Math100C IX

C23,34,35,26

Shikun Nie, PhD student, Department of Mathematics, UBC

Topic: Newton’s Method

Description of Newton's Method

Suppose you want to find the root (or x-intercept of a function $f(x)$ where you have an algebraic expression for $f(x)$ and $f'(x)$, but where you cannot easily "solve" $f(x) = 0$.

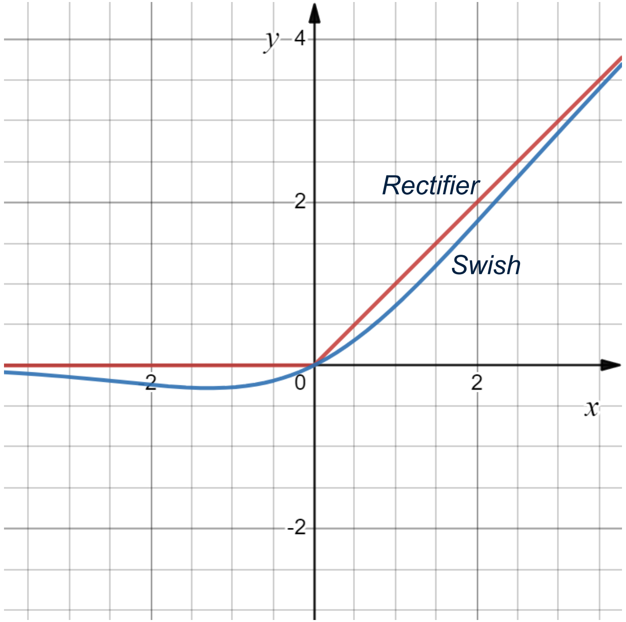
Recall from assignment 3, we have

$r(x) = \max(0, x)$

$s(x) = \frac{x}{1+e^{-x}}$

$$F(x) = r(x) - s(x) = \begin{cases} -\frac{x}{1+e^{-x}}, x < 0 \\ x - \frac{x}{1+e^{-x}}, x \geq 0 \end{cases}$$

$f(x) = \frac{d}{dx}F(x)$



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One example of this is the derivative $f(x) = F'(x)$ of the vertical distance function

$$F(x) = r(x) - s(x) = -\frac{x}{1 + e^{-x}}$$

defined on $(-\infty, 0)$, from Assignment 3. In that assignment you narrowed down the root of $f(x)$ (so the maximum of $F(x)$) to the interval $[-2, -1]$, without actually finding the root.

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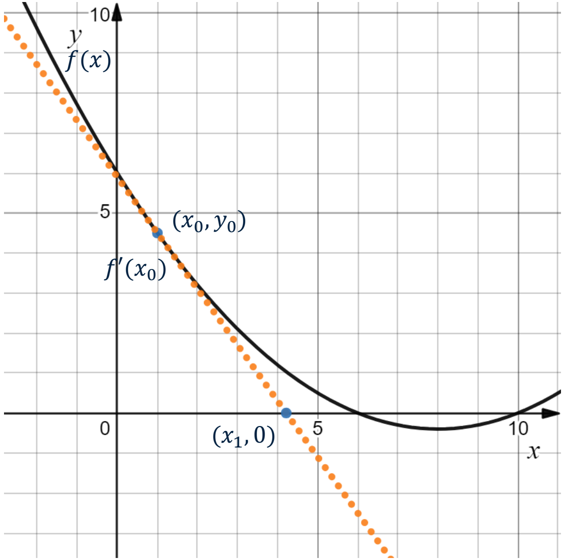
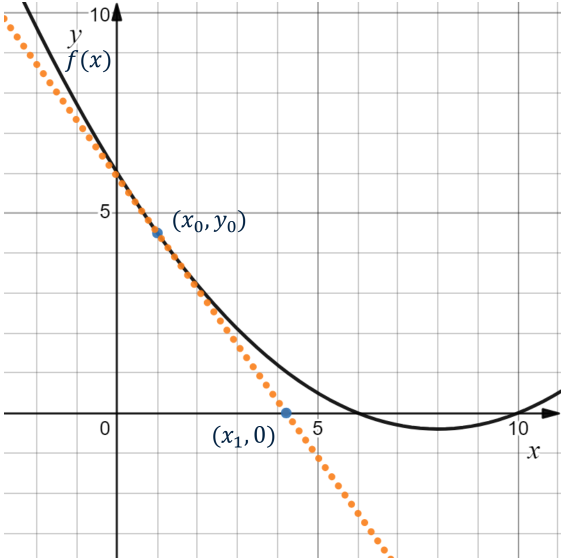
1. Use the picture below to write x_1 in terms of the other quantities labelled in the picture.



Description of Newton's Method

Suppose you want to find the root (or x-intercept of a function $f(x)$ where you have an algebraic expression for $f(x)$ and $f'(x)$, but where you cannot easily "solve" $f(x) = 0$.

2. Is it possible that x_1 is the root?



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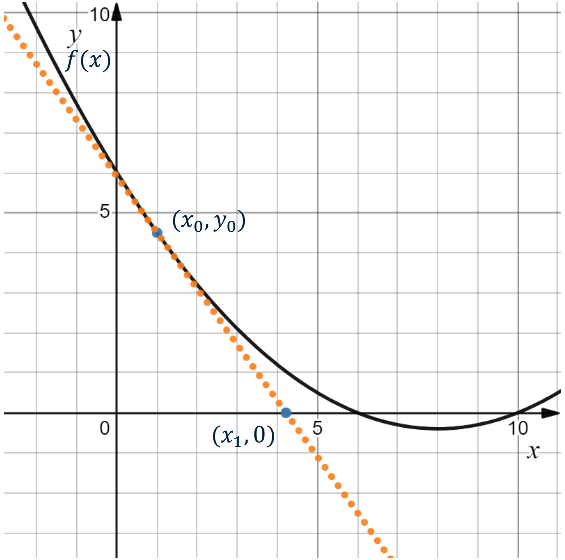
3. If x_1 is not the root, what is an idea to get closer to the root? (Hint, we found x_1 using a linear approximation. Can we use the same idea again?) Draw a picture to explain the idea.



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3. **Definition**: This method, of successively using the intercepts of linear approximations

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

to estimate the root, is known as *Newton's method*.

Applications of Newton's Method

Let $f(x)$ be the derivative of the vertical distance function from Assignment 3. Then

$$f(x) = -\frac{e^x(x + e^x + 1)}{(e^x + 1)^2}$$

$$f'(x) = \frac{(x-2)e^{-x}}{(1+e^{-x})^2} - \frac{2xe^{-2x}}{(1+e^{-x})^3}$$

1. Write an Excel program to estimate the root of $f(x)$ on $[-2, -1]$. The columns of your Excel spreadsheet should be labelled: "Guess number", "Guess (x_n)", "Function at x_n $\left(f(x_n)\right)$", "Derivative at x_n $\left(f'(x_n)\right)$", and "Intercept of linear approximation about guess (x_{n+1})".

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2. Use a few iterations and an initial guess of $x_0 = -2$. What approximation do you converge to?

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3. Try some other initial guesses. What do you observe, and how can you explain it?

Additional problems

1. CLP-1 Appendix C: Examples C.1.2-C.1.5.

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