GTN EXCL Program

1. Problems

Problema 1. Let $x_1, x_2, \cdots, x_n > 0$ such that $\frac{1}{1+x_1} + \ldots + \frac{1}{1+x_n} = 1$. Prove that

$$x_1 x_2 \ldots x_n \geq (n-1)^n.$$

Solución. We see that

$$1 - \frac{1}{1+x_1} = \frac{1}{1+x_2} + \ldots + \frac{1}{1+x_n} \geq (n-1)\sqrt[n-1]{\frac{1}{1+x_2} \times \ldots \times \frac{1}{1+x_n}} = \frac{n-1}{\sqrt[n-1]{(1+x_2)\ldots(1+x_n)}}$$

So

$$\frac{x_1}{1+x_1} \geq \frac{n-1}{\sqrt[n-1]{(1+x_2)\ldots(1+x_n)}}$$

Generally

$$\frac{x_k}{1+x_k} \geq \frac{n-1}{\sqrt[n-1]{\prod_{i\neq k}(1+x_i)}}$$

Multiply this result for $k = 1, 2, \ldots, n$ we find that

$$\frac{x_1}{1+x_1} \cdot \ldots \cdot \frac{x_n}{1+x_n} \geq \frac{(n-1)^{n-1}}{\sqrt[n-1]{(1+x_1)^{n-1} \cdot (1+x_2)^{n-1} \ldots (1+x_n)^{n-1}}}$$
$$\frac{x_1}{1+x_1} \cdot \ldots \cdot \frac{x_n}{1+x_n} \geq \frac{(n-1)^{n-1}}{(1+x_1)\cdot(1+x_2)\ldots(1+x_n)}$$
$$\implies x_1 x_2 \ldots x_n \geq (n-1)^n. \qquad \blacksquare$$

Problema 2. For positive a, b, c prove that

$$\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} > 2$$

Solución. Given real numbers $x_1, \ldots, x_n$ we have

$$\sqrt[n]{x_1 \cdot x_2 \cdot \ldots \cdot x_n} \geq \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}}.$$

where the case of equality occurs when $x_1 = x_2 = \ldots = x_n$. So, we find that

$$\sqrt{\frac{a}{b+c}} = \sqrt{1 \cdot \frac{a}{b+c}} \geq \frac{2}{1 + \frac{b+c}{a}} \geq \frac{2a}{a+b+c}$$

Similary $\sqrt{\frac{b}{c+a}} \geq \frac{2b}{a+b+c}$ and $\sqrt{\frac{c}{a+b}} \geq \frac{2c}{a+b+c}$ where the equality occurs when $1 = \frac{a}{b+c} = \frac{b}{c+a} = \frac{c}{a+b}$ it's which no posible. Then $\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} > 2$. $\blacksquare$

Problema 3. For positive a, b, c, d prove that

$$1 < \frac{a}{a+b+c} + \frac{b}{b+c+d} + \frac{c}{c+d+a} + \frac{d}{d+a+b} < 2.$$

Solución. Since a, b, c, d are positive, we find

$$1 < \frac{a^2}{a^2+ab+ac} + \frac{b^2}{b^2+bc+bd} + \frac{c^2}{c^2+cd+ac} + \frac{d^2}{d^2+ad+bd} < 2.$$

and we take $A = \frac{a^2}{a^2+ab+ac} + \frac{b^2}{b^2+bc+bd} + \frac{c^2}{c^2+cd+ac} + \frac{d^2}{d^2+ad+bd}$. By Cauchy-Schwars we find that

$$\left(a^2+b^2+c^2+d^2+ab+2ac+ad+bc+2bd+cd\right) \cdot A \geq (a+b+c+d)^2$$

$$\iff \left[(a+c)^2+(a+c)(b+d)+(b+d)^2\right]\cdot A \geq (a+c)^2+2(a+c)(b+d)+(b+d)^2$$
$$\iff A \geq \frac{(a+c)^2+2(a+c)(b+d)+(b+d)^2}{(a+c)^2+(a+c)(b+d)+(b+d)^2}$$

Since $(a+c)^2+2(a+c)(b+d)+(b+d)^2 > (a+c)^2+(a+c)(b+d)+(b+d)^2$, hence $A > 1$.

It's clear that

$$-2 < -\frac{a}{a+b+c}-\frac{b}{b+c+d}-\frac{c}{c+d+a}-\frac{d}{d+a+b} < -1$$
$$2 < \frac{b+c}{a+b+c}+\frac{c+d}{b+c+d}+\frac{d+a}{c+d+a}+\frac{a+b}{d+a+b} < 3.$$

Set $B = \left(\frac{b+c}{a+b+c}+\frac{c+d}{b+c+d}+\frac{d+a}{c+d+a}+\frac{a+b}{d+a+b}\right)$. By Cauchy-Schwars we find that

$$[(b+c)(a+b+c)+(c+d)(b+c+d)+(d+a)(c+d+a)+(a+b)(d+a+b)]\cdot B \geq [2(a+b+c+d)]^2$$
$$\left[(a+b+c+d)^2+a^2+b^2+c^2+d^2+(a+c)(b+d)\right]\cdot B \geq [2(a+b+c+d)]^2$$

We know that $(a+b+c+d)^2 > a^2+b^2+c^2+d^2+(a+c)(b+d)$ because this inequality is reducible to $(a+c)(b+d)+2(ac+bd) > 0$ which is true because all number are positive. Hence

$$2(a+b+c+d)^2 > (a+b+c+d)^2+a^2+b^2+c^2+d^2+(a+c)(b+d).$$

Thus

$$B \geq \frac{4(a+b+c+d)^2}{(a+b+c+d)^2+a^2+b^2+c^2+d^2+(a+c)(b+d)} > \frac{4(a+b+c+d)^2}{2(a+b+c+d)^2} = 2.$$

Hence $B > 2$. ■