In [349…

```
import numpy as np
import math
import scipy as scp
import scipy.stats as ss
import matplotlib.pyplot as plt
import pandas as pd
```

Q1

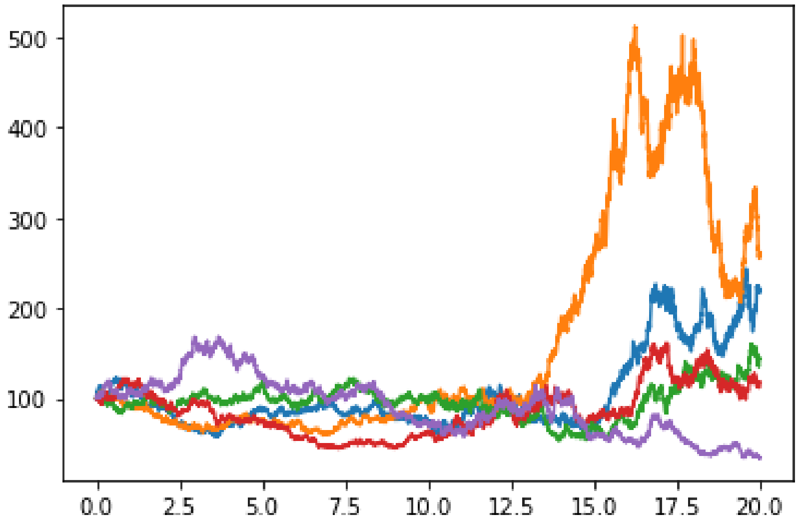
1)

In [333…

```
def Q1_1(Asset_Value=100, Debt_Value=80, T=20, r=0.031, Volatility_1=(


    np.random.seed(0)

    dt = T/(N)           # time interval
    df = np.exp(-r * dt)  # discount factor per time time interval

    X0 = np.zeros((paths,1))
    increments_1 = ss.norm.rvs(loc=(r - Volatility_1**2/2)*dt, scale=n
    increments_2 = ss.norm.rvs(loc=(r - Volatility_2**2/2)*dt, scale=n
    X = np.concatenate((X0,increments_1, increments_2), axis=1).cumsur
    A = Asset_Value * np.exp(X)

    dt_array = np.concatenate((0, np.array(N*[dt])), axis=None).cumsur

    for i in np.arange(np.shape(A)[0]):
        plt.plot(dt_array, A[i])

    plt.show()

    return A[:,-1]
    # return("Equity", A_T[-1] - Debt_Value)
```

In [334…

```
Q1_1()
```



Out[334…

```
array([220.87401444, 258.62456041, 144.73840758, 118.58170687,
        34.31593205])
```

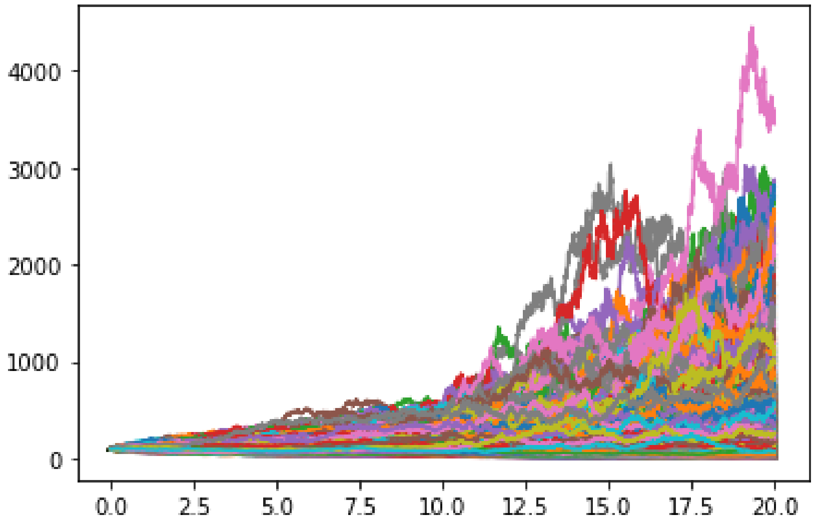
2) & 3)

In [335…

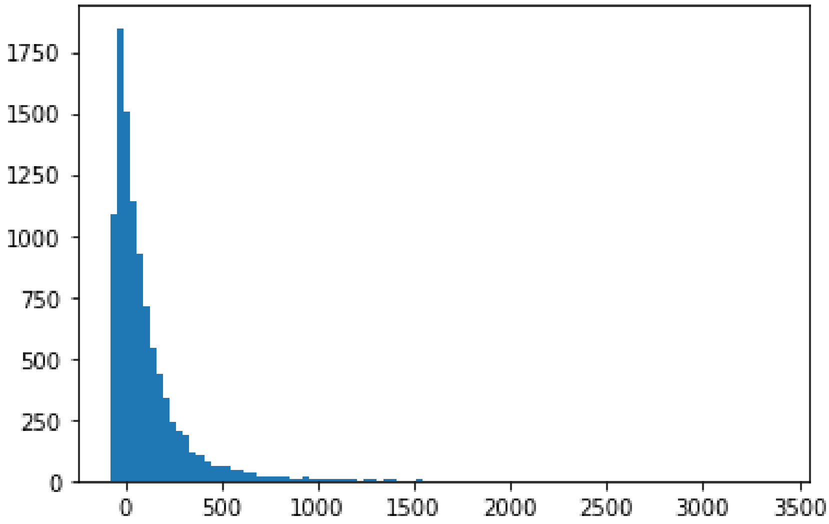
```
def Q1_23(Asset_Value=100, Debt_Value=80, T=20, r=0.031, Volatility_1=


    np.random.seed(0)

    dt  = T/(N)             # time interval
    df  = np.exp(-r * dt)   # discount factor per time time interval

    X0  = np.zeros((paths,1))
    increments_1 = ss.norm.rvs(loc=(r - Volatility_1**2/2)*dt, scale=r
    increments_2 = ss.norm.rvs(loc=(r - Volatility_2**2/2)*dt, scale=r
    X = np.concatenate((X0,increments_1, increments_2), axis=1).cumsur
    A = Asset_Value * np.exp(X)

    dt_array = np.concatenate((0, np.array(N*[dt])), axis=None).cumsur

    residual  = A[:,-1] - Debt_Value
    Default_Probability = np.sum(A[:,-1] < Debt_Value)/paths

    for i in np.arange(np.shape(A)[0]):
        plt.plot(dt_array, A[i])

    plt.show()

    plt.hist(residual,bins = 100)



    return("Default Probability = ", Default_Probability)
```

In [336…

```
Q1_23()
```




```
Default Probability =  0.3263
```



In [ ]:

```

```

4)

In [338…

```
A = 100
D = 80
r = 0.031
sig = 0.20
T = 20



prob = ss.norm.cdf(-(np.log(A/D) + (r - 1/2 * sig**2)*T)/(sig*np.sqrt(

print("Analytical Default Probability = ", prob)
```

```
Analytical Default Probability =  0.31014141195633205
```

For the analytical default probability under the risk neutral measure, we got default probability = 31.01%

For the simulation result to the analytical default probability for model 1, we got default probability = 32.63%

In [ ]:

5)

In [331…

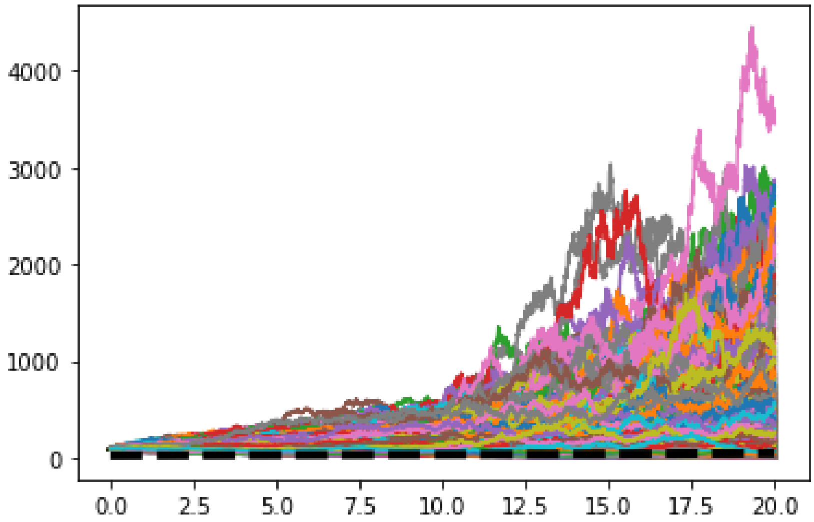
```
def Q1_5(Asset_Value=100, Debt_Value=80, T=20, r=0.031, Volatility_1=(


    np.random.seed(0)

    dt = T/(N)          # time interval
    df = np.exp(-r * dt)  # discount factor per time time interval

    X0 = np.zeros((paths,1))
    increments_1 = ss.norm.rvs(loc=(r - Volatility_1**2/2)*dt, scale=r
    increments_2 = ss.norm.rvs(loc=(r - Volatility_2**2/2)*dt, scale=r
    X = np.concatenate((X0,increments_1, increments_2), axis=1).cumsur
    A = Asset_Value * np.exp(X)


    tem = np.concatenate((0, np.array(N*[-r * dt])), axis=None).cumsur
    barrier_array = barrier*np.exp(tem)[::-1]

    dt_array = np.concatenate((0, np.array(N*[dt])), axis=None).cumsur

    knock_out = np.zeros(paths)

    for i in np.arange(np.shape(A)[0]):
        for j in np.arange(np.shape(A)[1]):
            if A[i,j] < barrier_array[j]:
                knock_out[i] = 1

    Equity_Value = np.zeros(paths)

    for i in np.arange(paths):
        if knock_out[i] == 1:
            Equity_Value[i] = 0

        elif A[i,-1] < Debt_Value:
            Equity_Value[i] = 0

        else:
            Equity_Value[i] = A[i,-1] - Debt_Value


    residual = A[:,-1] - Debt_Value

    Default_Probability = np.sum(Equity_Value == 0)/paths

    for i in np.arange(np.shape(A)[0]):
        plt.plot(dt_array, A[i])

    plt.plot(dt_array, barrier_array, alpha=1, linewidth=4, color='k',
    plt.show()

    plt.hist(residual,bins = 100)
```

```
    return("Default Probability = ", Default_Probability)
```

Set the barrier at maturity to 50, and took into account the discount factor to obtain the default threshold before maturity

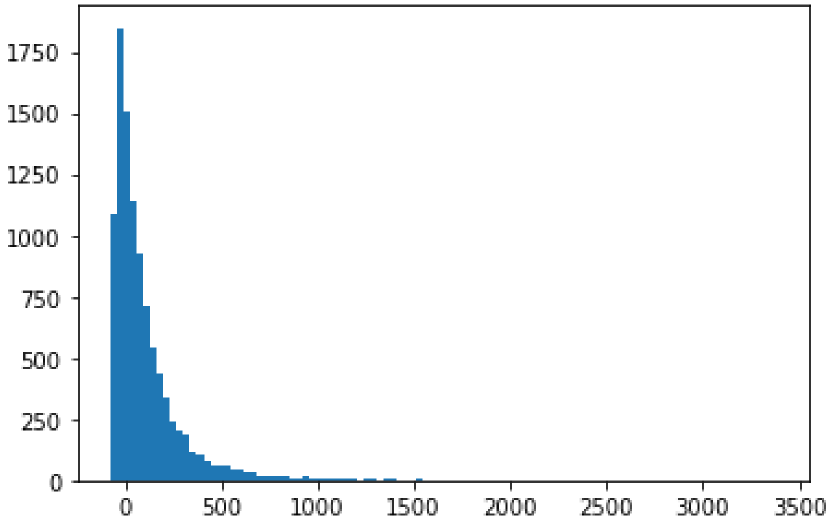
In [332…

```
Q1_5()
```



Out[332…

```
('Default Probability = ', 0.3563)
```



6)

Model 1 is Merton Model
Model 2 is Black-Cox Model

With the same simulation path (Same randon seed):
The simulation default probabilities assuming model 1 = 32.63%
The simulation default probabilities assuming model 2 = 35.63%

The default probabilities assuming model 2 is 3% higher than assuming model 1, becasue the default will be happen at any time up to maturity whenever the asset value drop below a predetermined threshold. As a result, if the asset value drop below a threshold at anytime before maturity, the firm will default immediately even the asset price may rise higher than the debt payable at maturity. Therefore, The default probabilities assuming model 2 will always be higher than assuming model 1.

In this case the different is not significant at only 3%, because the barrier threshold is set very low at 50.

In [ ]:

In [ ]:

Q2

In [350…

```
class Option_param():
    """
    Option class wants the option parameters:
    S0 = current stock price
    K = Strike price
    T = time to maturity

    """
    def __init__(self, A=120, D=95, T=10, payoff="Bond"):
        self.A = A
        self.D = D
        self.T = T
        self.payoff = payoff
```

In [351…

```
class Diffusion_process():
    """
    Class for the diffusion process:
    r = continuously compounded risk-free interest rate (% p.a.)
    sig = volatility (% p.a.)
    """
    def __init__(self, r=0.09, theta=0.05, k=0.3, sigma_r=0.03, sigma_
        self.r = r
        self.theta = theta
        self.k = k
        self.sigma_r = sigma_r
        self.sigma_a = sigma_a
        self.p = p
        self.lambda_r = lambda_r
```

In [352…

```
class Equity_and_Bond_pricer():

    def __init__(self, Option_info, Process_info):
        """
        Option_info:  of type Option_param. It contains (A,D,T) i.e. c
        Process_info:  of type Diffusion_process. It contains (r,mu, s
```

```
        """

        # Option_info:
        self.r = Process_info.r           # interest rate
        self.theta = Process_info.theta
        self.k = Process_info.k
        self.sigma_r = Process_info.sigma_r
        self.sigma_a = Process_info.sigma_a        # diffusion coeffic
        self.p = Process_info.p
        self.lambda_r = Process_info.lambda_r

        # Process_info:
        self.A = Option_info.A           # current asset price
        self.D = Option_info.D             # strike
        self.T = Option_info.T             # maturity in years
        self.payoff = Option_info.payoff









    def Black_Scholes(self):
        """ Black Scholes closed formula:
            payoff: Equity or Bond.
            A: float.    initial Asset level.
            D: float strike price.
            T: float maturity (in year fractions).
            r: float constant risk-free short rate.
            sigma: volatility factor in diffusion term. """

        d1 = (np.log(self.A/self.D) + (self.r + self.sigma_a**2 / 2) *
        d2 = (np.log(self.A/self.D) + (self.r - self.sigma_a**2 / 2) *

        Equity = self.A * ss.norm.cdf( d1 ) - self.D * np.exp(-self.r
        Bond = self.A - (self.A * ss.norm.cdf( d1 ) - self.D * np.exp(

        if self.payoff=='Equity':
            return ("Equity = ", Equity)
        elif self.payoff=='Defaultable Bond':
            return ("Defaultable Bond = ", Bond, "Yield = ", -1/self.T
        else:
            raise ValueError("invalid type. Set 'Equity' or 'Defaultab




    def Black_Scholes_with_Stochastic_Interest_Rates(self):
        """ Black Scholes closed formula:
            payoff: Equity or Bond.
            A: float.    initial Asset level.
            D: float strike price.
            T: float maturity (in year fractions).
            r: float constant risk-free short rate.
            sigma: volatility factor in diffusion term. """
```

```
        B = 1/self.k * (1 - np.exp(-self.k*self.T))
        A = (self.theta - self.lambda_r*self.sigma_r/self.k - 1/2*(sel
                + (self.sigma_r**2)/(4*self.k) * B**2

        risk_free_bond_price = np.exp(-A - B*self.r)


        total_variance = self.sigma_a**2 * self.T + \
                        (self.sigma_r**2/self.k**3) * (self.k*self.T + (1
                        + 2 * self.p * self.sigma_a * self.sigma_r/self.k*

        total_volatility = np.sqrt(total_variance)

        h1 = (np.log(self.A/(risk_free_bond_price*self.D)) + 1/2 * tot
        h2 = h1 - total_volatility

        Equity = self.A * ss.norm.cdf(h1) - risk_free_bond_price * sel
        Bond = self.A - Equity

        if self.payoff=='Equity':
            return ("Equity = ", Equity)

        elif self.payoff=='Defaultable Bond':
            return ("Defaultable Bond = ", Bond, "Yield = ", -1/self.T
        elif self.payoff=='Risk-Free Bond':
            return ("Risk-Free Bond = ", risk_free_bond_price, "Yield
        else:
            raise ValueError("invalid type. Set 'Equity' or 'Defaultab
```

In [ ]:

(1)

In [353…

```
Table_1 = np.zeros((20, 4),dtype=int)

# Option parameter
A = np.array([80,90,125,180])
D = 95
T = np.arange(0.5, 10.5, 0.5)

# Diffusion process parameter
r = 0.028
sigma_a = 0.23
```

In [354…

```
for i, t in enumerate(T):
    for j, a in enumerate(A):

        opt_param = Option_param(A=a, D=D, T=t, payoff="Defaultable Bo
        diff_param = Diffusion_process(r=r, sigma_a=sigma_a)
        Defaultable_Bond = Equity_and_Bond_pricer(opt_param, diff_para
```

```
        credit_spread = (Defaultable_Bond.Black_Scholes()[3] - r)*1000
        Table_1[i,j] = math.ceil(credit_spread)
```

In [355…

```
df = pd.DataFrame(Table_1, columns=A, index=T)
df
```

Out[355…

	80	90	125	180
0.5	3469	1779	58	1
1.0	1846	1097	119	2
1.5	1299	833	145	7
2.0	1020	687	157	14
2.5	848	592	161	20
3.0	731	525	162	26
3.5	646	474	161	32
4.0	581	433	159	36
4.5	529	401	157	40
5.0	487	373	154	43
5.5	452	350	151	46
6.0	422	330	148	48
6.5	396	313	145	50
7.0	374	298	142	51
7.5	354	284	139	52
8.0	337	272	137	54
8.5	321	261	134	54
9.0	307	251	132	55
9.5	295	242	129	56
10.0	283	233	127	56

In [ ]:

In [ ]:

(2)

In [356…

```
Table_2 = np.zeros((20, 2),dtype=int)
```

```
# Option parameter
A = 120
D = 95
T = np.arange(0.5, 10.5, 0.5)

# Diffusion process parameter
r = np.array([0.03, 0.09])
sigma_a = 0.18
```

In [357…

```
for i, t in enumerate(T):
    for j, rate in enumerate(r):

        opt_param = Option_param(A=A, D=D, T=t, payoff="Defaultable Bo
        diff_param = Diffusion_process(r=rate, sigma_a=sigma_a)
        Defaultable_Bond = Equity_and_Bond_pricer(opt_param, diff_para

        credit_spread = (Defaultable_Bond.Black_Scholes()[3] - rate)*1
        Table_2[i,j] = math.ceil(credit_spread)
```

In [359…

```
df = pd.DataFrame(Table_2, columns=r, index=T)
df
```

Out[359…

	0.03	0.09
0.5	28	15
1.0	66	31
1.5	84	35
2.0	92	35
2.5	95	33
3.0	96	30
3.5	95	28
4.0	94	25
4.5	92	23
5.0	90	21
5.5	88	19
6.0	86	18
6.5	84	16
7.0	82	15
7.5	80	14
8.0	78	12
8.5	76	11
9.0	74	11

	0.03	0.09
9.5	73	10
10.0	71	9

In [ ]:

(3)

In [ ]:

In [ ]:

In [ ]:

Q3

In [ ]:

In [360…

```
Table_1 = np.zeros((20, 4),dtype=int)

# Option parameter
A = np.array([80,90,125,180])
D = 95
T = np.arange(0.5, 10.5, 0.5)

# Diffusion process parameter
r = 0.028
theta = 0.05
k = 0.3
sigma_r = 0.03
sigma_a = 0.23
p = 0
lambda_r = 0
```

In [361…

```
for i, t in enumerate(T):
    for j, a in enumerate(A):

        # Creates the Defaultable Bond
        opt_param = Option_param(A=a, D=D, T=t, payoff="Defaultable Bo
        diff_param = Diffusion_process(r=r, theta=theta, k=k, sigma_r=
        Defaultable_Bond = Equity_and_Bond_pricer(opt_param, diff_para

        # Creates the risk-Free Bond
        opt_param = Option_param(A=a, D=D, T=t, payoff="Risk-Free Bond
        diff_param = Diffusion_process(r=r, theta=theta, k=k, sigma_r=
```

```
        Risk_Free_Bond = Equity_and_Bond_pricer(opt_param, diff_param)


        credit_spread = (Defaultable_Bond.Black_Scholes_with_Stochasti
        Table_1[i,j] = math.ceil(credit_spread)
```

In [363…

```
df = pd.DataFrame(Table_1, columns=A, index=T)
df
```

Out[363…

	80	90	125	180
0.5	3457	1771	57	1
1.0	1827	1083	117	2
1.5	1276	816	142	7
2.0	993	668	152	13
2.5	820	571	155	20
3.0	701	502	154	25
3.5	615	449	152	30
4.0	548	408	150	34
4.5	496	375	146	38
5.0	453	347	143	40
5.5	418	323	139	43
6.0	388	303	136	44
6.5	362	286	132	46
7.0	340	270	129	47
7.5	320	256	126	48
8.0	303	244	123	49
8.5	287	233	120	49
9.0	273	223	117	49
9.5	261	213	114	50
10.0	249	205	112	50

In [ ]:

In [ ]:

In [365…

```
Table_2 = np.zeros((20, 2),dtype=int)
```

```
# Option parameter
A = 120
D = 95
T = np.arange(0.5, 10.5, 0.5)

# Diffusion process parameter
r = np.array([0.03, 0.09])
theta = 0.05
k = 0.3
sigma_r = 0.03
sigma_a = 0.18
p = 0
lambda_r = 0
```

In [366…

```
for i, t in enumerate(T):
    for j, rate in enumerate(r):

        # Creates the Defaultable Bond
        opt_param = Option_param(A=A, D=D, T=t, payoff="Defaultable Bo
        diff_param = Diffusion_process(r=rate, theta=theta, k=k, sigma
        Defaultable_Bond = Equity_and_Bond_pricer(opt_param, diff_para

        # Creates the risk-Free Bond
        opt_param = Option_param(A=A, D=D, T=t, payoff="Risk-Free Bond
        diff_param = Diffusion_process(r=rate, theta=theta, k=k, sigma
        Risk_Free_Bond = Equity_and_Bond_pricer(opt_param, diff_param)



        credit_spread = (Defaultable_Bond.Black_Scholes_with_Stochasti
        Table_2[i,j] = math.ceil(credit_spread)
```

In [367…

```
df = pd.DataFrame(Table_2, columns=r, index=T)
df
```

Out[367…

	0.03	0.09
0.5	28	16
1.0	65	34
1.5	82	41
2.0	89	44
2.5	92	45
3.0	92	45
3.5	91	44
4.0	90	44
4.5	88	44

	0.03	0.09
5.0	86	43
5.5	83	42
6.0	81	42
6.5	79	41
7.0	77	41
7.5	75	40
8.0	73	40
8.5	71	39
9.0	69	39
9.5	67	38
10.0	65	37

In [ ]:

Question 1) In a few paragraphs discuss the main empirical failings of the benchmark Merton model. 2) discuss how the extension to the Merton model given by equations 2-3 can resolve these failings 3) and discusses the economic mechanism of the model with equations and derivations

Answer 1) The main empirical failing of the benchmark Merton model is that the assumption of constant interest short rate violates the real market observation of interest rate, which is empirically a mean-reversing non-constant rate. The failings lead to severe mispricing of assets, both equity and bond. This failing can be resolved with Merton Model with Stochastic Interest Rates.

```
Another failing is that the debt is assumed to able to
debt only at matury while in real-life the debt can be
default before the maturity if the asset value fall below
a predetermined threshold, and this failing can be solve
with a black-Cox Model.
```

2) The interest short rate observed empirically in the market is a mean-reversion diffusion process like a Vasicek or CIR process. The dynamic of the interest rate is essential when we price a risk-free bond, while the price of a risk-free bond is needed for pricing a defaultable bond.

```
The diffusion process of interest rate is correlated to
the diffusion of asset price that both “the volatility of
the interest rate” and “the correlation between the
interest rate and asset price” would affect the total
```

variance of A/PD. The total variance is the implied volatility in the Black-Scholes equation when pricing equity as long as a call option, bond as a risk-free bond and short as a put option.

Therefore, defining the dynamics of the interest rate as a Vasicek process in Equation 3, and the correlation of the diffusion process in Equation 4 is crucial when we price a bond. With Equation 3 and 4, it can resolve the failings that lead to severe mispricing of assets.

We can compare the credit spread with 1) Merton Model and 2) Merton Model with mean-reverse stochastic interest rates, from table 2 that has a interest at 9% and the long-term interest rate mean at 5%. For Merton Model, We can observe that the credit spread is negatively correlated to the time to maturity, because the draft of asset value is very high at 9% that even with volatility the asset value will have higher chance to be larger than the debt payable if enough time to maturity is given. On the other hand for Merton Model with mean-reverse stochastic interest rates, We can observe that the credit spread is positive correlated to the time to maturity, because the 9% interest rate draft for asset price could not be sustain and will eventually reverse to the long-term mean 5% if enough time to maturity is given. As a result for a larger time to matury, the long-term 5% interest rate do not provide enough draft to overcome the larger volatility due to the larger time to matury. Therefore, the credit spread is positive correlated to the time to maturity for stochastic interest rate model in this setting. All in all, we can conclude that with the equation 3 & 4 defined the mean-reverse dynamic and the correlation between asset price and interest rate, the main failings of Merton Model can be resolved.

3) The economic mechanism for the non-constant interest rate is that the central bank increases the interest short rate to reduce the inflation to cool down the overheated economy while decreasing the interest short rate to encourage corporate financing and reduce the cost of capital to boost the economy. This economic mechanism of using interest rate as a tool to stimulate and cool the economy is implied by the mean-reversion process of interest rate.

In [ ]:

In [ ]:

In [ ]:

In [ ]:

In [ ]:

In [ ]:

In [ ]: