Robust Optimization in Portfolio Management

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1 Introduction

A portfolio is a collection of financial instruments that an investor holds, and portfolio management involves building and overseeing a selection of investments that help investors to balance risk and reward. The optimal asset allocation for each investor is usually different because of diverse goals, risk tolerance, and investment horizon.

The most common approach for asset allocation is the mean–variance model [1]. It expresses risk as variance and allows investors to decide how much risk they are willing to take on in exchange for different levels of reward. This processing of measuring an asset's risk against its expected return usually leads to building an efficient frontier, which calculates the biggest reward at a given level of risk or the least risk at a given level of return.

However, such framework requires estimation for the future return and expected volatility, and inaccurate estimation can largely affect the allocation results. In this project, we propose using robust optimization to reduce the high sensitivity of asset allocation to inputs such that the allocation results are more resilient to changes in market expectation.

Portfolio optimization is an important problem in the financial sector, and potential decision makers include both retail investors without much trading experience and experienced portfolio managers. Our robust portfolio optimization will benefit both groups. For retail investors, it helps maximize the investment return and diversify the risk, and thus increase personal wealth. For experienced portfolio managers who make asset allocation decisions based on their expectation for future economy, a robust portfolio optimization provides a more resilient solutions to better handle unexpected market movement. It also largely reduces the trading cost, as less rebalancing is needed.

2 Data

Although the portfolio optimization problem generally includes different asset classes, we only consider stock allocation in this project for simplicity. Referring to the Global Industry Classification Standard [2], we selected 10 stocks from five different industry sectors, hoping to capture the diversification across sectors. The five sectors are healthcare (Johnson & Johnson, UnitedHealth Group Inc), financials (JP Morgan Chase, Citigroup), industrials

(Delta, Allegiant Travel Co.), technology (Apple, Meta), and consumer discretionary (Target, BestBuy). We obtained the data using Python API, yfinance.

We calculated the stock daily return by subtracting its previous day's closing price from today's closing price and dividing the difference by the previous close, i.e., $\frac{(P_t - P_{t-1})}{P_{t-1}}$, where P_t denotes the closing price on day t. Figure 1 is an example of Delta's daily return from January 1, 2019 to April 1, 2020. We can see that its average daily return was around zero over time, but its variance increased dramatically after the start of Covid-19 pandemic. Other stocks that we selected showed similar patterns. We used the daily returns from January 1, 2019 to Dec 31, 2019 (before Covid) as our training set, and the daily returns from January 1, 2020 to April 1, 2020 (after Covid) as test set. Portfolio return and risk are estimated by mean and variance, respectively.

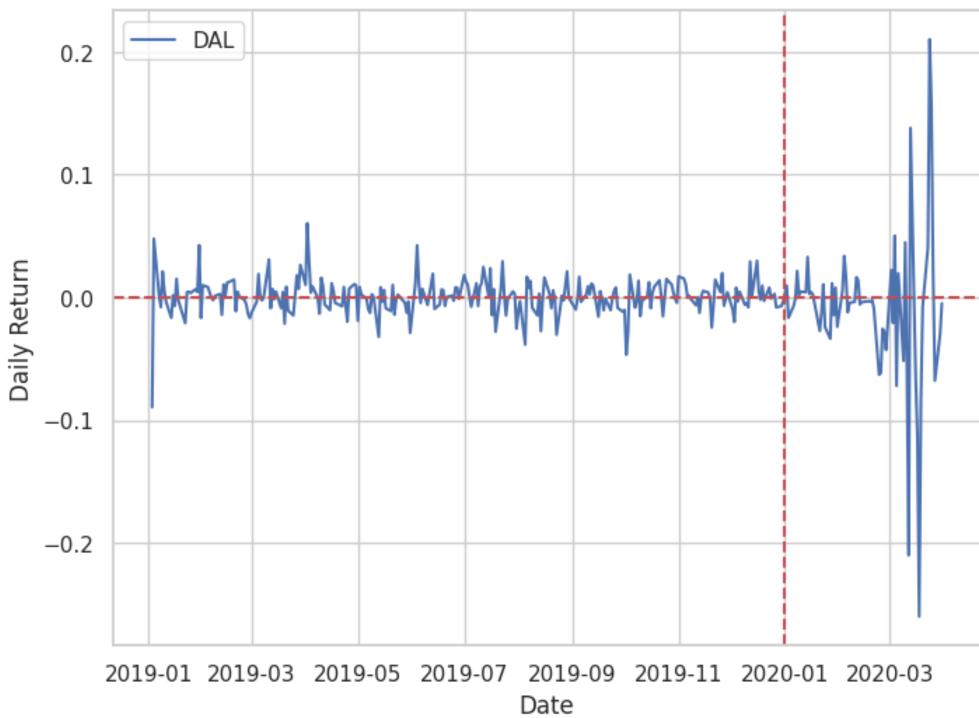

Figure 1: Daily Return of Delta Stock Before and After Covid

Note that this setup implies that we expect the past returns to be good predictions for future returns, as our asset allocation was optimized over the past returns. However, that's usually not true in reality. In the result section, we will evaluate how our robust portfolio reacts to this inaccurate estimation, compared with the static baseline model. Specifically, we will apply the allocation results (optimized over past returns) on the actual, future daily returns in test set, and see whether it is more resilient to market changes.

3 Model

Portfolio management is a "multi-objective" optimization problem, as it aims to maximize expected total return and minimize the risk. In reality, investors usually have a required return, or minimum acceptable rate of return. Therefore, we frame our problem using "goal programming" approach, which is a branch of multi-objective optimization problem. Our objective becomes minimizing the portfolio volatility [3], and we put the required return in constraint. The problem can be formulated as the following,

$$
\begin{aligned}
\min_{w} \quad & w^T \Sigma w && (1)\\
\text{s.t.} \quad & \tilde{r}^T w \geq \beta && (2)\\
& w_i \leq t, \ \forall i = 1, ..., n && (3)\\
& \sum w_i = 1 && (4)\\
& w_1, ..., w_n \geq 0 && (5)
\end{aligned}
$$

we are optimizing over portfolio weights, w, with w_i indicating the weight corresponding to the ith asset, and the parameter Σ denotes the covariance matrix of the stocks in our portfolio. Constraint (2) specifies the required return of investor, where β denotes the minimum acceptable rate of return, and $\tilde{r}$ represents the expected return with uncertainty. We set $\beta = 0.04\%$ in this project. The constraint (3) limits the allocation in a single position by threshold t ($t = 0.4$), which is constant for every stock. The constraint (5) restricts short shelling so that we can only "buy" stocks. Note that we ignore risk free asset in our problem.

We reformulated our problem using ellipsoidal uncertainty sets on expected asset returns. Mathematically,

$$
\begin{aligned}
\min_{w} \quad & w^T \Sigma w && (6)\\
\text{s.t.} \quad & \overline{r}^T w - \rho \|(\Sigma^{-\frac{1}{2}})^T w\|_2 \geq \beta && (7)\\
& w_i \leq t, \ \forall i = 1, ..., n && (8)\\
& \sum w_i = 1 && (9)\\
& w_1, ..., w_n \geq 0 && (10)
\end{aligned}
$$

The $\overline{r}$ in constraint (7) indicates mean return over the period from January 1, 2019 to Dec 31, 2019, and ρ decides the importance of robustness over optimality. We made an assumption here that the covariance matrix is static over time, so the only uncertainty is in our required-return constraint (7). In this project, we define "infeasibility" to be violating the constraint (7), meaning that the minimum acceptable return is not met.

To better understand the feasibility and optimality trade-off of our robust optimization, we also implemented a baseline model based on the average of past daily returns. In other words, it considers no uncertainty in the expected return, thus $\tilde{r} = \overline{r}$.

4 Results

4.1 Asset Allocation

We summarize the results of asset allocations for both models in Figure 2, where the two dotted lines represent the weights of each stock and the bars represent the corresponding Sharpe Ratio. On the top of each bar, there are two numbers where the upper one denotes the variance and the lower one denotes the mean return of each stock.

The first interesting observation is that our robust model captures the overall pattern of Sharpe Ratio, even though we did not specify it in the formulation. One possible explanation is that with uncertainty in returns, the robust model has to diversify more across all stocks, rather than betting on one single position. Otherwise, the risk of failing to meet the required return constraint is high. As a result, the robust model indirectly pays more attention to profitable stocks to increase return besides the original goal of minimizing the variance.

The baseline model, however, allocates more weights to less risky stocks. For example, the least risky but least profitable stock "ALGT" accounts for 30% of the baseline portfolio whereas the second most profitable stock "JNJ" is assigned little weight because its risk is high. The baseline model cares more about lowering risk as that is the objective in the formulation, and the mean return of each stock is deterministic, which makes the minimum target return constraint much easier to be met. Also, note that the asset allocations using robust optimization are more evenly distributed than the deterministic baseline model due to the effect of regularization.

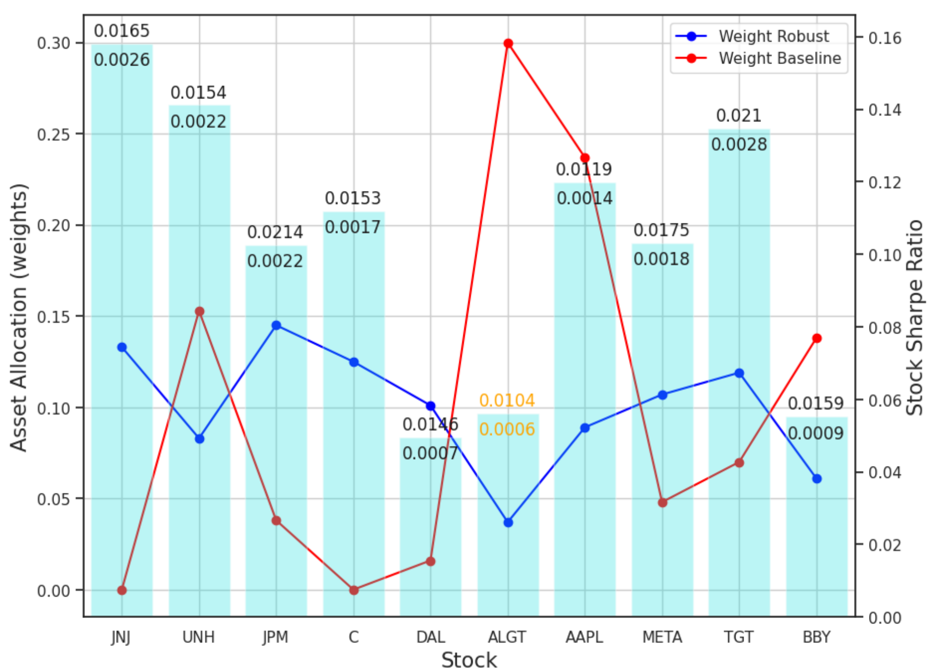

Figure 2: Robust versus baseline asset allocation before covid

4.2 Robustness and Optimality Trade-off

Figure 3 shows the trade-off between robustness and optimality. As is shown in the upper line graph, when we increase the uncertainty measured by ρ, the feasible probability of the robust model has an upward trend and is always higher than the baseline. Meanwhile, the objective value, which is the portfolio variance, also increases as ρ gets bigger.

This sheds light on some insights into the investment business. By achieving the same target return, the robust model possibly results in less portfolio turnover with slightly higher risk, as the perturbations in the market increases. In addition, since both models feed on data from stable years before Covid-19 and are tested on data during the hit, it further shows the robust model should be more resilient to the crisis.

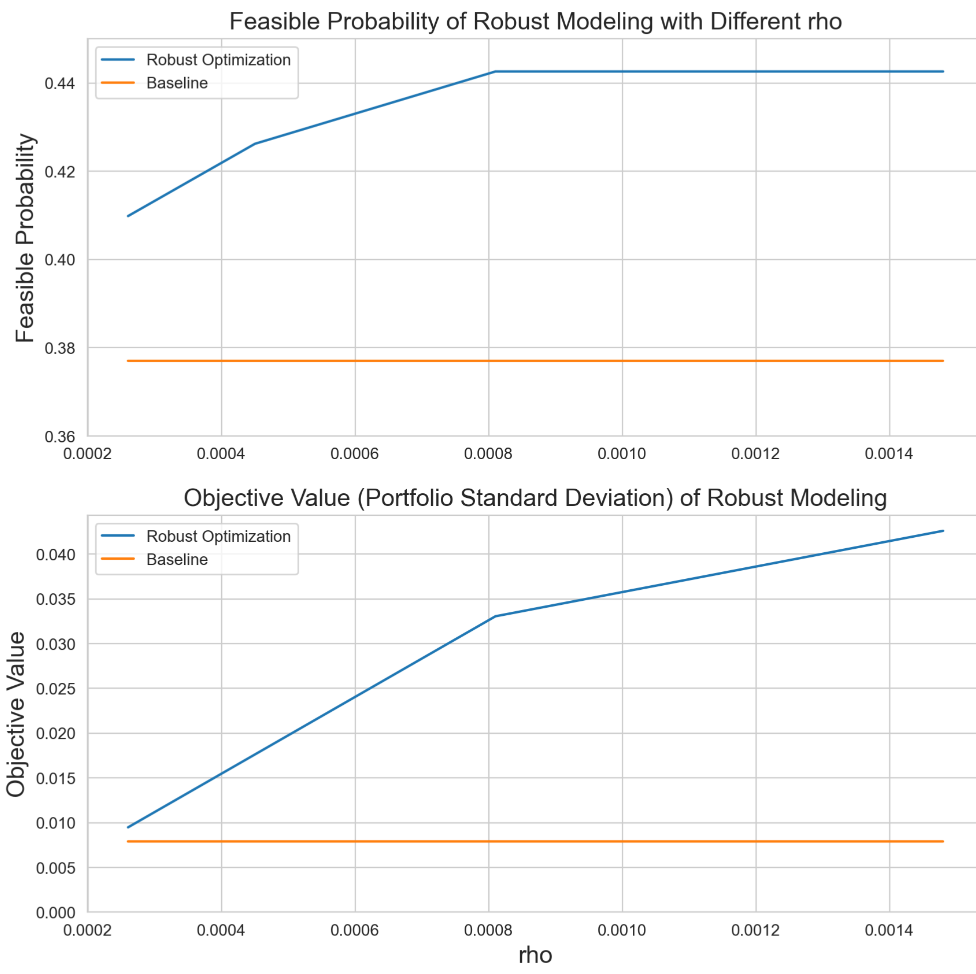

Figure 3: The trade-off of robustness (feasibility probability) and optimality

5 Conclusion

In this project, we applied robust optimization technique with ellipsoidal uncertainty set on the asset allocation problem. Robust portfolio optimization takes into considerations the worst-case estimations and hence accounts for worst situations. Our experiment on 10 stocks supports this claim, as our robust portfolio achieves a higher feasible probability than the baseline model in the turbulent market during pandemic.

For future work, we plan to model the uncertainty at factor level. Factor better captures the relationship between stocks and the market; decomposing portfolio return into reasonable broad factors helps us better understand what drives the returns than the ambiguity at the individual asset level [4]. Moreover, we simplified the portfolio management problem in current work, as we ignored all possible costs generated during the portfolio building and overseeing process. Transaction cost, market impact, and execution risk should be considered. To better capture the real-world scenario, we plan to reformulate our portfolio objective to incorporate the optimal control of execution cost [5] in future work.

References

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