Math322 Midterm2 Solutions[1]
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1. **Question:** Let H be the group S_5.

   (a) 2 marks Show that every 5 Sylow subgroup of H is cyclic.

   (b) 2 marks Show that H has exactly 6 5-Sylow subgroups.

   (c) 3 marks Let X denote the set of 5-Sylow subgroups of H, and let G act on X by conjugation. Show that this defines an injective homomorphism $\phi : G \to S_X \cong S_6$, and that $\phi(H) \cong H \cong S_5$.

   (d) 2 marks Let $G = S_6$. If $1 \leq i \leq 6$, let $G_i = \{\sigma \in G : \sigma(i) = i\} \in G$. Show that $G_i \cong S_5 \cong H$.

   (e) 3 marks Observe that all the subgroups $G_i, 1 \leq i \leq 5$ constructed above, as well as the subgroup $G_7 = \phi(H)$ from part (c), are isomorphic to S_5 and hence to each other. However, they are all distinct subgroups of S_6: prove that if $1 \leq i, j \leq 7$, and $G_i = G_j$ (as subgroups of S_6) then $i = j$. **Solution:**

   (a) Since $5! = 5 \times 24$, the 5-Sylow subgroup of H has an order 5. Since 5 is a prime number, there exists an element of order 5 of this 5-Sylow subgroup. Therefore it is cyclic.

   (b) Denote the number of 5-Sylow subgroups as n_5. By Sylow's theorem, $n_5|24$ and $n_5 = 1 \mod 5$. Thus, $n_5 = 1, 6$. If $n_5 = 1$, then this unique 5-Sylow subgroup must be a normal subgroup of S_5. However we know that the only proper nontrivial normal subgroup of S_5 is the alternating group A_5, which is a contradiction. Therefore, $n_5 = 6$.

   (c) To prove homomorphism, we need to show that $\phi(e) = \mathrm{id}_X$ and for $\forall g, h \in H$, $\phi(gh) = \phi(g) \circ \phi(h)$. Notice that $\forall P \in \mathrm{Syl}_5(H)$

   $$ePe^{-1} = P$$
   $$ghP(gh)^{-1} = g(hPh^{-1})g^{-1}$$

   Therefore, ϕ is a homomorphism. The kernel of ϕ is a normal subgroup of $H = S_5$. The only normal subgroups of S_5 are the trivial subgroup $\{e\}$, A_5 and S_5. From Sylow's theorem, all 5-Sylow subgroups are conjugates to each other. So $\phi(H)$ is a transitive subgroup of S_6. So $\ker(\phi) \neq H, A_5$. We conclude that $\ker(\phi) = \{e\}$ and by first isomorphism theorem $\phi(H) \cong \frac{H}{\{e\}} \cong H \cong S_5$.

   (d) G_i is a subgroup of G since $\forall g, g' \in G_i$, $gg'(i) = g(i) = i$, which shows $gg' \in G_i$. Let $\tau = (i6)$ if $i \neq 6$ or $\tau = e$ if $i = 6$ and define the following map:

   $$\begin{aligned} f :& G_i \to S_5 \\ & g \mapsto f(g) = \tau g \tau^{-1} \end{aligned}$$

   It is a homomorphism since $\forall g, g' \in G_i$,

   $$f(gg') = \tau gg' \tau^{-1} = \tau g \tau^{-1} \tau g' \tau^{-1} = f(g)f(g')$$

   and $\forall s \in S_5$, $f(\tau^{-1} s \tau) = s$. Therefore, it is an isomorphism. We have

   $$G_i \cong S_5 \cong H$$

   (e) This question was worded incorrectly on the test. It asked whether $G_i = G_j$ for $0 \leq i, j \leq 6$. The intention of the question was to include the case of $G_7 = \phi(H)$. Most students answered the question as worded, and showed that if $0 \leq i, j \leq 6$, then $G_i = G_j$ only when $i = j$. This answer was accepted. If $1 \leq i] \neq j \leq 6$, then $G_i \cap G_j$ is the set of elements fixing both i, j and there are only 24 such elements.

   However, to deal with the remaining case of $G_7 = \phi(H)$, we argue as follows. Sylow's theorem states all p-Sylow subgroups are conjugates to each other. So $\phi(H)$ is transitive as a subgroup of S_6, while G_i is not transitive for $1 \leq i \leq 6$, since it fixes the position i. Thus $G_7 = \phi(H) \neq G_i, 1 \leq i \leq 6$.

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[1] If you find any typos, please send an email to kennethnye@math.ubc.ca

2. **Question:** In this question, let $G = S_9$.

(a) 2 marks Show that the order of a 3-Sylow subgroup of G is 81.

(b) 2 marks Let $H \subset G$ denote the set of elements $\{(123)^a(456)^b(789)^c, 0 \le a, b, c \le 2\}$. Show that H is a subgroup of G of order 27.

(c) 2 marks Let $K \subset G$ denote the subgroup generated by $(147)(258)(369) \in G$. Let $HK = \{\sigma\tau : \sigma \in H, \tau \in K\}$. Show that HK has cardinality 81.

(d) 4 marks Show that HK is a subgroup of G.

(e) 4 marks Recall that the exponent of a group X is the smallest positive integer n such that $x^n = 1, \forall x \in X$. Show that the exponent of $X = HK$ is $n = 9$.

(f) 4 marks Give an example of groups $G_1 \subset G_2 \subset G_3$ where G_1 is normal in G_2 and G_2 is normal in G_3, but G_1 is not normal in G_3.

Solution:

(a) Since $9! = 3^4 \times m$ with $\gcd(3, m) = 1$, by Sylow's theorem, the 3-Sylow subgroup has an order 81.

(b) If $(123)^a(456)^b(789)^c = (123)^{a'}(456)^{b'}(789)^{c'}$, then $a = a' \mod 2$, $b = b' \mod 2$ and $c = c' \mod 2$. Each unique combination of a, b, c is an unique element and thus there are in total 27 elements in H. To prove H is a subgroup, clearly H is nonempty and notice that if $(123)^a(456)^b(789)^c$, $(123)^{a'}(456)^{b'}(789)^{c'} \in H$, $(123)^{a-a'}(456)^{b-b'}(789)^{c-c'} \in H$.

(c) The subgroup generated by $(147)(258)(369)$ is $H = \{(147)^a(258)^a(369)^a, 0 \le a \le 2\}$. It is clear that $H \cap K = \{e\}$, which implies that if $h_1, h_2 \in H$ and $k_1, k_2 \in K$, then $h_1k_1 = h_2k_2 \implies h_1 = h_2$ and $k_1 = k_2$. Thus, we have

$$|HK| = |H||K| = 81.$$

Note that you can't quote the fundamental theorems of homomorphisms here, as we don't know that HK is a group or that H, K are normal, or anything like that.

(d) To show HK is a subgroup, we need to prove $HK = KH$. In our case, it suffices to show

$$(147)(258)(369)(123)((147)(258)(369))^{-1} = (456) \in H$$

$$(147)(258)(369)(456)((147)(258)(369))^{-1} = (789) \in H$$

$$(147)(258)(369)(789)((147)(258)(369))^{-1} = (123) \in H$$

The point of this problem is that if $k = (147)(258)(369)$ is the generator of K, and $h = (123)^a(456)^b(789)^c \in H$, then $khk^{-1} = (123)^b(456)^c(789)^a \in H$, so K is in the normalizer of H.

(e) The group S_9 does not contain any elements of order 27 or 81, so the exponent is 3 or 9. The following element has order 9

$$(456)(147)(258)(369) = (158269347)$$

so the exponent must be 9. One could also do this calculation without any calculation, since HK is a 3-Sylow subgroup of S_9, and every element of order 9 is contained is some Sylow 3-subgroup (by Sylow 2). Since all 3-Sylow subgroups are conjugate and isomorphic, they must all contain elements of order 9.

(f) Four examples are given below

$$\langle(123)\rangle \triangleleft H \triangleleft HK$$

$$\{(1), (12)(34)\} \triangleleft V \triangleleft A_4$$

$$\{(1), (12)(34)\} \triangleleft A_4 \triangleleft S_4$$

$$\langle s \rangle \triangleleft \langle r^2, s \rangle \triangleleft D_8 = \langle r, s | r^4 = 1, s^2 = 1, rs = sr^{-1} \rangle$$

where the last is a minimal example.