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Deciphering robust portfolios

Woo Chang Kim[a,*], Jang Ho Kim[b], and Frank J. Fabozzi[c]

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JEL classification: C44; C61; G11

Keywords: Robust portfolio optimization; Mean-variance model; Fundamental factors

[a] Department of Industrial and Systems Engineering, Korea Advanced Institute of Science and Technology (KAIST), Yuseong-gu, Daejeon 305-701, Republic of Korea Email: wkim@kaist.ac.kr

[b] Department of Industrial and Systems Engineering, Korea Advanced Institute of Science and Technology (KAIST), Yuseong-gu, Daejeon 305-701, Republic of Korea Email: janghokim@kaist.ac.kr

[c] EDHEC Business School, Nice, France Email: frank.fabozzi@edhec.edu

[*] Corresponding author

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Abstract

Robust portfolio optimization has been developed to resolve the high sensitivity to inputs of the Markowitz mean-variance model. Although much effort has been put into forming robust portfolios, there have not been many attempts to analyze the characteristics of portfolios formed from robust optimization. We investigate the behavior of robust portfolios by analytically describing how robustness leads to higher dependency on factor movements. Focusing on the robust formulation with an ellipsoidal uncertainty set for expected returns, we show that as the robustness of a portfolio increases, its optimal weights approach the portfolio with variance that is maximally explained by factors.

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1. Introduction

As a result of the global financial crisis of 2008, asset managers have placed greater emphasis on managing portfolio uncertainty. The distinction between risk and uncertainty is made by classifying risk as events with unforeseen outcomes but attached probability distributions to the outcomes (Knight, 1921). In these terms, financial crises clearly falls under uncertainty since there are too many factors, mostly unforeseen, that lead to financial disasters. Furthermore, the existence of uncertainty directly affects decision-making, often resulting in behavior that cannot be explained by aversion to risk alone (Savage, 1954, and Ellsberg, 1961). This notion of uncertainty appears frequently in many studies in economics and finance (see, for example, Camerer and Weber, 1992).

One of the earlier approaches in forming optimal portfolios under parameter uncertainty was to utilize Bayesian methods (Klein and Bawa, 1976). Gilboa and Schmeidler (1989) axiomatize the maxmin expected utility decision rule with uncertainty aversion assumptions (see, also, Dow and Werlang, 1992, and Epstein and Wang, 1994). Alternatively, a number of studies apply methods from robust control theory for asset pricing and model robust decision-making by allowing model misspecifications (Hansen, Sargent, and Tallarini, 1999, Hansen, Sargent, and Wang, 2002, Anderson, Sargent, and Hansen, 2003, and Maenhout, 2004, 2006). Moreover, Hansen *et al*. (2002) illustrate similarities between the control theory approach and the maxmin expected utility theory of Gilboa and Schmeidler (1989).

Another method for forming robust portfolios that has gained momentum in the last decade is robust optimization. Robust portfolio optimization formulates robust counterparts within the mean-variance framework (Markowitz, 1952, and Elton and Gruber, 1997) and this development has been motivated to resolve the high sensitivity of mean-variance portfolios to its input parameters (Michaud, 1989, Best and Grauer, 1991a, 1991b, Chopra and Ziemba, 1993, and Broadie, 1993). The method optimizes the worst case by defining uncertainty sets of uncertain parameters (Lobo and Boyd, 2000, Halldórsson and Tütüncü, 2003, Goldfarb and Iyengar, 2003, and Tütüncü and Koenig, 2004). The worst-case approach of robust portfolio optimization not only constructs portfolios that perform well under uncertainty but also results in efficiently solved formulations.[1]

The 2008-2010 global financial crisis has made it clear that robustness of portfolios is extremely important and consequently a more thorough understanding of robust portfolios is required to motivate its proper use. There has not been much work on deciphering robust portfolios for the purpose of analyzing any noticeable attributes. Therefore, we analyze the

[1] For a thorough review on the development of robust portfolio optimization, please refer to Fabozzi *et al*. (2007a, 2007b), Fabozzi, Huang, and Zhou (2010), and Kim, Kim, and Fabozzi (2013a).

behavior of stock portfolios formed from the robust formulation with an ellipsoidal uncertainty set for the expected returns. Specifically, we look into how robust portfolios tilt their exposure to market factors. Controlling the exposure to factors is especially important because portfolio managers often manage the overall risk of portfolios by setting the amount of risk impacted by the movement in fundamental factors. We find that robust portfolios depend more on fundamental factor movements compared to classical mean-variance portfolios. In this paper, we provide a mathematical framework and analytic explanation along with empirical analyses as to why higher robustness of portfolios from robust optimization leads to increased dependency on market factors.

There have been several notable studies that extend our analytic findings. Kim *et al*. (2013a) empirically find that there is a high correlation of robust portfolio returns with factor returns, and Kim *et al*. (2013b) present revised formulations that control the factor exposure of robust portfolios. Furthermore, Kim, Kim, and Fabozzi (2013b) analyze weights given to individual stocks that compose robust portfolios. Finally, we note that our results are related to the findings of Maenhout (2006) who derives how the optimal robust portfolio weights depend on the volatility of the state variable, which is comparable to the factor variance in our work. His approach is similar to robust optimization in that it guards against the worst case; uncertainty exists in the state equation and alternative state equations measured by relative entropy are considered in order to gain robustness. However, Maenhout's model differs from ours since the state vector follows a diffusion and the risk premium follows a mean-reverting process, thereby making our contribution unique and noteworthy.

The remainder of the paper is organized as follows. In Section 2, a quadratic programming problem that has an equivalent effect on optimal portfolios as the robust portfolio optimization problem with ellipsoidal uncertainty is presented. Section 3 builds our mathematical arguments on the dependency of robust portfolios. The observations are further empirically confirmed with simulations and historical stock returns in Section 4, and Section 5 concludes.

2. Robust formulation as quadratic programming representation

We begin by reviewing how robust optimization is applied to portfolio selection and introduce the formulation with an ellipsoidal uncertainty set on the expected stock returns. Since this robust formulation results in a second-order cone program, we find a quadratic program with similar behavior that can be analytically observed for studying factor exposures of robust portfolios.

2.1. Robust formulation with ellipsoidal uncertainty

In the classical Markowitz problem (1952), the optimal portfolio is found by computing the tradeoff between risk and return. A portfolio that invests in *n* stocks is represented as a vector of weights, $\omega \in \mathbb{R}^n$, where each weight represents the proportion of wealth allocated to a stock. Then portfolio risk and return become $\omega^T \Sigma \omega$ and $\omega^T \mu$, respectively, where $\Sigma \in \mathbb{R}^{n \times n}$ is the covariance matrix of returns and $\mu \in \mathbb{R}^n$ is the expected returns of *n* stocks. The mean-variance model solves a portfolio problem with a quadratic objective function,

$$\min_{\omega} \left\{ \frac{1}{2} \omega^T \Sigma \omega - \lambda \omega^T \mu \;\middle|\; \omega \in \Omega \right\}$$

and investors can adjust the framework to fit their risk levels by changing the value of λ. In the above formulation, λ is the risk-seeking coefficient where setting it to $\lambda = 0$ finds the portfolio with minimum risk. The set Ω defines the universe of allowable portfolios and constraints on portfolio weights are often employed. Throughout our analyses, we set $\Omega = \{\omega \in \mathbb{R}^n \mid \omega^T \iota = 1\}$ where $\iota \in \mathbb{R}^n$ is the vector of ones, which is a requirement for fully investing in stocks.

One of the main shortcomings of the mean-variance model is that the inputs μ and Σ are not known with certainty; robust models look for portfolios that are less sensitive to changes in the input values. In robust optimization, a set of possible values for the uncertain parameters is defined and the optimal solution must be feasible regardless of which value is realized. Since robust optimization takes the worst-case approach, the robust counterpart of the classical problem finds a robust portfolio by looking at the worst case within the uncertainty set. We only consider uncertainty in expected returns because it is known to affect portfolio performance much more than errors in variance or covariance (Chopra and Ziemba, 1993). The robust counterpart of the classical formulation can therefore be written as

$$\min_{\omega}\left\{\max_{\mu\in U}\frac{1}{2}\omega^T\Sigma\omega-\lambda\omega^T\mu\;\middle|\;\omega^T\iota=1\right\}$$

where the uncertainty set U determines the possible values of expected returns. The maximization represents the inner problem of finding the worst case within U while assuming ω to be fixed. One of the most studied uncertainty sets for expected returns is an ellipsoid around $\hat{\mu}\in\mathbb{R}^n$ that is defined as (Goldfarb and Iyengar, 2003)

$$U_\delta(\hat{\mu})=\left\{\mu\;\middle|\;(\mu-\hat{\mu})^T\Sigma_\mu^{-1}(\mu-\hat{\mu})\le\delta^2\right\}$$

where $\delta\in\mathbb{R}$ sets the size of the uncertainty set and $\Sigma_\mu\in\mathbb{R}^{n\times n}$ is the covariance matrix of estimation errors for the expected returns. With the uncertainty set $U_\delta(\hat{\mu})$, the robust problem can be reformulated as a second-order cone programming problem,[2]

$$\min_{\omega}\left\{\frac{1}{2}\omega^T\Sigma\omega-\lambda\omega^T\hat{\mu}+\lambda\delta\sqrt{\omega^T\Sigma_\mu\omega}\;\middle|\;\omega^T\iota=1\right\}. \tag{1}$$

This paper focuses on observing portfolios from this robust portfolio optimization problem with ellipsoidal uncertainty set.

2.2. A quadratic program for analyzing robust behavior

The main goal of our study is to examine the behavior of portfolios as their robustness is increased. In other words, we analyze optimal portfolios while increasing the value of δ,

[2] The derivation of the ellipsoidal model is explained by Fabozzi *et al.* (2007b) in pages 371 to 372.

which results in expanding the uncertainty set. However, since the second-order cone program given by (1) cannot be analytically solved, it is not a trivial task to reveal properties of robust portfolios generated directly from (1). Therefore, we instead find a quadratic program with an extra parameter similar to δ where increasing this extra parameter has the equivalent effect on portfolios as expanding the uncertainty set of the robust formulation. Investigating the analytic solution of this quadratic program will provide behavioral patterns of robust portfolios. The existence of such a portfolio selection problem with a quadratic objective function is shown by the following lemma.

Lemma 1. There exists an $a \in \mathbb{R}$ such that the optimal portfolio for the robust formulation given by (1) coincides with the optimal solution of the quadratic program given by (2),

$$\min_{\omega} \left\{ \frac{1}{2}\omega^T \Sigma \omega - \lambda \omega^T \hat{\mu} + \frac{1}{2}\lambda a \omega^T \Sigma_\mu \omega \;\middle|\; \omega^T \iota = 1 \right\}. \tag{2}$$

Proof. Appendix A.

This shows that problems (1) and (2) can result in the equivalent optimal portfolio by properly setting the value of a in terms of δ. More importantly, it proves that increasing the value of δ in (1) has the same effect on portfolios as increasing the value of a in (2).

The comparability between problems (1) and (2) are further confirmed by plotting the resulting portfolios on the mean-standard deviation plane, which is the standard approach for expressing efficient frontiers.[3] Using a 3-year rebalancing period from 1970 to 2012, portfolios are formed every three years by solving the original robust formulation (1) with various levels of δ. Every three years, another set of portfolios are also constructed by solving (2) with various levels of a. As shown in Figure 1, increasing the level of δ in (1) influences portfolios in the same manner as increasing the level of a in (2) in terms of

[3] Industry portfolios (Fama and French, 1997) are used to represent the stock market. We elaborate on the use of industry portfolios in Section 4.2.

annualized risk and return of portfolios; they both modify portfolios to move to the lower-left region (lower risk and lower return) and the frontiers show similar curvature. The range of values for a is scaled for this demonstration so that the smallest value of a in (2) results in portfolios with risk levels similar to portfolios with the smallest value of δ in (1), and this clearly displays how the two frontiers almost overlap. In the following sections, we use this finding to analyze the quadratic program given by (2) but conclude by applying the developed arguments to the original robust formulation given by (1).

PLACE FIGURE 1 ABOUT HERE

3. Stylized analytical approach

In this section, we analytically emphasize that robust portfolios become more dependent on factor movements as robustness is increased. We investigate how increasing the value of a in problem (2) changes the optimal portfolio, which has the same effect as increasing the robustness. The analysis is carried out in two steps. In the first step, the portfolio that depends the most on factor variance is derived. In the second step, it is shown that the optimal portfolio asymptotically approaches the portfolio from the first step when increasing the level of a. Throughout the paper, the portfolio with maximum dependency on factors is referred to as the *factor portfolio* and denoted by ω_{max}.

3.1. Assumptions

We make the following general assumptions on stock returns. There are a total of *n* risky stocks where the covariance matrix of returns, $\Sigma \in \mathbb{R}^{n\times n}$, is positive-definite. Stock returns, $r \in \mathbb{R}^n$, are explained by a factor model, $r = \alpha + Bf + \varepsilon$, with the returns of *m* ($\leq n$) factors, $f \in \mathbb{R}^m$, and the variance of factor returns is denoted by $\Sigma_f \in \mathbb{R}^{m\times m}$. The vector $\alpha \in \mathbb{R}^n$ is the intercept, and the factor loadings and error term of the factor model are $B \in \mathbb{R}^{n\times m}$ and $\varepsilon \in \mathbb{R}^n$, respectively. Moreover, we assume that the error term is uncorrelated between stocks and therefore its covariance matrix, $\Sigma_\varepsilon \in \mathbb{R}^{n\times n}$, becomes a diagonal matrix. Similarly, the estimation errors of expected returns between stocks are assumed to be uncorrelated.

In addition to the above standard assumptions, we include the following stylized assumptions that Σ has the same diagonal terms and the same off-diagonal terms where $\rho > 0$, and Σ_ε also has the same diagonal values,

$$\Sigma = \begin{bmatrix} \sigma^2 & & \rho\sigma^2 \\ & \ddots & \\ \rho\sigma^2 & & \sigma^2 \end{bmatrix} \text{ and } \Sigma_\varepsilon = \begin{bmatrix} \sigma_\varepsilon^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_\varepsilon^2 \end{bmatrix}. \tag{3}$$

Finally, suppose the estimation error covariance matrix has a simplified diagonal form as well,

$$\Sigma_\mu = \begin{bmatrix} \sigma_\mu^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_\mu^2 \end{bmatrix}. \tag{4}$$

3.2. Portfolio with maximum dependency on factors

Since our main goal is to show that increasing portfolio robustness increases its dependency on factor movements, we first look at the factor portfolio and later show that improving robustness results in the optimal portfolio to converge to this factor portfolio.

Proposition 1. The factor portfolio is

(a) $\omega_{max} = \frac{1}{\iota^T \Sigma_\varepsilon^{-1/2} x_{max}} \Sigma_\varepsilon^{-1/2} x_{max}$ where x_{max} is the eigenvector corresponding to the largest eigenvalue of the matrix $\Sigma_\varepsilon^{-1/2} B \Sigma_f B^T \Sigma_\varepsilon^{-1/2}$,

(b) the equally-weighted portfolio when the covariance matrices Σ and Σ_ε follow the simplified structures as given by (3).

Proof. See Appendix B.

Proposition 1 states that the factor portfolio is the equally-weighted portfolio when the simplified structures for the covariance matrices are assumed and the assumptions are carried throughout our mathematical arguments. However, note that Proposition 1(a) holds even without the simplifications shown in (3) and the factor portfolio derived here is used in

Section 4 for analyzing robust portfolios in the generic case. So far, we looked at the factor portfolio with maximum dependency on factor movements and we next investigate the portfolio with maximum robustness.

3.3. Portfolio with maximum robustness

We decompose robust portfolios and study how increasing the robustness affects the composition of portfolios. We also derive the portfolio that is asymptotically reached when a portfolio has maximum robustness. For the remainder of this section, the structure of covariance matrices given by (3) and (4) is assumed.

Let us define $\varSigma_a = \varSigma + a\varSigma_\mu$ for $a \geq 0$. Then, since $\varSigma_a$ has identical diagonal terms and identical off-diagonal terms, its inverse $\varSigma_a^{-1}$ also has the same structure and can be expressed as

$$\varSigma_a^{-1} = \begin{bmatrix} \alpha_a & & \beta_a \\ & \ddots & \\ \beta_a & & \alpha_a \end{bmatrix}. \tag{5}$$

Note that because $\varSigma$ and $\varSigma_\mu$ are positive-definite from (3) and (4), so is $\varSigma_a$ and thus $\varSigma_a^{-1}$ is also positive-definite. From the definition of $\varSigma_a$, the problem given by (2) can be reformulated as

$$\min_\omega \left\{ \frac{1}{2} \omega^T \varSigma_a \omega - \lambda \omega^T \hat{\mu} \,\middle|\, \omega^T \iota = 1 \right\} \tag{6}$$

where λa from (2) is represented simply by a in (6) since λ is a constant term. The composition of robust portfolio selected from the formulation given by (6) is characterized below.

Proposition 2. The optimal robust portfolio constructed from the problem given by (6) is

(a) a weighted sum of two portfolios *u* and *v*, where *u* is a portfolio with weights based on the expected excess returns where the excess return is obtained by subtracting the average expected return of *n* stocks from the individual expected returns, $\left(\hat{\mu} - \frac{1}{n}(\hat{\mu}^T \iota)\iota\right)$, and v is the equally weighted portfolio, $\frac{1}{n}\iota$, and

(b) approaches the equally-weighted portfolio as the value of a is increased.

Proof. See Appendix C.

Increasing the value of a has the same effect as increasing the robustness. Therefore, Proposition 2 shows that higher robustness results in portfolios that deviate less from the portfolio with equal weights until asymptotically approaching the equally-weighted portfolio for maximum robustness.[4]

3.4. Convergence of robust portfolios

We finally reach the conclusion of our argument that robust portfolios approach the factor portfolio as robustness is increased. The findings for the portfolio problem (2) are summarized in the following statement.

Proposition 3. As $a > 0$ increases for the problem given by (2), the optimal portfolio converges to the factor portfolio.

Proof. The proof follows from Propositions 1 and 2, and the equivalence between problems (2) and (6). □

We have demonstrated, under the assumptions made in (3) and (4), that the robust portfolio bets more on the factors than its non-robust version. Moreover, as the robustness parameter a increases, the portfolio depends more on the variance of factors. Even though structural simplifications are assumed for analytically solving the robust portfolio problem, we next provide empirical evidence that our claims hold without these assumptions. Recall that Proposition 1(a), which presents the closed-form solution of the factor portfolio, holds even without the stylized assumptions of (3) and (4). This provides insight on analyzing the

[4] Similar to our findings, Pflug, Pichler, and Wozabal (2012) focus on the Kantorovich metric for defining uncertainty sets and show that uniform diversification is the optimal investment decision in situations of high uncertainty.

dependency of robust portfolios on factors under the generic settings, and this will be explored in Section 4.

4. Empirical approach

We now observe that increasing robustness forms portfolios that are more dependent on factors without imposing the stylized structures. Unfortunately, since it becomes difficult to approach analytically when the assumptions are relaxed, we instead conduct several empirical analyses. We investigate robust portfolios from the problem given by (6) as well as the original robust problem given by (1).

4.1. Simulation with generated returns

Before eliminating the structural assumptions on the covariance matrices, we first confirm through simulation that the optimal portfolio from solving the robust problem given by (6) converges to the factor portfolio when the robustness is increased under the assumptions (3) and (4).[5] The following steps describe a single iteration of the simulation but multiple iterations are performed to verify the observed behavior.

Step 1: Generate a positive-definite matrix Σ that has identical diagonal elements and identical off-diagonal elements

Step 2: Generate diagonal matrices Σ_ε and Σ_μ where each matrix has identical diagonal elements that are strictly positive

Step 3: Compute the optimal solution $\overline{\omega}_a$ of (6) for $\Sigma_a = \Sigma + a\Sigma_\mu$

Step 4: Conduct eigenvalue decomposition on $\Sigma_\varepsilon^{-1/2} B \Sigma_f B^T \Sigma_\varepsilon^{-1/2}$ and derive ω_{max}

[5] For the two simulations in Section 4.1, we set $\lambda = 1$ because the primary objective is to observe the effect of increasing the value of a. However, in Section 4.2, values of λ between 0.01 and 0.09 are used because classical mean-variance portfolios constructed from the 49 industry data show annualized risk between 5% and 30%.

Step 5: Plot the distance between the optimal portfolio and the factor portfolio,

$$d(a) = \|\overline{\omega}_a - \omega_{max}\|_2$$

Step 6: Repeat Steps 3-5 by varying the value of a

In Step 4, ω_{max} is simply the equally-weighted portfolio for this simulation due to the assumptions. Figure 2(a) shows that as the value of a is increased from 0 to 100, the distance $d(a)$ measured by 2-norm approaches zero.

PLACE FIGURE 2 ABOUT HERE

Similar simulations are performed with randomly generated data but without the assumptions given by (3) and (4).

Step 1: Generate a symmetric positive-definite matrix Σ_f

Step 2: Generate a factor-loading matrix B

Step 3: Generate diagonal matrices Σ_ε and Σ_μ with strictly positive diagonal elements

Step 4: Repeat Steps 3-5 from the first simulation by varying the value of a

The above iteration is repeated multiple times and Figure 2(b) clearly displays that the Euclidean distance between the optimal portfolio and ω_{max} decreases as the value of a is increased. Even though the distance does not asymptotically reach zero, the decreasing pattern clearly demonstrates that increasing robustness moves portfolios closer to the factor portfolio even in the generic case.

4.2. Analysis with historical returns

The analysis is further extended by relaxing not only all stylized assumptions but even the diagonality of the estimation error covariance matrix.[6] Moreover, data from the US equity market is used to confirm our arguments with historical stock market returns; industry-

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[6] In our empirical analyses, we consider stock returns to be a stationary process and samples to be independent and identically distributed which allows estimating the error covariance matrix as $\Sigma_\mu = \frac{1}{T}\Sigma$, where T is the sample size.

level and stock-level returns are used for forming portfolios. Since industries are good representative building blocks for stock portfolios (Kim and Mulvey, 2009), we mainly present results from using the 49 industry portfolios introduced by Fama and French (1997). In addition, for the fundamental factors, we use the three-factor model proposed by Fama and French (1993, 1995). Daily returns for the 49 industries and the three factors from 1970 to 2012 are collected.[7] The observations are not restricted to these values, but the results using a 3-year rebalancing period, 90% confidence level, and a risk-seeking coefficient level of 0.03 are primarily discussed.

The portfolio problem given by (6), which is equivalent to solving (2), is solved using historical returns. The curves in Figure 3 clearly confirm our pattern for the optimal portfolio for all 3-year periods; the curves indicate that the optimal portfolio becomes closer to the factor portfolio as the magnitude of penalization increases.

PLACE FIGURE 3 ABOUT HERE

Even though it is illustrated in Section 2 that the quadratic program given by (2) can be used to analyze the behavior of the original robust formulation with an ellipsoidal uncertainty set given by (1), we confirm our findings by directly solving the original problem using historical returns. For this experiment, we change the confidence of the uncertainty set, which has the same effect of changing the value of a in the previous empirical tests; a higher confidence level expands the uncertainty set and is represented by a higher value of δ in the objective function.[8] From Figure 4, the relationship between the confidence level and the distance from the factor portfolio is consistent also in the robust formulation given by (1); the

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[7] Data for the three factors and the industry returns are obtained from the Kenneth R. French online data library (http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html).

[8] A υ% confidence level is represented by setting δ^2 as the value of the υth percentile of a χ^2 distribution with the number of stocks as its degrees of freedom (Fabozzi *et al*., 2007b).

distance decreases as the confidence increases from 0% to 99%. In particular, Figure 4(b), which focuses on the results for confidence between 0% and 10%, clearly displays a sharp decrease in distance when the confidence level is increased from zero. This demonstrates that robust portfolios are more dependent on the Fama-French factors compared to the classical mean-variance portfolios.

PLACE FIGURE 4 ABOUT HERE

The analyses in this section demonstrate several important points. First, our argument that was initially presented with assumptions is shown to hold empirically even without those simplifications. Second, observations not only show that increasing the robustness of robust portfolios increases dependency on factors, but they also reveal that the increase in dependency is large between non-robust portfolios and robust portfolios with even a small uncertainty set.

5. Conclusion

Robust portfolio optimization has had a major impact on resolving the sensitivity issue of the mean-variance model. Although the worst-case approach to portfolio selection proposes a method for forming robust portfolios, not much is known on how robust portfolios behave. Focusing on the robust portfolio formulation with an ellipsoidal uncertainty set for expected returns, we show that an increase in robustness results in the optimal portfolio being more dependent on factor movements. Due to the limitations of analytically solving a second-order cone problem, we find a quadratic program with equivalent behavior and provide mathematical proofs on the pattern of the relationship between the magnitude of the penalized matrix and the distance from the factors. In addition, we present several empirical results which support our findings even without simplified assumptions using simulated and historical stock market returns. The main contribution of this paper is revealing the factor exposure of robust equity portfolios and providing evidence that robust portfolios might be robust since they are betting more on market factors.

Appendix A

Proof of Lemma 1. We first introduce two portfolio selection problems with extra inequality constraints,

$$\min_{\omega}\left\{\frac{1}{2}\omega^T\Sigma\omega-\lambda\omega^T\hat{\mu}\;\middle|\;\sqrt{\omega^T\Sigma_\mu\omega}\le\sqrt{\delta_0},\ \omega^T\iota=1\right\} \tag{A1}$$

and

$$\min_{\omega}\left\{\frac{1}{2}\omega^T\Sigma\omega-\lambda\omega^T\hat{\mu}\;\middle|\;\omega^T\Sigma_\mu\omega\le\delta_0,\ \omega^T\iota=1\right\} \tag{A2}$$

where δ_0 is non-negative. The problems given by (A1) and (A2) are considered identical because the estimation error covariance matrix is positive-definite.

Next, we discuss the relationship between problems (1) and (A1) and also between problems (2) and (A2).

(i) (1) and (A1):

For the robust formulation given by (1), the Lagrangian is written as

$$L_{(1)}(\omega,\gamma)=\frac{1}{2}\omega^T\Sigma\omega-\lambda\omega^T\hat{\mu}+\lambda\delta\sqrt{\omega^T\Sigma_\mu\omega}+\gamma(\omega^T\iota-1),$$

and its first-order conditions for the optimal solution $(\bar{\omega},\bar{\gamma})$ are

$$\Sigma\bar{\omega}-\lambda\hat{\mu}+\lambda\delta\frac{\Sigma_\mu\bar{\omega}}{\sqrt{\bar{\omega}^T\Sigma_\mu\bar{\omega}}}+\bar{\gamma}\iota=0\quad\text{and}\quad\bar{\omega}^T\iota-1=0.$$

Similarly, the Lagrangian function for (A1) is

$$L_{(\mathrm{A1})}(\omega,\xi,\gamma)=\frac{1}{2}\omega^T\Sigma\omega-\lambda\omega^T\hat{\mu}+\xi\left(\sqrt{\omega^T\Sigma_\mu\omega}-\sqrt{\delta_0}\right)+\gamma(\omega^T\iota-1),$$

and the Karush-Kuhn-Tucker (KKT) conditions for the optimal solution $\left(\bar{\omega}_{\mathrm{A}},\bar{\xi}_{\mathrm{A}},\bar{\gamma}_{\mathrm{A}}\right)$

includes[9]

$$\Sigma\overline{\omega}_{\mathrm{A}} - \lambda\hat{\mu} + \bar{\xi}_{\mathrm{A}}\frac{\Sigma_{\mu}\overline{\omega}_{\mathrm{A}}}{\sqrt{\overline{\omega}_{\mathrm{A}}^{T}\Sigma_{\mu}\overline{\omega}_{\mathrm{A}}}} + \bar{\gamma}_{\mathrm{A}}\iota = 0 \quad \text{and} \quad \overline{\omega}_{\mathrm{A}}^{T}\iota - 1 = 0.$$

We see that problems (1) and (A1) will have the same optimal portfolio, $\overline{\omega} = \overline{\omega}_A$, when $\lambda\delta = \bar{\xi}_A$ and $\bar{\gamma} = \bar{\gamma}_A$.

(ii) (2) and (A2):

By taking the same approach as in (i), the first-order conditions of problem (2) for the optimal solution $(\overline{\omega}, \bar{\eta})$ are

$$\Sigma\overline{\omega} - \lambda\hat{\mu} + \lambda a\Sigma_{\mu}\overline{\omega} + \bar{\eta}\iota = 0 \quad \text{and} \quad \overline{\omega}^{T}\iota - 1 = 0,$$

and the following should hold for the optimal solution $(\overline{\omega}_A, \bar{\zeta}_A, \bar{\eta}_A)$ of (A2) from its KKT conditions,

$$\Sigma\overline{\omega}_A - \lambda\hat{\mu} + 2\bar{\zeta}_A\Sigma_{\mu}\overline{\omega}_A + \bar{\eta}_A\iota = 0 \quad \text{and} \quad \overline{\omega}_A^{T}\iota - 1 = 0.$$

Again, problems (2) and (A2) will find the identical optimal portfolio, $\overline{\omega} = \overline{\omega}_A$, when $\lambda\delta = 2\bar{\zeta}_A$ and $\bar{\eta} = \bar{\eta}_A$.

In summary, since problems (A1) and (A2) are identical, it follows that solving the revised formulation given by (2) becomes equivalent to solving the original robust problem given by (1) with proper choices of parameters. Therefore, there exists an a in (2) that finds the same portfolio as (1). □

Appendix B

The following lemma is introduced before proving Proposition 1.

Lemma 2. For a matrix with identical diagonal terms and identical off-diagonal terms

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[9] We omit the rest of the KTT conditions to demonstrate how the optimal ω of (A1) satisfies the first-order conditions of (1). This is also the case when analyzing problem (A2).

expressed as $Y = \gamma\mathbb{I} + \theta I$, where $\mathbb{I} \in \mathbb{R}^{n\times n}$ is the matrix of ones, $I \in \mathbb{R}^{n\times n}$ is the identity matrix, and $\gamma, \theta \in \mathbb{R}$,

(i) the characteristic polynomial is $\det(Y - \lambda I) = (-1)^n(\lambda-\theta)^{n-1}(\lambda - \theta - n\gamma)$,

(ii) if $\gamma \geq 0$, the largest eigenvalue of Y is $\theta + n\gamma$,

(iii) if $\gamma \geq 0$, the eigenvector corresponding to the largest eigenvalue of Y is $\frac{1}{\sqrt{n}}\iota$.

Proof of Lemma 2.

(i) From the structure of matrix Y, we can write

$$Y - \lambda I = A + u_\gamma u_\gamma^T \quad \text{where } A = \begin{bmatrix} \theta-\lambda & & 0 \\ & \ddots & \\ 0 & & \theta-\lambda \end{bmatrix} \text{ and } u_\gamma = \begin{bmatrix} \sqrt{\gamma} \\ \vdots \\ \sqrt{\gamma} \end{bmatrix}.$$

Since A is invertible, from the matrix determinant lemma (Ding and Zhou, 2007),

$$\begin{aligned} \det(Y-\lambda I) &= \left(1 + u_\gamma^T A^{-1} u_\gamma\right)\det(A) \\ &= \left(1 + n\frac{\gamma}{\theta-\lambda}\right)(\theta-\lambda)^n = (-1)^n(\lambda-\theta)^{n-1}(\lambda-\theta-n\gamma). \end{aligned} \tag{B1}$$

(ii) From the definition of characteristic polynomials, the eigenvalues of Y are the solutions to $\det(Y-\lambda I) = 0$. From (B1), the eigenvalues of Y are θ and $\theta + n\gamma$. Since $\gamma \geq 0$, $\theta + n\gamma \geq \theta$ for all θ and thus the largest eigenvalue is $\theta + n\gamma$.

(iii) For the largest eigenvalue $\theta + n\gamma$, since

$$(Y - (\theta+n\gamma)I)\iota = (\gamma\mathbb{I} + \theta I - (\theta + n\gamma)I)\iota = \gamma(\mathbb{I} - nI)\iota = 0,$$

the vector of ones is the corresponding eigenvector and the normalized solution is $\frac{1}{\sqrt{n}}\iota$. □

We now present the proof of the proposition on the factor portfolio.

Proof of Proposition 1.

(a) For a portfolio ω, its variance can be decomposed from the factor model as

$$\text{Var}(\omega^T r) = \text{Var}(\omega^T Bf + \omega^T\varepsilon) = \omega^T B\Sigma_f B^T\omega + \omega^T\Sigma_\varepsilon\omega.$$

Note that $\omega^T B\Sigma_f B^T\omega$ is the variance of the portfolio due to the factors, whereas $\omega^T\Sigma_\varepsilon\omega$ is the variance attributable to the errors. Thus, the portfolio with variance that is the most dependent on f is the solution to

$$\max_{\omega}\left\{\frac{\omega^T B\Sigma_f B^T\omega}{\omega^T\Sigma_\varepsilon\omega}\,\middle|\,\omega^T\iota=1\right\}. \tag{B2}$$

The maximization problem without the constraint of (B2) becomes

$$\max_{\omega_0}\frac{\left(\omega_0^T\Sigma_\varepsilon^{1/2}\right)\Sigma_\varepsilon^{-1/2}B\Sigma_f B^T\Sigma_\varepsilon^{-1/2}\left(\Sigma_\varepsilon^{1/2}\omega_0\right)}{\left(\omega_0^T\Sigma_\varepsilon^{1/2}\right)\left(\Sigma_\varepsilon^{1/2}\omega_0\right)} \tag{B3}$$

where ω_0 represents the unconstrained weight, and the value of $\Sigma_\varepsilon^{1/2}\omega_0$ that maximizes (B3) is the eigenvector x_{max} corresponding to the largest eigenvalue of the matrix $\Sigma_\varepsilon^{-1/2}B\Sigma_f B^T\Sigma_\varepsilon^{-1/2}$. Then, the optimal unconstrained portfolio is $\bar{\omega}_0 = \Sigma_\varepsilon^{-1/2}x_{max}$ and thus the factor portfolio becomes

$$\omega_{max}=\frac{1}{\iota^T\Sigma_\varepsilon^{-1/2}x_{max}}\Sigma_\varepsilon^{-1/2}x_{max}.$$

(b) From the factor model and our assumptions on the structure of matrices Σ and Σ_ε, the matrix $B\Sigma_f B^T$ also has the same diagonal terms and also the same off-diagonal terms. Moreover, since $\Sigma_\varepsilon^{-1/2}$ is a diagonal matrix with identical values, the matrix $\Sigma_\varepsilon^{-1/2}B\Sigma_f B^T\Sigma_\varepsilon^{-1/2}$ can be written in the form $\gamma\mathbb{I}+\theta I$ for proper choices of $\gamma \geq 0$ and θ. It follows from Lemma 2 that the eigenvector of $\Sigma_\varepsilon^{-1/2}B\Sigma_f B^T\Sigma_\varepsilon^{-1/2}$ corresponding to the largest eigenvalue is $\frac{1}{\sqrt{n}}\iota$, and ω_{max} becomes the equally-weighted portfolio, $\frac{1}{n}\iota$, that sums to one. □

Appendix C

Proof of Proposition 2.

(a) The optimal portfolio for (6) can be found from the first-order optimality conditions. From the Lagrangian function

$$L_{(6)}(\omega,\gamma) = \frac{1}{2}\omega^T \Sigma_a \omega - \lambda\omega^T \hat{\mu} + \gamma(\omega^T \iota - 1),$$

the optimality conditions for the equality-constrained problem (6) are

$$\Sigma_a \bar{\omega} - \lambda\hat{\mu} + \bar{\gamma}\iota = 0 \quad \text{and} \quad \bar{\omega}^T \iota - 1 = 0$$

for the optimal values $(\bar{\omega}, \bar{\gamma})$. The optimal portfolio is

$$\bar{\omega} = \lambda \Sigma_a^{-1}\hat{\mu} - \frac{\lambda\hat{\mu}^T \Sigma_a^{-1}\iota - 1}{\iota^T \Sigma_a^{-1}\iota}\Sigma_a^{-1}\iota = \lambda(\alpha_a - \beta_a)\left(\hat{\mu} - \frac{1}{n}(\hat{\mu}^T \iota)\iota\right) + \frac{1}{n}\iota$$

due to (5) and this proves our claim by letting $u = \hat{\mu} - \frac{1}{n}(\hat{\mu}^T \iota)\iota$ and $v = \frac{1}{n}\iota$.

(b) From (a), the weights of u sum to zero and the weights of v sum to one. Thus, the value of $\lambda(\alpha_a - \beta_a)$ determines how much the weights given to each stock deviate from the equally-weighted portfolio. Since λ is a constant, the value of a affects both the value of $\alpha_a - \beta_a$ and the optimal portfolio.

First, note that $\alpha_a - \beta_a > 0$ since Σ_a^{-1} is a positive-definite matrix. Then, it is sufficient to show that $f(a) = \alpha_a - \beta_a$ is a decreasing function of $a > 0$. The matrix Σ_a can be represented from (3) and (4) as

$$\Sigma_a = \Sigma + a\Sigma_\mu = \rho\sigma^2 \mathbb{I} + \left(a\sigma_\mu^2 - (\rho - 1)\sigma^2\right)I.$$

By defining $A = \left(a\sigma_\mu^2 - (\rho - 1)\sigma^2\right)I$ and $\iota_\sigma = |\sigma|\iota$, the Woodbury matrix identity (Woodbury, 1950, and Henderson and Seale, 1981),

$$(A + \rho\iota_\sigma\iota_\sigma^T)^{-1} = A^{-1} - \frac{\rho}{1 + \rho\iota_\sigma^T A^{-1}\iota_\sigma}A^{-1}\iota_\sigma\iota_\sigma^T A^{-1}$$

and the expression given by (5) result in

$$\alpha_a - \beta_a = \frac{1}{a\sigma_\mu^2 + (1 - \rho)\sigma^2}.$$

It is shown that $f(a) = \alpha_a - \beta_a$ is a decreasing function of $a > 0$ and thus the optimal portfolio approaches $\frac{1}{n}\iota$ as a is increased. □

References

Anderson, E. W., Hansen, L. P., Sargent, T. J., 2003. A quartet of semigroups for model specification, robustness, prices of risk, and model detection. Journal of the European Economic Association, 1(1), 68-123.

Best, M. J., Grauer, R. R., 1991a. On the sensitivity of mean-variance-efficient portfolios to changes in asset means: Some analytical and computational results. Review of Financial Studies, 4(2), 315-342.

Best, M. J., Grauer, R. R., 1991b. Sensitivity analysis for mean-variance portfolio problems. Management Science, 37(8), 980-989.

Broadie, M., 1993. Computing efficient frontiers using estimated parameters. Annals of Operations Research, 45(1), 21-58.

Camerer, C., Weber, M., 1992. Recent developments in modeling preferences: Uncertainty and ambiguity. Journal of Risk and Uncertainty, 5, 325-370.

Chopra, V., Ziemba, W. T., 1993. The effect of errors in mean and co-variance estimates on optimal portfolio choice. Journal of Portfolio Management, 19(2), 6-11.

Ding, J., Zhou, A., 2007. Eigenvalues of rank-one updated matrices with some applications. Applied Mathematics Letters, 20, 1223-1226.

Dow, J., Werlang, S. R. C., 1992. Uncertainty aversion, risk aversion, and the optimal choice of portfolio. Econometrica, 60(1), 197-204.

Ellsberg, D., 1961. Risk, ambiguity, and the Savage axioms. Quarterly Journal of Economics, 75, 643–669.

Elton, E. J., Gruber, M. J., 1997. Modern portfolio theory, 1950 to date. Journal of Banking and Finance, 21, 1743-1759.

Epstein, L. G., Wang, T., 1994. Intertemporal asset pricing under Knightian uncertainty. Econometrica, 62(3), 283-322.

Fabozzi, F. J., Huang, D., Zhou, G., 2010. Robust portfolios: Contributions from operations research and finance. Annals of Operations Research, 176, 191-220.

Fabozzi, F. J., Kolm, P. N., Pachamanova, D. A., Focardi, S. M., 2007a. Robust portfolio optimization. Journal of Portfolio Management, 33, 40-48.

Fabozzi, F. J., Kolm, P. N., Pachamanova, D. A., Focardi, S. M., 2007b. Robust Portfolio Optimization and Management. Hoboken, NJ: Wiley.

Fama, E. F., French, K. R., 1993. Common risk factors in the returns on stocks and bonds. Journal of Financial Economics, 33(1), 3-56.

Fama, E. F., French, K. R., 1995. Size and book-to-market factors in earnings and returns. Journal of Finance, 50(1), 131-155.

Fama, E. F., French, K. R., 1997. Industry costs of equity. Journal of Financial Economics, 43(2), 153-193.

Gilboa, I., Schmeidler, D., 1989. Maxmin expected utility with non-unique prior. Journal of Mathematical Economics, 18, 141-153.

Goldfarb, D., Iyengar, G., 2003. Robust portfolio selection problems. Mathematics of Operations Research, 28(1), 1-38.

Halldórsson, B.V., Tütüncü, R.H., 2003. An interior-point method for a class of saddle-point problems. Journal of Optimization Theory and Applications, 116(3), 559-590.

Hansen, L. P., Sargent, T. J., Tallarini, T. D., 1999. Robust permanent income and pricing. Review of Economic Studies, 66, 873-907.

Hansen, L. P., Sargent, T. J., Turmuhambetova, G. A., Williams, N., 2002. Robustness and uncertainty aversion. Manuscript, University of Chicago.

Hansen, L. P., Sargent, T. J., Wang, N. E., 2002. Robust permanent income and pricing with filtering. Macroeconomic Dynamics, 6, 40-84.

Henderson, H. V., Searle, S. R., 1981. On deriving the inverse of a sum of matrices. SIAM

Review, 23(1), 53-60.

Kim, J. H., Kim, W. C., Fabozzi, F. J., 2013a. Recent developments in robust portfolios with a worst-case approach. Journal of Optimization Theory and Applications, forthcoming.

Kim, J. H., Kim, W. C., Fabozzi, F. J., 2013b. Composition of robust equity portfolios. Finance Research Letters, 10(2), 72-81.

Kim, W. C., Kim, J. H., Ahn, S. H., Fabozzi, F. J., 2013a. What do robust equity portfolio models really do? Annals of Operations Research, 205(1), 141-168.

Kim, W. C., Kim, M. J., Kim, J. H., Fabozzi, F. J., 2013b. Robust portfolios that do not tilt factor exposure. European Journal of Operational Research, 234(2), 411-421.

Kim, W. C., Mulvey, J. M., 2009. Evaluating style investment. Quantitative Finance, 9, 637-651.

Klein, R. W., Bawa, V. S., 1976. The effect of estimation risk on optimal portfolio choice. Journal of Financial Economics, 3, 215-231.

Knight, F. H., 1921. Risk, Uncertainty and Profit. Boston, MA: Hart, Schaffner and Marx.

Lobo, M. S., Boyd, S., 2000. The worst-case risk of a portfolio. Technical report, Stanford University, http://www.stanford.edu/~boyd/papers/pdf/risk_bnd.pdf.

Maenhout, P. J., 2004. Robust portfolio rules and asset pricing. Review of Financial Studies, 17(4), 951-983.

Maenhout, P. J., 2006. Robust portfolio rules and detection-error probabilities for a mean-reverting risk premium. Journal of Economic Theory, 128, 136-163.

Markowitz, H., 1952. Portfolio selection. Journal of Finance, 7(1), 77-91.

Michaud, R. O., 1989. The Markowitz optimization enigma: Is ‘‘optimized’’ optimal? Financial Analysts Journal, 45, 31-42.

Pflug, G., Pichler, A., Wozabal, D., 2012. The 1/N investment strategy is optimal under high model ambiguity. Journal of Banking and Finance, 36, 410-417.

Savage, L. J., 1954. The Foundations of Statistics. New York: Wiley.

Tütüncü, R. H., Koenig, M., 2004. Robust asset allocation. Annals of Operations Research, 132, 157-187.

Woodbury, M. A., 1950. Inverting modified matrices. Memorandum Report 42, Statistical Research Group, Princeton, N.J.

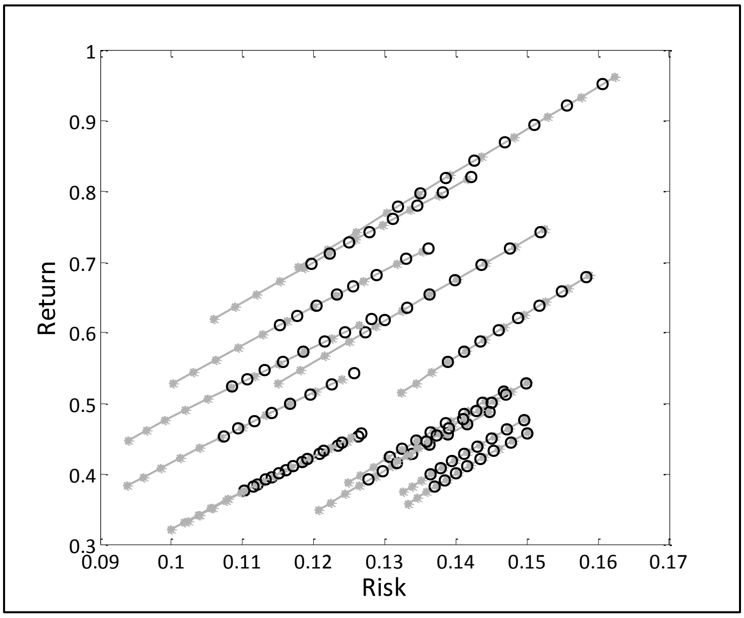
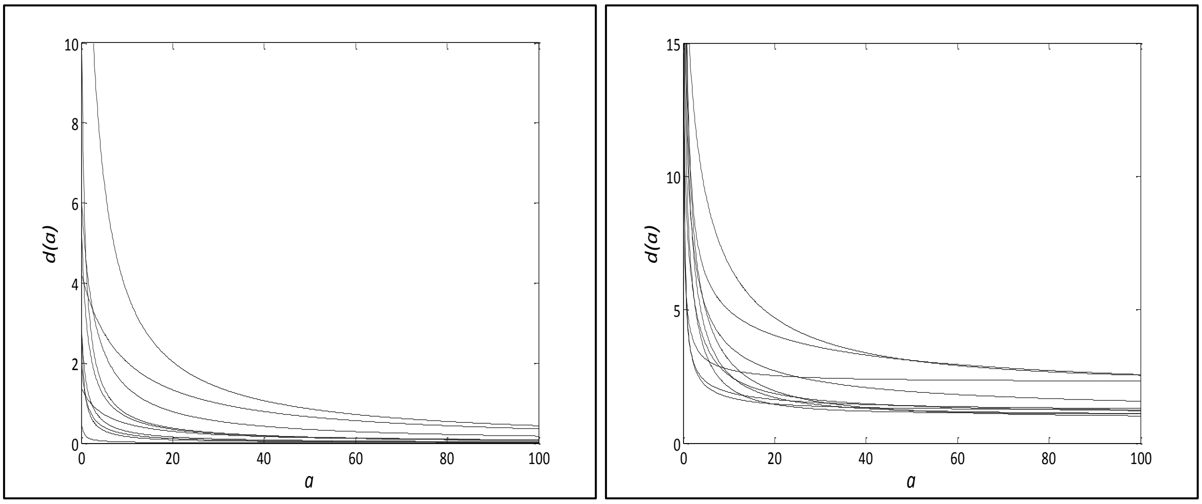


Figure 1 Portfolios from solving problem (1) (in gray) and problem (2) (in black) for increasing values of δ and a (from upper-right to lower-left)

Portfolios from solving (1) during the same period (and differ only in the value of δ) are connected in gray.



(a) (b)

Figure 2 Distance between $\boldsymbol{\omega_{max}}$ and the optimal portfolio from simulation

Results for 10 simulations with 100 stocks and four factors are shown. Figure 3(a) and 3(b) are performed with and without stylized assumptions, respectively.

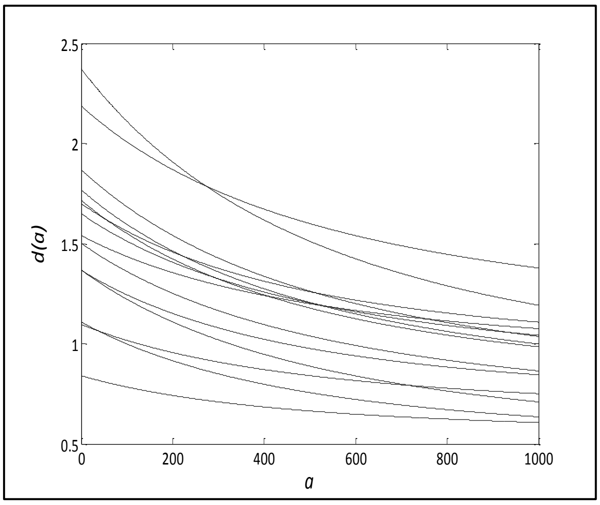
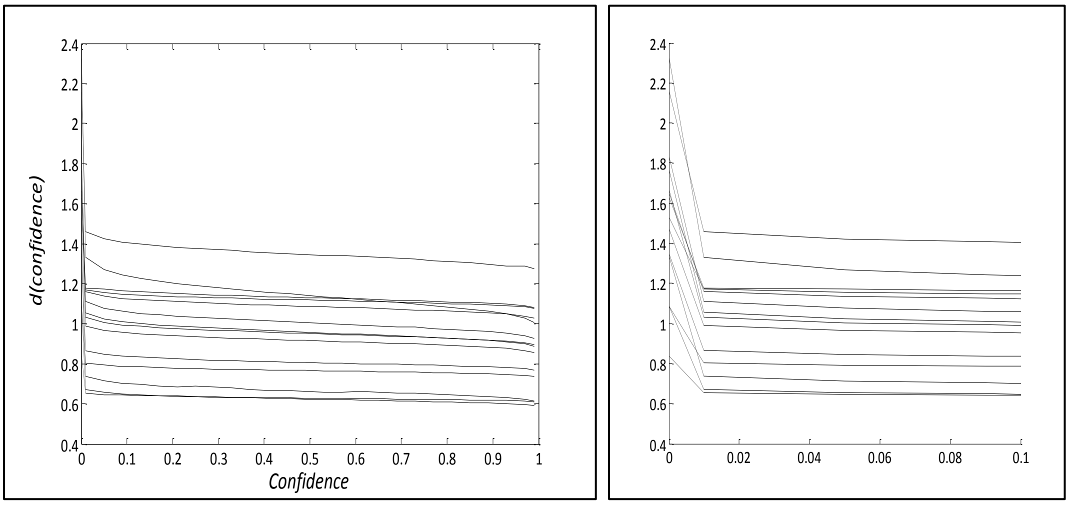


Figure 3 Distance between $\boldsymbol{\omega}_{\boldsymbol{max}}$ and the industry-level optimal portfolio

Each curve represents portfolios during the same 3-year period that differ only in the value of a. For a, the range of 1 to 100 is used because it forms portfolios with annualized risk roughly between 10% and 20%.



(a) (b)

Figure 4 Distance between $\boldsymbol{\omega_{max}}$ and the industry-level robust optimal portfolio when varying the robustness (confidence level)

Dotted lines connect values for 0% and 1% confidences to present how using even a small uncertainty set (1% level) shows higher dependency on factors than the case without incorporating uncertainty (0% level). Figure 5(b) zooms into results between 0% and 10% confidence levels.