Math101C: Integral Calculus

Power series and radius of convergence

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Small Class IX for C15,18,22,24

Outline

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Power series

Examples

1. Recall the geometric series $1 + r + r^2 + r^3 + \ldots = \sum_{n=0}^{\infty} r^n$.
   For what values of r does this converge?
   When it converges what does the series equal?
2. If $r = \frac{1}{2}$ what is the value of the series?
   If $r = -\frac{1}{3}$ what is the value of the series?
   What's your favourite number between -1 and 1?
   What value does that give for the series?
   This is starting to sound like something.

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Power series

3 Takeaway

We have more than we thought here, we have a series representation for the *function* $\frac{1}{1-x}$. That is if $|x| < 1$

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Power series

Examples

4 Consider a new function define via series

$$g(x) = \sum_{n=0}^{\infty} (x-3)^n$$

For what values of x does this series converge?

Assuming the series converges, what's the familiar form of this function are we dealing with here?

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Power series

5 Takeaway

A *power series* is a series of the form

$$A_0 + A_1(x-c) + A_2(x-c)^2 + A_3(x-c)^3 + \ldots = \sum_{n=0}^{\infty} A_n(x-c)^n$$

The constant c is the *center* of this series. The A_n's are the *coefficients*.

Outline

Radius of convergence

Examples

1. For which values of x does the series

$$x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots = \sum_{n=1}^{\infty} \frac{1}{n}x^n$$

   converge? Use the ratio test.

2. **Note**: the ratio test says nothing about what happens at the end point. Analysis at the endpoint can be delicate and anything can happen (absolute convergence, conditional convergence, divergence). In the above example we get our familiar harmonic series and alternating harmonic series at the endpoints: the series diverges for $x = 1$ and converges (conditionally) when $x = -1$.

Radius of convergence

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Radius of convergence

Examples

3 Apply the ratio test to the general power series $\sum_{n=0}^{\infty} A_n(x-c)^n$. For what values of x does this converge? Suppose that $A = \lim_{n\to\infty} \left|\frac{A_{n+1}}{A_n}\right|$ exists.

Definition (4)

When the limit $A = \lim_{n\to\infty} \left|\frac{A_{n+1}}{A_n}\right|$ exists, the quantity

$$R = \frac{1}{A} = \left(\lim_{n\to\infty} \left|\frac{A_{n+1}}{A_n}\right|\right)^{-1}$$

is called the *radius of convergence*.

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Examples

5 What happens when $A = 0$? When $A = \infty$?

6 Takeaway

- When $A = 0$ (or $R = \infty$) we say that the series has an infinite radius of convergence (converges for all x).
- When $A = \infty$ (or $R = 0$) we say that the series has a radius of convergence of zero (converges only when $x = c$).

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Radius of convergence

Examples

7 (Example 3.5.7) Determine the radius of convergence for the power series $\sum_{n\to\infty} A_n x^n$ where A_n is defined as $A_n = 1$ for even n and $A_n = 2$ for odd n, so

$$\sum_{n=0}^{\infty} A_n x^n = 1 + 2x + x^2 + 2x^3 + \ldots$$

What happens when you try to apply the ratio test? What other tests can you use for particular values of x?

Radius of convergence

8 Takeaway

We won't prove ths theorem but:

Theorem (3.5.9)

Given a power series $\sum_{n=0}^{\infty} A_n(x-c)^n$, one of the following cases must hold.

- i *The power series converges for all x (infinite radius of convergence).*
- ii *There is a radius $0 < R < \infty$ such that the series converges for $|x-c| < R$ and diverges for $|x-c| > R$ (R is the radius of convergence).*
- iii *The series converges for $x = c$ but diverges for all $x \neq c$ (Radius of convergence is 0).*

Addtional Problems

1 CLP-2 Problem book section 3.5: Q1-Q11, Q13, Q14, Q15, Q18, Q21, Q22

For Additional Problems I

E. Yeager, J. Feldman, A. Rechnitzer
CLP-2 Integral Calculus Exercise
https://personal.math.ubc.ca/~CLP/CLP2/clp_2_ic_problems.pd