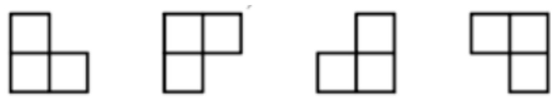
# GTN EXCL Program

## Problemas

## 1. Problemas

**Problema 1.1.** Find the number of rational numbers $r$, $0 < r < 1$, such that when $r$ is written as a fraction in the lowest terms, the numerator and the denominator have a sum of 1000.

**Problema 1.2.** Each unit square of $4 \times 4$ square grid is colored either red, green or blue. Over all possible coloring of the grid, what is the maximun possible number of L-trominos that contain exactly one square of each color? (L-trominos are made up of three unit squares sharing a corner, as shown below.)



**Problema 1.3.** You see four statements. Select which of them are true.

- If the number $2^n - 1$ is prime for some positive integer $n$ then $n$ is prime.
- Call positive integer number $k$ *good* if for any integer $a$ the number $a^k - a$ is divisible by 1001. The smallest good number greater than 1 is 721.
- Given prime number $p = 2017$. Numbers $0^{21}$, $1^{21}$, $2^{21}$, $\cdots$, $(p-1)^{21}$ have distinct remainders modulo $p$.
- For any positive integer $k$ there are infinitely prime numbers $p$ such that $k$ is a square residue modulo $p$.

**Problema 1.4.** Briefly tell what is wrong with this solution.

**Problem.** The residential are has the shape of a rectangle divided by $a$ vertica and $b$ horizontal lines into $(a+1)(b+1)$ rectangular plots. The inspector can find out the area of any small rectangle. Whate is the smallest number of questions enough to findf out the area of the entire area?

**Answer:** $a + b + 1$.

**Solution.** We are going to prove the following statement: For an integer $n \geq 0$, the following is true: for any non-negative integers $a$ and $b$ whose sum is $n$, at least $n+1$ question is needed.

Base $n = 0$. We have only one small rectangle, we need to ask a question about its are.

Step $n \to n+1$. Let's take any $a$ and $b$ with the sum of $n+1$. Since $n+1 > 0$, then one of the numbers $a$ or $b$ is greater than zero. We can assume that $a > 0$. Consider the last column. Obviously, it is necessary to ask about at least one section from this column. Consider the remaining rectangle $(a) \times (b+1)$. Acoording to the assumption of induction, it needs at least $(a-1) + b + 1 = n$ questions. Another question is needed for the last column, so a total of $n+1$ question is needed.

**Problema 1.5.** Bob was given a problem. *Integer $a$ and $b$ satisfy $(\sqrt{3}+2)^n = a+b\sqrt{3}$. Prove that* $a^2-3b^2=1$. Bob wrote the following solution. Denote $a_n$ and $b_n$ as integers satisfying $(\sqrt{3}+2)^n = a_n + b_n\sqrt{3}$. We'll prove $a_n^2 - 3b_n^2 = 1$ using induction. Base case for $n=1$ is obvious because $a_1 = 2, b_1 = 1$. For induction step you need to note that

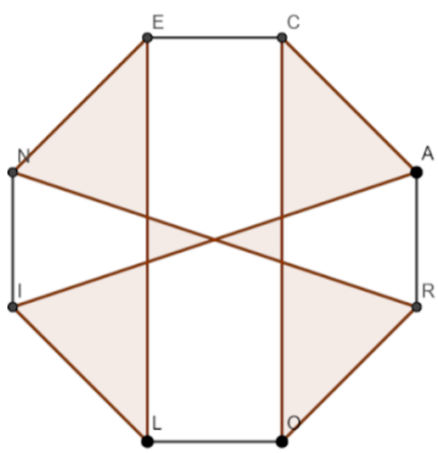
$$\left(\sqrt{3}+2\right)^{n+1} = \left(a_n + b_n\sqrt{3}\right)\left(2+\sqrt{3}\right) = 2a_n + 3b_n + (a_n+2b_n)\sqrt{3}.$$

Hence $a_{n+1} = 2a_n + 3b_n$ and $b_{n+1} = a_n + 2b_n$. Let's prove that $(2a_n+3b_n)^2 - 3\left(a_n+2b_n\right)^2 = 1$.

$$(2a_n+3b_n)^2 - 3\left(a_n+2b_n\right)^2 = (4-3)a_n^2 + (9-12)b_n^2 + (12-12)a_nb_n = a_n^2 - 4b_n^2 = 1.$$

The teacher told that this solution is incomplete without lemma that $a_n$ and $b_n$ are always unique. Is the teacher correct? Try to explain your answer briefly.

**Problema 1.6.** An equiangular octagon $CAROLINE$, $CA = RO = LI = NE = \sqrt{2}$ and $AR = OL = IN = EC = 1$. The self-intersecting octagon $CORNELIA$ enclosed six non-overlapping triangular regions. Let $K$ be the area enclosed by $CORNELIA$, that is, the total area of the six triangular regions. Then $K = \frac{a}{b}$, where $a$ and $b$ are relatively prime positive integers. Find $a+b$.




**Problema 1.7.** Let $x_1 < x_2 < x_3$ be the three real roots of the equation $\sqrt{2014}x^3 - 4029x^2 + 2 = 0$. Find $x_2\left(x_1+x_3\right)$.