Mathematics 322 — Midterm 2 — 80 minutes

November 23rd, 2023

- The test consists of 12 pages and 2 questions worth a total of 30 marks.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)
- No work on this page will be marked.
- Fill in the information below before turning to the questions.

Student number								
Section								
Name	. . . . .							
Signature								

1. Let H be the group S_5.

   (a) [2 marks] Show that every 5 Sylow subgroup of H is cyclic.

(b) [2 marks] Show that H has exactly 6 5-Sylow subgroups.

(c) [3 marks] Let X denote the set of 5-Sylow subgroups of H, and let H act on X by conjugation. Show that this defines an injective homomorphism $\phi : H \to S_X \cong S_6$, and that $\phi(H) \cong H \cong S_5$.

(d) [2 marks] Let $G = S_6$. If $1 \leq i \leq 6$, let $G_i = \{\sigma \in G : \sigma(i) = i\} \subset G$. Show that $G_i \cong S_5 \cong H$.

(e) [3 marks] Prove or disprove: $\phi(H) \neq G_i$, for $\forall i$ with $1 \leq i \leq 6$.

2. In this question, let $G = S_9$.

   (a) [2 marks] Show that the order of a 3-Sylow subgroup of G is 81.

(b) [2 marks] Let $H \subset G$ denote the set of elements $\{(123)^a(456)^b(789)^c, 0 \leq a, b, c \leq 2\}$. Show that H is a subgroup of G of order 27.

(c) [2 marks] Let $K \subset G$ denote the subgroup generated by $(147)(258)(369) \in G$. Let $HK = \{\sigma\tau : \sigma \in H, \tau \in K\}$. Show that HK has cardinality 81.

(d) [4 marks] Show that HK is a subgroup of G.

(e) [4 marks] Recall that the exponent of a group X is the smallest positive integer n such that $x^n = 1, \forall x \in X$. Show that the exponent of $X = HK$ is $n = 9$.

(f) [4 marks] Give an example of groups $G_1 \subset G_2 \subset G_3$ where G_1 is normal in G_2 and G_2 is normal in G_3, but G_1 is not normal in G_3.