Computational Uncertainty Quantification for parametrized Magnetic Resonance Electrical Impedance Tomography

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Abstract: We apply Uncertainty Quantification to characterize the noise in the solution to the inverse problem in Magnetic Resonance Electrical Impedance Tomography. Using Bayesian inversion and MCMC methods we infer the parameters of a known inclusion embedded in a background medium from measurements of its emitted magnetic field.

1 MREIT

In MREIT, an MRI machine is used to measure the interior magnetic flux density induced by the injection current from the EIT technique. Having access to interior magnetic field data instead of boundary current-voltage measurements overcomes the severe ill-posedness of EIT [1].

2 The forward problem

Let Ω be an isotropic volume conductor with conductivity σ. The forward problem is the map from σ to the interior magnetic field $\mathbf{B}$. A voltage is applied to the boundary $\partial\Omega$. The electric potential u is governed by the Poisson equation

$$\begin{aligned} -\nabla \cdot \sigma \nabla u &= 0 \quad \text{in } \Omega, \\ u(x, y, z) &= x \quad \text{on } \partial\Omega. \end{aligned}$$

In a practical setup, an alternating 1 mA current with a frequency of 6 Hz may be used as injection current [2]. The current field can be computed via the vector form of Ohm's law $\mathcal{F}(\sigma) = \mathbf{J} = -\sigma\nabla u$. The current field $\mathbf{J}$ induces a magnetic field $\mathbf{B}$ which is given by the Biot-Savart law [3]

$$\mathcal{G}(\mathcal{F}(\sigma)) = \mathbf{B} = \frac{1}{4\pi} \int_\Omega \mathbf{J}(x') \times \frac{x - x'}{|x - x'|^3} \mathrm{d}x'.$$

The MRI machine measures only the z-component of $\mathbf{B}$.

3 The inverse problem

We seek to estimate σ from noisy magnetic field measurements assuming i.i.d. Gaussian noise

$$B_z = \mathcal{G}(\mathcal{F}(\sigma)) + E, \quad E \sim N(0, \epsilon^2 I).$$

We use an $8 \times 8 \times 8$ grid and use FEniCS for the forward computations [4]. We add 10% relative noise to the discrete magnetic field data. We define σ as an ellipsoidal inclusion rotated of constant conductivity in the xy-plane, embedded in a 1 cm^3 cube with a constant conductivity of 1 S/cm. The inclusion is defined by the following eight parameters

Semi-axes (r_x, r_y, r_z)	$(0.15, 0.3, 0.2)$ cm
Center coordinates (c_x, c_y, c_z)	$(0.43, 0.48, 0.41)$ cm
Conductivity of inclusion κ	2 S/cm
Rotation angle δ	$45° \approx 0.785$ rad

We pose the inverse problem as one of Bayesian inference. The idea is to infer the parameters given magnetic field data by using the *posterior* probability density via Bayes' rule

$$\pi_{\text{post}}(r, c, \kappa, \delta | B_z) \propto \pi_{\text{like}}(B_z | r, c, \kappa, \delta)\pi_{\text{pr}}(r, c, \kappa, \delta),$$

where π_{like} quantifies the data fidelity and π_{pr} quantifies the prior knowledge. The posterior is the solution to the Bayesian inverse problem. By having a probability density we can reason about the uncertainty. The challenge is to explore the posterior by estimating its moments and statistics.

4 Numerical results

We use a Metropolis-within-Gibbs method to sample the posterior [5]. We draw 5000 samples $(r_k, c_k, \kappa_k, \delta_k)$ and discard the first 1000 as burn-in. The CPU time was 35.1 hours and 136 MB of memory was used on a High Performance Computing cluster.

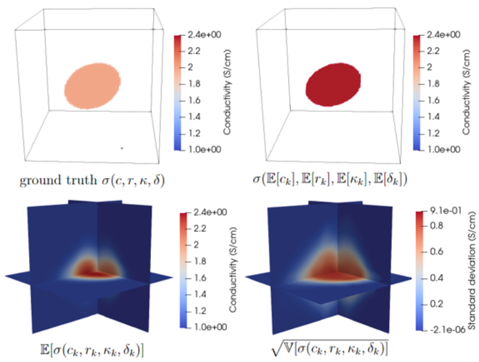

Figure 1: Top: Ground truth conductivity and mapped sample means. Bottom: Mean and standard deviation of mapped samples.

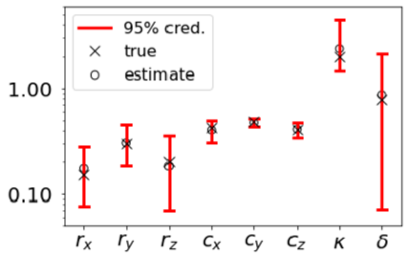

Figure 2: Parameter UQ for σ with 95% credibility intervals.

5 Conclusion

We pose a Bayesian version of the conductivity reconstruction problem in MREIT and solve it using MCMC methods. Our findings indicate that uncertainty is concentrated around the edges of the inclusion. The parameter-wise UQ reveals that there is less uncertainty involved with estimating the location of an inclusion compared with its shape. A finer meshing of the domain reduces discretization errors and uncertainty but is impractical due to time concerns.

References

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