Calculus II

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We hope you have fun with this resource and find it helpful! It is a beautiful subject, and we tried to honor that with a beautiful text. The text is still in its infancy, and we welcome any and all feedback you could give us. Thank you in advance for any comments, complaints, suggestions, and questions.

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Chapter 1

Overview

A Three-Part Course

The topics of Calculus II fall into three parts that each have an appropriate place in the story of the calculus sequence.

- **Part I: Integration.** The first part of the course ties loose ends from Calculus I. The ending of Calculus I showed that antiderivatives can be used to evaluate integrals via the Fundamental Theorem of Calculus. However, by the end of Calculus I, only the very simplest antiderivatives can actually be computed. Part one expands the student's knowledge of **techniques of antidifferentiation**. These techniques are subsequently put to use computing **length, area, volume, and center of mass**.
- **Part II: Sequences and Series.** This is the topic that makes up the body of Calculus II. **Sequences and series** embody the beauty of mathematics; from simple beginnings (a sequence is just a list... a series is just adding up a list of numbers...) it quickly leads to incredible structure, surprises, complexity, and open problems. **Power series** redefine commonly used transcendental functions (functions that are not computed using algebra, e.g. cosine). If you've ever wondered **what your calculator does when you press the cosine button**, this is where you find out! (**Hint:** It does not have a circle of radius one spinning around with a team of elves that measure x coordinates.)
- **Part III: Coming Attractions.** By the end of Calculus II, the student is ready for a *lot* of other classes. The end of Calculus II thus ends with a sampler platter of topics that show the vast knowledge base built upon the foundation laid in Calculus II. Here the text takes a bite out of **Differential Equations**, serves some polar and parametric coordinates as a palate cleanser before **Calculus III**, and tastes some **Complex Analysis** to aid in digestion of Differential Equations. For dessert, it serves a scoop of **Probability** with both discrete and continuous colored sprinkles.

How to Use This Book

This book is meant to facilitate *Active Learning* for students, instructors, and learning assistants. Active Learning is the process by which the student participates directly in the learning process by reading, writing, and interacting with peers. This contrasts the traditional model where the student passively listens to lecture while taking notes. This book is designed as a self-guided step-by-step exploration of the concepts. The text incorporates theory and examples together in order to lead the student to discovering new results while still being able to relate back to familiar topics of mathematics. Much of this text can be done independently by the student for class preparation. During class sessions, the instructor and/or learning assistants may find it advantageous to encourage group work while being available to assist students, and work with students on a one-on-one or small group basis. This is highly desirable as extensive research has shown that active learning improves student success and retention. (For example, see `www.pnas.org/content/111/23/8410` for Scott Freeman's metaanalysis of 225 studies supporting this claim.)

What is Different about this Book

If you leaf through the text, you'll quickly notice two major structural differences from many traditional calculus books:

1. The exercises are very intermingled with the readings. Gone is the traditional separation into "section" versus "exercises".

2. Whitespace was included for the student to write and work through exercises. Parts of pages have indeed been intentionally left blank.

A consequence of this structure is that the readings and exercise are closely linked. It is intended for the student to do the readings and exercises concurrently.

Ok... Why?

The goal of this structure is to help the student simulate the process by which a mathematician reads mathematics. When a mathematician reads a paper or book, he/she always has a pen in hand and is constantly working out little examples alongside and scribbling incomprehensible notes in bad handwriting. It takes a *long* time and a lot of experience to know how to come up with good questions to ask oneself or to know what examples to work out in order to to help oneself absorb the subject. Hence, the exercises sprinkled throughout the readings are meant to mimic the margin scribbles or side work a mathematician engages in during the act of reading mathematics.

The Legend of Coffee

A potential hazard of this self-guided approach is that while most examples are meant to be simple exercises to help with absorption of the topics, there are some examples that students may find quite difficult. To prevent students from spinning their wheels in frustration, we have labeled the difficulty of all exercises using coffee cups as follows:

Coffee Cup Legend		
Symbol	Number of Cups	Description of Difficulty
☕	A One-Cup Problem	Easy warm-up suitable for class prep.
☕☕	A Two-Cup Problem	Slightly harder, solid groupwork exercise.
☕☕☕	A Three-Cup Problem	Substantial problem requiring significant effort.
☕☕☕☕	A Four-Cup Problem	Difficult problem requiring effort and creativity!

Glossary of Symbols

In Precalculus and Calculus I, there is a wide range of how much notation from Set Theory gets used. To get everyone on the same page, here is a short list of some notation we will use in this text.

Sets and Elements

Often in mathematics, we construct collections of objects called *sets*.

- If an object x is in a set A, we say x is an *element* of A and write $x \in A$.
- If an object x is not in a set A, we say x is not an element of A and write $x \notin A$.

Any particular object is either an element of a set or it is not. We do not allow for an object to be partially contained in a set, nor do we allow for an object to appear multiple times in a set. Often we use curly braces around a comma-separated list to indicate what the elements are.

Example 1.0.0.1. A Prime Example

Suppose P is the set of all prime numbers. We write

$$P = \{2, 3, 5, 7, 11, 13, 17, \ldots\}$$

For example, $2 \in P$ and $65,537 \in P$, but $4 \notin P$.

Some Famous Sets of Numbers

The following are fundamental sets of numbers used in Calculus 2.

- **Natural Numbers:** The set $\mathbb{N}$ of natural numbers is the set of all positive whole numbers, along with zero. That is,

  $$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \ldots\}$$

  Note that in many other sources, zero is not included in the natural numbers. Both are widely used; be aware the choice on this convention will change throughout your mathematical travels!

- **Integers:** The set of integers $\mathbb{Z}$ is the set of all whole numbers, whether they are positive, negative, or zero. That is,

  $$\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

- **Rational Numbers:** The set of rational numbers $\mathbb{Q}$ is the set of all numbers expressible as a fraction whose numerator and denominator are both integers.

- **Real Numbers:** The set of real numbers $\mathbb{R}$ is the set of all numbers expressible as a decimal.

- **Complex Numbers:** The set $\mathbb{C}$ of complex numbers is the set of all numbers formed as a real number (called the real part) plus a real number times i (called the imaginary part), where i is a symbol such that $i^2 = -1$.

Set-Builder Notation

The most common notation used to construct sets is *set-builder notation*, in which one specifies a name for the elements being considered and then some property $P(x)$ that is the membership test for an object x to be an element of the set. Specifically,

$$A = \{x \in B : S(x)\}$$

means that an object x chosen from B is an element of the set A if and only if the claim $S(x)$ is true about x. Sometimes the "$\in B''$ gets dropped if it is clear from context what set the elements are being chosen from. The set-builder notation above gets read as "the set of all x in B such that $S(x)$". One can think of this as running through all elements of B and throwing away any that do not meet the condition described by S.

Example 1.0.0.2. Interval Notation

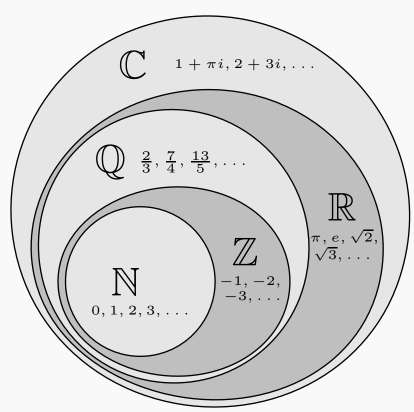
Interval notation can be expressed in set-builder notation as follows:

- $(a, b) = \{x \in \mathbb{R} : a < x < b\}$
- $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$
- $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$
- $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$

Example 1.0.0.3. Rational, Real, and Complex in Set-Builder Notation

Set-builder notation is often used to express the sets of rational, real, and complex numbers as follows:

- $\mathbb{Q} = \left\{\frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0\right\}$
- $\mathbb{R} = \{0.a_0a_1a_2a_3a_4 \ldots \times 10^n : n \in \mathbb{N}, a_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ where $i \in \mathbb{N}\}$ Note this is essentially scientific notation; the concatenation of the a_i's represents the digits in a base-ten decimal expansion.
- $\mathbb{C} = \{a + bi : a \in \mathbb{R}, b \in \mathbb{R}\}$



Part I

Sequences and Series

1.1 Infinite Series

Well here's an interesting question.

What does it mean to add up infinitely many numbers?
-Lots of people

We provide the most commonly used modern definition.

Definition 1.1.0.1. Infinite Series, Convergence, and Divergence

Let a_n be a sequence of real numbers. Then the *infinite sum* of all terms of a_n is defined to be the limit of partial sums A_N. That is,

$$\sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} A_N = \lim_{N\to\infty} \sum_{n=0}^{N} a_n.$$

If the limit exists, we say the infinite series *converges* to the value of the limit. If the limit is infinity or does not exist, then we say the infinite series *diverges*.

The idea is simple; if you want to add up infinitely many numbers, a good place to start is by just adding up finitely many of them. However, if you only add up finitely many, your answer has some error to it. If you want that error to go down, add up more and more of them! The limit of the values of these partial sums will be the exact answer.

Exercise 1.1.0.2. The Return of the Discrete/Continuous Analogy ☕☕☕

In what way is the definition of an infinite series analagous to the definition of a horizontally unbounded improper integral?

A Solution: In the same way that the definition of the infinite series involves taking the limit as the last term approaches infinity, the definition for a horizontally unbounded improper integral also involves taking the limit of the upper bound of integration: $\int_a^\infty f(x)\,\mathtt{d}x = \lim_{c\to\infty} \int_a^c f(x)\,\mathtt{d}x$.

Exercise 1.1.0.3. The Definitions in Words ☕☕

We have defined three very important interconnected structures:

- A sequence a_n.
- A sequence of partial sums A_N.
- An infinite series $\sum_{n=0}^{\infty} a_n$.

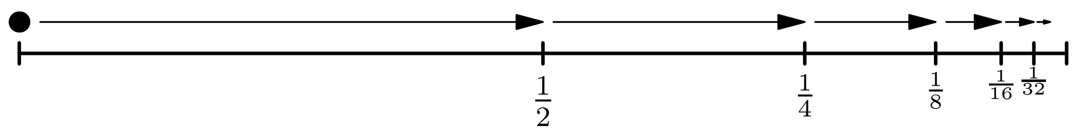
Describe in words how the three structures are related and are built from one another.

A Solution: Taking the sum of some consecutve elements of the sequence a_n produces the partial sums A_N. If we take an infinite number of the elements of a_n and sum them, we generate the infite series $\sum_{n=1}^{\infty}$, which is equivalent to taking the limit $\lim_{N\to} A_N$.

1.1.1 Zeno's Paradox, Resolution, and Consequences

The next example is traceable back to the writings of Aristotle in the third century BC! Specifically, he states Zeno's Paradox of *Dichotomy* as:

> *That which is in locomotion must arrive at the half-way stage before it arrives at the goal.*



This was meant to be a "proof" that an object (say an arrow in flight) could never reach its target. This paradox is resolved with our notion of infinite series.

Exercise 1.1.1.1. A Classic Infinite Series

Consider the sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$. This can be interpreted as the sequence of distances the arrow must travel in the Dichotomy paradox (supposing it was fired one meter from its target and all lengths are measured in meters). Since it were fired from one meter away, we expect that the total distance traveled is one.

- Find an explicit formula a_n that describes the sequence above.

A Solution: $a_n = \frac{1}{2^{n+1}}$.

- Compute the corresponding sequence of partial sums A_N.

A Solution: Taking the Geometric Series Formula, with first term $\frac{1}{2}$ and common ration $\frac{1}{2}$, we get

$$A_N = \frac{1}{2} \cdot \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} = \frac{1}{2} \cdot \frac{1 - \frac{1}{2^{n+1}}}{\frac{1}{2}} = 1 - \frac{1}{2^{n+1}}.$$

- Evaluate the infinite sum

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

by taking the limit of the sequence of partial sums. Verify the total is in fact one.

A Solution:

$$\lim_{N\to\infty} 1 - \frac{1}{2^{N+1}} = 1 - \lim_{N\to\infty} \frac{1}{2^{N+1}} = 1 - 0 = 1.$$

In the above case, we were able to provide a resolution for the paradox and verify the total with our geometric series formula, but the answer was not particularly surprising. Here is a more interesting example!

Exercise 1.1.1.2. An Alternating Geometric Series ☕☕☕

Suppose a bug moves forward half a meter. It then moves backwards one-fourth of a meter. It then moves forward one-eighth of a meter. It then moves backwards one-sixteenth of a meter. This pattern of moving forwards, then backwards, by half the previous distance each time, continues forever. At the end of time, where does the bug end up?
To solve this problem, we notice that it is equivalent to adding up all terms in the sequence $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \cdots$. Following the method of Example **??.??**, this sequence can be expressed as a geometric sequence with initial term $a_0 = \frac{1}{2}$ and common ratio $r = -\frac{1}{2}$ as follows:

$$a_n = \frac{1}{2}\left(-\frac{1}{2}\right)^n.$$

- Let $A_N = \sum_{n=0}^{N} a_n$ be the sequence of partial sums. Find a formula for A_N.

A Solution: Since this is a geometric series, we again use the Geometric Series Formula:

$$A_N = \frac{1}{2} \cdot \frac{1-\left(-\frac{1}{2}\right)^{N+1}}{1-\left(-\frac{1}{2}\right)} = \frac{1}{2} \cdot \frac{1-\left(-\frac{1}{2}\right)^{N+1}}{\frac{3}{2}} = \frac{1-\left(-\frac{1}{2}\right)^{N+1}}{3}.$$

- Compute $\sum_{n=0}^{5} a_n$.

A Solution:

$$A_5 = \frac{1-\left(-\frac{1}{2}\right)^{5+1}}{3} = \frac{1-\left(-\frac{1}{2}\right)^{6}}{3} = \frac{1-\frac{1}{64}}{3} \approx 0.4219.$$

- Compute $\sum_{n=0}^{10} a_n$.

$$A_{10} = \frac{1-\left(-\frac{1}{2}\right)^{10+1}}{3} = \frac{1-\left(-\frac{1}{2}\right)^{11}}{3} = \frac{1+\frac{1}{2^{11}}}{3} \approx 0.3335.$$

- Compute $\sum_{n=0}^{\infty} a_n$ from the definition of an infinite series.

A Solution: Take the limit of A_N as N approaches infinity.

$$\lim_{N\to\infty} A_N = \lim_{N\to\infty} \frac{1-\left(-\frac{1}{2}\right)^{N+1}}{3} = \frac{1-\left(\lim_{N\to\infty}\left(-\frac{1}{2}\right)^{N+1}\right)}{3} = \frac{1}{3}.$$

So, where does the bug end up?

A Solution: It ends up one-third of a meter forward from where it started.

1.1.2 Infinite Geometric Series

The notion of an infinite series can be used to give a rigorous interpretation to the infinite decimal expansions as well!

Exercise 1.1.2.1. Repeating Decimal Expansion ☕☕

- Notice that the decimal expansion 0.333 can be written as a geometric series with three terms and common ratio 1/10 using the definition of place value. In particular,

$$0.333 = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000}.$$

Compute its value via the finite geometric series formula.

A Solution: This is a geometric series with first term $\frac{3}{10}$ and common ration $r = \frac{1}{10}$, and 3 terms. By the geometric series formula we get

$$\frac{3}{10} \cdot \frac{1 - \frac{1}{10^3}}{1 - \frac{1}{10}} = \frac{3}{10} \cdot \frac{1 - \frac{1}{1000}}{\frac{9}{10}} = \frac{\frac{999}{1000}}{3} = \frac{333}{1000} = 0.333$$

.

- Write 0.3333 as a geometric series with four terms and common ratio 1/10. Compute its value via the finite geometric series formula.

A Solution:

$$0.3333 = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000}$$

. Evaluating this with the geometric series formula is almost exactly the same as the previous problem.

$$\frac{3}{10} \cdot \frac{1 - \frac{1}{10^4}}{1 - \frac{1}{10}} = \frac{3}{10} \cdot \frac{1 - \frac{1}{10000}}{\frac{9}{10}} = \frac{\frac{9999}{10000}}{3} = \frac{3333}{10000} = 0.3333$$

.

- Write 0.33333 as a geometric series with five terms and common ratio 1/10. Compute its value via the finite geometric series formula.

A Solution:

$$0.33333 = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \frac{3}{100000}$$

. Repeat the process above.

$$\frac{3}{10} \cdot \frac{1 - \frac{1}{10^5}}{1 - \frac{1}{10}} = \frac{3}{10} \cdot \frac{1 - \frac{1}{100000}}{\frac{9}{10}} = \frac{\frac{99999}{100000}}{3} = \frac{33333}{100000} = 0.33333$$

.

- Write

$$\underbrace{0.3333\ldots3}_{n \text{ threes}}$$

as a geometric series with n terms and common ratio $1/10$. Compute its value in terms of n via the finite geometric series formula.

A Solution:

$$0.33333\ldots3 = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \frac{3}{100000} + \cdots + \frac{3}{10^n}$$

. Repeat the process above.

$$\frac{3}{10} \cdot \frac{1 - \frac{1}{10^{n+1}}}{1 - \frac{1}{10}} = \frac{1 - \frac{1}{10^{n+1}}}{3}.$$

- Take the limit as n approaches infinity of your formula from the previous part to prove that "point three repeating" really does equal one-third.

A Solution:

$$\lim_{n\to\infty} \frac{1 - \frac{1}{10^{n+1}}}{3} = \frac{1 - \lim_{n\to\infty}\left(\frac{1}{10^{n+1}}\right)}{3} = \frac{1}{3}.$$

We can generalize the previous examples. Notice in all cases, the geometric series formula let us calculate an explicit formula for the sequence of partial sums. As long as the common ratio $|r| < 1$, the limit as $n \to \infty$ will exist, as the r^{N+1} term will go to zero and we will be left with just $a_0 \frac{1}{1-r}$. This brings us to the *infinite geometric series* formula (also sometimes just referred to as the geometric series formula).

Theorem 1.1.2.2. Infinite Geometric Series Formula

If a and r are real numbers and $|r| < 1$, then

$$a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}.$$

Notice in the above formula, the number a represents the first term of the series and r represents the common ratio.

Example 1.1.2.3. A Messier Repeating Decimal Expansion

Suppose we wish to write the repeating decimal

$$1.\overline{615384}$$

as a fraction. We repeat (heh) the method of Exercise 1.1.2.1, where we use base-ten place value to express the decimals as sums of terms with common ratio equal to a negative power of ten.

$$\begin{aligned}1.\overline{615384} &= 1.615384615384615384\ldots \\ &= 1 + \frac{615384}{10^6} + \frac{615384}{10^{12}} + \frac{615384}{10^{18}} + \cdots\end{aligned}$$

The very first term, 1, clearly does not fit the pattern given by the rest of the terms. So, we won't worry about that term, and instead just work on evaluating the rest while we leave the 1 out front. The rest of the terms form an infinite geometric series with initial term $a = \frac{615384}{10^6}$ and common ratio $r = \frac{1}{10^6}$. We now apply the infinite geometric series formula, noting that r, being one over a million, is comfortably between -1 and 1 as required. Note that we resolve the compound fraction below by multiplying the top and bottom by 10^6.

$$\begin{aligned}1.\overline{615384} &= 1 + \frac{\frac{615384}{10^6}}{1 - \frac{1}{10^6}} \\ &= 1 + \frac{615384}{10^6 - 1} \\ &= 1 + \frac{615384}{999999} \\ &= 1 + \frac{8}{13} \\ &= \frac{21}{13}\end{aligned}$$

Exercise 1.1.2.4. Using the Geometric Series Formula ☕☕

Consider the following series:

$$\sum_{n=5}^{\infty} \frac{3^n}{2^{2n+1}}$$

- Write out the first few terms of the above series. That is, expand the sigma notation by plugging in $n = 5, 6, 7, 8, \ldots$ and evaluating the summand in each case.

A Solution:

n	a_n	A_n	Total
5	$\frac{3^5}{2^{11}}$	$\frac{3^5}{2^{11}}$	≈ 0.1187
6	$\frac{3^6}{2^{13}}$	$\frac{3^5}{2^{11}} + \frac{3^6}{2^{13}}$	≈ 0.2076
7	$\frac{3^7}{2^{15}}$	$\frac{3^5}{2^{11}} + \frac{3^6}{2^{13}} + \frac{3^7}{2^{15}}$	≈ 0.2744
8	$\frac{3^8}{2^{17}}$	$\frac{3^5}{2^{11}} + \frac{3^6}{2^{13}} + \frac{3^7}{2^{15}} + \frac{3^8}{2^{17}}$	≈ 0.3244

- Is the above series geometric? Explain why or why not. If so, what is the common ratio r? What is the first term a?

A Solution: Yes it is, because each term is a ratio of the last term. The common ratio

$r = \frac{3}{4}$ and the first term is $\frac{3^5}{2^{11}}$.

- Find the value of the above series.

A Solution: Plugging what we have into the infinite geometric series formula, we get

$$\sum_{n=5}^{\infty} \frac{3^n}{2^{2n+1}} = \frac{\frac{3^5}{2^{11}}}{1-\frac{3}{4}} = \frac{\frac{3^5}{2^{11}}}{\frac{1}{4}} = \frac{3^5}{2^{11}} \cdot 4 = \frac{3^5}{2^9} \approx 0.4746.$$

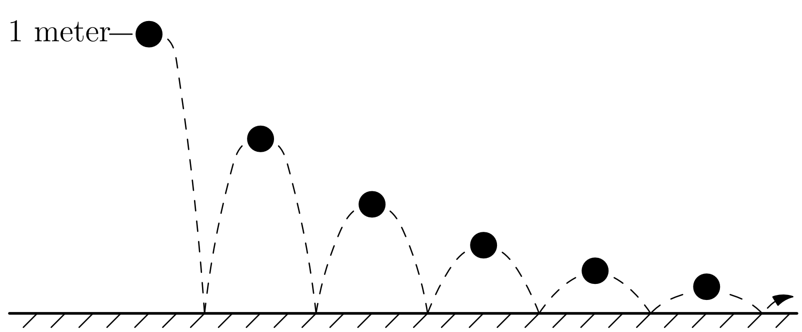
Exercise 1.1.2.5. Not Using the Geometric Series Formula ☕

Explain why the following calculation is not valid according to our definition of infinite series:

$$\begin{aligned} 1 + 2 + 2^2 + 2^3 + 2^4 + \cdots &= \frac{1}{1-2} \\ &= -1 \end{aligned}$$

A Solution: The common ratio $r = 2$, and so $|r| > 1$. Thus, we cannot use our definition above, which only applies to $|r| < 1$.

Exercise 1.1.2.6. The Bouncing Ball ☕☕



A magical bouncy ball is bounced from a height of 1 meter. On each bounce, it always rebounds to exactly five-eighths of the height it fell from. What is the ball's total vertical distance traveled from now until the end of time?

A Solution: In this case, we are dealind with a geometric series with first term 1 and common ratio $\frac{5}{8}$. Applying our infinite series formula, we get

$$\frac{1}{1-\frac{5}{8}} = \frac{8}{3} = 2.7\overline{3}.$$

So the ball travels $2.7\overline{3}$ meters until the end of time.

Exercise 1.1.2.7. Evaluating Another Infinite Series ☕☕

Consider the constant sequence $a_n = 2$. Now consider the corresponding infinite sum:

$$\sum_{n=0}^{\infty} a_n$$

- Write out the first five terms of the sequence a_n. Also write out the first five terms of the corresponding sequence of partial sums.

  A Solution: The first five terms of a_n are $2, 4, 8, 16, 32$. The first five terms of the corresponding partial sums are $2, 6, 14, 30, 62$.

- Find an explicit formula for the sequence of partial sums.

  A Solution: $A_N = 2(N+1)$.

- Does the infinite series converge? If so, what value does it converge to?

  A Solution: Taking the limit, we can clearly see that the infinite series $\lim_{N\to\infty A_N} = \lim_{N\to\infty} 2(N+1) = \infty$ does not converge.

Exercise 1.1.2.8. A Telescoping Sum ☕☕☕

Consider the following sequence:

$$a_n = \frac{2}{n^2+5n+6}$$

- Compute the first five terms of the sequence.

A Solution:

n	a_n
0	$\frac{2}{6} = \frac{1}{3}$
1	$\frac{2}{1+5+6} = \frac{1}{6}$
2	$\frac{2}{4+10+6} = \frac{1}{10}$
3	$\frac{2}{9+15+16} = \frac{1}{15}$
4	$\frac{2}{16+20+6} = \frac{1}{21}$

- Compute the first five partial sums of the sequence.

A Solution: We can make the process a little faster by transferring the solution from one line down to the next, and then just adding on the new term from a_N by referring to the table above.

N	A_N
0	$\frac{1}{3}$
1	$\frac{1}{3} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$
2	$\frac{1}{2} + \frac{1}{10} = \frac{3}{5}$
3	$\frac{3}{5} + \frac{1}{15} = \frac{2}{6}$
4	$\frac{2}{6} + \frac{1}{21} = \frac{5}{7}$

- Based on your data, conjecture a formula for

$$A_N = \sum_{n=0}^{N} \frac{2}{n^2 + 5n + 6}$$

A Solution: We need to play around with the terms of A_N a bit to get what we want. Since we want something that follows some pattern as N increases, we can try to convert the fractions so that the numerator and the denominator both increase as N gets larger. This requires some experimentation to figure out what denominator to choose, but playing around eventually gives us $A_N = \frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \frac{4}{6}, \frac{5}{7}, \ldots$.

Then we can conjecture that the pattern is $A_N = \frac{N+1}{N+3}$.

- Prove your answer is correct via a partial fraction decomposition. Specifically, perform a PFD on $a_n = \frac{2}{n^2+5n+6}$ and then notice that when you add the terms in a partial sum, all but two terms cancel! (This lucky happening is what is referred to as a series *telescoping*, as it is collapsing in on itself much like a retractable telescope would.)

A Solution: To do a PFD on $a_n = \frac{2}{n^2+5n+6}$, first factor the denominator to get $n^2+5n+6 = (n+2)(n+3)$. This gives us the decomposition

$$\frac{A}{n+2} + \frac{B}{n+3} = \frac{A(n+3) + B(n+2)}{(n+2)(n+3)}.$$

This yields the system of equations:

$$An + Bn = (A+B)n = 0, \qquad 3A + 2B = 2$$

so that,

$$A = -B \quad \Longrightarrow \quad -3B + 2B = -B = 2,$$
$$A = 2, B = -2.$$

So this means we can rewrite a_n as

$$a_n = \frac{2}{n+2} - \frac{2}{n+3}.$$

Now, if we examine the partial sums using this expansion, we will see the telescoping:

$$\begin{aligned}
A_N &= \left(\frac{2}{2} - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{2}{4}\right) + \left(\frac{2}{4} - \frac{2}{5}\right) + \cdots + \left(\frac{2}{(N+2-1)} - \frac{2}{(N+3-1)}\right) + \left(\frac{2}{(N+2)} - \frac{2}{(N+3)}\right) \\
&= \frac{2}{2} - \frac{2}{3} + \frac{2}{3} - \frac{2}{4} + \frac{2}{4} - \frac{2}{5} + \cdots + \frac{2}{(N+1)} - \frac{2}{(N+2)} + \frac{2}{(N+2)} - \frac{2}{(N+3)} \\
&= 1 - \frac{2}{(N+3)} \\
&= \frac{N+3-2}{N+3} \\
&= \frac{N+1}{N+3}.
\end{aligned}$$

This agrees with our conjecture.

- Use your formula for the partial sums and the definition of an infinite series to write an $N-\epsilon$ proof for the value of
$$\sum_{n=0}^{\infty} \frac{2}{n^2+5n+6}.$$

A Solution: Let $A_N = \frac{2}{n^2+5n+6}$. Then the limit $\lim_{N\to\infty} A_N = \sum_{n=0}^{\infty} \frac{2}{n^2+5n+6}$. We wish to show that the limit is 1. So, we need to show that, for all $\epsilon > 0$, there exists some $N' > 0$ such that for all $N > N'$, we have $|A_N - 1| < \epsilon$. Note, we are just using L'hopital's rule to get 1 as our limit candidate.

Let $N' = \frac{-2}{\epsilon} - 3$. Then we have

$$\begin{aligned}
|A_N - 1| &= \left| \sum_{n=0}^{N} \frac{2}{n^2 + 5n + 6} - 1 \right| \\
&= \left| \frac{N+1}{N+3} - 1 \right| \\
&= \left| \frac{N+1}{N+3} - 1 \right| \\
&= \left| \frac{N+1-N-3}{N+3} \right| \\
&= \left| \frac{-2}{N+3} \right| \\
&< \left| \frac{-2}{N'+3} \right| \\
&= \left| \frac{-2}{\frac{-2}{\epsilon} - 3 + 3} \right| \\
&= \left| \frac{-2}{\frac{-2}{\epsilon}} \right| \\
&= \epsilon.
\end{aligned}$$

Thus, for all $\epsilon > 0$, we can find some $N' > 0$, so that for any $N > N'$ $|A_N - 1| < \epsilon$ and thus, $\lim_{N\to\infty} A_N = 1$

.

Exercise 1.1.2.9. Practice with Infinite Series

For each of the following sequences a_n, carry out the following steps:

- Write out the first five terms of the sequence a_n. Also write out the first five terms of the sequence of partial sums A_N for the corresponding series.
- Find a formula for the sequence of partial sums $A_N = \sum_{n=0}^{N} a_n$.
- Does the infinite series $\sum_{n=0}^{\infty} a_n$ appear to converge? If so, what value does it appear to converge to?

And now, the sequences:

- The sequence defined by
$$a_n = 2n$$

- The sequence defined by
$$a_n = 2^n$$

- The sequence defined by
$$a_n = \left(\frac{2}{3}\right)^n$$

- The sequence defined by
$$a_n = \left(\frac{-1}{2}\right)^n$$

- The sequence defined by

$$a_n = (-1)^n$$

- The sequence defined by

$$\begin{aligned} a_0 &= 3 \\ a_n &= \frac{-1}{3} a_{n-1} \end{aligned}$$

- The sequence defined by

$$\begin{aligned} a_0 &= 5 \\ a_n &= a_{n-1} + 1 \end{aligned}$$

- The sequence defined by

$$a_n = \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{otherwise} \end{cases}$$

Part II

Coming Attractions

Selected Answers and Hints

Exercise 1.1.0.2. Think about an integral of the form $\int_{x=0}^{x=\infty} f(x)\,\mathrm{d}x$. How does one handle that infinity in the bounds?

Exercise 1.1.1.1. The sequence is $a_n = \frac{1}{2^{n+1}}$. Since this is a geometric sequence, the finite geometric series formula can be applied to then find the sequence of partial sums A_N.

Exercise 1.1.1.2. It ends up one-third of a meter forward from where it started.

Exercise 1.1.2.4. Yes, the series is geometric with initial term $\frac{3^5}{2^{11}}$ and common ratio 3/4. The infinite series totals to $\frac{3^5}{2^9}$.

Exercise 1.1.2.5. Think about what the value of r would be for that series. What restrictions did we have on r in the statement of the infinite geometric series formula?

Exercise 1.1.2.6. $1 + 2\frac{5}{8} + 2\left(\frac{5}{8}\right)^2 + 2\left(\frac{5}{8}\right)^3 + \cdots = 1 + 2\frac{5/8}{1-5/8} = 1 + 2\frac{5/8}{3/8} = 13/3 = 4.\overline{3}$ meters.

Exercise 1.1.2.7. The partial sums are $A_N = 2(N+1)$. The infinite series is the limit of A_N as N goes to infinity, which here is clearly again infinity. Thus, the infinite series diverges.

Exercise 1.1.2.8. The partial sums are

$$A_N = \frac{N+1}{N+3}$$

for an infinite sum of 1.

Exercise 1.1.2.9. The infinite series $\Sigma_{n=0}^{\infty} a_n$ are •Divergent •Divergent •3 •$\frac{2}{3}$ •Divergent •$\frac{9}{4}$ •Divergent •1

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