Mathematics 322 — Midterm 0 — 80 minutes

September 14th 2023

- The test consists of 10 pages and 4 questions worth a total of 35 marks.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)
- No work on this page will be marked.
- Fill in the information below before turning to the questions.

Student number								
Section								
Name								
Signature								

1. Let $f : A \to B$ and $g : B \to C$ be functions.

   (a) $\boxed{\text{2 marks}}$ Prove or disprove: If $g \circ f$ is injective, then f is injective.

   (b) $\boxed{\text{2 marks}}$ Prove or disprove: If $g \circ f$ is surjective, then f is surjective.

(c) $\boxed{\text{2 marks}}$ Prove or disprove: $\forall U \subseteq C$, $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$. Note that the superscripts -1 in this question refer to preimages, not inverse functions. The quantities on either side of the equality are sets.

(d) $\boxed{\text{2 marks}}$ Prove or disprove: $\forall U \subseteq B$, $f^{-1}(U) = (g \circ f)^{-1}(g(U))$. Again, the superscripts -1 in this question refer to preimages, not inverse functions.

2. Let X denote the set of lines in $\mathbf{R}^2$. Define a binary relation on X by saying that if $L_1, L_2 \in X$, then $L_1 \sim L_2 \iff \exists p_1, q_1, p_1 \neq q_1 \in L_1, \exists p_2, q_2 \in L_2, p_2 \neq q_2$, such that $p_1 - q_1 = p_2 - q_2$. Note that points on the line are just vectors $(a, b) \in \mathbf{R}^2$.

   (a) $\boxed{\text{3 marks}}$ Show that $\sim$ is an equivalence relation. Describe in words what it means for two lines to be equivalent.

(b) $\boxed{\text{5 marks}}$ Construct a bijection between the set of equivalence classes of X under $\sim$ to the set $[0, \infty) \subset \mathbf{R}$.

This page has been left blank for your workings and solutions.

3. (a) $\boxed{\text{2 marks}}$ State the principle of mathematical induction

(b) $\boxed{\text{5 marks}}$ Use induction to prove that

$$n \in \mathbf{Z}, n \geq 3 \implies \frac{2 \cdot 4 \cdot 6 \cdot (2n)}{1 \cdot 3 \cdot 5 \cdot (2n-1)} \geq \sqrt{3n+1}.$$

Hint: You may have to carefully expand a couple of cubic polynomials.

This page has been left blank for your workings and solutions.

4. (a) $\boxed{\text{4 marks}}$ Find all values $a, b \in \mathbf{R}$ such that the function $\mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^2 + ax + b$ is surjective, or prove that no such a, b exist.

(b) $\boxed{\text{4 marks}}$ Find all values $a, b \in \mathbf{R}$ such that the function $\mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^2 + ax + b$ is injective, or prove that no such a, b exist.

(c) $\boxed{\text{4 marks}}$ Find the set $U \subset \mathbf{R}$ defined by $U = \{s \in \mathbf{R} | \exists (a,b) \in \mathbf{R}^2 \text{ such that } \forall x \in \mathbf{R}, x^2 + ax + b < s, \}$.