Mathematics 322 — Midterm 1 — 80 minutes

October 19th 2023

- The test consists of 11 pages and 5 questions worth a total of 30 marks.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)
- No work on this page will be marked.
- Fill in the information below before turning to the questions.

Student number								
Section								
Name								
Signature								

1. Let G be the group S_6.

   (a) $\boxed{\text{2 marks}}$ Prove or disprove: G contains an element of order 8.

   (b) $\boxed{\text{2 marks}}$ Prove or disprove: the element $(12)(14)(23456)$ has order 4.

2. Let G be a group and K a subgroup.

   (a) $\boxed{\text{3 marks}}$ Show that the sets G/K and $K\backslash G$ of left and right cosets have the same cardinality. (Note: the cardinalities may be infinite. Do not assume any sets in this question are finite. Try to construct a bijection between left cosets and right cosets.)

   (b) $\boxed{\text{3 marks}}$ Prove that if $gKg^{-1} \subset K$ for all $g \in G$, then $gKg^{-1} = K$ for all $g \in G$. Again, do not assume that G or K is finite.

3. Let G be a finite abelian group of order n, written with additive notation. Let q be any positive integer, and let $G_q = \{x \in G : qx = 0\}$ and $G^q = \{qx : x \in G\}$.

   (a) $\boxed{\text{2 marks}}$ Show that G_q and G^q are subgroups of G, for all q.

(b) $\boxed{\text{3 marks}}$ Show that $(q, n) = 1 \implies G_q = \{0\}$.

(c) $\boxed{\text{3 marks}}$ Prove that $(q, n) = 1 \implies G^q = G$.

This page has been left blank for your workings and solutions.

4. (a) $\boxed{\text{4 marks}}$ Let G be a finite group and let H, K be subgroups. Let $x \in G$ and set $HxK = \{hxk | h \in H, k \in K\}$. Show that the cardinality of HxK is equal to $\frac{\#H \cdot \#K}{\#(H \cap xKx^{-1})}$.

(b) $\boxed{\text{3 marks}}$ Let G be a group and let K, H be subgroups of G with $H \subset K$. Let π_K denote the map $G \mapsto G/K$ given by $g \mapsto gK$ and similarly define $\pi_H : G \mapsto G/H$. Show that there exists a unique map $\pi : G/H \to G/K$ such that $\pi_K = \pi \circ \pi_H$.

This page has been left blank for your workings and solutions.

5. $\boxed{\text{5 marks}}$ Let p denote a prime number and let G be a finite abelian group of order p^2. Show that G has a unique subgroup of order p if G cyclic and that G has $p + 1$ subgroups of order p if G is not cyclic. If you quote any theorem from the text or class, you should state the theorem that you use.