Math312 Test #2 Sols

July 27, 2023

1a Suppose $d = (a, b)$, then $d|a$ and $d|b$. So $d|(a + cb)$. Let's denote $d' = (a + cb, b)$, clearly $d \leq d'$. Now $d'|(a + cb)$ and $d'|b$, then $d'|a$ since $a = (a + cb) - cb$. So $d' \leq d$ as well. In conclusion, $d = d'$.

1b By applying results from 1(a), we have $(n+1, n^2 - n+1) = (n+1, -2n+1) = (n + 1, 3)$. Since the only divisors of 3 is 1 or 3, we conclude that $(n + 1, n^2 - n + 1) = 1$ or 3.

2a

$$1001 = 289 \times 3 + 134$$

$$289 = 134 \times 2 + 21$$

$$134 = 21 \times 6 + 8$$

$$21 = 8 \times 2 + 5$$

$$8 = 5 \times 1 + 3$$

$$5 = 3 \times 1 + 2$$

$$3 = 2 \times 1 + 1$$

$$2 = 2 \times 1$$

So $(289, 1001) = 1$

2b

$$1 = 3 - 2 \times 1$$

$$= 3 - (5 - 3 \times 1) \times 1$$

$$= 3 \times 2 - 5 \times 1$$

$$= (8 - 5 \times 1) \times 2 - 5 \times 1$$

$$= -5 \times 3 + 8 \times 2$$

$$= -(21 - 8 \times 2) \times 3 + 8 \times 2$$

$$= 8 \times 8 - 21 \times 3$$

$$
\begin{aligned}
&= (134 - 21 \times 6) \times 8 - 21 \times 3 \\
&= 134 \times 8 - 21 \times 51 \\
&= 134 \times 8 - (289 - 134 \times 2) \times 51 \\
&= 134 \times 110 - 289 \times 51 \\
&= (1001 - 289 \times 3) \times 110 - 289 \times 51 \\
&= 1001 \times 110 - 289 \times 381
\end{aligned}
$$

3a Start by $\lceil 99400891 \rceil = 9970$.

$$99400891 - 9970^2 = -9$$

Therefore, we have

$$99400891 = 9970^2 - 3^2 = (9970 + 3)(9970 - 3) = 9973 \times 9967$$

3b Start by $\lceil 6411023 \rceil = 2532$

$$6411023 - 2532^2 = -1$$

Threrefore, we have

$$6411023 = 2532^2 - 1^2 = (2532 + 1)(2532 - 1) = 2533 \times 2531$$

4a By extended Euclidean algorithm, we have $11 \times 30 - 7 \times 47 = 1$, thus the solution is

$$x = (-11) \times 11 + 47t, y = (-11) \times (-7) - 30t$$

4b By extended Euclidean algorithm, we have $4 \times 25 - 1 \times 95 = 5$, thus the solution is

$$x = 194 \times 4 + 19t, y = 194 \times (-1) - 5t$$

5 We prove $\sqrt{35}$ is irrational by contradiction. Suppose $\sqrt{35}$ is rational, then there exist positive integers p, q such that $(p, q) = 1$ and $\sqrt{35} = \frac{p}{q}$. By squaring both sides of the equations, we have

$$35 = \frac{p^2}{q^2}$$

$$35q^2 = p^2$$

Since $5|35q^2$, then $5|p^2$. Use the following fact: let p be a prime and a be a positive integer then $p|a^2 \Leftrightarrow p|a \Leftrightarrow p^2|a^2$, we see that $5|p^2 \Leftrightarrow 5|p \Leftrightarrow 25|p^2$. Therefore, $25|35q^2$. It must be that $5|q^2 \Leftrightarrow 5|q$. Then $(p, q) \geq 5$, which is a contradiction.