Math101C: Integral Calculus

Separable differential equations

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Small Class VI for C15,18,22,24

Outline

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The logistic equation

Examples

1. Consider the differential equation $\frac{dy}{dt} = y(1-y)$. You may recognise this equation from the MATH 100 week 6 and week 8 small classes. This frequently used equation is called the *logistic equation*.
2. Find the general solution to the logistic equation.

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Examples

3. What is the long-time behaviour of the solution?
4. Consider the initial value problem $y(0) = y_0 > 0$. Find A in terms of y_0.
5. What value of y_0 gives us a time independent $y(t)$?
6. Sketch the solution $y(t)$ for $y_0 = \frac{1}{2}$ and $y_0 = \frac{3}{2}$.

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Takeaway

The equilibrium state $y = 1$ functions as a capacity for population. This is what the environment will allow.

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1. Problems and takeaways
   - The logistic equation
   - The logistic equation with harvesting
2. Additional Problems

with harvesting

Examples

1. Let's turn to the logistic equation with constant proportional harvesting $\frac{dy}{dt} = y(1-y) - ky$ where the constant k represents the per capita harvesting rate (fishing/hunting/gathering) at a particular t.
2. What do we expect to happen to the population if the constant k is high? What if the constant k is low?
3. Solve the differential equation.

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Examples

4. What value of k have we missed? How can we know that we missed it?
5. Solve the differential equation for the parameter value $k = 1$.
6. Once again set $y(0) = y_0 > 0$ and find the constants A and C.
7. We have 3 interesting situations depending on k. Find the long time behaviour, $\lim_{t\to\infty} y(t)$, for $k < 1$, $k = 1$, and $k > 1$.

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Takeaways

$k = 1$ is an important parameter value. The qualitative behaviour of the system is different depending on whether $k < 1$ (stable steady state $y = 1 - k$), $k = 1$ (population death as $t \to \infty$ which is asymptotically like $\frac{1}{t}$), $k > 1$ (population death as $t \to \infty$ which is asymptotically like $e^{-(k-1)t}$ so much faster than $\frac{1}{t}$).

Addtional Problems

1 Recall the logistic equation with constant (not proportional) harvesting $\frac{dy}{dt} = y(1-y) - H$. Here the H term represents harvesting at a constant rate (constant fishing/hunting/gathering).

a In MATH 100 small class week 8, you explored the phase lines associated with this equation for different values of H. Performing this analysis once more will give you intuition but is not strictly speaking necessary.

b Begin solving this differential equation. You'll want to complete the square to arrive at

$$-\int \frac{dy}{(y-\frac{1}{2})^2 + (H-\frac{1}{4})} = t + C$$

c The actual value of H will change how we do this integral. We'll have afew cases.

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Additional Problems

d. Solve for $y(t)$ in each of the three cases: $H = \frac{1}{4}, H > \frac{1}{4}, H < \frac{1}{4}$. Explain how you knew that these were the correct values to consider.

e. Some of these solutions have multiple branches so you'll need to select the correct one. A sketch will help, as will selecting a value for $y(0)$. In particular, when $H = \frac{1}{4}$ the value of $y(0)$ will change the fate of the population (consider $y(0) < \frac{1}{2}$ and $y(0) > \frac{1}{2}$).

f. Immediately taking $\lim_{t\to\infty} y(t)$ can be misleading. Instead, interpret $y(T) = 0$ as population death and don't consider $t > T$.

g. There are a few issues from a modeling perspective of this equation – the population will go negative ($y = 0$ is not an equilibrium solution). Part of this is noting that we insist on harvesting, say 10 fish, even if there is insufficient population, say only 8 fish. Writing down a model and interpreting whether it has the properties we desire is a difficult, but important, skill.

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For Additional Problems I

E. Yeager, J. Feldman, A. Rechnitzer
CLP-2 Integral Calculus Exercise
https://personal.math.ubc.ca/~CLP/CLP2/clp_2_ic_problems.pd