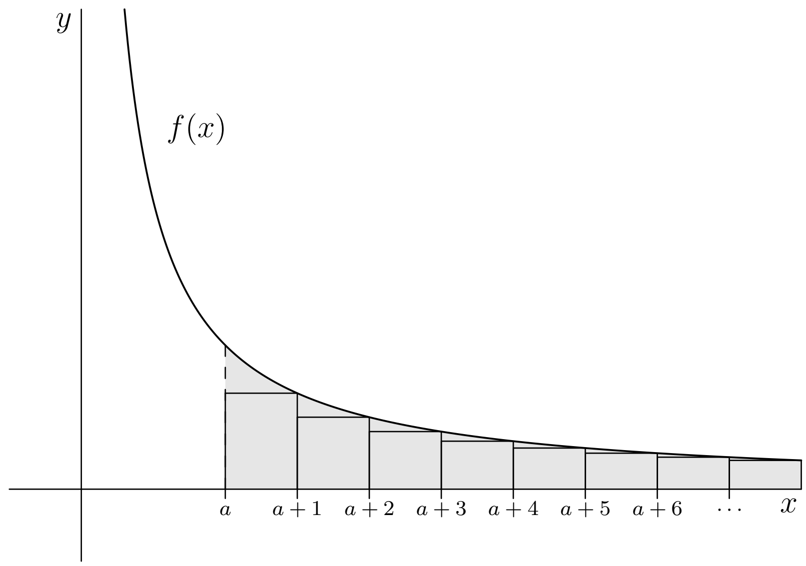
- Explain why the diagram below justifies the inequality

$$\int_{x=a}^{x=\infty} f(x)\,\mathtt{d}x \leq \sum_{n=a}^{\infty} f(n).$$



- Explain why the diagram below justifies the inequality

$$\sum_{n=a+1}^{\infty} f(n) \leq \int_{x=a}^{x=\infty} f(x)\,\mathtt{d}x.$$



- Add $f(a)$ to both sides of the previous inequality to conclude

$$\sum_{n=a}^{\infty} f(n) \leq f(a) + \int_{x=a}^{x=\infty} f(x)\,\mathtt{d}x.$$

- Putting both inequalities together, we now have that

$$\int_{x=a}^{x=\infty} f(x)\,\mathtt{d}x \leq \sum_{n=a}^{\infty} f(n) \leq f(a) + \int_{x=a}^{x=\infty} f(x)\,\mathtt{d}x. \qquad (\bowtie)$$

  Explain why this inequality shows that the infinite series converges if and only if the corresponding improper integral converges.

Example 2.6.3.3. Using the Integral Test

- The infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges because the corresponding improper integral is

$$\begin{aligned}\int_{x=1}^{x=\infty} \frac{1}{x^2}\,\mathtt{d}x &= \lim_{c\to\infty} -\frac{1}{x}\Bigg]_{x=1}^{x=c} \\ &= \lim_{c\to\infty} -\frac{1}{c} - -\frac{1}{1} \\ &= 1.\end{aligned}$$

  Since the improper integral converges to a finite value, the Integral Test says that the corresponding infinite series converges as well.

- The infinite series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges because the corresponding improper integral is

$$\begin{aligned}\int_{x=1}^{x=\infty} \frac{1}{\sqrt{x}}\,\mathtt{d}x &= \lim_{c\to\infty} 2\sqrt{x}\Big]_{x=1}^{x=c} \\ &= \lim_{c\to\infty} 2\sqrt{c} - 2\sqrt{1} \\ &= \infty.\end{aligned}$$

  Since the improper integral diverges, the Integral Test says that the corresponding infinite series diverges as well.

Exercise 2.6.3.4. Practice with the Integral Test ☕☕

Use the Integral Test to decide if the following infinite series converge or diverge.

- $\sum_{n=2}^{\infty} \frac{1}{n\,\mathtt{ln}(n)}$

- $\sum_{n=2}^{\infty} \frac{1}{n(\mathtt{ln}(n))^2}$

- $\sum_{n=2}^{\infty} \frac{1}{n^2+1}$

The next example shows why the assumption of $f(x)$ being a decreasing function is necessary for the Integral Test.

Exercise 2.6.3.5. An Interesting Example ☕☕☕

Consider the function

$$f(x) = |\sin(\pi x)|$$

- Graph the function $f(x)$ over the positive x-axis.
- Explain why the integral $\int_{x=0}^{x=\infty} f(x)\,\mathtt{d}x$ is equal to infinity.
- Compute the infinite sum $\sum_{n=0}^{\infty} f(n)$.
- In this case, the integral diverged, while the infinite sum converged. Why does this example not contradict the Integral Test?

2.6.4 p-Test

The next result is just a special case of the Integral Test. However, it comes up often enough that it is worth stating on its own.

Theorem 2.6.4.1. The p-Test

Let p be a real number. Then the sum $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges otherwise.

Exercise 2.6.4.2. Justifying the p-Test Using the Integral Test ☕

- Apply the Integral Test to the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ in the case where $p > 1$. Show the corresponding improper integral converges, and thus the series does as well.

- Apply the Integral Test to the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ in the case where $p < 1$. Show the corresponding improper integral diverges, and thus the series does as well.

- What does the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ do when $p = 1$?

2.6.5 Alternating Series Test and Error Bound

Theorem 2.6.5.1. Alternating Series Test (AST)

Let a_n be a decreasing sequence of positive numbers that approaches zero as n approaches infinity. The summation

$$\sum_{n=0}^{\infty} (-1)^n a_n$$

converges.

Intuitively, this test says that the positives and negatives will cancel each other out and the partial sum pendulum will eventually stabilize at some well-defined limit.

Example 2.6.5.2. Using AST

Consider the series $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n$. This series can be rewritten as $\sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2}\right)^n$. Since the sequence $\left(\frac{1}{2}\right)^n$ is positive and decreasing to zero, AST tells us that it converges.

Exercise 2.6.5.3. Conditional vs Absolute ☕

Did the above series converge conditionally or absolutely? Explain.

Notice that the above series also converges by the Geometric Series Test. It is very common that more than one convergence test will apply to the same series.

Exercise 2.6.5.4. Tests Not Applying ☕

- Explain why the convergence of $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n$ cannot be analyzed using the No Hope Test.

- Explain why the convergence of $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n$ cannot be analyzed using the Integral Test.

- Explain why the convergence of $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n$ cannot be analyzed using the p-Test.

Exercise 2.6.5.5. The AST in Action ☕☕

For each of the series below, explain why it converges by AST or explain why AST does not apply to that series.

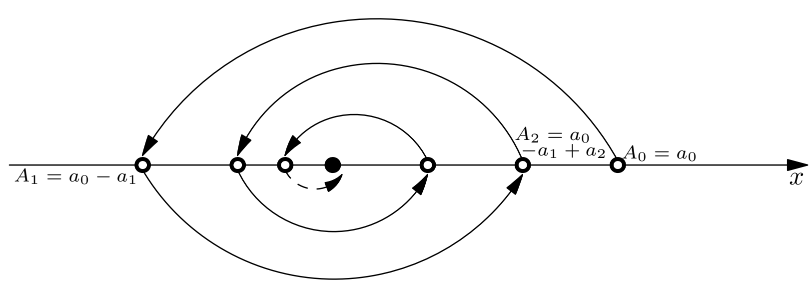
- $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$

- $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n}$

- $\sum_{n=1}^{\infty} (\pi/2 - \arctan(n))^n$

- $\sum_{n=1}^{\infty} (\arctan(n) - \pi/2)^n$

In a convergent alternating series, the partial sums always “leapfrog” back and forth over the limiting value the series converges to. This implies that the value of the infinite series is always no further away than the next unused term in any partial sum.



Theorem 2.6.5.6. Alternating Series Error Bound

Let a_n be a sequence of positive decreasing terms. Let $\sum_{n=0}^{\infty}(-1)^n a_n$ be a convergent series. Then the error in a partial sum is always less than the first unused term in that partial sum. That is,

$$\left|\sum_{n=0}^{\infty}(-1)^n a_n - \sum_{n=0}^{N}(-1)^n a_n\right| < |a_{N+1}|.$$

Example 2.6.5.7. Alternating Series Error Bug

Consider again the bug described in Exercise **??.??**. Suppose we wish to know how close the bug is to its ending destination after it has reversed course two times. Since the series is a convergent alternating series, we can use the Alternating Series Error Bound to determine how close it is to its final location. In particular, the Alternating Series Error Bound implies

$$\left|\left(\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\cdots\right)-\left(\frac{1}{2}-\frac{1}{4}+\frac{1}{8}\right)\right| < \frac{1}{16}.$$

To independently verify this claim, notice that we had already concluded in Exercise **??.??** that the infinite series totaled to $\frac{1}{3}$. If we plug in this value, we obtain

$$\begin{aligned}\left|\left(\frac{1}{3}\right)-\left(\frac{1}{2}-\frac{1}{4}+\frac{1}{8}\right)\right| &= \left|\frac{1}{3}-\left(\frac{3}{8}\right)\right| \\ &= \left|\frac{8}{24}-\frac{9}{24}\right| \\ &= \left|-\frac{1}{24}\right|\end{aligned}$$

which is indeed less than one-sixteenth.

Exercise 2.6.5.8. Alternating Series Error Bug ☕☕

- After the bug has reversed course three times, how close does the Alternating Series Error Bound guarantee the bug is to its final location?

- How many times would the bug have to reverse course for the Alternating Series Error Bound to guarantee the bug is within one one-thousandth of its final location? After you figure this out, compute the corresponding partial sum and verify that it is within 0.001 of one-third.

2.6.6 Limit Comparison Test

This test says that if two sequences have the same growth order, then the corresponding infinite series either both converge or both diverge. This works because if two sequences have the same growth order, then in the long term they are just a nonzero constant factor apart, and multiplying by a nonzero constant factor cannot change convergence or divergence. Furthermore, having larger magnitude terms than a divergent series implies divergence, and smaller magnitude terms than a convergent series implies convergence.

Theorem 2.6.6.1. Limit Comparison Test (LCT)

Let a_n and b_n be sequences of nonnegative terms.

- Suppose a_n has larger growth order than b_n, and $\sum_{n=0}^{\infty} b_n$ diverges. Then $\sum_{n=0}^{\infty} a_n$ also diverges.
- Suppose a_n has smaller growth order than b_n, and $\sum_{n=0}^{\infty} b_n$ converges. Then $\sum_{n=0}^{\infty} a_n$ also converges.
- Suppose a_n and b_n have the same growth order. Then $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ either both converge or both diverge.

A short intuitive way to state LCT is as follows:

A series with smaller growth order than a convergent series must also be convergent. A series with larger growth order than a divergent series must also be divergent.

The Limit Comparison Test is particularly useful if a_n is expressed as a fraction with the numerator and denominator both algebraic (expressed just with polynomials and radicals). In this case, one can build a comparison sequence by taking the ratio of leading terms from the numerator and denominator.

Example 2.6.6.2. Using Leading Terms from the Numerator and Denominator

Suppose we wish to determine the convergence/divergence of

$$\sum_{n=1}^{\infty} \frac{7n+3}{n\sqrt{n^2+n+1}}.$$

Call the summand $a_n = \frac{7n+3}{n\sqrt{n^2+n+1}}$. The idea is that in the numerator, the "plus three" is insignificant as n approaches infinity. Thus, we keep only the $7n$ in the numerator. In the denominator, we observe that the $n+1$ is insignificant compared to the n^2 it is being added to. Again, we keep only the $n\sqrt{n^2} = n \cdot n = n^2$, the leading term of the denominator. We have built our comparison function

$$b_n = \frac{7n}{n^2} = \frac{7}{n}.$$

Sometimes it is nice to visualize this as just crossing out all lower order terms. For large n,

$$\frac{7n + \cancel{3}}{n\sqrt{n^2 + \cancel{n} + \cancel{1}}} \approx \frac{7n}{n^2} = \frac{7}{n}.$$

Next, we verify that a_n and b_n have the same growth order. Proceeding with the limit, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{7n+3}{n\sqrt{n^2+n+1}} \frac{n}{7} \\
&= \lim_{n\to\infty} \frac{7n+3}{7\sqrt{n^2+n+1}} \\
&= \lim_{n\to\infty} \frac{7n^2+3n}{7n\sqrt{n^2+n+1}} \\
&= \lim_{n\to\infty} \frac{7n^2+3n}{7\sqrt{n^2}\sqrt{n^2+n+1}} \\
&= \lim_{n\to\infty} \frac{7n^2+3n}{7\sqrt{n^4+n^3+n^2}} \\
&= \lim_{n\to\infty} \frac{7n^2+3n}{7\sqrt{n^4+n^3+n^2}} \frac{\frac{1}{n^2}}{\frac{1}{\sqrt{n^4}}} \\
&= \lim_{n\to\infty} \frac{7+\frac{3}{n}}{7\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}} \\
&= \frac{7}{7\sqrt{1}} \\
&= 1.
\end{aligned}$$

Thus, a_n and b_n have the same growth order, since their ratio is a nonzero constant. Furthermore,

$$\sum_{n=1}^{\infty} \frac{7}{n} = 7 \sum_{n=1}^{\infty} \frac{1}{n}$$

which diverges by Bernoulli's argument in Example 2.6.1.2. By LCT, $\sum_{n=1}^{\infty} \frac{7n+3}{n\sqrt{n^2+n+1}}$ diverges as well.

Exercise 2.6.6.3. Practice with LCT

Use LCT to determine whether the series converge or diverge.

- $\sum_{n=1}^{\infty} \frac{n+2}{n^3+1}$

- $\sum_{n=1}^{\infty} \frac{n^5}{n\sqrt{n^7+3n+1}}$

Note that one can attempt to apply LCT can be applied to essentially any function, not just algebraic functions. However, when transcendental functions are involved, it can be more difficult to identify choice of comparison function.

Example 2.6.6.4. Cosine of Reciprocals

Consider the series $\sum_{n=1}^{\infty} \left(1 - \cos\left(\frac{1}{n}\right)\right)$. We admittedly pull a rabbit out of a hat and decide to compare to the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. We demonstrate the summands have the same growth order by taking a limit of their ratios. In particular,

$$\begin{aligned}
\lim_{n\to\infty} \frac{1/n^2}{1-\cos\left(\frac{1}{n}\right)} &= \lim_{n\to\infty} \frac{-2/n^3}{\sin\left(\frac{1}{n}\right)(-1/n^2)} \\
&= \lim_{n\to\infty} \frac{2/n}{\sin\left(\frac{1}{n}\right)} \\
&= \lim_{n\to\infty} \frac{-2/n^2}{\cos\left(\frac{1}{n}\right)(-1/n^2)} \\
&= \lim_{n\to\infty} \frac{-2}{\cos\left(\frac{1}{n}\right)} \\
&= \lim_{n\to\infty} \frac{2}{\cos\left(\frac{1}{n}\right)} \\
&= \frac{2}{1} \\
&= 2.
\end{aligned}$$

Since the limit is a nonzero constant, we conclude that $\frac{1}{n^2}$ and $\left(1-\cos\left(\frac{1}{n}\right)\right)$ have the same growth order. Since the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, we use LCT to conclude that $\sum_{n=1}^{\infty}\left(1-\cos\left(\frac{1}{n}\right)\right)$ also converges.

The above example may be a bit unsatisfying in the sense that we gave no indication where the choice of comparison function came from. Hang in there though! Once we have techniques of power series in our toolbox, we will revisit such examples and see why this was in fact a very natural choice of comparison.

Exercise 2.6.6.5. In and Out of L'Hospital ☕

- In the above example, LHR was applied twice. Identify which two lines it was used on. In each case, make a brief note as to why it was valid to use LHR.
- In the above example, we compared to the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$. How did we know that series was convergent?

Exercise 2.6.6.6. Sine of Reciprocals ☕☕☕

Determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$.

2.6.7 Ratio Test

This test is essentially a LCT against a geometric series.

Exercise 2.6.7.1. Ratio of Consecutive Terms ☕

If a_n is a geometric sequence, what is a_{n+1}/a_n?

For a sequence that is not geometric, there may not be a constant ratio of consecutive terms, but we can still look at the limit of ratios of successive terms!

Theorem 2.6.7.2. Ratio Test

Consider the series

$$\sum_{n=0}^{\infty} a_n.$$

- If $\lim_{n\to\infty} |a_{n+1}/a_n| < 1$ then the series converges absolutely.
- If $\lim_{n\to\infty} |a_{n+1}/a_n > 1|$ then the series diverges.
- If $\lim_{n\to\infty} |a_{n+1}/a_n = 1|$ then the ratio test gives no information.

The Ratio Test is particularly helpful in analyzing series involving factorials, since so much cancellation will occur when computing the ratio of consecutive terms.

Example 2.6.7.3. A Series with Factorials

Here we analyze the convergence of the series

$$\sum_{n=0}^{\infty} \frac{2^n}{n!}.$$

Call $a_n = \frac{2^n}{n!}$. We now compute the limit of the ratio of consecutive terms. Proceeding, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{\frac{2^{(n+1)}}{(n+1)!}}{\frac{2^n}{n!}} \\
&= \lim_{n\to\infty} \frac{2^{(n+1)}}{2^n} \frac{n!}{(n+1)!} \\
&= \lim_{n\to\infty} 2 \frac{n\cdots 3\cdot 2\cdot 1}{(n+1)\cdot n\cdots 3\cdot 2\cdot 1} \\
&= \lim_{n\to\infty} \frac{2}{n+1} \\
&= 0 \\
&< 1.
\end{aligned}$$

Thus, the series converges by the Ratio Test!

Exercise 2.6.7.4. Practice with Ratio Test

Use the Ratio Test to prove the following series converge or diverge, or explain why the Ratio Test provides no information in that case.

- $\sum_{n=1}^{\infty} \frac{n+1}{n!}$

- $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$

- $\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$

- $\sum_{n=1}^{\infty} \frac{2^n}{n^{10}}$

2.6.8 Direct Comparison Test

The Direct Comparison Test (DCT) is similar to LCT, except we are directly comparing magnitudes of terms instead of growth orders. We already used the idea of DCT in both Bernoulli's analysis of the Harmonic series and the Integral Test.

Theorem 2.6.8.1. Direct Comparison Test

Let a_n and b_n be sequences. If for all natural numbers n, $|a_n| \leq |b_n|$, then

- $\sum_{n=0}^{\infty} b_n$ converges implies that $\sum_{n=0}^{\infty} a_n$ also converges.
- $\sum_{n=0}^{\infty} a_n$ diverges implies that $\sum_{n=0}^{\infty} b_n$ also diverges.

A short intuitive way to state DCT is as follows:

A series smaller than a convergent series must also be convergent. A series larger than a divergent series must also be divergent.

The trick when using DCT is to pick an easy series to compare to, for example a p-series.

Example 2.6.8.2. Redirect the Trig Function

Here we analyze the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n(3+\sin(n))}$. Notice that $-1 < \sin(n) < 1$ for all $n \in \mathbb{N}$. Adding three to all sides of that inequality produces

$$2 < 3 + \sin(n) < 4.$$

Thus, we can bound our summand as

$$\frac{1}{n\,(4)} < \frac{1}{n\,(3+\sin(n))} < \frac{1}{n\,(2)}.$$

The right-hand side bound gives no information, but on the left-hand side we have something useful! In particular, the series $\sum_{n=1}^{\infty} \frac{1}{4n} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n} = \infty$. Since our series in question is larger than a series that totals to infinity, it must also diverge to infinity.

Exercise 2.6.8.3. Practice with DCT

Use the Direct Comparison Test to prove the following series converge or diverge.

- $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$

- $\sum_{n=1}^{\infty} \frac{2+\cos(n)}{n}$

- $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

2.6.9 Mixed Practice with Convergence Tests

In practice, when you encounter a series, there are usually many tests that will apply. Try the following, using any valid applicable test you like!

Exercise 2.6.9.1. Mixed Practice ☕☕☕

Determine if each of the following infinite series converges absolutely, converges conditionally, or diverges. In each case, explain what tests you used and how!

- $\sum_{n=0}^{\infty} \frac{n}{n+2}$

- $\sum_{n=0}^{\infty} \frac{n}{n^2+2}$

- $\sum_{n=0}^{\infty} \frac{(-1)^n n}{n^2+2}$

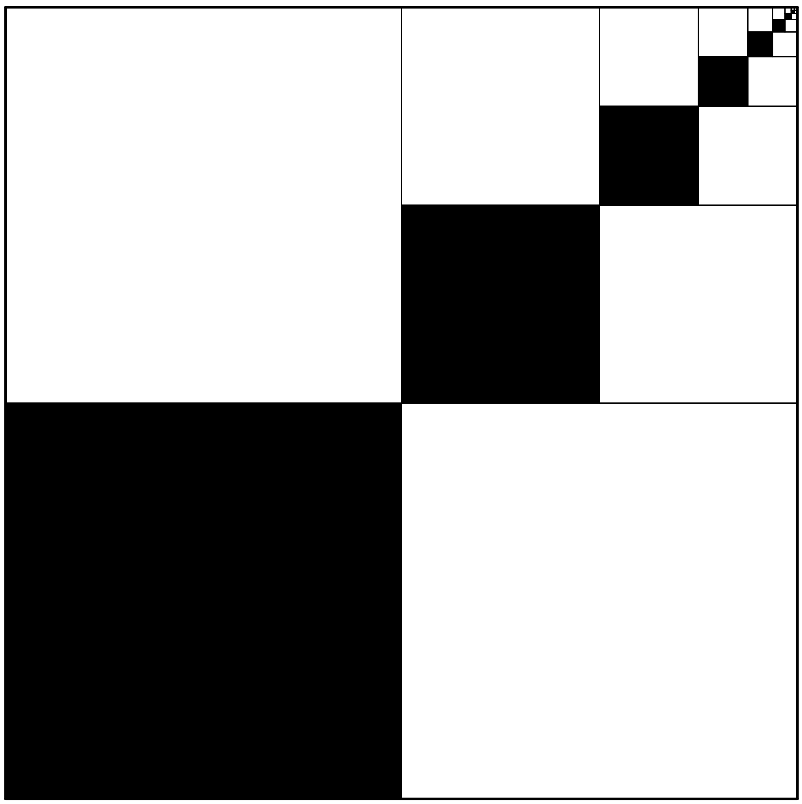
- $\sum_{n=0}^{\infty} \frac{(-1)^n n}{n^3+2}$

2.6.10 A Classical Infinite Series

Having played with formalism for far too long at this point, it is time to visit an infinite series coming from the geometry of Archimedes!

Exercise 2.6.10.1. An Infinite Series of Archimedes ☕☕☕

Consider the following diagram.



Assume the entire square has side length 1 and that each further subdivision into squares uses side lengths that are half the previous.

- What proportion of the whole large square is colored black? As a consequence, what is the total area of all the black squares added up?

- Write the area of each individual black square and build an infinite series that represents the total black area.

- Find the sum of the series using the Geometric Series Formula. Verify it agrees with your total for the area above.

- Can you show the series converges using the No Hope Test?
- Can you show the series converges using the Geometric Series Test?
- Can you show the series converges using the Ratio Test?
- Can you show the series converges using the Alternating Series Test?
- Can you show the series converges using the Limit Comparison Test?
- Can you show that the series converges using the Integral Test?

2.7 Chapter Σmary

In this chapter, we introduced three main concepts: sequences, series, and infinite series. Under each, we put a list of tasks you want to be able to complete for each by the end of this chapter.

1. **Sequences:** Lists of numbers.

$$a_0, a_1, a_2, a_3, \dots$$

   (a) **Convert** sequences between our three forms of writing them:

      i. List of terms.

      ii. Explicit formula.

      iii. Recursive formula.

   (b) Identify **geometric** and **arithmetic** sequences and know their explicit and recursive formulas.

	Arithmetic Sequence	Geometric Sequence
Defining Feature	Common difference d	Common ratio r
Recursive Formula	$a_n = a_{n-1} + d$	$a_n = a_{n-1}r$
Explicit Formula	$a_n = a_0 + dn$	$a_n = a_0 r^n$

   (c) Memorize the $N - \epsilon$ **definition of the limit of a sequence**, namely

$$\lim_{n\to\infty} a_n = L$$

   if and only if

$$\forall \epsilon > 0, \exists N \in \mathbb{R}, \forall n \in \mathbb{N}, n > N \implies |a_n - L| < \epsilon.$$

   Be able to use the definition to inform the steps in **writing an** $N - \epsilon$ **proof** of sequential convergence.

2. **Series:** A sum of a finite list of numbers.

$$a_0 + a_1 + a_2 + a_3 + \cdots + a_n$$

   (a) Given a sequence a_n, build a new sequence $A_N = \sum_{n=0}^{N} a_n$ called the **sequence of partial sums**. To find A_N from a_n, we discussed the following three strategies:

      i. If a_n is an arithmetic sequence, use the **arithmetic series formula** to calculate the sequence of partials sums as

$$A_N = (\text{Number of Terms}) \cdot (\text{Average of First and Last})\,.$$

      ii. If a_n is a geometric sequence, use the **geometric series formula** to calculate the sequence of partials sums as

$$A_N = \text{Initial Term} \cdot \frac{1 - \text{Common Ratio}^{\text{Number of Terms}}}{1 - \text{Common Ratio}}.$$

      iii. If a_n is neither arithmetic nor geometric, write out a table of values of A_N for the first few N values and see if you notice a pattern.

   (b) Given a partial sum A_N, find the sequence a_n from which it came by taking the **difference of consecutive terms**, namely

$$A_n - A_{n-1} = a_n.$$

3. **Infinite Series:** A sum of an infinite list of numbers.

$$a_0 + a_1 + a_2 + a_3 + \cdots$$

(a) Understand Cauchy's **definition of infinite series** as the limit of the sequence of partial sums. More formally, we define

$$\sum_{n=0}^{\infty} a_n = \lim_{N\to\infty}\left(\sum_{n=0}^{N} a_n\right).$$

(b) Use the above definition along with the geometric series formula to build the **infinite geometric series formula**, the fact that

$$\sum_{n=0}^{\infty} a_0 r^n = \frac{a_0}{1-r}$$

as long as $|r| < 1$.

(c) Understand the distinction between a **conditionally convergent series** and an **absolutely convergent series**, as well as the key reason why we care. You might destroy all of mathematics and the universe as we know it if you perform harmless-looking rearrangements on a conditionally convergent series. Absolutely convergent series, on the other hand, can be rearranged as you would for any finite sum.

(d) Given an infinite series $\sum_{n=0}^{\infty} a_n$, determine if it converges absolutely, converges conditionally, or diverges using one or more of our eight **convergence tests**. We list these tests with short descriptions below. Note these descriptions do not necessarily include every detail and precondition of the test; these are intended only as a short phrase to help remember the essence of the test.

i. **No Hope Test:** If the summand does not approach zero, the series has no hope of converging. If the summand does approach zero, the series has some hope of converging and another test is needed.

ii. **Geometric Series Test:** A geometric series converges if and only if the common ratio is between 1 and -1.

iii. **Integral Test:** A summation converges if and only if the corresponding improper integral converges.

iv. p **Test:** A series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for $p \in \mathbb{R}$ converges if and only if $p > 1$.

v. **Alternating Series Test :** A summation of terms that approach zero and alternate sign will converge.

vi. **Limit Comparison Test:** Having larger growth order than a divergent series implies divergence. Having smaller growth order than a convergent series implies convergence. If two summands have the same growth order, then either both series converge or both diverge.

vii. **Ratio Test:** If the absolute value of the ratio between consecutive terms in the series approaches a value...

- ...less than 1, the series converges.
- ...greater than 1, the series diverges.
- ...equal to 1, the test gives no information.

Note that this test is just doing LCT against a geometric series and seeing if the given series has the same growth order as a convergent or as a divergent geometric series.

viii. **Direct Comparison Test:** Being greater than a divergent series implies divergence. Being smaller than a convergent series implies convergence.

(e) Be able to apply the **Alternating Series Error Bound** to determine an upper bound for how far an approximation via a partial sum can be from the true value of an infinite alternating series.

2.8 Mixed Practice

2.8.1 Warm Ups

These are good problems for reinforcing the vocabulary and foundational concepts of this chapter.

Exercise 2.8.1.1. ☕

Consider the sequence

$$a_n = \frac{1}{2n}$$

and the claim that

$$\lim_{n\to\infty} \frac{1}{2n} = L = 0.$$

a.) For the limit above, find minimum N values for each of the following ϵ. That is, for each ϵ, find the *smallest* value of N such that all $|a_n - L| < \epsilon$ for $n > N$.

- $\epsilon = 0.1$
- $\epsilon = 0.01$
- $\epsilon = 0.001$

A Solution: Note:

n	a_n
1	$\frac{1}{2}$
2	$\frac{1}{4}$
3	$\frac{1}{6}$
4	$\frac{1}{8}$
5	$\frac{1}{10}$
$\cdots$	$\cdots$
n	$\frac{1}{2n}$

The progression shows that when $n = 5$, $a_n = 0.1$ so we have the smallest value of N to yield and $\epsilon = 0.1$ is 5.
Similarly, we have $a_n = \frac{1}{100}$ when $n = 50$ and $a_n = \frac{1}{1000}$ when $n = 500$.

---

b.) Find a formula for N in terms of ϵ. Plug in $\epsilon = 0.01$ and confirm your answer above.

A Solution: $\left|\frac{1}{2n} - 0\right| < \epsilon \Longrightarrow \frac{1}{2n} < \epsilon \Longrightarrow \frac{1}{2\epsilon} < n \Longrightarrow \frac{1}{2\epsilon} = N$
Test with $\epsilon = 0.001$: $N = \frac{1}{2\cdot 0.001} = \frac{1}{2}\frac{1}{\frac{1}{100}} = \frac{100}{2} = 50$

---

c.) Write an $N - \epsilon$ proof to verify that the above limit is correct.

A Solution: Let $\epsilon > 0$ also, let $N = \frac{1}{2n}$ and let $n \in \mathbb{N}$ Assume $n > N$ We wish to show that under these circumstances, the distance from $a_n = \frac{1}{2n}$ to $L = 0$ will be less than ϵ.

$|a_n - L| = \left|\frac{1}{2n} - 0\right| = \frac{1}{2n} < \frac{1}{2N}$ since $N < n$ by our assumptions $\frac{1}{2N} = \frac{1}{2\frac{1}{2\epsilon}} = \frac{1}{\epsilon} = \epsilon$
Thus the terms will be within ϵ or 0 past the index $\frac{1}{2\epsilon}$, no matter how small ϵ is chosen.
Therefore, $\lim\limits_{n\to\infty} \frac{1}{2n} = 0$

Exercise 2.8.1.2. ☕

In the game *Clash of Clans*, a Barbarian King can be upgraded using Dark Elixir. Suppose your king is currently at level 10. To upgrade from level 10 to level 11 requires 40,000 Dark Elixir. Every upgrade past that requires 5,000 more Dark Elixir than what the previous upgrade cost. For example, to upgrade from level 11 to level 12 will require 45,000. To upgrade from level 12 to level 13 requires 50,000, and so on. What is the total amount of Dark Elixir required to upgrade your level 10 king to level 40?

A Solution: Total upgrade cost $= 40,000 + 45,000 + 50,000 + \cdots + (40,000 + 5,000(40 - 11)) = 40000 + 45000 + \cdots + 185000$
This is an arithmetic series which is the number of terms times the average of the first and last terms $= 30 \cdot \frac{185000+40000}{2} = 15 \cdot 225000 = 3,375,000$ dark elixir.

Exercise 2.8.1.3. ☕☕

An Italian math professor confesses he has a pizza-eating problem. He decides to change his usual policy of "Each minute, I eat all the pizza I see, until it's all gone", to his new rule: "Each minute, I eat 1/4 of all the pizza I see." He orders an 8 slice pan of pizza.

a.) After one minute, how much pizza is left?

A Solution: $8 - \frac{1}{4}8 = \frac{3}{4}8 = 6$ slices.

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b.) After two minutes, how much pizza is left?

A Solution: $\frac{3}{4}\left(\frac{3}{4}8\right) = 4.5$ slices.

---

c.) After twenty minutes, how much pizza is left?

A Solution: $\left(\frac{3}{4}\right)^{20} 8$ slices.

---

d.) No matter how long he spends with the pan, how much of the pan will never be eaten? Explain.

A Solution: None because as time goes to ∞ the number of slices left goes to 0, since $\left(\frac{3}{4}\right)^n 8$ represents the number of slices left after n minutes and $\lim_{n\to\infty} \left(\frac{3}{4}\right)^n 8 = 0$ slices.

Exercise 2.8.1.4.

For each of the following infinite series, determine if it converges absolutely, converges conditionally, or diverges. Explain clearly what your reasoning is, citing any tests you use.

a.) $\sum\limits_{n=0}^{\infty} \frac{(-2)^n+n^2}{n!}$

A Solution: Apply the ratio test $\lim\limits_{n\to\infty} \left| \frac{(-2)^{n+1}n+(n+1)^2}{(n+1)!} \cdot \frac{n!}{(-2)^n+n^2} \right| = \lim\limits_{n\to\infty} \left| \frac{1}{n+1} \frac{(-2)^{n+1}+(n+1)^2}{(-2)^n+n^2} \right| = \lim\limits_{n\to\infty} \left| \frac{1}{n+1} \frac{(-2)+\frac{(n+1)^2}{(-2)^n}}{1+\frac{n^2}{(-2)^n}} \right| = \lim\limits_{n\to\infty} \frac{2}{n+1}$ because $\frac{(n+1)^2}{(-2)^n}$ and $\frac{n^2}{(-2)^n}$ goes to zero as n goes to ∞ but since $\lim\limits_{n\to\infty} \frac{2}{n+1} = 0 < 1$, it converges absolutely.

b.) $\sum\limits_{n=0}^{\infty} F_n$

A Solution: Use the No Hope test: $\lim\limits_{n\to\infty} \sum\limits_{n=0}^{\infty} F_n \neq 0$ since $F_n = 0, 1, 1, 2, 3, 5, 8, \cdots \infty$ So $\sum\limits_{n=0}^{\infty} F_n$ Diverges.

c.) $\sum_{n=0}^{\infty}(-1)^n F_n$

A Solution: Use the No Hope test: $\lim\limits_{n\to\infty} \sum\limits_{n=0}^{\infty} (-1)^n F_n \neq 0$ since $F_n = 0, -1, 1, -2, 3, -5, 8, \cdots$ does not approach zero. So $\sum\limits_{n=0}^{\infty} (-1)^n F_n$ Diverges by the No Hope Test

d.) $\sum_{n=0}^{\infty} \sqrt{\frac{2}{n^3+n+1}}$

A Solution: We can use the Limit Comparison Test with $\frac{1}{n^{3/2}}$ since $\lim\limits_{n\to\infty} \frac{\sqrt{\frac{2}{n^3+n+1}}}{\frac{1}{n^{3/2}}} = \lim\limits_{n\to\infty} \sqrt{\frac{\frac{2}{n^3+n+1}}{\frac{1}{n^{3/2}}}} = \lim\limits_{n\to\infty} \sqrt{\frac{2}{n^3+n+1} \frac{n^3}{1}} = \lim\limits_{n\to\infty} \sqrt{\frac{2}{1+\frac{1}{n^2}+\frac{1}{n^3}}} = \sqrt{2}$ So they have the same growth order. According to the Limit Comparison Test, if two functions have the same growth order, then the sums of them both with either converge or diverge. So we can use $\sum\limits_{0}^{\infty} \frac{1}{n^3/2}$ to determine the behavior of $\sum_{n=0}^{\infty} \sqrt{\frac{2}{n^3+n+1}}$ and since $\sum\limits_{0}^{\infty} \frac{1}{n^3/2}$ converges by the p-test since $p = \frac{3}{2} > 1$ So $\sum_{n=0}^{\infty} \sqrt{\frac{2}{n^3+n+1}}$ converges absolutely.

---

e.) $\sum\limits_{n=0}^{\infty} \frac{2^n}{2^n+1}$

A Solution: Apply the No Hope Test $\lim\limits_{n\to\infty} \frac{2^n}{2^n+1} = 1 \neq 0$ So it diverges.

---

f.) $\sum\limits_{n=0}^{\infty} \frac{n}{2^n+1}$

A Solution: First note that $\frac{n}{2^n+1} < \frac{n}{2^n}$ so we can apply the Direct Comparison Test using $\frac{n}{2^n}$. The no Hope test yields no information since $\lim\limits_{n\to\infty} \frac{n}{2^n} = 0$. We can instead apply the Ratio Test and get $\lim\limits_{n\to\infty} \left| \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} \right| = \lim\limits_{n\to\infty} \left| \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} \right| = \lim\limits_{n\to\infty} \frac{1}{2} < 1$ So it converges absolutely.

---

g.) $\sum\limits_{n=0}^{\infty} \frac{e^n}{e^{2n}+1}$

A Solution: Use the Integral Test: $\int_0^{\infty} \frac{e^x}{e^{2x}+1}\, dx$ Let $u = e^x$ then $du = dx$ and we get $\int_{u=1}^{\infty} \frac{u}{u^2+1}\, du = \lim\limits_{b\to\infty} \int_{u=1}^{b} \frac{u}{u^2+1}\, du = \lim\limits_{b\to\infty} \arctan u \Big|_1^b \lim\limits_{b\to\infty} \arctan b - \arctan 1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$
Since the integral converges the series also converges and converges absolutely.
You could also use the ratio test to get the same result.

2.8.2 Sample Test Problems

Exercise 2.8.2.1. ☕☕☕

Define the following recursive sequence:

$$a_0 = 2$$
$$a_{n+1} = a_n - \frac{1}{2^n}$$

a.) Compute the first few values of the sequence a_n. Fill them in the table below:

n	0	1	2	3	4	5
a_n						

A Solution:

n	0	1	2	3	4	5
a_n	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$

b.) Define A_N to be the sequence of partial sums of a_n. Find the first few values of the sequence A_N.

N	0	1	2	3	4	5
A_N						

A Solution:

N	0	1	2	3	4	5
A_N	2	3	3.5	3.75	3.875	3.9375

c.) Compute the following infinite sum:

$$\sum_{n=1}^{\infty} a_n$$

How does this quantity relate to your work in part b)?

A Solution: Notice a_n is a geometric sequence with initial term $a_0 = 2$ and the ratio $r = \frac{1}{2}$. We can then use the infinite geometric series formula $\sum\limits_{n=0}^{\infty} a_n = a \cdot \frac{1}{1-r}$ so we have $\sum\limits_{n=0}^{\infty} 2 \cdot \left(\frac{1}{2}\right)^n = 2 \cdot \frac{1}{1-\frac{1}{2}} = 2\frac{1}{\frac{1}{2}} = 2 \cdot 2 = 4$ So the partial sums converge to 4.

Exercise 2.8.2.2. ☕☕☕

a.) Formally define what it means for a sequence a_n to converge to a limit L.

A Solution: $\lim\limits_{n\to\infty} a_n = L \iff$ For all $\epsilon > 0$ there exists N such that for all $n > N$, $|a_n - L| < \epsilon$

b.) Consider the sequence $a_n = \frac{n}{3n+1}$. What is $\lim_{n\to\infty} a_n$?

A Solution: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{3n+1} = \lim_{n\to\infty} \frac{1}{3+\frac{1}{n}} = \frac{1}{3}$

c.) Write an $N-\epsilon$ proof of your claim in part b.

A Solution: First do the side work:
$\left|\frac{n}{3n+1} - \frac{1}{3}\right| = \left|\frac{3n-3n-1}{3(3n+1)}\right| = \left|\frac{-1}{3(3n+1)}\right| < \epsilon \iff \frac{1}{3\epsilon} < 3n+1 \iff \frac{1}{3\epsilon} - 1 < 3n \iff \frac{1}{9\epsilon} - \frac{1}{3} < n$
Let $\epsilon > 0$ then choose $N = \frac{1}{9n} - \frac{1}{3}$ Then choose $n \in \mathbb{N}$ with $n > N$. We now show that any a_n for such n is no more than ϵ away from $\frac{1}{3}$.
$\left|\frac{n}{3n+1} - \frac{1}{3}\right| = \left|\frac{3n}{3(3n+1)} - \frac{3n+1}{3(3n+1)}\right| = \left|\frac{-1}{3(3n+1)}\right| = \frac{1}{3(3n+1)} < \frac{1}{3(3N+1)}$
note here we made the denominator smaller by introducing N for n
$\frac{1}{3(3N+1)} = \frac{1}{3\left(3\left(\frac{1}{9\epsilon}-\frac{1}{3}\right)+1\right)} = \frac{1}{3\left(\left(\frac{1}{3\epsilon}-1\right)+1\right)} = \frac{1}{3\left(\frac{1}{3\epsilon}\right)} = \frac{1}{\frac{1}{\epsilon}} = \epsilon$

Exercise 2.8.2.3. ☕☕☕

Consider the following infinite series;

$$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots$$

a.) Apply the Divergence Test/No Hope Test to the above series. What does it tell you about its convergence or divergence?

A Solution: First rewrite the series as $\sum_{n=0}^{\infty} (-1)^n \frac{1}{n!}$ Now we can apply the No Hope Test $\lim_{n\to\infty} (-1)^n \frac{1}{n!} = 0$ So the no hope test gives no information since it requires the limit to be not equal to zero.

b.) Apply the Alternating Series Test to the above series. What does it tell you about its convergence or divergence?

A Solution: According to the Alternating Series test, we just need to determine if $\frac{1}{n!}$ approaches zero and it does so $\sum_{n=0}^{\infty} (-1)^n \frac{1}{n!}$ converges absolutely by the Alternating Series Test.

c.) Apply the Ratio Test to the above series. What does it tell you about its convergence or divergence?

A Solution: $\lim\limits_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}}{(n+1)!}}{\frac{(-1)^n}{n!}}\right| = \lim\limits_{n\to\infty}\left|\frac{(-1)^{n+1}}{(n+1)!}\cdot\frac{n!}{(-1)^n}\right| = \lim\limits_{n\to\infty}\frac{1}{(n+1)} = 0 < 1$ So $\sum\limits_{n=0}^{\infty}(-1)^n\frac{1}{n!}$ converges absolutely by the Ratio Test

Exercise 2.8.2.4. ☕☕

Consider the sequence given by the following recurrence relation:

$$a_0 = 0$$
$$a_n = a_{n-1} + 3n^2 - 3n + 1$$

a.) Write out the first five terms of a_n.

A Solution: $a_0 = 0, a_1 = 1, a_2 = 1+3\cdot 2^2 - 3\cdot 2+1 = 8, a_3 = 8+3\cdot 3^2 - 3\cdot 3+1 = 27, a_4 = 1 + 3\cdot 4^2 - 3\cdot 4 + 1 = 64$

b.) Find an explicit formula for a_n.

A Solution: These are consecutive perfect cubes so $a_n = n^3$

c.) Does $\sum_{n=0}^{\infty} a_n$ converge or diverge? Explain why, clearly indicating any tests you use in the process.

A Solution: It diverges by the No Hope Test since $\lim\limits_{n\to\infty} n^3 \neq 0$

Part III

Coming Attractions

Selected Answers and Hints

Exercise 2.3.1.4. $\bullet \frac{1}{n+1}$ $\bullet n+1$ $\bullet (n+2)(n+1)$ $\bullet (2n+2)(2n+1)$

Exercise 2.4.0.3. Think about what happens if the common difference d is zero and if the common ratio r is 1.

Exercise 2.4.1.2. The common ratio r is what we multiply by to get from term to term. Listing out the terms $a_0, a_0r, a_0r^2, a_0r^3, \cdots$ shows that a_0r^n is the explicit formula.

Exercise 2.5.0.7. It is absolutely convergent, since the series of corresponding positive terms is $0.1 + 0.02 + 0.004 + 0.0008 + 0.00016 + \cdots$ which converges to one-eighth.

Exercise 2.6.1.4. The first one-half would come from the first term itself. But since the total is one, it means the terms $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ must themselves total to be the other one-half. Thus, we can try to group terms to form batches that total to one-half, but the second batch uses up all infinitely many remaining terms!

Exercise 2.6.1.7. The first two summands have limits of $\sqrt{2}$ and 1, respectively. Since these limits are nonzero, the series has no hope of converging and thus diverges. The third summand does approach zero as n goes to infinity, so it gives no information.

Exercise 2.6.2.2. $\bullet$The summation $\Sigma_{n=1}^{\infty} \frac{1}{n^2}$ has no common ratio r and thus is not a geometric series. For example, the first three terms are $1, 1/4$, and $1/9$. Thus, the first two ratios between consecutive terms are $1/4$ and $4/9$, which are not equal. $\bullet$The given geometric series has common ratio $r = -1/3$. After taking the absolute value of each term, it becomes the series $18 + 6 + 2 + \frac{2}{3} + \frac{2}{9} + \cdots$ which still converges as it now has common ratio $r = 1/3$. $\bullet$It is not possible to build a conditionally convergent geometric series. If we are given a convergent geometric series, then the common ratio r satisfies $|r| < 1$. Taking the absolute value of each term in the series might flip the sign on r, but it will not change the magnitude. Thus, any convergent geometric series must converge absolutely.

Exercise 2.6.4.2. For $p > 1$ or $p < 1$, one can repeat the corresponding calculations from Example 2.6.3.3. If $p = 1$, the series is the harmonic series, which diverges.

Exercise 2.6.5.3. Taking term-by-term absolute values produces the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$. Since it totals to a finite value, the original series converges absolutely.

Exercise 2.6.5.4. $\bullet$Since $\lim_{n\to\infty} \left(-\frac{1}{2}\right)^n = 0$, the No Hope Test gives no information. $\bullet$The Integral Test does not apply since the terms are not positive and decreasing. In this case, it is actually even worse than that, as the function $\left(-\frac{1}{2}\right)^x$ is undefined for all half-integer values of x. $\bullet$The summand is not of the form $1/n^p$, so the very narrow p-Test does not apply.

Exercise 2.6.5.5. The first two and last converge by AST. It does not apply to the third.

Exercise 2.6.5.8. After three reversals, the bug is within $\frac{1}{32}$ of its final location. The bug would have to reverse course nine times to be guaranteed by the Alternating Series Error Bound to be within one one-thousandth of its final location.

Exercise 2.6.6.3. The first converges by comparison to $\Sigma\frac{1}{n^2}$. The second diverges by comparison to $\Sigma\frac{1}{n^{-1/2}}$.

Exercise 2.6.6.6. Use the comparison function $\frac{1}{n}$ to show the series diverges.

Exercise 2.6.7.4. •Converges, ratio 0. •No info, ratio 1. •Converges, ratio 0. •Diverges, ratio 2.

Exercise 2.6.8.3. These series are convergent by DCT against $\frac{1}{n^2}$, divergent by DCT against $\frac{1}{n}$, and convergent by DCT against $\frac{1}{n^2}$.

Exercise 2.6.9.1. •Divergent by NHT or Integral Test. •Divergent by Integral Test or LCT against $\frac{1}{n}$. •Convergent by AST, but only conditionally since the absolute value is the previous summand whose series diverged. •Absolutely convergent since taking term-by-term absolute value produces a convergent series (which can be shown convergent via LCT with $\frac{1}{n^3}$).

Exercise 2.6.10.1. The black region is one-third of the total square and thus must total to one-third. The infinite series for the black square areas is $\frac{1}{4}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\cdots$. NHT and AST give no information here, but all the rest of the tests work to determine convergence! Use $\frac{1}{n^2}$ as a comparison function for LCT.

Exercise 2.8.1.1. a. For $\epsilon = 0.1, n = 5,\quad \epsilon = 0.01, n = 05,\quad \epsilon = 0.001, n = 500$ b. $\frac{1}{2\epsilon} = N$ c. Let $\epsilon > 0$ also, let $N = \frac{1}{2n}$ and let $n \in \mathbb{N}$ Assume $n > N$ We wish to show that under these circumstances, the distance from $a_n = \frac{1}{2n}$ to $L = 0$ will be less than ϵ. $|a_n - L| = \left|\frac{1}{2n} - 0\right| = \frac{1}{2n} < \frac{1}{2N}$ since $N < n$ by our assumptions $\frac{1}{2N} = \frac{1}{2\frac{1}{2\epsilon}} = \frac{1}{\epsilon} = \epsilon$ Thus the terms will be within ϵ or 0 past the index $\frac{1}{2\epsilon}$, no matter how small ϵ is chosen. Therefore, $\lim\limits_{n\to\infty}\frac{1}{2n} = 0$

Exercise 2.8.1.2. $3,375,000$

Exercise 2.8.1.3. a.) 6 Slices, b.) 4.5 Slices, c.)$\left(\frac{3}{4}\right)^{20}8$ Slices, d.) None because as time goes to ∞ the number of slices left goes to 0, since $\left(\frac{3}{4}\right)^{n}8$ represents the number of slices left after n minutes and $\lim\limits_{n\to\infty}\left(\frac{3}{4}\right)^{n}8 = 0$ slices.

Exercise 2.8.1.4. a.) Converges Absolutely by the Ratio Test.
b.) Diverges by the No Hope Test.
c.) Diverges by the No Hope Test.
d.) Converges by the Limit Comparison Test and the p-test.
e.) Diverges by the No Hope Test
f.) Converges Absolutely by the Ratio Test.
g.)Converges Absolutely by the Integral Test or the Ratio Test.

Exercise 2.8.2.1. Notice a_n is a geometric sequence with initial term $a_0 = 2$ and the ratio $r = \frac{1}{2}$. We can then use the infinite geometric series formula $\Sigma_{n=0}^{\infty}a_n = \frac{a}{1-r}$ so we have $\Sigma_{n=0}^{\infty}2\cdot\left(\frac{1}{2}\right)^n = 2\cdot\frac{1}{1-\frac{1}{2}} = 2\frac{1}{\frac{1}{2}} = 2\cdot 2 = 4$. Thus, the sequence of partial sums converges to 4.

Exercise 2.8.2.2. a.) $\lim\limits_{n\to\infty} a_n = L \iff$ For all $\epsilon > 0$ there exists N such that for all $n > N, |a_n - L| < \epsilon$

b.) $\frac{1}{3}$
c.) Let $\epsilon > 0$ then choose $N = \frac{1}{9n} - \frac{1}{3}$ Then choose $n \in \mathbb{N}$ with $n > N$. We now show that any a_n for such n is no more than ϵ away from $\frac{1}{3}$.
$\left|\frac{n}{3n+1} - \frac{1}{3}\right| = \left|\frac{3n}{3(3n+1)} - \frac{3n+1}{3(3n+1)}\right| = \left|\frac{-1}{3(3n+1)}\right| = \frac{1}{3(3n+1)} < \frac{1}{3(3N+1)}$
note here we made the denominator smaller by introducing N for n
$\frac{1}{3(3N+1)} = \frac{1}{3\left(3\left(\frac{1}{9\epsilon}-\frac{1}{3}\right)+1\right)} = \frac{1}{3\left(\left(\frac{1}{3\epsilon}-1\right)+1\right)} = \frac{1}{3\left(\frac{1}{3\epsilon}\right)} = \frac{1}{\frac{1}{\epsilon}} = \epsilon$

Exercise 2.8.2.3. a.) $\lim_{n\to\infty} (-1)^n \frac{1}{n!} = 0$ So the no hope test gives no information since it requires the limit to be not equal to zero.
b.)It converges absolutely by the Alternating Series Test.
c.)It converges absolutely by the Ratio Test

Exercise 2.8.2.4. a.)$a_0 = 0, a_1 = 1, a_2 = 1 + 3 \cdot 2^2 - 3 \cdot 2 + 1 = 8, a_3 = 8 + 3 \cdot 3^2 - 3 \cdot 3 + 1 = 27, a_4 = 1 + 3 \cdot 4^2 - 3 \cdot 4 + 1 = 64$
b.)$a_n = n^3$
c.)It diverges by the No Hope Test