Math101C: Integral Calculus

Intro to probabilities

S. Nie[1]

[1]Department of Mathematics
University of British Columbia

Small Class XI for C15,18,22,24

Outline

Outline

Intro to probability

Definitions

1. The *sample space* is the set of all possible outcomes of a random experiment. An *event* can be roughly thought as a subset of the *sample space*. A *probability* is a number between 0 and 1 that we interpret as a likelihood of an *event*.

Intro to probability

Examples

- A *sample space* for a coin toss is a set $\{\text{Head}, \text{Tail}\}$; a *sample space* for rolling a six-sided die is a set {⚀, ⚁, ⚂, ⚃, ⚄, ⚅}.
- Sometimes we assign each outcome from the *sample space* to a numerical *value*, for example we assign the outcome of head as 1 while the outcome of tail as 0. The outcome from rolling a dice can be assigned numerically as $\{1, 2, 3, 4, 5, 6\}$ and let's use a *random variable* to denote these values. e.g. $X = 1, 2, ..., 6$.
- In the experiment of rolling a dice, an example *event* would be the outcomes are even numbers, {⚁, ⚃, ⚅}, which is equivalent to $\{X = 2, 4, 6\}$.

Intro to probability

Examples

- A *sample space* for a coin toss is a set $\{\text{Head}, \text{Tail}\}$; a *sample space* for rolling a six-sided die is a set $\{$⚀, ⚁, ⚂, ⚃, ⚄, ⚅$\}$.
- Sometimes we assign each outcome from the *sample space* to a numerical *value*, for example we assign the outcome of head as 1 while the outcome of tail as 0. The outcome from rolling a dice can be assigned numerically as $\{1, 2, 3, 4, 5, 6\}$ and let's use a *random variable* to denote these values. e.g. $X = 1, 2, ..., 6$.
- In the experiment of rolling a dice, an example *event* would be the outcomes are even numbers, $\{$⚁, ⚃, ⚅$\}$, which is equivalent to $\{X = 2, 4, 6\}$.

Intro to probability

Examples

- A *sample space* for a coin toss is a set $\{\text{Head}, \text{Tail}\}$; a *sample space* for rolling a six-sided die is a set {⚀, ⚁, ⚂, ⚃, ⚄, ⚅}.
- Sometimes we assign each outcome from the *sample space* to a numerical *value*, for example we assign the outcome of head as 1 while the outcome of tail as 0. The outcome from rolling a dice can be assigned numerically as $\{1, 2, 3, 4, 5, 6\}$ and let's use a *random variable* to denote these values. e.g. $X = 1, 2, ..., 6$.
- In the experiment of rolling a dice, an example *event* would be the outcomes are even numbers, {⚁, ⚃, ⚅}, which is equivalent to $\{X = 2, 4, 6\}$.

Intro to probability

Examples

2. What is the probability, event, and sample space in the statement "a fair coin when flipped has a $\frac{1}{2}$ change of turning up heads."
3. What does a probability 1 mean? What does a probability 0 mean? What does a probability of $\frac{1}{2}$ mean?
4. Suppose a particular event occurs with probability $\frac{1}{3}$. What do we expect that happen if we perform the trial 21 times? Are we guaranteed the event will occur 7 times? What if we perform the trial 105 times?

Intro to probability

Examples

2 What is the probability, event, and sample space in the statement "a fair coin when flipped has a $\frac{1}{2}$ change of turning up heads."

3 What does a probability 1 mean? What does a probability 0 mean? What does a probability of $\frac{1}{2}$ mean?

4 Suppose a particular event occurs with probability $\frac{1}{3}$. What do we expect that happen if we perform the trial 21 times? Are we guaranteed the event will occur 7 times? What if we perform the trial 105 times?

Intro to probability

Examples

2 What is the probability, event, and sample space in the statement "a fair coin when flipped has a $\frac{1}{2}$ change of turning up heads."

3 What does a probability 1 mean? What does a probability 0 mean? What does a probability of $\frac{1}{2}$ mean?

4 Suppose a particular event occurs with probability $\frac{1}{3}$. What do we expect that happen if we perform the trial 21 times? Are we guaranteed the event will occur 7 times? What if we perform the trial 105 times?

Intro to probability

Note

5 We aren't learning this in order to gamble, we want to understand large data sets. And yes, the integrals are coming, you'll do lots in the next large class.

Outline

PMFs

Examples

1. What is the probability of rolling a (fair) 6-sided die and getting the value 3?

PMFs

Examples

1. What is the probability of rolling a (fair) 6-sided die and getting the value 3?

PMFs

Definition

2 We write

$$\Pr(R = 3) = \frac{1}{6}$$

to mean the probability of the event R (roll) taking the value 3 is $\frac{1}{6}$.

Note

3 We can write

$$f(x) = \Pr(R = x) = \begin{cases} \frac{1}{6} & x \in \{1, 2, 3, 4, 5, 6\} \\ 0 & \text{otherwise} \end{cases}$$

It is not uncommon to exclude the "otherwise" line. Probabilities are assumed zero if not listed. Our new function $f(x)$ is the Probability Mass Function (PMF).

PMFs

Examples

4 Consider the following PMF defined for an **unfair** (weighted) 6-sided die. Compute the following probabilities:

x	$f(x) = \Pr(R = x)$
1	$\frac{1}{24}$
2	$\frac{2}{24}$
3	$\frac{3}{24}$
4	$\frac{5}{24}$
5	$\frac{6}{24}$
6	$\frac{7}{24}$

- $\Pr(R = 4)$
- $\Pr(R \leq 3)$
- $\Pr(R = 0)$
- $\Pr(R \leq 7)$
- $\sum_{x=1}^{6} f(x)$

PMFs

Takeaway

5 The probability mass function takes values on $[0, 1]$. The sum of the probabilities of all possible values must be 1.

Definition

6 So far we have worked with *discrete* values. These values can be listed separately. There are also continuous values. These values exist on a *continuum*, i.e. the nondenumerable set of real numbers.

PMFs

Examples

7 Which of the following values are discrete and which are continuous? What are the possible values?

- Choose an integer in $[1, 10]$.
- Choose a real number in $[1, 10]$.
- Rolling 3 6-sided dice and adding their values.
- The amount of force imparted on the dice you rolled.
- Rolling a die with each hand and choosing the hand that imparted greater force.
- Number of pets you have.
- Your exact age at noon today.
- Volume of a box.

PMFs

Examples

8 Let's see what happens when discrete values start to look continuous.

i Pick a number at random, either 0 or 1. What is $f(1) = \Pr(X = 1)$? Sketch the PMF.

ii Pick a number in $[0, 1]$ whose decimal expansion has one digit (so 0.1, or 0.2, or 0.3, etc). What is $f(0.5)$? Sketch the PMF.

iii Pick a number in $[0, 1]$ whose decimal expansion has 2 digits. Sketch the PMF.

iv Pick a number in $[0, 1]$ whose decimal expansion has 5 digits. Sketch the PMF.

v Pick a real number uniformly at random from $[0, 1]$. Sketch the PMF. What is $f(0.7)$? What is $f(1)$?

PMFs

Examples

8 Let's see what happens when discrete values start to look continuous.

   i Pick a number at random, either 0 or 1. What is $f(1) = \Pr(X = 1)$? Sketch the PMF.

   ii Pick a number in $[0, 1]$ whose decimal expansion has one digit (so 0.1, or 0.2, or 0.3, etc). What is $f(0.5)$? Sketch the PMF.

   iii Pick a number in $[0, 1]$ whose decimal expansion has 2 digits. Sketch the PMF.

   iv Pick a number in $[0, 1]$ whose decimal expansion has 5 digits. Sketch the PMF.

   v Pick a real number uniformly at random from $[0, 1]$. Sketch the PMF. What is $f(0.7)$? What is $f(1)$?

PMFs

Examples

8. Let's see what happens when discrete values start to look continuous.
   - i Pick a number at random, either 0 or 1. What is $f(1) = \Pr(X = 1)$? Sketch the PMF.
   - ii Pick a number in $[0, 1]$ whose decimal expansion has one digit (so 0.1, or 0.2, or 0.3, etc). What is $f(0.5)$? Sketch the PMF.
   - iii Pick a number in $[0, 1]$ whose decimal expansion has 2 digits. Sketch the PMF.
   - iv Pick a number in $[0, 1]$ whose decimal expansion has 5 digits. Sketch the PMF.
   - v Pick a real number uniformly at random from $[0, 1]$. Sketch the PMF. What is $f(0.7)$? What is $f(1)$?

PMFs

Examples

8 Let's see what happens when discrete values start to look continuous.

i Pick a number at random, either 0 or 1. What is $f(1) = \Pr(X = 1)$? Sketch the PMF.

ii Pick a number in $[0, 1]$ whose decimal expansion has one digit (so 0.1, or 0.2, or 0.3, etc). What is $f(0.5)$? Sketch the PMF.

iii Pick a number in $[0, 1]$ whose decimal expansion has 2 digits. Sketch the PMF.

iv Pick a number in $[0, 1]$ whose decimal expansion has 5 digits. Sketch the PMF.

v Pick a real number uniformly at random from $[0, 1]$. Sketch the PMF. What is $f(0.7)$? What is $f(1)$?

PMFs

Examples

8 Let's see what happens when discrete values start to look continuous.

i Pick a number at random, either 0 or 1. What is $f(1) = \Pr(X = 1)$? Sketch the PMF.

ii Pick a number in $[0, 1]$ whose decimal expansion has one digit (so 0.1, or 0.2, or 0.3, etc). What is $f(0.5)$? Sketch the PMF.

iii Pick a number in $[0, 1]$ whose decimal expansion has 2 digits. Sketch the PMF.

iv Pick a number in $[0, 1]$ whose decimal expansion has 5 digits. Sketch the PMF.

v Pick a real number uniformly at random from $[0, 1]$. Sketch the PMF. What is $f(0.7)$? What is $f(1)$?

PMFs

Takeaway

The probability mass function stops being useful when we have many possible values (or a continuous variable). What we really want is some kind of distribution that tells us which ranges are likely and which ranges aren't. Ideally a curve, with lots of area in some regions and not in others. Area. Area we can compute. With integrals!

Addtional Problems

1. Optimal, Integral, Likely `https://personal.math.ubc.ca/~elyse/OIL/` Practice book section 4.1: Q1, Q4, Q5
2. Optimal, Integral, Likely `https://personal.math.ubc.ca/~elyse/OIL/` Practice book section 4.2: Q1-Q6

For Additional Problems I

E. Yeager, J. Feldman, A. Rechnitzer
CLP-2 Integral Calculus Exercise
https://personal.math.ubc.ca/~CLP/CLP2/clp_2_ic_problems.pd