Math322 Midterm0 Solutions

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1(a) Suppose $x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$, then

$$g \circ f(x_1) = g(f(x_1)) = g(f(x_2)) = g \circ f(x_2)$$

By the injectivity of $g \circ f$, we have $x_1 = x_2$. Therefore, f is injective.

1(b) Let's disprove it by counter example. Let

$$f : \{1\} \to \{1, 2\}$$

$$x \mapsto x$$

and

$$g : \{1, 2\} \to \{1\}$$

$$x \mapsto 1$$

It is clear in this example, $g \circ f$ is surjective but f is not.

1(c) Let's prove it. For any a, we have

$$a \in f^{-1}(g^{-1}(U)) \Leftrightarrow f(a) \in g^{-1}(U) \Leftrightarrow g(f(a)) \in U \Leftrightarrow g \circ f(a) \in U \Leftrightarrow a \in (g \circ f)^{-1}(U)$$

Therefore, $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$.

1(d) Let's disprove it by counter-example. Take f, g from 1(b) and choose $U = \{2\}$, then $f^{-1}(U) = \emptyset$ while

$$(g \circ f)^{-1}(g(U)) = f^{-1}\left(g^{-1}\left(g\left(\{2\}\right)\right)\right) = f^{-1}\left(g^{-1}\left(\{1\}\right)\right) = f^{-1}(\{1, 2\}) = \{1\}$$

2(a) Reflexivity: for any $L_1 \in X$, choose two arbitary points $p_1, q_1 \in L_1$ such that $p_1 \neq q_1$ and let $p_2 = p_1, q_2 = q_1$, then $p_1 - q_1 = p_2 - q_2$. So $L_1 \sim L_1$. Symmetry: if $L_1 \sim L_2$, then $\exists p_1, q_1 \in L_1, p_1 \neq q_1$ and $p_2, q_2 \in L_2, p_2 \neq q_2$ such that $p_1 - q_1 = p_2 - q_2$. By commutativity of equality, we have $p_2 - q_2 = p_1 - q_1$. Therefore, $L_2 \sim L_1$. Transitivity: if $L_1 \sim L_2$,then $\exists p_1, q_1 \in L_1, p_1 \neq q_1$ and $p_2, q_2 \in L_2, p_2 \neq q_2$ such that $p_1 - q_1 = p_2 - q_2$. Similarly, if $L_2 \sim L_3$,then $\exists p_2\prime, q_2\prime \in L_2, p_2\prime \neq q_2\prime$ and $p_3, q_3 \in L_3, p_3 \neq q_3$ such that $p_2\prime - q_2\prime = p_3 - q_3$. It is obvious to see that $\lambda\left(p_2 - q_2\right) = p_2\prime - q_2\prime$ for some $\mathbb{R} \ni \lambda \neq 0$. Let's choose the position vector p_1 and direction vector $p_1 - q_1$ for line L_1, then the vector equation of L_1 is $r(t) = p_1 + t(p_1 - q_1), t \in \mathbb{R}$. Choose

$$L_1 \ni q_1^\star = r(-\lambda) = p_1 - \lambda(p_1 - q_1) = p_1 - \lambda(p_2 - q_2)$$

then

$$p_1 - q_1^\star = \lambda(p_2 - q_2) = p_2\prime - q_2\prime = p_3 - q_3$$

Therefore, $L_1 \sim L_3$. In conclusion, $\sim$ is an equivalent realtion on the set X. Lines with the same slope will be in the same equivalent class.

2(b) Define the equivalent class of X as the angle θ between the line and the x-axis, then $\theta \in [0, \pi)$. Let f be the following function

$$f : [0, \pi) \to [0, \infty)$$
$$x \mapsto \tan\left(\frac{\theta}{2}\right)$$

3(a) Given a propositional function $P(n)$ defined for integers n, and a fixed integer a. Then, if these two conditions are true: (1) $P(a)$ is true; (2) if $P(k)$ is true for some integer $k \geq a$, then $P(k+1)$ is also true. Then $P(n)$ is true for all integers $n \geq a$.

3(b) For the base case when $n = 3$ we see $\frac{16}{5} = \sqrt{\frac{256}{25}} > \sqrt{10}$. Let's assume for $n = k$, the statement is true, that is to say

$$\frac{2 \times 4 \times \cdots \times 2k}{1 \times 3 \times \cdots \times (2k-1)} \geq \sqrt{3k+1}$$

Then, in the case $n = k + 1$,

$$\frac{2 \times 4 \times \cdots \times 2(k+1)}{1 \times 3 \times \cdots \times (2(k+1)-1)} \geq \frac{2k+2}{2k+1}\sqrt{3k+1} = \sqrt{\frac{12k^3 + 28k + 20k + 4}{4k^2 + 4k + 1}}$$

$$= \sqrt{3k + 4 + \frac{k}{4k^2 + 4k + 1}} \geq \sqrt{3k+4}$$

Hence, by the principle of mathematical induction, the statement is true for all $n \geq 3$.

4(a) No such a, b exist since for $\forall a, b \in \mathbb{R}$, let $y_0 = -\frac{a^2}{4} + b - 1$, there is no $x \in \mathbb{R}$ such that $f(x) = x^2 + ax + b = y_0$.

4(b) No such a, b exist since for $\forall a, b \in \mathbb{R}$, $f(-\frac{a}{2} + 1) = f(-\frac{a}{2} - 1)$, however $-\frac{a}{2} + 1 \neq -\frac{a}{2} - 1$.

4(c) $U = \emptyset$. For $\forall s \in \mathbb{R}$, either $-\frac{a^2}{4} + b \geq s$ or $-\frac{a^2}{4} + b < s$. For the first case, $\forall x \in \mathbb{R}, x^2 + ax + b \geq s$ while for the later case, choose $\mathbb{R} \ni x_0 = \frac{-a+\sqrt{a^2-4(b-s)}}{2}$, then $x^2 + ax + b = s$.