Math312 Section 101 Test #1 Solutions

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1a For $n = 4$, since $3^4 = 81 > 64 = 4^3$, the statemenet is true.

Assume the statement is true for $n = k$, that is to say $3^k > k^3$.

For $n = k + 1$, we have

$$
\begin{aligned}
&3^{k+1} \\
&= 3^k \times 3 \\
&> 3k^3 \\
&= k^3 + k \cdot k^2 + (k^2 - 1) \cdot k + k \\
&> k^3 + 3k^2 + 3k + 1 \\
&= (k + 1)^3
\end{aligned}
$$

Note that we have used the assumption for the first inequality and the fact $k > 3, k^2 - 1 > 3$ for the second inequality. By the principle of induction, the statement is true for $\{n \in \mathbb{N} : n > 3\}$

1b By calculating $\{j \cdot e\}, j = 1, 2, \cdots, 8$, we see that $a = 7$, $b = 19$ and $|7 \cdot e - 19| \approx 0.028 < \frac{1}{8}$.

2a $p_1p_2 \cdots p_{n-1} + 1$ is either a prime or a composite numer. In the case that it is a prime number, it is obvious that $p_n \leq p_1p_2 \cdots p_{n-1} + 1$. In the case that it is a composite number, since it is not disivisble by any $p_i, i = 1, 2, \cdots, n - 1$ and there must be a prime number that divides it, hence $p_{n-1} < p_n < p_1p_2 \cdots p_{n-1} + 1$. In conclusion, $p_n \leq p_1p_2 \cdots p_{n-1} + 1$.

2b By the recurrence relation of Fibonacci sequence, we have

$$
\begin{aligned}
f_{n+3} &= f_{n+2} + f_{n+1} \\
&= (f_{n+1} + f_n) + f_{n+1} \\
&= 2f_{n+1} + f_n
\end{aligned}
$$

Hence,

$$f_{n+3} - f_n = 2f_{n+1}$$

3a Since $a^3 - a = (a - 1)a(a + 1)$ is a product of three consecutive integers, $3|(a^3 - a)$.

3b We need to show $\forall n \in \mathbb{Z}, 3 | \left(n^3 + (n+1)^3 + (n+2)^3\right)$. We start with the base case where $n = 0$. Since $0^3 + 1^3 + 2^3 = 9$, which is divisible by 3, the statement is true. For $n = k$, assume the statement is true, that is to say $3 | \left(k^3 + (k+1)^3 + (k+2)^3\right)$. Based on this assumption, we need to prove the statement is also true for both $n = k - 1$ and $n = k + 1$. For $n = k - 1$

$$
\begin{aligned}
&(k-1)^3 + k^3 + (k+1)^3 \\
&= k^3 + (k+1)^3 + (k+2)^3 + (k-1)^3 - (k+2)^3 \\
&= k^3 + (k+1)^3 + (k+2)^3 - 3\left((k-1)^2 + (k-1)(k+2) + (k+2)^2\right)
\end{aligned}
$$

Notice that we have used the factorization trick $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ in the last equality. Hence it is divisible by 3. For $n = k + 1$, the strategy is similar and thus omitted. In conclusion, by the principle of induction, the statement is true.

4a
$$
\begin{array}{r}
54321 \\
-12345 \\
\hline
41643
\end{array}
$$

4b
$$
\begin{array}{r}
\mathrm{E\,B\,E} \\
\times \quad \mathrm{B\,C} \\
\hline
\mathrm{B\,0\,E\,8} \\
\mathrm{A\,2\,2\,A\ \ } \\
\mathrm{A\,D\,3\,8\,8}
\end{array}
$$

5a Using the following two statements: f_n is even if and only if n is divisible by 3. f_n is divisible by 5 if and only if n is divisible by 5. We conclude that f_n has unit digit 0 when n is a multiple of 15. For example $f_{15} = 610$.

5b The unit digit of a Fibonacci number depends on the sum of the unit digits of the previous two Fibonacci numbers. Thus, unit digits are periodic with a period at most 100. In fact the exact period is 60 since f_{60} ends with 0 while f_{61} ends with 1. Within the first 60 numbers, $f_{21}, f_{39}, f_{42}, f_{48}$ end with 6. So $n \in \{60k + 21, 60k + 39, 60k + 42, 60k + 48\}$ where $k \in \mathbb{N} \cup \{0\}$.