Calculus II

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Contents

Acknowledgements

Thank you so much for choosing this text and a big thanks to all who made this possible!

Kenneth would like to thank Front Range Community College for their relentless dedication to excellence, inclusivity, and student success. To help Ken help our students, they supported this project with an Innovation Grant. In particular, thanks to FRCC president Andy Dorsey for believing in the project and Christy Gomez for her vision, structure, and pragmatism. He would like to thank his family Ken, Gina, Maria, Bryan, and Keenan for the (even in a family that loves combinatorics) countless mathematical adventures throughout the years. He is also grateful for the training he received from generous mentors while he was just a lowly graduate student at Colorado State University – especially from Alexander Hulpke and Tim Penttila. Lastly, thanks to his wife Faith for all the love and support throughout the project. She learned to make coffee just so she could help, despite thinking it's the most disgusting liquid in the visible universe.

Jenna would like to thank the fantastic faculty at Front Range Community College – Boulder County Campus and University of Colorado Boulder for their training and dedication to helping her become a better mathematician. She would also like to thank the Monks family who has done so much to help her mathematically and encouraged her to continue to grow. Finally, thanks to her family for brewing the coffee and putting up with her hermit tendencies.

Thanks to Joe Darschewski, FRCC-BCC's excellent Calculus instructor, for his being the first instructor to use the text and offer an enormous amount of helpful feedback. He also typeset the new Mixed Practice sections, which will be very helpful to the students!

We hope you have fun with this resource and find it helpful! It is a beautiful subject, and we tried to honor that with a beautiful text. The text is still in its infancy, and we welcome any and all feedback you could give us. Thank you in advance for any comments, complaints, suggestions, and questions.

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Chapter 1

Overview

A Three-Part Course

The topics of Calculus II fall into three parts that each have an appropriate place in the story of the calculus sequence.

- **Part I: Integration.** The first part of the course ties loose ends from Calculus I. The ending of Calculus I showed that antiderivatives can be used to evaluate integrals via the Fundamental Theorem of Calculus. However, by the end of Calculus I, only the very simplest antiderivatives can actually be computed. Part one expands the student's knowledge of **techniques of antidifferentiation**. These techniques are subsequently put to use computing **length, area, volume, and center of mass**.

- **Part II: Sequences and Series.** This is the topic that makes up the body of Calculus II. **Sequences and series** embody the beauty of mathematics; from simple beginnings (a sequence is just a list... a series is just adding up a list of numbers...) it quickly leads to incredible structure, surprises, complexity, and open problems. **Power series** redefine commonly used transcendental functions (functions that are not computed using algebra, e.g. cosine). If you've ever wondered **what your calculator does when you press the cosine button**, this is where you find out! (**Hint:** It does not have a circle of radius one spinning around with a team of elves that measure x coordinates.)

- **Part III: Coming Attractions.** By the end of Calculus II, the student is ready for a *lot* of other classes. The end of Calculus II thus ends with a sampler platter of topics that show the vast knowledge base built upon the foundation laid in Calculus II. Here the text takes a bite out of **Differential Equations**, serves some polar and parametric coordinates as a palate cleanser before **Calculus III**, and tastes some **Complex Analysis** to aid in digestion of Differential Equations. For dessert, it serves a scoop of **Probability** with both discrete and continuous colored sprinkles.

How to Use This Book

This book is meant to facilitate *Active Learning* for students, instructors, and learning assistants. Active Learning is the process by which the student participates directly in the learning process by reading, writing, and interacting with peers. This contrasts the traditional model where the student passively listens to lecture while taking notes. This book is designed as a self-guided step-by-step exploration of the concepts. The text incorporates theory and examples together in order to lead the student to discovering new results while still being able to relate back to familiar topics of mathematics. Much of this text can be done independently by the student for class preparation. During class sessions, the instructor and/or learning assistants may find it advantageous to encourage group work while being available to assist students, and work with students on a one-on-one or small group basis. This is highly desirable as extensive research has shown that active learning improves student success and retention. (For example, see `www.pnas.org/content/111/23/8410` for Scott Freeman's metaanalysis of 225 studies supporting this claim.)

What is Different about this Book

If you leaf through the text, you'll quickly notice two major structural differences from many traditional calculus books:

1. The exercises are very intermingled with the readings. Gone is the traditional separation into "section" versus "exercises".

2. Whitespace was included for the student to write and work through exercises. Parts of pages have indeed been intentionally left blank.

A consequence of this structure is that the readings and exercise are closely linked. It is intended for the student to do the readings and exercises concurrently.

Ok... Why?

The goal of this structure is to help the student simulate the process by which a mathematician reads mathematics. When a mathematician reads a paper or book, he/she always has a pen in hand and is constantly working out little examples alongside and scribbling incomprehensible notes in bad handwriting. It takes a *long* time and a lot of experience to know how to come up with good questions to ask oneself or to know what examples to work out in order to to help oneself absorb the subject. Hence, the exercises sprinkled throughout the readings are meant to mimic the margin scribbles or side work a mathematician engages in during the act of reading mathematics.

The Legend of Coffee

A potential hazard of this self-guided approach is that while most examples are meant to be simple exercises to help with absorption of the topics, there are some examples that students may find quite difficult. To prevent students from spinning their wheels in frustration, we have labeled the difficulty of all exercises using coffee cups as follows:

Coffee Cup Legend

Symbol	Number of Cups	Description of Difficulty
☕	A One-Cup Problem	Easy warm-up suitable for class prep.
☕☕	A Two-Cup Problem	Slightly harder, solid groupwork exercise.
☕☕☕	A Three-Cup Problem	Substantial problem requiring significant effort.
☕☕☕☕	A Four-Cup Problem	Difficult problem requiring effort and creativity!

Glossary of Symbols

In Precalculus and Calculus I, there is a wide range of how much notation from Set Theory gets used. To get everyone on the same page, here is a short list of some notation we will use in this text.

Sets and Elements

Often in mathematics, we construct collections of objects called *sets*.

- If an object x is in a set A, we say x is an *element* of A and write $x \in A$.
- If an object x is not in a set A, we say x is not an element of A and write $x \notin A$.

Any particular object is either an element of a set or it is not. We do not allow for an object to be partially contained in a set, nor do we allow for an object to appear multiple times in a set. Often we use curly braces around a comma-separated list to indicate what the elements are.

Example 1.0.0.1. A Prime Example

Suppose P is the set of all prime numbers. We write

$$P = \{2, 3, 5, 7, 11, 13, 17, \ldots\}$$

For example, $2 \in P$ and $65,537 \in P$, but $4 \notin P$.

Some Famous Sets of Numbers

The following are fundamental sets of numbers used in Calculus 2.

- **Natural Numbers:** The set $\mathbb{N}$ of natural numbers is the set of all positive whole numbers, along with zero. That is,

  $$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \ldots\}$$

  Note that in many other sources, zero is not included in the natural numbers. Both are widely used; be aware the choice on this convention will change throughout your mathematical travels!

- **Integers:** The set of integers $\mathbb{Z}$ is the set of all whole numbers, whether they are positive, negative, or zero. That is,

  $$\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

- **Rational Numbers:** The set of rational numbers $\mathbb{Q}$ is the set of all numbers expressible as a fraction whose numerator and denominator are both integers.

- **Real Numbers:** The set of real numbers $\mathbb{R}$ is the set of all numbers expressible as a decimal.

- **Complex Numbers:** The set $\mathbb{C}$ of complex numbers is the set of all numbers formed as a real number (called the real part) plus a real number times i (called the imaginary part), where i is a symbol such that $i^2 = -1$.

Set-Builder Notation

The most common notation used to construct sets is *set-builder notation*, in which one specifies a name for the elements being considered and then some property $P(x)$ that is the membership test for an object x to be an element of the set. Specifically,

$$A = \{x \in B : S(x)\}$$

means that an object x chosen from B is an element of the set A if and only if the claim $S(x)$ is true about x. Sometimes the “$\in B''$ gets dropped if it is clear from context what set the elements are being chosen from. The set-builder notation above gets read as “the set of all x in B such that $S(x)$”. One can think of this as running through all elements of B and throwing away any that do not meet the condition described by S.

Example 1.0.0.2. Interval Notation

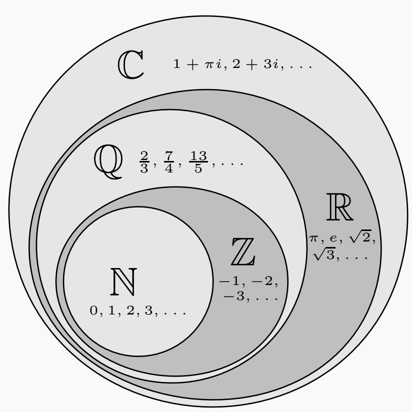
Interval notation can be expressed in set-builder notation as follows:

- $(a, b) = \{x \in \mathbb{R} : a < x < b\}$
- $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$
- $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$
- $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$

Example 1.0.0.3. Rational, Real, and Complex in Set-Builder Notation

Set-builder notation is often used to express the sets of rational, real, and complex numbers as follows:

- $\mathbb{Q} = \left\{\frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0\right\}$
- $\mathbb{R} = \{0.a_0a_1a_2a_3a_4 \ldots \times 10^n : n \in \mathbb{N}, a_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ where $i \in \mathbb{N}\}$ Note this is essentially scientific notation; the concatenation of the a_i's represents the digits in a base-ten decimal expansion.
- $\mathbb{C} = \{a + bi : a \in \mathbb{R}, b \in \mathbb{R}\}$



Part I

Integration

Part II

Sequences and Series

Chapter 2

Sequences and Series: Commas and Plus Signs Run Amok

2.1 Definition of Sequences

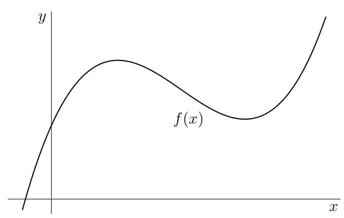
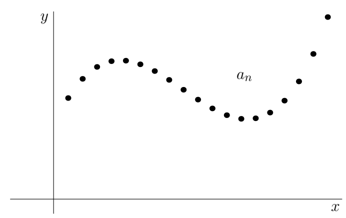
Definition 2.1.0.1. Sequence

A *sequence* is a function whose domain is a subset of $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, the set of natural numbers.

Typically for $n \in \mathbb{N}$, we write a_n as the output corresponding to the input n. The output could technically be an object of any type, but in this course we usually use real numbers or complex numbers as our outputs. Technically the sequence itself is the map $n \mapsto a_n$, though since this is a bit cumbersome to write, we often write just a_n to refer to the entire sequence, similar to how we write $f(x)$ for a function on the real numbers. When the outputs are real numbers, we can graph sequences as a collection of points of the form (input,output) just as we would for functions on the real numbers.

2.1.1 Sequences vs Functions on the Real Numbers

The chart below compares sequences (functions whose domain is the natural numbers) and functions whose domain is the real numbers.

Trait or Notation	**Function on Reals**	**Sequence**
Default Independent Variable	x	n
Default Formula Notation	$f(x)$	a_n
Domain D	Subset of $\mathbb{R}$	Subset of $\mathbb{N}$
Graph	$\{(x, f(x)) : x \in D\}$ 	$\{(n, a_n) : n \in D\}$ 

Less formally, a sequence is simply a list of objects. The correspondence between sequences as maps and sequences as lists of objects is that the k^{th} object in the list is the output corresponding to $k-1$ under the map. That is, the map $n \mapsto a_n$ corresponds to the list $a_0, a_1, a_2, a_3, \ldots$.

Example 2.1.1.1. The Sequence of Even Natural Numbers

Consider the list of nonnegative even numbers: $0, 2, 4, 6, 8, 10, \ldots$. We can view this as the map $n \mapsto 2n$, which we can see as a function on the naturals with the following inputs and outputs:

$$\begin{gathered} 0 \mapsto 0 \\ 1 \mapsto 2 \\ 2 \mapsto 4 \\ 3 \mapsto 6 \\ 4 \mapsto 8 \\ \vdots \end{gathered}$$

2.2 Notation for Sequences

There are two main ways that we define sequences: as explicit formulas and as recursive formulas. The next two subsections describe these methods.

2.2.1 Explicit Formulas

Often times we define a sequence via what is called an *explicit formula* or a *closed formula*. This is a formula given in terms of n that shows explicitly how to compute the output corresponding to an input n in finitely many steps expressed in our usual language of algebraic and transcendental functions.

Example 2.2.1.1. The Sequence of Even Natural Numbers: Explicit Formula

The sequence of even natural numbers defined above has

$$a_n = 2n$$

as its explicit formula.

Exercise 2.2.1.2. Practice with Explicit Formulas

Find an explicit formula for the sequence of...

- ...odd natural numbers.

$$1, 3, 5, 7, \ldots$$

A Solution: $a_n = 2n + 1, n \in \{0, 1, 2, \ldots\}$

- ...even integers starting at -4 and counting upwards, two at a time.

$$-4, -2, 0, 2, \ldots$$

A Solution: $-4 + 2n, n \in \{0, 1, 2, \ldots\}$

- ...all multiples of 5, starting from 20 and counting downwards.

$$20, 15, 10, 5, 0, -5, \ldots$$

A Solution: $a_n = 20 - 5n, n \in \{0, 1, 2, \ldots\}$

- ...all natural numbers that are one more than a multiple of 3.

$$1, 4, 7, 10, \ldots$$

A Solution: $a_n = 3n + 1, n \in \{0, 1, 2, \ldots\}$

- ...consecutive powers of 2, starting from 1.

$$1, 2, 4, 8, \ldots$$

A Solution: $a_n = 2^n, n \in \{0, 1, 2, \ldots\}$

- ...terms that alternate forever between positive and negative one.

$$1, -1, 1, -1, \ldots$$

A Solution: $a_n = (-1)^n, n \in \{0, 1, 2, \ldots\}$

- ...fractions whose numerators are all even natural numbers starting from zero and whose denominators are all odd natural numbers starting from 1.

$$\frac{0}{1}, \frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \ldots$$

A Solution: $a_n = \frac{2n}{2n+1}, n \in \{0, 1, 2, \ldots\}$

2.2.2 Recursive Formulas

Recursion is a beautiful, powerful, and useful concept. It is the idea of defining a structure in terms of smaller instances of that same type of structure. In the case of sequences, we want to define a later term a_n as a formula given in terms of a_k for some k values strictly less than n. This definition of later terms built out of previous terms is called the *recursion.* Additionally, a recursive definition requires the definition of some initial term or terms to get the process rolling. These early terms are called the *base cases* or *initial terms.*

Returning to our favorite little example once again, we ask how we can find a recursive formula for the sequence of even numbers. Notice how the later terms relate to the earlier terms; each term is exactly two more than the previous term. We build a recursive formula out of this observation.

Example 2.2.2.1. The Sequence of Even Natural Numbers: Recursive Formula

$$\begin{aligned} a_0 &= 0 \\ a_n &= 2 + a_{n-1} \text{ for } n \geq 1 \end{aligned}$$

Exercise 2.2.2.2. Absorbing the Language ☕

In the recursive formula above, which expression is the base case? Which part is the recursion?

A Solution: $a_0 = 0$ is the base case and $a_n = 2 + a_{n-1}$ is the recursion.

Exercise 2.2.2.3. Practice with Recursive Formulas ☕☕

Find a recursive formula for the sequence of...

- ...odd natural numbers.

$$1, 3, 5, 7, \ldots$$

A Solution:

$$a_0 = 1$$

$$a_n = a_{n-1} + 2$$

- ...even integers starting at -4 and counting upwards, two at a time.

$$-4, -2, 0, 2, \ldots$$

A Solution:

$$a_0 = -4$$

$$a_n = a_{n-1} + 2$$

- ...all multiples of 5, starting from 20 and counting downwards.

$$20, 15, 10, 5, 0, -5, \ldots$$

A Solution:

$$a_0 = 20$$

$$a_n = a_{n-1} - 5$$

- ...all natural numbers that are one more than a multiple of 3.

$$1, 4, 7, 10, \ldots$$

A Solution:

$$a_0 = 1$$
$$a_n = a_{n-1} + 3$$

- ...consecutive powers of 2, starting from 1.

$$1, 2, 4, 8, \ldots$$

A Solution:

$$a_0 = 1$$
$$a_n = a_{n-1} \cdot 2$$

- ...terms that alternate forever between positive and negative one.

$$1, -1, 1, -1, \ldots$$

A Solution:

$$a_0 = 1$$
$$a_n = a_{n-1} \cdot (-1)$$

- ...fractions whose numerators are all even natural numbers starting from zero and whose denominators are all odd natural numbers starting from 1.

$$\frac{0}{1}, \frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \ldots$$

A Solution:

$$a_0 = \frac{0}{1}, a_n = \frac{1}{n^2 + n} + a_{n-1}.$$

This is obtained by calculating the difference $a_n - a_{n-1}$ and then rearranging the equatin to solve for a_n

.

2.3 Factorials

Some sequences have no simple explicit formula and are most easily thought of recursively. The sequence of factorials is a famous example of this type.

2.3.1 Recursive Formula for Factorials

Example 2.3.1.1. Factorials, Defined Recursively

Consider the following recursively defined sequence:

$$\begin{aligned} a_0 &= 1 \\ a_n &= n \cdot a_{n-1} \text{ for } n \geq 1 \end{aligned}$$

We can unwind this recursion a bit to obtain a more accessible expression for factorials. Observe the following calculations based on the base case and recursion given above:

$$\begin{aligned} a_0 &= 1 = 1 \\ a_1 &= 1 \cdot a_0 = 1 \\ a_2 &= 2 \cdot a_1 = 2 \cdot 1 = 2 \\ a_3 &= 3 \cdot a_2 = 3 \cdot 2 \cdot 1 = 6 \\ a_4 &= 4 \cdot a_3 = 4 \cdot 3 \cdot 2 \cdot 1 = 24 \\ a_5 &= 5 \cdot a_4 = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \end{aligned}$$

This sequence comes up so frequently that we give it its own symbol, the exclamation point! Since the factorial of n always amounts to the product of all natural numbers greater than or equal to 1 but less than or equal to n, we write the following:

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$$

Note that almost any expression involving a shady "$\cdots$" is truthfully a recursion in disguise!

Exercise 2.3.1.2. Why is the Factorial of Zero Equal to One? ☕

Looking carefully at the above definition, you will notice that

$$0! = 1$$

It is a common mistake to compute 0! as 0 instead. Here is one way to see why it should in fact be 1.

- If you compute 2^2, how many numbers are you multiplying together?
- If you compute 2^1, how many numbers are you multiplying together?
- If you compute 2^0, how many numbers are you multiplying together?

- Right, zero numbers are being multiplied together. A product like this is called an *empty product* and is always defined to be one, since that is the multiplicative identity.
- If you compute 3! as the product of all natural numbers greater than or equal to 1 and but less than or equal to 3, how many numbers are you multiplying together?
- If you compute 2! as the product of all natural numbers greater than or equal to 1 and but less than or equal to 2, how many numbers are you multiplying together?
- If you compute 1! as the product of all natural numbers greater than or equal to 1 and but less than or equal to 1, how many numbers are you multiplying together?
- If you compute 0! as the product of all natural numbers greater than or equal to 1 and but less than or equal to 0, how many numbers are you multiplying together?
- Since 0! is also an empty product (much like 2^0), what should we define it to be?

The following type of simplification will occur frequently throughout the our adventures in infinite series and power series. They all follow directly from the recursive definition of factorials.

Example 2.3.1.3. Simplifying Factorials

Let n represent a positive natural number. Consider the expression $\frac{(n+2)!}{n!}$. The numerator represents the product of all natural numbers between $n+2$ and 1, inclusive. The denominator represents the product of all natural numbers between n and 1, inclusive. We expand out these products and then cancel whatever factors they have in common.

$$\begin{aligned} \frac{(n+2)!}{n!} &= \frac{(n+2)(n+1)(n)(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{(n)(n-1)(n-2)\cdots 3\cdot 2\cdot 1} \\ &= (n+2)(n+1) \end{aligned}$$

Thus, that ratio of factorials cleans up to just a polynomial!

Exercise 2.3.1.4. Simplifying Factorials ☕☕

Let n be a natural number greater than or equal to 1. Reduce the following fractions! (Are they fractorials?)

- $\frac{n!}{(n+1)!}$

  A Solution:

  $$\frac{1}{n+1}$$

- $\frac{(n+1)!}{n!}$

  A Solution:

  $$n+1$$

- $\frac{(n+2)!}{n!}$

A Solution:

$$(n+1)(n+1)$$

- $\frac{(2n+2)!}{(2n)!}$

A Solution:

$$(2n+2)(2n+1)$$

2.3.2 Explicit Formula for Factorials: The Gamma Function

The gamma function, denoted Γ, is the most common smooth interpolation of the factorial function on the positive integers. The gamma function is defined using improper integrals.

Definition 2.3.2.1. The Gamma Function

For $n \in \mathbb{R}$, define

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} \, \mathtt{d}x$$

That is, $\Gamma(n)$ is the area under the graph of $x^{n-1}e^{-x}$ in the first quadrant.

Exercise 2.3.2.2. Computing Values of Gamma

- To see the manner in which the gamma function provides a continuous analog of the factorial function, fill out the values in the following table:

n	$n!$	$\Gamma(n)$
1		
2		
3		
4		

A Solution:

n	$n!$	$\Gamma(n)$
1	1	1
2	2	1
3	6	2
4	24	6

Computing the values of the gamma function will require quite a bit of work. You don't have to show all details of the integrals above, but make sure you are comfortable doing such manipulations by hand.

- Given the table above, conjecture an explicit formula for the factorials. It won't be a nice little algebraic formula, but express it in terms of an improper integral. Write the conjecture below.

$$n! = \int_0^\infty \qquad\qquad\qquad \mathtt{d}x$$

A Solution:

$$n! = \int_0^\infty x^n e^{-x}\,\mathtt{d}x$$

Observe that you were be able to use the smaller instances of the gamma function to help you compute the larger instances! That is, when you apply integration by parts to compute $\Gamma(n)$, it will produce an expression that involves the integral you computed for $\Gamma(n-1)$.

It turns out this relationship is exactly what shows that the Γ function will *always* match the values of the factorial function. For factorial, we have:

$$n! = n \cdot (n-1)!$$

Exercise 2.3.2.3. Gamma Recursion ☕☕☕

What is the corresponding relationship for the Gamma function? Specifically, how does $\Gamma(n)$ relate to $\Gamma(n-1)$? Write your answer below.

A Solution: We can compute this by looking at $\Gamma(n+1)$ since it's a little easier to work with, and computing it with integration by parts, with $u = x^n, \mathtt{d}u = nx^{n-1}\,\mathtt{d}x, v = e^{-x}, \mathtt{d}v = -e^{-x}$:

$$\begin{aligned}\Gamma(n+1) &= \int_0^\infty x^n e^{-x}\,\mathtt{d}x \\ &= -x^n e^{-x}\Big]_0^\infty + n\int x^{n-1}e^{-x}\,\mathtt{d}x \\ &= 0 + n\Gamma(n)\end{aligned}$$

So $\Gamma(n+1) = n\Gamma(n)$ and:

$$\Gamma(n) = (n-1)\Gamma(n-1).$$

2.3.3 An Approximation for the Factorials

As mentioned above, there does not exist a simple algebraic explicit formula for the factorial function. However **Stirling's Formula** gives a very nice explicit asymptotic formula.

Stirling's Formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

That is to say, as n approaches infinity, $n!$ approaches $\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. We don't have the tools to fully prove Stirling's Formula in this course, though it will be occasionally helpful to have a rough measure of the growth order of a factorial function.

2.4 Arithmetic and Geometric Sequences

Here we provide the definitions for two particularly famous families of sequences, arithmetic and geometric.

Definition 2.4.0.1. Arithmetic Sequence

A sequence a_n is called *arithmetic* if and only if there exists some real constant d such that $a_{n+1} - a_n = d$ for all natural numbers n. In such a sequence, the number d is called the *common difference*.

Definition 2.4.0.2. Geometric Sequence

A sequence a_n is called *geometric* if and only if there exists some real constant r such that $a_{n+1}/a_n = r$ for all natural numbers n. In such a sequence, the number r is called the *common ratio*.

Exercise 2.4.0.3. Playing with the Definition ☕☕

- Return to Exercise 2.2.1.2. Which of those sequences are arithmetic? For those that are, what is the common difference?

A Solution:

1. The sequence of odd natural numbers $1, 3, 5, 7, \ldots$ is arithmetic and has difference of 2.
2. The sequence of even integers $-4, -2, 0, 2, \ldots$ is arithmetic and has difference 2.
3. The sequence of multiples of 5 is arithmetic and has difference -5, since the later term is less than the previous term.
4. The sequence of natural numbers one more than a multiple of 3, given by $1, 4, 7, 10, \ldots$ is arithmetic and has difference of 3.

- Return again to Exercise 2.2.1.2. Which of those sequences are geometric? For those that are, what is the common ratio?

  A Solution:

  1. The sequence of consecutive powers of 2, give by $1, 2, 4, 8, \ldots$ is geometric, and has common ratio 2.
  2. The sequence of alternating ones, $1, -1, 1, -1, \ldots$ is geometric, with common ratio -1.

- Give an informal definition of an arithmetic sequence. (Think of what you would say if you had to explain what it was to a fifth grader).

  A Solution: An arithmetic sequence grows (or decreases) by the same amount at each step. The step size from one number to the next is always the same. You add each term to the same number to get the next term.

- Give an informal definition of a geometric sequence. (Think again of what you would say if you had to explain what it was to a fifth grader).

  A Solution: A geometric sequence grows (or shrinks) more and more at each step. You multiply each term by the same number to get the next term.

- Give an example of a sequence that is arithmetic but not geometric.

  A Solution: $1.1, 1.2, 1.3, 1.4, \ldots$.

- Give an example of a sequence that is geometric but not arithmetic.

  A Solution: $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \cdots$.

- Can a sequence simultaneously be both arithmetic and geometric? If it is possible, give an example of such a sequence. If it is not possible, explain why it is not possible.

  A Solution: The sequence $1, 1, 1, 1, \ldots$ is both geometric (common ratio is 1) and arithmetic (common difference 0).

Exercise 2.4.0.4. Converting Between Recursive and Explicit Definitions ☕☕

- Write a sentence that explains the difference between defining a sequence recursively vs defining a sequence explicitly.

A Solution: Defining a sequence recursively means defining each term in relation to the previous (or several previous) terms. To find a given term, you have to calculate all the ones before it.

Defining a sequence explicitly allows us to get any term on its own.

- Consider the following recursively-defined sequence:

$$a_0 = 5$$
$$a_n = 2 \cdot a_{n-1}$$

Write out the first five terms of this sequence. Can you find an explicit formula?

A Solution:

$$5, 10, 20, 40, 80, \ldots.$$

The explicit formula is given by $5 \cdot 2^n$.

- Consider the following explicitly-defined sequence:

$$a_n = 3n - 2$$

Write out the first five terms of this sequence. Can you find an recursive formula?

A Solution:

$$-2, 1, 4, 7, 10, 13, \ldots.$$

A recursive form is given by

$$a_0 = -2$$
$$a_{n+1} = a_n + 3.$$

2.4.1 Explicit Formulae for Arithmetic and Geometric Sequences

Example 2.4.1.1. Explicit Formula for Arithmetic Sequences

Since every arithmetic sequence starts with some initial term a_0 and then adds the same number d each time to get from term to term, we can say the explicit formula will always have the same form. In particular, we have the terms of the sequence as follows:

$$a_0, a_0 + d, a_0 + 2d, a_0 + 3d, \cdots .$$

Thus, a generic term of the sequence looks like

$$a_n = a_0 + dn$$

where a_0 is the initial term and d is the common difference.

Exercise 2.4.1.2. Explicit Formula for Geometric Sequences ☕☕

Repeat the process of the above example to demonstrate that every geometric sequence has explicit formula

$$a_n = a_0 r^n$$

where a_0 is the initial term and r is the common ratio.

A Solution: Since a geometric sequence starts with a_0 and then multiplies each term by the same common ratio, we get the sequence

$$a_0, a_0 \cdot r, a_0 \cdot r^2, a_0 \cdot r^3, \ldots .$$

Then some generic term is

$$a_n = a_0 r^n.$$

2.5 Convergence of Sequences

2.5.1 Intuitive and Formal Definitions

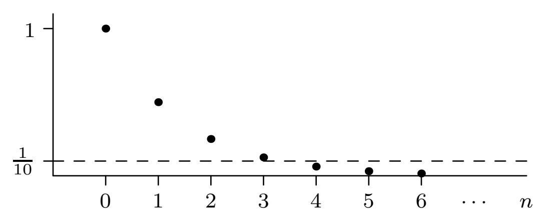
Consider the sequence $a_n = \frac{1}{2^n}$. Listing out a few terms, we see that a_n looks like:

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$$

We would like a way to describe the long-term behavior of such a sequence. Intuitively, we see that the numbers are becoming arbitrarily close to zero.

Exercise 2.5.1.1. The Idea of Convergence

- Label the y coordinates on the graph of the sequence below.



- How far into the sequence would you have to travel to find only terms that are no more than one-tenth from zero? That is to say, how large does n have to be to guarantee that a_n is between one-tenth and zero?

A Solution: The 4^{th} term gets us below one-tenth.

- How far into the sequence would you have to travel to find only terms that are no more than one-hundredth from zero?

A Solution: We want x such that $2^x \geq 100$. Since $2^7 = 128$ we need 7 terms.

- How far into the sequence would you have to travel to find only terms that are no more than one-thousandth from zero?

A Solution: By the same reasoning as above, we can go out to 10 terms, since $2^{10} = 1024 \geq 1000$.

No matter how small of a measurement we choose (one-tenth, one-hundredth, one-thousandth, etc), we could always find that after a certain point, all of our sequence terms are no further than that measurement from zero. This is exactly the notion we will reformulate in a more formal manner to define sequential convergence.

Recall the mathematical shorthands often used to help concisely state messy definitions: the symbol "$\forall$" means "for all" and the symbol "$\exists$" means "there exists". See Chapter 1 for more detail. Also recall that for any real numbers a and b, the distance between a and b can be written as $|a - b|$. Using these shorthands, we define the limit of a sequence!

Definition 2.5.1.2. Convergence of a Sequence

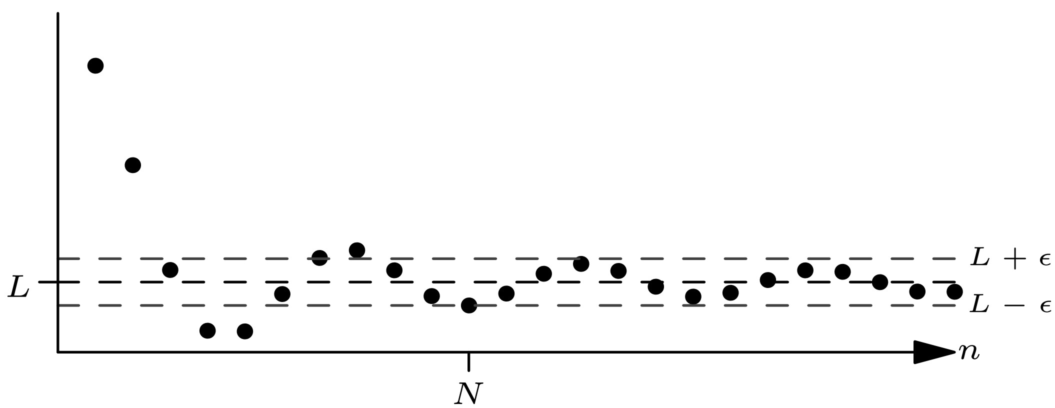
We say the sequence a_n converges to a limit $L \in \mathbb{R}$ and write

$$\lim_{n\to\infty} a_n = L$$

if and only if

$$\forall \epsilon > 0, \exists N \in \mathbb{R}, \forall n \in \mathbb{N}, n > N \implies |a_n - L| < \epsilon$$

If no such L exists, we say the sequence *diverges*.



Exercise 2.5.1.3. Digesting the Definition ☕☕

- In the definition of convergence, what role does ϵ play? Specifically, what is it bounding the distance between?

A Solution: ϵ acts a a margin of error around the target value.

- In the definition of convergence, what role does N play? What role does n play?

A Solution: N is acting like a cutoff value, above which the sequence should behave the way we want it to.

- Restate the formal definition of sequential convergence in words rather than symbols. The statement

$$\lim_{n\to\infty} a_n = L$$

means...

A Solution: As n gets very large, or rather, as we go farther into the sequence, the terms get closer to our target value, L.

Exercise 2.5.1.4. The Fibonacci Numbers ☕☕

Define the sequence of Fibonacci numbers F_n via the following recursive formula:

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_n &= F_{n-1} + F_{n-2} \end{aligned}$$

- Compute the first eight terms of the Fibonacci sequence using the above recursion. That is, compute F_0 through F_7.

A Solution:

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_2 &= F_0 + F_1 = 0 + 1 = 1 \\ F_3 &= F_1 + F_2 = 1 + 1 = 2 \\ F_4 &= F_2 + F_3 = 1 + 2 = 3 \\ F_5 &= F_3 + F_4 = 2 + 3 = 5 \\ F_6 &= F_4 + F_5 = 3 + 5 = 8 \\ F_7 &= F_5 + F_4 = 5 + 8 = 13 \end{aligned}$$

- Compute the following quantities:

$$\begin{aligned} F_2/F_1 &= \\ F_3/F_2 &= \\ F_4/F_3 &= \\ F_5/F_4 &= \\ F_6/F_5 &= \\ F_7/F_6 &= \end{aligned}$$

A Solution:

$$\begin{aligned}
F_2/F_1 &= \frac{1}{1} = 1 \\
F_3/F_2 &= \frac{2}{1} = 2 \\
F_4/F_3 &= \frac{3}{2} = 1.5 \\
F_5/F_4 &= \frac{5}{3} = 1.\overline{6} \\
F_6/F_5 &= \frac{8}{5} = 1.6 \\
F_7/F_6 &= \frac{13}{8} = 1.625
\end{aligned}$$

- What would you conjecture about

$$\lim_{n\to\infty} F_{n+1}/F_n$$

Does it seem to be going to infinity, zero, or stabilizing at something inbetween?

A Solution: It seems to be stabilizing at some value around 1.62. In actuality, it converges to the Golden Ratio: 1.61803....

It is difficult to tell exactly what that limit of ratios is without knowing an explicit formula for the Fibonacci numbers. Stay tuned, as we will find this in a later chapter!

Exercise 2.5.1.5. Comparing Growth Orders of Sequences ☕☕

Rank the following functions in growth order from smallest to largest:

$$\begin{aligned}
a_n &= n^n \\
b_n &= e^n \\
c_n &= n^2 \\
d_n &= n!
\end{aligned}$$

Note that for sequences we can compare growth orders in the same manner as we did in Subsection **??**. To compare the growth orders of two sequences a_n and b_n, we compute $\lim_{n\to\infty} \frac{a_n}{b_n}$ and conclude that a_n has larger growth order if the limit is infinity, b_n has larger growth order if the limit is zero, and the growth orders are the same if the limit is a nonzero constant. Here you will need Stirling's Formula from Subsection 2.3.3.

A Solution: In the context of computing limits, Stirling's Formula for factorials lets us use $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. Then we just need to compare the various expressions to each other.

$$c_n = n^2$$
$$b_n = e^n$$
$$d_n = n!$$
$$a_n = n^n$$

2.5.2 $N - \epsilon$ Proofs

This complicated definition can be unwound into a to-do list for what one must do to prove that a sequence converges to a particular limit. In particular, to show that the limit of a_n is equal to a number L, one must:

- Let ϵ be an arbitrary positive real number.
- Choose N, typically defined as a function of ϵ, since smaller values of ϵ will usually require a larger N to be chosen.
- Let n represent an arbitrary natural number greater than N.
- Using the definition of N and the assumption that $n > N$, prove that any corresponding a_n satisfies $|a_n - L| < \epsilon$.

Figuring out exactly what N should be in terms of ϵ usually requires a bit of algebra before the proof is written up. If the formula for a_n is clean enough, you might be able to just work backwards from the inequality $|a_n - L| < \epsilon$. If you solve it for n, you will find an expression that n must be larger than. Note here we are essentially just finding an inverse function for a_n.

Example 2.5.2.1. Solving for N

Let us solve for N with regards to our sequence $a_n = \frac{1}{2^n}$. Since here we suspect $L = 0$, we solve for n in the following inequality:

$$\begin{aligned} \left| \frac{1}{2^n} - 0 \right| &< \epsilon \\ \frac{1}{2^n} &< \epsilon \\ \frac{1}{\epsilon} &< 2^n \\ \ln\left(\frac{1}{\epsilon}\right) &< \ln\left(2^n\right) \\ \ln\left(\frac{1}{\epsilon}\right) &< n \ln(2) \\ \frac{\ln\left(\frac{1}{\epsilon}\right)}{\ln(2)} &< n \end{aligned}$$

Thus we determined our choice of N, namely

$$N = \frac{\ln\left(\frac{1}{\epsilon}\right)}{\ln(2)}.$$

Exercise 2.5.2.2. Justifying Our Work ☕

In words, annotate the above example to indicate why each line follows from the previous.

A Solution:

$$\left|\frac{1}{2^n} - 0\right| < \epsilon \qquad \text{(This relates } n, \epsilon \text{ and the limit 0)}$$

$$\frac{1}{2^n} < \epsilon$$

(We just drop the 0, and the sequence is always positive, so we can drop the absolute value signs.)

$$\frac{1}{\epsilon} < 2^n \qquad \text{(Multiply both sides by } 2^n \text{ and divide both by } \epsilon.\text{)}$$

$$\ln\left(\frac{1}{\epsilon}\right) < \ln(2^n)$$

(Take the natural log of both sides to bring down the exponent on the RHT.)

$$\ln\left(\frac{1}{\epsilon}\right) < n\ln(2) \qquad \text{(Exponent rule for logs lets us bring down } n \text{ on the RHT.)}$$

$$\frac{\ln\left(\frac{1}{\epsilon}\right)}{\ln(2)} < n \qquad \text{(Divide both sides by } \ln(2) \text{ to isolate } n.\text{)}$$

Now that we found our value for N, we are ready to follow the steps described above and construct our proof.

Example 2.5.2.3. Writing an $N - \epsilon$ Proof

Prove that

$$\lim_{n\to\infty} \frac{1}{2^n} = 0$$

Proof. Let ϵ be an arbitrary positive real number. Choose $N = \frac{\ln\left(\frac{1}{\epsilon}\right)}{\ln(2)}$. Let n be a natural number such that $n > N$. Under these circumstances, we wish to show that a corresponding a_n will be less than ϵ away from 0. Proceeding:

$$\begin{aligned}
\left|\frac{1}{2^n} - 0\right| &= \frac{1}{2^n} \\
&< \frac{1}{2^N} \\
&= \frac{1}{2^{\left(\frac{\ln\left(\frac{1}{\epsilon}\right)}{\ln(2)}\right)}} \\
&= \frac{1}{2^{\left(\log_2\left(\frac{1}{\epsilon}\right)\right)}} \\
&= \frac{1}{\frac{1}{\epsilon}} \\
&= \epsilon.
\end{aligned}$$

Thus, for indices n that are larger than our choice of N, the corresponding terms in our sequence are less than ϵ away from zero as desired. ☐

Exercise 2.5.2.4. Justifying Our Work

Once again in words, annotate the above example to indicate why each line follows from the previous. Pay particular attention to identify where we used the starting assumption that $n > N$.

A Solution:

$$\left|\frac{1}{2^n} - 0\right| = \frac{1}{2^n}$$

(We know that the sequence terms are always positive, so we clear out the absolute value bars.)

$< \frac{1}{2^N}$ (This is our starting assumption $n > N$.)

(Since n is in the denominator of a fraction, replacing it with a smaller number has the effect of)

(making the fraction bigger.)

$= \frac{1}{2^{\left(\frac{\ln\left(\frac{1}{\epsilon}\right)}{\ln(2)}\right)}}$ (Substitute in the value we chose for N.)

$= \frac{1}{2^{\left(\log_2\left(\frac{1}{\epsilon}\right)\right)}}$ (Change of base formula.)

$= \frac{1}{\frac{1}{\epsilon}}$ (Since the base of the log is the base of the exponent,)

(we can replace the whole expression with the log argument.)

$= \epsilon.$ (Simplify the fraction.)

As this course does not go through a general treatment of what constitutes a proof or how to come up with one, the example above could be taken as a template for how an $N - \epsilon$ proof should be written. In a more in-depth study of analysis, you will encounter more complicated situations where the above template may be too simplistic. It will be expanded upon when you have the right tools! For now, follow the above proof template for the following exercises:

Exercise 2.5.2.5. Verifying a Limit ☕☕☕

Consider the sequence given by the following explicit formula:

$$a_n = \frac{2n}{n+1}$$

- List the terms of the sequence corresponding to $n = 1$, $n = 10$, $n = 100$, and $n = 1000$. What do the terms appear to be converging to as n goes to ∞?

- If you choose $\epsilon = 0.1$, what could the corresponding N be?

- If you choose $\epsilon = 0.01$, what could the corresponding N be?

- Write an $N - \epsilon$ proof that verifies your guess above is correct.

Exercise 2.5.2.6. Writing $N - \epsilon$ Proofs ☕☕☕

Write $N - \epsilon$ proofs for each of the following limits:

- $\lim_{n\to\infty} \frac{n}{3n+1} = \frac{1}{3}$

- $\lim_{n\to\infty} \sqrt{9 + 1/n} = 3$

Part III

Coming Attractions

Selected Answers and Hints

Exercise 2.3.1.4. $\bullet\frac{1}{n+1}$ $\bullet n+1$ $\bullet(n+2)(n+1)$ $\bullet(2n+2)(2n+1)$

Exercise 2.4.0.3. Think about what happens if the common difference d is zero and if the common ratio r is 1.

Exercise 2.4.1.2. The common ratio r is what we multiply by to get from term to term. Listing out the terms $a_0, a_0r, a_0r^2, a_0r^3, \cdots$ shows that a_0r^n is the explicit formula.

Exercise 2.5.1.5. In the context of computing a limit to infinity, it is fine to replace $n!$ by $\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. Setting up limits of ratios and testing growth order with LHR and good old algebra will then verify that the order goes $n^2, e^n, n!, n^n$.