Math101C: Integral Calculus

Taylor Series

S. Nie[1]

[1]Department of Mathematics
University of British Columbia

Small Class X for C15,18,22,24

Outline

1. Problems and takeaways
   - Taylor series for $\arctan x$
   - Computing quantities with Taylor series
2. Additional Problems

Outline

1. Problems and takeaways
   - Taylor series for arctan x
   - Computing quantities with Taylor series
2. Additional Problems

Taylor series for arctan x

Examples

1. We're going to find the Taylor series for $\arctan x$. We need a few pieces:
   - i What's the derivative of $\arctan x$?
   - ii What's the Taylor series for $\frac{1}{1-x}$ (and radius of convergence)?
2. Use the above facts to find a Taylor series for $\arctan x$.
3. Our manipulations have shown that this series converges for $|x| < 1$ and diverges for $|x| > 1$. What happens when $x = \pm 1$?

Taylor series for arctan x

Examples

1. We're going to find the Taylor series for $\arctan x$. We need a few pieces:
   i. What's the derivative of $\arctan x$?
   ii. What's the Taylor series for $\frac{1}{1-x}$ (and radius of convergence)?
2. Use the above facts to find a Taylor series for $\arctan x$.
3. Our manipulations have shown that this series converges for $|x| < 1$ and diverges for $|x| > 1$. What happens when $x = \pm 1$?

Taylor series for arctan x

Examples

1. We're going to find the Taylor series for $\arctan x$. We need a few pieces:
   i. What's the derivative of $\arctan x$?
   ii. What's the Taylor series for $\frac{1}{1-x}$ (and radius of convergence)?
2. Use the above facts to find a Taylor series for $\arctan x$.
3. Our manipulations have shown that this series converges for $|x| < 1$ and diverges for $|x| > 1$. What happens when $x = \pm 1$?

Taylor series for arctan x

Examples

1. We're going to find the Taylor series for $\arctan x$. We need a few pieces:
   i. What's the derivative of $\arctan x$?
   ii. What's the Taylor series for $\frac{1}{1-x}$ (and radius of convergence)?
2. Use the above facts to find a Taylor series for $\arctan x$.
3. Our manipulations have shown that this series converges for $|x| < 1$ and diverges for $|x| > 1$. What happens when $x = \pm 1$?

Taylor series for arctan x

Examples

1. We're going to find the Taylor series for $\arctan x$. We need a few pieces:
   i. What's the derivative of $\arctan x$?
   ii. What's the Taylor series for $\frac{1}{1-x}$ (and radius of convergence)?
2. Use the above facts to find a Taylor series for $\arctan x$.
3. Our manipulations have shown that this series converges for $|x| < 1$ and diverges for $|x| > 1$. What happens when $x = \pm 1$?

Taylor series for arctan x

Takeaway

4. All together we have

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots$$

for $-1 \leq x \leq 1$.

5. With the Taylor series for arctan in hand, we can understand this mysterious function in terms of polynomials. We love polynomials!

Taylor series for arctan x

Takeaway

4. All together we have

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots$$

   for $-1 \leq x \leq 1$.

5. With the Taylor series for arctan in hand, we can understand this mysterious function in terms of polynomials. We love polynomials!

Outline

1. Problems and takeaways
   - Taylor series for arctan x
   - Computing quantities with Taylor series
2. Additional Problems

Computing quantities with Taylor series

Examples

1. What is $\tan \frac{\pi}{4}$?
2. Use the fact that $\pi = 4 \arctan 1$ to write down an infinite sum representing π. Evaluate the first 5 terms in the series. How close do we get to the true value?
3. In large class, we saw a formula to estimate the error in a Taylor polynomial approximation. For alternating series, however, it's even easier.

    Is the sixth term in the series positive or negative?

    Is it possible for π to be larger than our five term approximation? Why/why not?

    Is it possible for π to be smaller than the six term approximation? Why/why not?

Computing quantities with Taylor series

Examples

1. What is $\tan \frac{\pi}{4}$?
2. Use the fact that $\pi = 4 \arctan 1$ to write down an infinite sum representing π. Evaluate the first 5 terms in the series. How close do we get to the true value?
3. In large class, we saw a formula to estimate the error in a Taylor polynomial approximation. For alternating series, however, it's even easier.
   - Is the sixth term in the series positive or negative?
   - Is it possible for π to be larger than our five term approximation? Why/why not?
   - Is it possible for π to be smaller than the six term approximation? Why/why not?

Computing quantities with Taylor series

Examples

1. What is $\tan \frac{\pi}{4}$?
2. Use the fact that $\pi = 4 \arctan 1$ to write down an infinite sum representing π. Evaluate the first 5 terms in the series. How close do we get to the true value?
3. In large class, we saw a formula to estimate the error in a Taylor polynomial approximation. For alternating series, however, it's even easier.

   Is the sixth term in the series positive or negative?

   Is it possible for π to be larger than our five term approximation? Why/why not?

   Is it possible for π to be smaller than the six term approximation? Why/why not?

Computing quantities with Taylor series

Examples

1. What is $\tan \frac{\pi}{4}$?
2. Use the fact that $\pi = 4 \arctan 1$ to write down an infinite sum representing π. Evaluate the first 5 terms in the series. How close do we get to the true value?
3. In large class, we saw a formula to estimate the error in a Taylor polynomial approximation. For alternating series, however, it's even easier.

   Is the sixth term in the series positive or negative?

   Is it possible for π to be larger than our five term approximation? Why/why not?

   Is it possible for π to be smaller than the six term approximation? Why/why not?

Computing quantities with Taylor series

Examples

1. What is $\tan \frac{\pi}{4}$?
2. Use the fact that $\pi = 4 \arctan 1$ to write down an infinite sum representing π. Evaluate the first 5 terms in the series. How close do we get to the true value?
3. In large class, we saw a formula to estimate the error in a Taylor polynomial approximation. For alternating series, however, it's even easier.

   Is the sixth term in the series positive or negative?

   Is it possible for π to be larger than our five term approximation? Why/why not?

   Is it possible for π to be smaller than the six term approximation? Why/why not?

Computing quantities with Taylor series

Takeaway

4 For an alternating series $S = \sum_{n=0}^{\infty}(-1)^n A_n$ where $\{A_n\}_{n=1}^{\infty}$ is monotonically decreasing and partial sum $S_N = \sum_{n=0}^{N}(-1)^n A_n$, the error $E_n = S - S_N$, is between 0 and the first dropped term. Put another way, S is between S_N and S_{N+1}.

Computing quantities with Taylor series

Examples

5. Our previous approximation wasn't very good. In fact, we'd need around 200 terms to get 2 decimal places. Let's try another value. What is $\tan \frac{\pi}{6}$?

6. Use the fact that $\pi = 6 \arctan\left(\frac{1}{\sqrt{3}}\right)$ to write down an infinite sum representing π. Evaluate the first 5 terms in the series. How close do we get to the true value?

7. Find a new bound on the decimal expansion for π.

Computing quantities with Taylor series

Examples

5 Our previous approximation wasn't very good. In fact, we'd need around 200 terms to get 2 decimal places. Let's try another value. What is $\tan \frac{\pi}{6}$?

6 Use the fact that $\pi = 6 \arctan\left(\frac{1}{\sqrt{3}}\right)$ to write down an infinite sum representing π. Evaluate the first 5 terms in the series. How close do we get to the true value?

7 Find a new bound on the decimal expansion for π.

Computing quantities with Taylor series

Examples

5. Our previous approximation wasn't very good. In fact, we'd need around 200 terms to get 2 decimal places. Let's try another value. What is $\tan \frac{\pi}{6}$?
6. Use the fact that $\pi = 6 \arctan\left(\frac{1}{\sqrt{3}}\right)$ to write down an infinite sum representing π. Evaluate the first 5 terms in the series. How close do we get to the true value?
7. Find a new bound on the decimal expansion for π.

Computing quantities with Taylor series

Takeaway

8 We can use Taylor series to represent and approximate particular function values (sometimes values we really care about).

Addtional Problems

1. Use a spreadsheet to implement the two approximation schemes above to approximate π using 200 terms in the series. How many decimal places do you get in each case?
2. Find a special angle for which the tangent function is known exactly that is smaller than $\frac{\pi}{6}$. Will approximating π using this fact produce a faster or slower converging series?
3. CLP-2 Problem book section 3.5: Q19, Q26, Q27
4. CLP-2 Problem book section 3.6: Q4, Q15-Q17, Q25, Q26, Q27, Q28, Q29, Q30 3

For Additional Problems I

E. Yeager, J. Feldman, A. Rechnitzer
CLP-2 Integral Calculus Exercise
https://personal.math.ubc.ca/~CLP/CLP2/clp_2_ic_problems.pd