

The Atalanta effect: How high-powered compensation reduces risk-taking

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Abstract

Agents frequently compete for both relative-performance rewards—mutual fund inflows generated by rankings, executive promotions based on peer comparisons, rank-dependent social status—and absolute-performance rewards—bonus payments for meeting targets, performance fees based on “high-water marks,” wealth from portfolio returns. It is well known that bonus compensation can engender risk-taking. However, we show that introducing bonus rewards into rank-based competitions can *reduce* risk-taking through the “Atalanta effect”: bonus rewards distract superior competitors from rank competition, thereby leveling the playing field and consequently reducing downside risk taking by weak competitors.

1 Introduction

Frequently agents’ welfare depends on the ranking of their performance relative to others: mutual fund managers compete for rankings in league-tables, a key determinant of fund inflows and thus fund manager compensation (Chen et al., 2015; Chevalier and Ellison, 1997; Del Guercio and Tkac, 2008; Ma et al., 2019). Executives compete for promotions based on relative ranking (Goel and Thakor, 2008; Kini and Williams, 2012; Coles et al., 2017). In fact and estimated 20% of large US firms use “forced ranking systems,” that base promotion and termination entirely on relative performance (Bates, 2003). Ranking can also affect welfare through its effect on social status (Becker et al., 2005; Ray and Robson, 2012). Ranking concerns even drive the risk-taking decisions of financial professions when ranking is anonymous (and thus divorced from status) and has no effect on pecuniary rewards (Kirchler et al., 2018). Thus, for quite disparate reasons, a considerable body of research suggests that many financial agents are *rank-motivated*, i.e., their welfare is affected by their ranking relative to other agents. At the same time, agent welfare is obviously also affected by non-rank absolute performance rewards, e.g., bonus compensation, stock options, portfolio values.

A literally classic example of mixed rank and non-rank rewards is provided by the mythical race between the huntress Atalanta and her suitor Melanion. When pressured by her father to marry, Atalanta agreed to marry the first suitor who defeated her in a footrace, provided that losing suitors would be executed. Atalanta was an accomplished sprinter; many suitors died. Because the outcome of these races depended purely on relative performance, these races were pure rank-based competitions.

By tossing three golden apples, Melanion introduced non-rank based incentives into this rank-based competition. Atalanta now faced a mix of rank and non-rank based rewards, escaping marriage and picking up golden apples. The pursuit of the golden apples distracted Atalanta sufficiently for Melanion to eke out a victory in the rank-based marriage competition.

In this paper we consider financial contests which, like the Atalanta/Melanion race, feature mixed rank and non-rank rewards. Specifically, we consider agents who can devote their capacities either to capturing rank-based rewards or the absolute performance rewards provided by *challenging bonus packages*, bonus compensation packages with bonus targets sufficiently high to encourage risk taking in the absence of rank rewards. Theoretical models and empirical studies have linked both bonus rewards and rank-based rewards with risk taking (e.g., Strack, 2016; Palomino and Prat, 2003; Lee et al., 2016; Ma and Tang, 2019). The question we ask is how challenging bonus packages affect risk taking by rank-motivated agents. We establish the following characterizations:

- (i) *Atalanta effect*: challenging bonus packages always reduce downside risk taking by rank-motivated agents.
- (ii) In some cases, challenging bonus packages also reduce upside risk taking by rank-

motivated agents.

These results do not depend on the origins of rank rewards or the motivations for offering bonus compensation. Thus, this paper does not attempt to determine optimal bonus compensation policies. Rather, like Green (1984), Martynova and Perotti (2018), and Carpenter (2000), it aims to characterize the effect of given reward structures on behavior. As the literature discussed above notes, rank rewards, in practice, are generated by a variety of forces frequently outside of the control of firms. The motivations advanced for offering bonus compensation are even more varied.¹

We do not claim that bonus compensation packages are provided by firms to reduce risk taking. We do show that bonus compensation can make contestant performance more attractive to risk averse agents (Section 3.5). However, bonus compensation for rank-motivated agents can be motivated by considerations orthogonal to risk reduction. In fact, in the on-line supplement (Section F), we endogenize bonus compensation in a framework where all agents are indifferent to the risks created by rank competition.² Our positive predictions on the effects of bonus compensation on risk taking are invariant to the motivations underlying the design of bonus compensation.

In our baseline, single-period analysis, two contestants with unequal ability compete by submitting *performance*. Performance is a random variable. For example, mutual fund managers' performance might consist of the returns or net asset values of their portfolios at the end of the period. Unequal ability is modeled by a *capacity* constraint, an upper bound on mean performance that is not the same for the two contestants. This specification is consistent with portfolio construction in complete markets and with the "unlimited operating flexibility" assumption used in models of corporate and venture-capital finance (Admati and Pfleiderer, 1994; Ravid and Spiegel, 1997).³

Contestants are also able to capture a bonus payment if their performance exceeds a threshold level. This bonus scheme, although stylized, closely tracks many standardized incentive schemes, e.g., 80/120 bonus plans which are, by far, the most common bonus compensation schemes for managers in U.S. corporations (Murphy, 1999).⁴ In the baseline analysis, we assume that bonus packages are *challenging*, i.e., the contestants have insufficient capacity to capture the bonus with certainty.

In addition to bonus packages, contestants also receive rank-based rewards: they earn

¹Rationales range from managerial bargaining power (Bebchuk et al., 2001) to optimizing managerial multi-tasking (Matějka and Ray, 2017).

²In the online supplement, we endogenize bonus compensation in a framework in which all agents are risk neutral. Bonus compensation is used by firms to motivate increased mean performance.

³It is possible to impose further restrictions on contestants' choice spaces, i.e., caps on maximum performance without substantially affecting the results of this analysis, provided that the caps are not so restrictive that they make attaining challenging bonus targets impossible.

⁴Under an 80/120 bonus scheme, the firm sets a performance target and associated maximum bonus. If the manager's performance equals or exceeds 120% of the target, the manager receives the maximum bonus; if performance falls short of 80% of the target, no payment is made to the manager; between 80% and 120% of the target, the bonus payment increases linearly in performance.

a rank reward whenever their performance tops the performance of their rival. We model competition for rank as a risk-taking contest. The risk-taking contest framework has been used to model rank competitions in financial market contests (Seel and Strack, 2013; Strack, 2016), social status contests (Robson, 1992; Becker et al., 2005; Ray and Robson, 2012) and political contests (Myerson, 1993). The risk taking contest framework slightly modifies the zero-noise “all-pay auction” rank-competition framework by limiting contestant performance through an expectational capacity constraint rather than through bid or effort costs.⁵ The all-pay auction framework (e.g., Olszewski and Siegel, 2016) has been extensively analyzed and is perhaps the most widely used framework for modeling rank competition.⁶ Thus, the two component parts of our framework are quite standard.

Combining these two standard frameworks to model mixed incentives produces a surprising result—the Atalanta effect: Although both bonus and rank competition, in isolation, engender downside risk taking, introducing bonus compensation into rank competitions reduces downside risk taking. Although surprising, the logic behind the Atalanta effect is fairly straightforward. In rank contests, winning small, i.e., outperforming rivals by a tiny margin, reaps the same reward as winning big; losing big produces the same reward as losing small. Downside risk taking results because weak contestants adopt “lose-big/win-small” strategies that involve downside risk taking: either mimicking strong contestants’ performance or submitting the worst feasible performance, zero.

Because weak contestants’ performance is low on average, it is easy for strong contestant to top with high probability. Thus, the marginal gain to strong contestants from applying capacity to attain rank dominance is smaller. Hence, when bonus packages are introduced, bonus chasing is more attractive to strong contestants. Capturing bonuses requires strong contestant to divert capacity to reaching the bonus target. Because targets are challenging, strong contestants are unable to always submit performance above the bonus threshold. Consequently, bonus chasing makes strong contestants less effective rank competitors at performance levels less than the bonus target. This reduces the amount of downside risk the weak contestant must assume to effectively compete for rank. Because only weak contestants assume downside risk, the overall level of downside risk taking falls.

The effect of bonus compensation on upside risk taking, the upper bound for the con-

⁵The alternative to the zero-noise framework is the noisy contest framework, e.g., Lazear–Rosen tournaments (Lazear and Rosen, 1981) and Tullock auctions (Tullock, 1980), where performance plus an exogenous noise term determines ranking. When the noise level is small, equilibria in the noisy contest setting are fairly intractable: mixed strategies over a countable set of performance levels whose only accumulation point is zero (cf. Ewerhart, 2015, 2017; Fu and Lu, 2012). For this reason, almost all papers in the noisy contest literature assume that the noise level is sufficient to guarantee the existence of pure-strategy, no risk-taking, equilibria. For this reason, noisy contest models are not used to model rank-motivated risk taking.

⁶In fact, the risk taking and all-pay auction equilibria are related by a correspondence: the equilibrium outcomes of the risk-taking contest are the same as the equilibrium outcomes of an all-pay auction in which the bidders’ valuation of the good in the all-pay auction equilibrium equals the reciprocal of the contestants’ shadow price of capacity (i.e., Lagrange multipliers) in the risk-taking contest equilibrium. For more discussion, see Appendix Section B.

testants’ performance, is less clear cut than its effect on downside risk taking. However, we show that when bonus targets are challenging but not extremely high and either (a) strength asymmetry between the contestants is extreme or (b) bonus compensation is fairly generous, bonus compensation also reduces upside risk-taking.

Because the Atalanta effect is founded on very generic properties of rank contests, the effect is not dependent on the specific simplifying assumptions of the baseline single-period model, i.e., the number of contestants, the exact form of the bonus contract. The implication of the Atalanta effect—reduced downside risk taking—does however depend on its hypothesis—challenging bonus packages. When bonus packages are *easy*, i.e., the bonus can be attained with probability one at least by strong contestants, bonus compensation can *increase* downside risk taking. Thus, when bonus compensation, in isolation, does not induce risk taking, it can increase risk taking when introduced into rank competitions.

Although the Atlanta effect appears to flow from quite generic properties of single-period rank competitions, it is not obvious that it is robust to the incentives produced by dynamic competition. Dynamic contests generate continuation values, values of future payoffs that depend on current actions. These continuation values might make the marginal return from rank competition higher for stronger contestants, leading stronger contestant to focus more on rank competition and thus force weaker contestants to take even more downside risk. It might seem that this continuation value effect should be especially strong when the stronger contestant can “knock out” the weaker contestant by winning one more rank competition, thereby monopolizing all future rank and bonus rewards.

We show that the continuation value effects produced in the dynamic settings can actually favor the Atalanta effect: once a contestant has a big lead, the question of which contestant will eventually emerge victorious is largely resolved. Thus, the *marginal* increase in continuation value to the leading contestant from topping a trailing rival once more can be quite small. This low marginal benefit encourages the leader to “chase the golden apples” and divert capacity to bonus chasing, thereby reducing the leader’s efficiency as a rank competitor and consequently reducing risk taking by the weaker contestant.

Contributions and related literature

Our analysis makes positive predictions on the relationship between risk taking and compensation when agents compete in rank-order contests. The hypotheses tested in the extensive body of empirical financial economics research on relative performance incentives are also substantially founded on rank-order contest theory.⁷ Thus, our predictions are clearly relevant to empirical research. A basic implication of our analysis for this research is that the function linking risk taking with rank and absolute performance rewards is not linear, not separable, and not always monotone. Thus, tests that aim to identify the risk

⁷See, for example, Coles et al. (2017); Kini and Williams (2012); Kempf and Ruenzi (2008); Chevalier and Ellison (1997); Kale et al. (2009).

taking/rank competition relationship using reduced-form linear specifications to control for performance pay are misspecified.⁸

Experimental research can also benefit from a theoretical framework that encompasses the interaction between rank and absolute performance rewards. Kirchler et al. (2018), in an experiment using financial professionals, found underperforming professionals selected significantly more risky portfolios when the experimenters confidentially informed them of their relative performance rankings, even though no pecuniary rewards were associated with these rankings. Thus, Kirchler et al. (2018) suggests that agents, or at least financial professionals, will pursue rank dominance even when pecuniary rewards and social status are not tied to performance rankings.

Kirchler et al. (2018) also found that linear compensation packages had little effect on subject behavior. In our framework, linear incentive packages have no effect on risk-taking. So, at least in our setting, this result is not surprising. More importantly perhaps, our analysis can be used to calibrate non-linear compensation packages for laboratory experiments which can identify, from contestant strategies, contestants' marginal rate of substitution between monetary absolute-performance rewards and the non-pecuniary rewards of rank dominance.

Because this paper integrates incentive compensation into the analysis of asymmetric contests and tournaments, it is closely related both to research on asymmetric contests (e.g., Xiao, 2016; Siegel, 2010; Hillman and Samet, 1987; Baye et al., 1996), and research on the effects of incentive compensation on risk-taking (e.g., Drechsler, 2014; Rose-Ackerman, 1991; Van Wesep and Wang, 2014). Our results extend the contest literature by incorporating the effect of absolute performance rewards into a standard rank-competition framework, and extend the compensation literature by incorporating rank incentives.⁹

Our analysis of rank incentives is also, in a general sense, related to research based on “keeping up with the Joneses” (KUJ) utility, i.e., utility that depends on the compensation received by rival agents (e.g., Lazear, 1989; Levine, 1991; Akerlof and Yellen, 1990; DeMarzo and Kaniel, 2017; Roussanov, 2010). However, the effects of rank-based rewards and KUJ utility are quite different. For example, KUJ models cannot accommodate non-pecuniary, non status-based preferences for rank dominance (Kirchler et al., 2018). Moreover, KUJ models almost always assume a smooth relationship between an agent's utility and the compensation of rival agents. Thus, compensation just below and just above rival agents' average compensation produces approximately the same utility. For this reason, KUJ models, in contrast to rank-competition models, do not encourage “win small/lose big” risk taking strategies, the focus of our analysis.

⁸More discussion of the empirical applications of our results is provided in Section 6.

⁹Because, the pursuit of rank and absolute performance rewards can be thought of as multitasking, our work is also somewhat related to the multitasking literature (Hölmstrom and Milgrom, 1991; Schöttner, 2007).

2 Example

Before introducing the general model, we develop an example that captures the intuition behind its fundamental insight. The development of the example is heuristic. The formal analysis, which will fill in some of the technical details, commences in the next section.

Consider two rival money managers in a single-period world: a weak manager (he), W , and a strong manager (she), S . These managers trade at date 0 and buy assets that produce a liquidating dividend at date 1, this liquidating dividend is a non-negative random variable, represented by X_W and X_S , respectively. We term the realized dividend the manager's *performance*.

The risk-neutral managers receive a bonus, $b \geq 0$, if their performance weakly exceeds a bonus threshold, r . In addition, the managers may receive a rank-based reward if their performance exceeds the performance of their rival. In the event of a performance-tie, the rank reward is divided equally between the two managers. A manager's *payoff* is the sum of the manager's expected rank and bonus rewards.

Financial markets are perfect and valuation is risk neutral. This implies that the attainability of a performance distribution depends only on its expected value. Absent bonus rewards, this specification is a special case of the randomly complete financial-market contest model of Strack (2016). Thus, the only constraint on the manager's performance is that expected performance, $\mathbb{E}[X_i]$, $i = S, W$, is limited by a *capacity constraint*, an upper bound on expected performance that depends on managerial ability, termed the manager's *capacity*. We assume that the capacities of the two managers are common knowledge. The strong manager's capacity equals μ_S and the weak manager's capacity equals μ_W , where $0 < \mu_W \leq \mu_S$. We also assume that neither manager has sufficient capacity to capture the bonus with certainty, i.e., $r > \mu_S$. In our example, the weak manager's capacity, μ_W , is 1, and the strong manager's capacity, μ_S , is 3.

2.1 Pure bonus competition

Suppose the bonus payment, b , equals 1, and the bonus threshold, r , equals 15, and there are no rank-based rewards. Given the capacity constraint, neither managers can submit performance that always weakly exceeds the bonus threshold. Thus, both managers will choose performance distributions that maximize the probability of attaining the bonus threshold subject to their capacity constraints. Consequently, both managers will adopt a “bang-bang” strategy of placing all probability weight on 0 or r .

In this bonus competition game, let p_i^B represent the probability of a performance of 0 for a manager of type $i = S, W$. The capacity constraint is clearly binding. Thus, optimal performance strategies satisfy

$$p_i^B \times 0 + (1 - p_i^B) \times 15 = \mu_i, \quad i = S, W.$$

The assumed capacities of the managers, and the fact that $r = 15$, imply that $p_S^B = 4/5$ and $p_W^B = 14/15$. Consequently, bonus competition engenders significant downside risk taking.

2.2 Pure rank competition

Next, consider pure rank competition game. Specifically, assume that the bonus payment $b = 0$, and the rank reward equals 1. The equilibria of such games have been extensively analyzed in the literature (e.g., Xiao, 2016; Hillman and Riley, 1989). The equilibrium calls for the strong manager to randomize uniformly over $[0, 6]$, and the weak manager to submit a performance of 0 with probability $p_R^0 = 2/3$ and, with probability $1 - p_R^0 = 1/3$, also submit performance uniformly distributed over $[0, 6]$.

Verifying this equilibrium is straightforward. First, consider the weak manager. Given the uniform distribution played by the strong manager, all distributions over $[0, 6]$ featuring the same expected performance produce the same payoff. Performance in excess of 6 produces the same probability of winning the rank reward as 6, and uses up more capacity. Because the capacity constraint is binding, using more capacity to attain the same probability of winning the rank reward lowers a manager's payoff.

Similarly, for the strong manager, because the weak manager is randomizing between a point mass at 0 and a uniform distribution over $[0, 6]$, all performance distributions placing no probability mass at 0, supported by $[0, 6]$, and satisfying the capacity constraint, produce the same payoff. Any strategy placing a positive mass on 0 results in a performance tie between the weak and strong manager at 0. An arbitrarily small increase in performance breaks the tie and increases the strong manager's probability of winning from $1/2 \times 2/3$ to at least $2/3$. Since the shadow price of capacity is not infinite, a sufficiently small increase in performance increases the strong manager's payoff. Any performance level in excess of 6 produces the same probability of winning the rank contest, 1, as a performance of 6, and uses more capacity.

Thus, in this pure rank competitions, the weak manager aims to shadow the performance of the strong manager to the extent he is able and compensates for his inferior capacity by taking downside risk. The capacity difference between the two managers is reflected only in difference between their downside performance levels.¹⁰

2.3 Rank and bonus competition

Now add a bonus package to the rank-based competition. Thus, assume that, in addition to the rank reward of 1, the managers receive the same bonus compensation as they received in the pure bonus competition game: a bonus payment $b = 1$ with a bonus threshold $r = 15$.

¹⁰In fact, conditioned on winning the rank competition, the weak manager's performance stochastically dominates the strong manager's performance. See Noe (2019) for verification of this assertion in an all-pay effort-bidding auction setting.

Our first observation is that the pure rank competition equilibrium strategies are not equilibrium strategies in this rank and bonus competition. A necessary condition for the pure rank competition strategies to be best replies in the rank and bonus competition is that the strong manager cannot gain from deviating to *chasing the bonus*, i.e., choosing a strategy that captures the bonus reward with positive probability.

If the weak manager plays his pure rank competition equilibrium strategy, all possible strong-manager payoffs from randomizing between two performances—(a) a performance between 0 and 6, the boundaries of the support of her pure rank competition equilibrium distribution, and (b) a performance of $r = 15$ —lie on a line in performance/reward space whose endpoints are $(0, 2/3)$ and $(15, 2)$, where the first component of these vectors is performance and the second is the associated reward. We call this line the *bonus line*. This characterization follows because, as performance approaches 0, the strong manager’s probability of winning the rank contest approaches $p_R^0 = 2/3$, and the manager does not capture the bonus. Thus, the manager’s reward equals $2/3$. If the strong manager’s performance equals 15, she captures the bonus $b = 1$, and, because the highest performance under the pure rank competition strategies is 6, she also captures the rank reward, 1. Thus, her reward equals $1 + b = 2$. The slope of the bonus line, which we represent by Δ_S^{Bonus} , equals

$$\Delta_S^{\text{Bonus}} = \frac{2 - 2/3}{15 - 0} = \frac{4}{45} \approx 0.089. \quad (2.1)$$

Similarly, all possible rewards to the strong manager from randomizing between two points in $[0, 6]$, lie on the line, termed the *rank line*, which connects $(0, 2/3)$ and $(6, 1)$. Thus, the slope of the rank competition line, which we represent by Δ_S^{Rank} , is given by

$$\Delta_S^{\text{Rank}} = \frac{1 - 2/3}{6 - 0} = \frac{1}{18} \approx 0.056. \quad (2.2)$$

For the strong manager, the slope of the bonus line is greater than the rank line and the two lines have the same intercept. Consequently, all randomizations between a point in $[0, 6]$ and $r = 15$, produce a higher payoff to the strong manager than randomizations between points in $[0, 6]$, implying *a fortiori* that when both randomizations satisfy the strong manager’s capacity constraint with equality, chasing the bonus produces a higher payoff than pure rank competition. Thus, when the bonus is introduced, the pure rank-competition equilibrium cannot be sustained. As illustrated in Figure 1, under the managers’ pure rank competition equilibrium strategies, the rank line is “too flat” to make rank competition attractive to the strong manager.

However, an equilibrium in which the strong manager chases the bonus and completely ignores rank competition also cannot be sustained: if the strong manager devotes all capacity to bonus chasing and thus randomizes between 0 and $r = 15$, given the capacity constraint $\mu_S = 3$, the strong manager will win the rank and bonus competition with probability $1/5$ and otherwise receive 0, yielding a payoff of $2 \times 1/5 = 0.40$. The weak

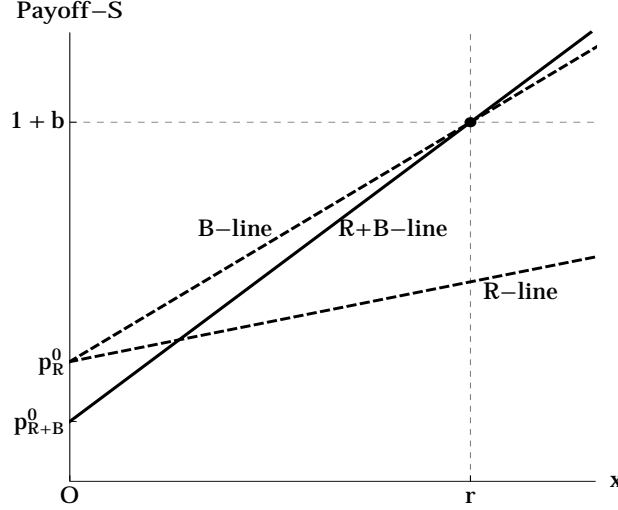


Figure 1: *Introducing bonus packages into rank competitions.*

manager would, in this case, win the rank competition with probability $4/5$ and would have capacity to spare to chase the bonus, implying a much higher payoff for the weak manager, which is not possible.¹¹ So, in equilibrium, the strong manager must both compete for rank dominance and chase the bonus.

Let p_{R+B}^0 represent the probability the weak manager submits 0 performance in the rank and bonus competition equilibrium. For the strong manager to both chase the bonus and compete for rank dominance, the managers' strategies must adjust to equate the strong manager's rank and bonus lines. If, in the rank and bonus competition equilibrium, the weak manager does not chase the bonus, the rank line is determined by two points in performance/reward space—(i) the point associated with zero performance, $(0, p_{R+B}^0)$, and (ii) the point associated with performance equal to bonus threshold, $(r, 1 + b) = (15, 2)$.

Given that the second point, $(r, 1 + b) = (15, 2)$, is fixed, and that the slope of the rank line must increase to make rank competition attractive to the strong manager, in the rank and bonus competition equilibrium, $(0, p_{R+B}^0)$ must lie below $(0, p_R^0)$. Consequently, in the rank and bonus competition equilibrium, the probability of zero performance by the weak manager, p_{R+B}^0 , must be lower than p_R^0 , the probability under the pure rank competition. This adjustment is illustrated in Figure 1.

Using the tools developed in the next section (Lemma 3), we can compute the equilibrium strategies of the two managers: with probability $p_{R+B}^r = 1/5 - 4/5\sqrt{31} \approx 0.06$, the strong manager targets the bonus, i.e., sets $x = r$ and, with probability $1 - p_{R+B}^r$, uniformly randomizes over $[0, u_{R+B}]$ where $u_{R+B} = \sqrt{31} - 1 \approx 4.57$; The weak manager, with probability $p_{R+B}^0 = 1/15(14 - \sqrt{31}) \approx 0.56$ sets $x = 0$ and, with probability $1 - p_{R+B}^0$, also uniformly randomizes over $[0, u_{R+B}]$.

Thus, as expected, $p_{R+B}^0 < p_R^0$, i.e., the probability of downside risk taking, is reduced

¹¹It is always feasible for the strong manager to simply copy the performance distribution selected by the weak manager, which will result in the strong manager receiving the same payoff as the weak manager.

by the introduction of bonus compensation. The reduction in downside risk taking results because the strong manager diverts $p_{R+B}^r \times r \approx 0.84$ of her capacity, $\mu_S = 3$, to bonus chasing. This diversion reduces the upper bound for the strong contestant's performance below the bonus threshold from $u_R = 6$ to $u_{R+B} \approx 4.57$. Because of this reduction, the downside risk, p_{R+B}^0 , the weak manager must assume to compete for rank dominance falls.

3 Risk Taking in Equilibrium

3.1 Formalization of the assumptions

As discussed in Section 2, contestants compete for both rank and bonus rewards, where performance is represented by the realization of random variables, X_S and X_W . Contestants' strategies can be characterized by the *performance distributions* they pick, denoted by F_i ($i = S$ or W). We represent the supports of the performance distribution of S and W by Supp_S and Supp_W respectively. As in the example in the previous section, we assume the bonus threshold, r is greater than the strong contestant's capacity, μ_S , the bonus payment, b , is greater than 0, and the rank reward equals 1.

Remark 1. A very standard result in the all-pay and risk taking literature is that, in equilibrium, contestants never submit performance distributions that result in a positive probability of tied performance. If the contestants tie with positive probability, a contestant could shift probability mass from the tie point to a higher performance level arbitrarily close to the tie point. This shift would cause the contestant's payoff to jump up, while the capacity used to make the shift could be made arbitrarily small. In Appendix Section D we provide a derivation of this result in the context of our model.

Because the rank-based reward is normalized to 1, and, per Remark 1, tied performance cannot occur in equilibrium, the expected rank-based rewards given performance x for contestant S and W are given by

$$\mathbb{P}(x > X_W) = \mathbb{P}(x \geq X_W) = F_W(x), \quad \mathbb{P}(x > X_S) = \mathbb{P}(x \geq X_S) = F_S(x),$$

respectively. The non-rank compensation for the contestants takes the form of a *bonus package* consisting of a bonus payment b and a bonus threshold r , where $b > 0$ and the bonus package is challenging, i.e., $r > \mu_S$. The bonus payment, b , is captured if and only if performance weakly exceeds the bonus threshold, r . Thus, the manager's bonus reward given performance x is given by $b \mathbb{1}_r(x)$. In this definition and in the sequel, for any real number c , we represent the indicator function for the set $[c, \infty)$ by $\mathbb{1}_c$, i.e.,

$$\mathbb{1}_c(x) = \begin{cases} 0 & x < c, \\ 1 & x \geq c. \end{cases}$$

We denote the reward from submitting a performance level of x to the strong and weak contestant by $\Pi_S(x)$ and $\Pi_W(x)$, and term Π_S and Π_W the *contest reward functions*. The reward functions are given by

$$\Pi_S(x) = F_W(x) + b \mathbb{1}_r(x), \quad \Pi_W(x) = F_S(x) + b \mathbb{1}_r(x). \quad (3.1)$$

Because contestants are risk neutral, their payoffs are given by the expectation of the rewards produced by the performance distributions. Thus, the payoffs to the two contestants are given by

$$\int_{\mathbb{R}^+} \Pi_i(x) dF_i(x), \quad i = S, W.$$

A contestant's problem can be formulated as choosing performance distribution over the non-negative real line to maximize her payoff. Each contestant faces a *capacity constraint* which restricts her choice of performance distribution. The capacity constraint for contestant $i = S, W$, is given by

$$\int_{\mathbb{R}^+} x dF_i(x) \leq \mu_i, \quad i = S, W.$$

We assume that the capacity, μ_i , of the strong contestant exceeds the capacity of the weak contestant, i.e., $\mu_S < \mu_W$.¹²

A pair of performance distributions, (F_S^*, F_W^*) , is an equilibrium if the probability of tied performance under (F_S^*, F_W^*) is zero and each performance distribution is a best reply to the other, i.e.,

$$\int_{\mathbb{R}^+} \Pi_i(x) dF_i^*(x) = \max \left\{ \int_{\mathbb{R}^+} \Pi_i(x) dF_i(x) : F_i(0-) = 0 \text{ and } \int_{\mathbb{R}^+} x dF_i(x) \leq \mu_i \right\}, \quad i = S, W, \text{ where}$$

$$\Pi_S(x) = F_W(x) + b \mathbb{1}_r(x), \quad \Pi_W(x) = F_S(x) + b \mathbb{1}_r(x).$$

In Appendix Section D we show that this definition identifies the same set of equilibria as would be identified if distribution pairs permitting tied performance were permitted and a tie-breaking rule was imposed that divided the rank reward equally between the contestants.

Our aim is to characterize equilibrium contestant performance distributions. Our characterization builds on a large literature on rank competitions and somewhat smaller but still significant literature on risk-taking rank contests. In order to avoid exhausting readers' patience, we will start by stating, in Remark 2, a few fairly obvious, known restrictions

¹²This assumption rules out the symmetric capacity, $\mu_S = \mu_W$. The relation between model parameters and equilibrium strategic behavior in our analysis is continuous. So, substituting μ for μ_S and μ_W recovers characterizations of equilibrium strategies in the symmetric case. However, because there is no downside risk taking in the symmetric case, all of our strict characterizations of downside risk taking with asymmetric capacity are weak and trivial in the symmetric case. We do not think it is worthwhile to expend space to provide such characterizations in our named results.

that these frameworks place on performance distributions. In Appendix Section D, we provide formal derivations of these restrictions.

Remark 2.

- (a) With the possible exception of performance $x = r$, the supports of the contestants' performance distributions will coincide.
- (b) $\text{Supp}_i \cup [0, r)$ and $\text{Supp}_i \cup [r, \infty)$, $i = S, W$, are connected.
- (c) Contestant performance distributions are continuous except perhaps at 0 and r .
- (d) 0 must be in the support of both contestants' performance distributions.

The logic behind these restrictions is that, when small changes in positive performance levels do not affect the probability of receiving the bonus reward, contestants' incentives are determined by the following characteristics of rank competitions: first, in rank competitions, topping rivals' performance by a small amount garners the same reward as topping by a large amount, and topping by a large amount uses more capacity. Thus, when competing for rank dominance, contestants "shadow" each other's performance. Second, an increase in performance not accompanied by an increase in the probability of topping rivals is never optimal. So over regions where the rank reward does not vary, the supports of the contestant's performance distributions are connected. Third, because discontinuities in rivals' performance strategies imply an infinite marginal gain from arbitrarily small increases in a contestant's performance, such discontinuities cannot occur in equilibrium.

Remark 3. The basic tool for identifying equilibrium distributions is a multiplier characterization of best responses. This characterization shows that the rewards produced by all performance levels, x , included in the support of a contestant's best reply are collinear. In other words, here exist *multipliers*, $\alpha_i \geq 0$ and $\beta_i > 0$, such that, if x is in the support of i 's equilibrium performance distribution, then $\Pi_i(x) = \alpha_i + \beta_i x$. We call the map $x \mapsto \alpha_i + \beta_i x$ contestant i 's *support line*. This implies that if x' and x'' are two performance levels in the support of i 's performance distribution, $\Pi_i(x'') - \Pi_i(x') = \beta_i (x'' - x')$, i.e., for a given contestant, the marginal gain from submitting any two performance levels in the support of the performance distribution is constant and equals to β_i . The multiplier characterization simply generalizes the marginal slope condition used to develop the example in Section 2. Since submitting a constant performance distribution equal to capacity is always feasible, the expected payoff to a contestant equals $\alpha_i + \beta_i \mu_i$. In Appendix Section D, we provide a formal justification for this approach.

3.2 Configurations

3.2.1 Configuration Eq0: Neither S nor W chases the bonus

Eq0 is the simplest configuration. In this configuration, the contest reduces to a purely rank-based contest. In Eq0, the bonus is not attractive to either contestant and both contestants play the pure rank-based competition strategies illustrated in Section 2.2, i.e.,

the same strategies that they would play in the absence of bonus compensation. Lemma 1 characterizes equilibrium strategies in this configuration. The proofs of this lemma and all subsequent results is deferred to Appendix Section A.

Lemma 1 (Eq0: Neither contestant chases the bonus) *When neither S nor W chases bonus, Eq0, the unique equilibrium can be characterized by the following performance distributions:*

$$F_S^* = \text{Unif}[0, 2\mu_S],$$

$$F_W^* = \left(1 - \frac{\mu_W}{\mu_S}\right) \mathbb{1}_0 + \frac{\mu_W}{\mu_S} \text{Unif}[0, 2\mu_S].$$

Both contestants' capacity constraints bind, and the weak contestant's probability of downside risk taking equals $(\mu_S - \mu_W)/\mu_S$.

In this equilibrium configuration, the strong contestant randomizes uniformly over $[0, 2\mu_S]$. The weak contestant does not have sufficient capacity to emulate this performance distribution. The weak contestant mimics the strong contestant's performance distribution to the extent his capacity permits. This requires "paying" for high performance associated with the strong contestant by putting probability mass at 0, i.e., taking downside risk. The support line for the strong contestant is $\alpha_S + \beta_S x$, where $\alpha_S = (\mu_S - \mu_W)/\mu_S$ and $\beta_S = \mu_W/(2\mu_S^2)$ and, for the weak contestant, $\beta_W x$ with $\beta_W = 1/(2\mu_S)$. Note that, because $\mu_W/\mu_S < 1$, the marginal value of capacity for the strong contestant, β_S is less than the marginal value for the weak contestant, β_W .

3.2.2 Configuration Eq1: Only S chases the bonus

As b increases or r decreases, bonus chasing could become attractive for one of the contestants. Our first result is when only one contestant chases the bonus, then the bonus chasing contestant must be the strong contestant.

Lemma 2 *There do not exist equilibria in which the weak contestant chases the bonus (i.e. the probability that the weak contestant captures the bonus is positive) and the strong contestant does not chase the bonus.*

The logic for this result is simple. If the strong contestant ignores the bonus and simply pursues rank competition, the strong contestant must be applying more capacity to rank competition than the weak contestant, implying that the marginal gain from applying capacity to rank competition is lower for the strong contestant. The marginal gains from chasing the bonus are the same for both weak and strong contestants. Thus, whenever bonus chasing is attractive to the weak contestant, it is also attractive to the strong contestant.

Our next result, Lemma 3, specifies equilibrium strategies for the two contestants when only one contestant, S , chases the bonus.

Lemma 3 (Eq1: S but not W chases bonus) *When only S chases the bonus, the equilibrium can be characterized by the following performance distributions:*

$$\begin{aligned} F_S^* &= p_S^r \mathbb{1}_r + (1 - p_S^r) \text{Unif}[0, u], \\ F_W^* &= p_W^0 \mathbb{1}_0 + (1 - p_W^0) \text{Unif}[0, u], \end{aligned}$$

where

$$u = \frac{2r\mu_W}{\mu_W + \sqrt{2br\mu_W + \mu_W^2}} < r, \quad p_S^r = \frac{2\mu_S - u}{2r - u} \in (0, 1), \quad p_W^0 = 1 - \frac{2}{u}\mu_W \in (0, 1).$$

In this equilibrium, both contestants' capacity constraints bind.

In Eq1, the strong contestant chooses a uniform distribution over $[0, u]$, with probability $1 - p_S^r$, and a point mass at the bonus threshold, r , with probability p_S^r ; the weak contestant randomizes by choosing a point mass at 0, with probability p_W^0 , and the same uniform distribution as the strong contestant, with probability $1 - p_W^0$.

Because the bonus payment, b , is positive, performance just below the bonus threshold, r , is dominated by performance equal to the bonus threshold. Thus, the upper bound of the region over which the contestants compete purely for rank dominance, u , is always less than the bonus threshold, r . Consequently, one can think of the region $(0, u]$ as the *pure rank-competition* region. In this region, the bonus is not captured and lowering performance lowers the probability of topping the rival contestant.

The distributions specified in Lemma 3 are best replies if and only if the graphs of the contestants' reward functions lie weakly below their support lines and, at all performance levels, x , in the support of their performance distributions, the support line evaluated at x equals the contest reward function evaluated at x . These optimality conditions are illustrated in Figure 2. Because the rank-based reward is earned if and only if the performance tops rival's performance and ties do not occur in equilibrium, the probability that a contestant wins the rank reward for a performance level of x is $F_j(x)$ where j represents the performance distribution of the rival. The contest reward functions of the contestants jump up by b at $x = r$, the point where the bonus threshold is attained.

Note that part of the strong contestant's capacity is diverted to targeting the bonus. Thus, the upper bound on the uniform distribution, u , over the pure rank competition region must be less than $2\mu_S$, the upper bound when all of the strong contestant's capacity is devoted to rank competition. Because $u < 2\mu_S$, the definition of p_W^0 provided in Lemma 3 implies that p_W^0 , the probability that the weak contestant places point mass on 0 in configuration Eq1, is less than $(\mu_S - \mu_W)/\mu_S$, the probability when both contestants

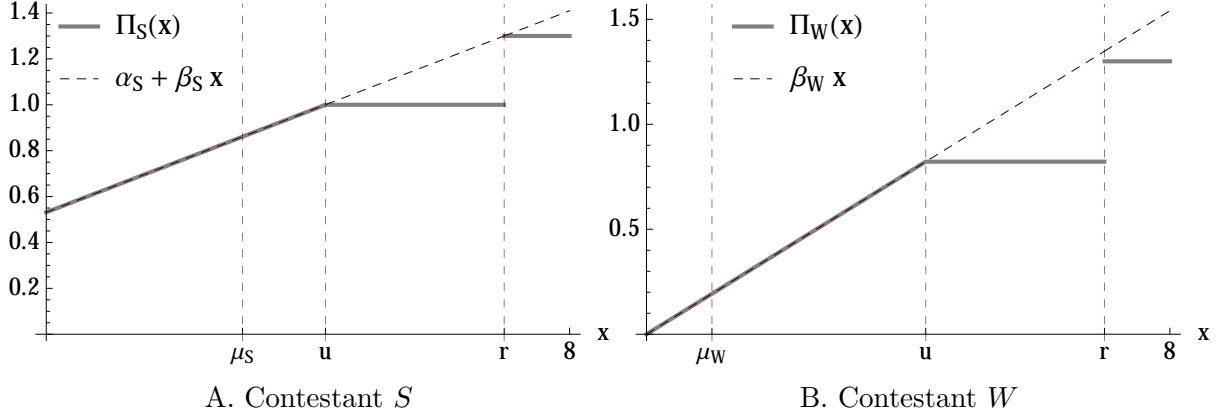


Figure 2: *Equilibrium rewards in Eq1 equilibria, when only S chases the bonus.* The figure illustrates the reward functions, Π_i and support lines, $\alpha_i + \beta_i x$ for contestant S (Panel A) and contestant W (Panel B). The horizontal axis represents performance, x . The parameters are $\mu_S = 3$, $\mu_W = 1$, $r = 7$ and $b = 0.3$.

ignore the bonus.

Proposition 1 (Atalanta effect—Eq1) *In configuration Eq1, (i.e., when only one contestant chases the bonus), the probability of downside risk taking, p_W^0 is less than the probability of downside risk taking in the absence of bonus compensation, $(\mu_S - \mu_W)/\mu_S$.*

In configuration Eq1, the reduction in downside risk taking by the weak contestant can be quite dramatic when the bonus threshold is very high and the size of the bonus is correspondingly large. This case is illustrated in Figure 3. When the bonus threshold is high, attaining the bonus requires a large diversion of capacity from rank competition by the strong contestant, resulting in large reduction in downside risk taking by the weak contestant. In the example in the figure, the weight placed on zero by the weak contestant is reduced from 0.67 to 0.17 by the introduction of bonus compensation.

3.2.3 Configuration Eq2: Both S and W chase the bonus

When the conditions for configuration Eq1 is violated, bonus chasing is tempting for the weak contestant as well as the strong contestant. When this occurs, our third configuration, Eq2, characterizes equilibrium behavior. In the Eq2 configuration, contestant performance distributions must satisfy the following conditions: (i) Each contestant's reward at the bonus threshold must meet her support line. (ii) The slope of each contestant's support line above and below the bonus threshold must be the same. (iii) For both contestants, the capacity constraint is binding. (iv) For both contestants, all points in the support of their performance distributions lie on their support lines. These conditions yield the following characterization of Eq2.

Lemma 4 (Eq2: Both chase bonus) *When both S and W chase the bonus, the unique*

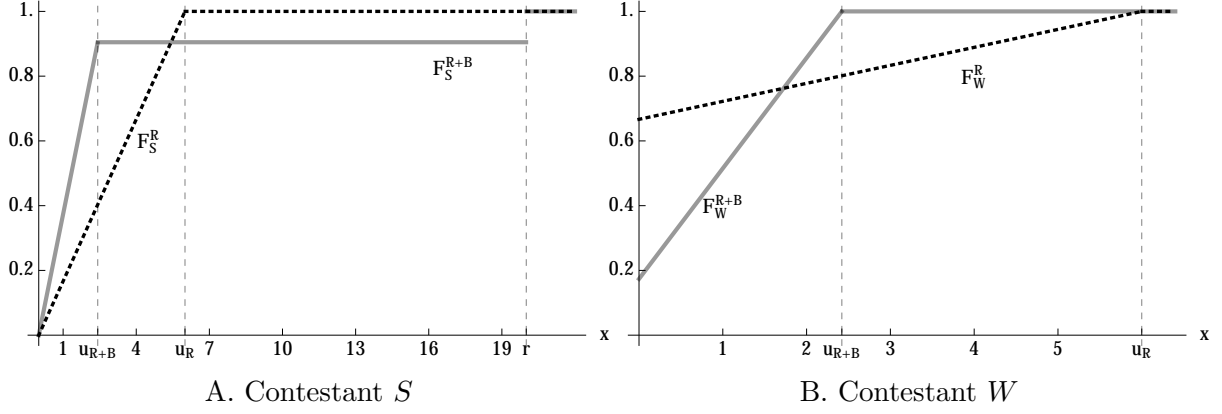


Figure 3: *Effect of bonus compensation on risk taking.* The figure presents the effect of bonus compensation equilibrium performance distributions for the strong contestant, S (Panel A), and the weak contestant, W (Panel B), in configuration Eq1, in which only S chases the bonus. In this figure the parameters $\mu_S = 3$, $\mu_W = 1$ are fixed. F_W^R (F_S^R) represents the performance distribution of W (S) in the absence of bonus competition. F_W^{R+B} (F_S^{R+B}) represents the performance distribution of W (S) when the bonus threshold $r = 20$ and bonus $b = 6$. u_R (u_{R+B}) represents the upper bound of the pure rank competition region under pure rank (rank and bonus) competition.

equilibrium can be characterized by the following performance distributions:

$$\begin{aligned} F_S^* &= p_S^h \text{Unif}[r, u_H] + p_S^r \mathbb{1}_r + (1 - p_S^h - p_S^r) \text{Unif}[0, u_L], \\ F_W^* &= p_W^h \text{Unif}[r, u_H] + p_W^0 \mathbb{1}_0 + (1 - p_W^h - p_W^0) \text{Unif}[0, u_L]; \end{aligned}$$

where

$$\begin{aligned} p_S^r &= \frac{b(\mu_S - \mu_W)}{br + \mu_W} \in (0, 1), & p_S^h &= \frac{(1+b)(u_H - r)}{u_H} \in (0, 1), \\ p_W^0 &= \frac{(1+b)(\mu_S - \mu_W)}{br + \mu_S} \in (0, 1), & p_W^h &= \frac{b(u_H - r)}{r - u_L} \in (0, 1), \\ u_L &\in (0, r), & u_H &> r. \end{aligned}$$

The specific functional forms of u_H and u_L are specified in Appendix Section A. Both contestants' capacity constraints bind.

In configuration Eq2, both contestants place positive probability weight on the region $(r, u_H]$ where performance strictly exceeds the bonus threshold. Because, over this region, sufficiently small reductions in performance always lower the probability of winning the rank reward but never affect the probability of winning the bonus reward, we term the region $(r, u_H]$, the *rank and bonus competition region*.

When performance is insufficient to capture the bonus, i.e., $x < r$, the weak contestant places probability mass p_W^0 on 0 and, with probability $1 - p_W^0 - p_W^h$, chooses a uniform distribution over $[0, u_L]$. The strong contestant, with probability $1 - p_S^r - p_S^h$, chooses a

uniform distribution over $[0, u_L]$. As in the Eq1 configuration, the weak contestant places positive weight on zero performance and the strong contestant does not and, over the pure rank competition region $(0, u_L]$, both contestants choose the same distribution of performance.

When performance is sufficient to capture the bonus, the strong manager *targets the bonus* by placing positive probability weight p_S^r exactly on the bonus threshold, while the weak contestant does not target the bonus and both contestants choose the same performance distribution over the rank and bonus competition region but target this region with different probabilities, p_S^h for the strong contestant and p_W^h for the weak contestant. Contest reward functions for this equilibrium configuration are illustrated by Figure 4.

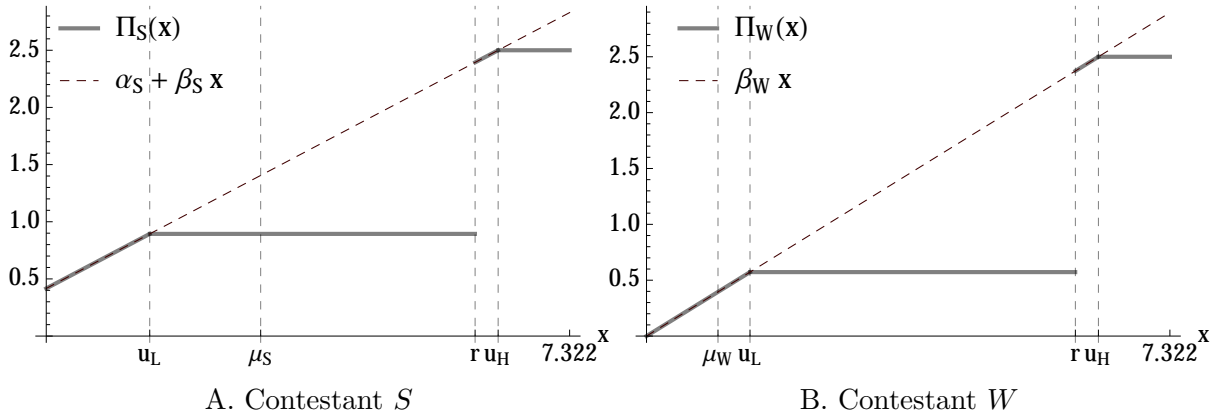


Figure 4: *Equilibrium rewards in Eq2 equilibria, when both S and W chase the bonus.* The figure illustrates the reward functions, Π_i and support lines, $\alpha_i + \beta_i x$ for contestant $i = S$ (Panel A) and contestant $i = W$ (Panel B). The horizontal axis represents performance, x . The parameters are $\mu_S = 3$, $\mu_W = 1$, $r = 6$ and $b = 1.5$.

The next result shows that the Atalanta effect is verified in all Eq2 configurations.

Proposition 2 (Atalanta effect—Eq2) *In configuration Eq2, (i.e., when both contestants chase the bonus), the probability of downside risk taking, p_W^0 , is less than the probability of downside risk taking in the absence of bonus compensation, $(\mu_S - \mu_W)/\mu_S$.*

3.3 Upside risk taking

In this section, we will show that the effect of bonus packages on upside risk taking depends on the degree of strength asymmetry between the contestants, measured by μ_S/μ_W , the size of the bonus, and the equilibrium configuration characterizing contestant behavior.

To make these observations precise, we need to map out the regions in r – b space that sustain the three configurations, Eq0, Eq1, and Eq2. The next result shows that the three equilibrium configurations, in fact, do partition the space of admissible bonus packages.

Lemma 5 *Any choice of admissible bonus packages, i.e., (r, b) , such that $r > \mu_S$ and $b > 0$, sustains one and only one equilibrium configuration, Eq0, Eq1, or Eq2.*

In the appendix we show that, although the regions supporting the three configurations are defined by implicit polynomial equations of fairly high order, fairly simple characterizations can be obtained using parametric curves that map out the boundaries between the regions. Using this approach, the boundary between Eq0 and Eq1 in r - b space and the boundary between Eq1 and Eq2, can be represented by the image of a parametric curve, with parameter $y > 0$, as follows:

$$\begin{aligned}\mathcal{B}_0^1(y) &= \frac{\sqrt{2y\mu_W + \mu_W^2} - \mu_W}{2\mu_S}, & \mathcal{R}_0^1(y) &= \frac{2y\mu_S}{\sqrt{2y\mu_W + \mu_W^2} - \mu_W}; \\ \mathcal{B}_1^2(y) &= y \left(\mu_S + \frac{y + \mu_S}{y + \mu_W} \sqrt{2y\mu_W + \mu_W^2} \right)^{-1}, & \mathcal{R}_1^2(y) &= \mu_S + \frac{y + \mu_S}{y + \mu_W} \sqrt{2y\mu_W + \mu_W^2}.\end{aligned}$$

The Eq1/Eq2 boundary equals the (r, b) pairs that satisfy $r = \mathcal{R}_1^2(y)$ and $b = \mathcal{B}_1^2(y)$, for some $y > 0$. The Eq0/Eq1 boundary equals the (r, b) pairs that satisfy $r = \mathcal{R}_0^1(y)$ and $b = \mathcal{B}_0^1(y)$, for some $y > 0$.

In Figure 5, we provide graphs of these curves. In Panel A, the ratio between the strength of the strong and weak contestant is 5:1; in Panel B, the ratio is 7.5:1. We see that, in both panels, at $y = 0$, $(r, b) = (2\mu_S, 0)$ and that, at $y = 0$, tangent slope of the Eq1/Eq2 boundary equals $1/(2\mu_S) = 1/6$. It is also easy to verify that, in both panels, the asymptotic slope of the Eq1/Eq2 boundary is given by $1/(2\mu_W)$. Thus, the initial behavior and asymptotic behavior of both curves are quite similar in the two panels.

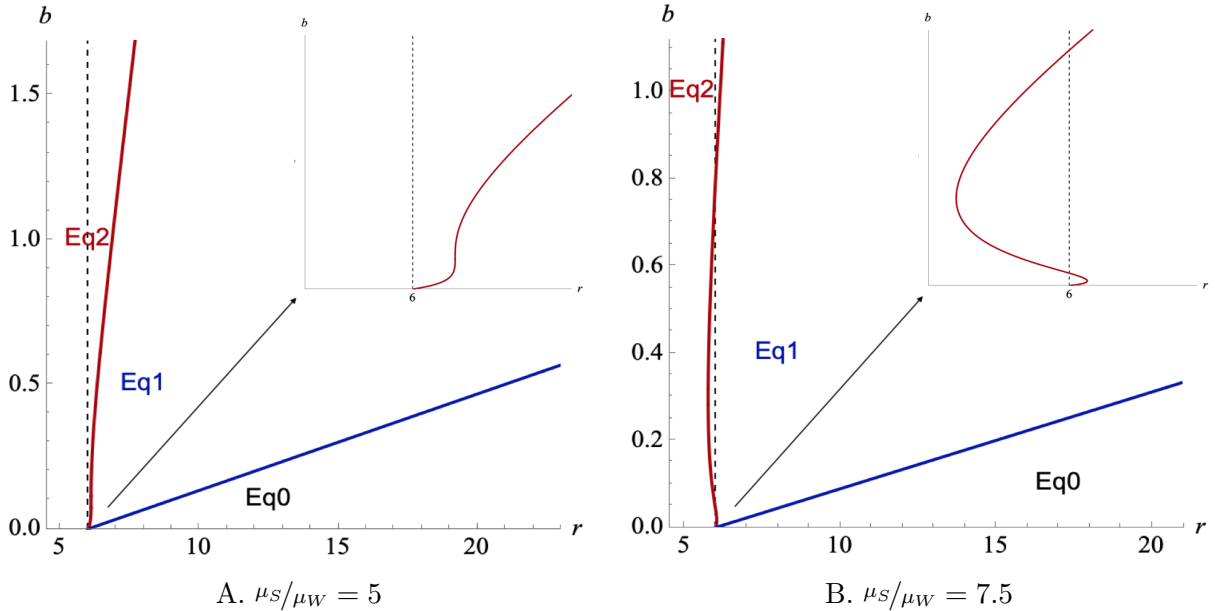


Figure 5: *Parametric curves for equilibrium configurations.* In the figure, the blue line represents the Eq1/Eq2 boundary and the red line represents the Eq0/Eq1 boundary. In Panel A, $(\mu_S, \mu_W) = (3, 0.6)$. In Panel B, $(\mu_S, \mu_W) = (3, 0.4)$. The dotted vertical line represents points where $r = 2\mu_S$, maximum performance in pure rank contests. In the top right corner of each graph, we provide a magnified view of Eq1/Eq2 boundary when b is small.

However, in Panel A, where strength asymmetry is not very extreme, the curve always

moves to the north-east, away from the dashed line representing the $r = 2\mu_S$ line. In Panel B, where strength asymmetry is very extreme, the curve bends back and transversally intersects the $r = 2\mu_S$ line. Thus, in Panel A, where strength asymmetry is less extreme, Eq1 configurations can be sustained only when the bonus threshold, r exceeds maximum contestant performance under pure rank competition, $2\mu_S$. However, in Panel B, where strength asymmetry is extreme, for some choices of the bonus level, b , Eq1 equilibria can be sustained even when at bonus thresholds, r , is less than $2\mu_S$.

In Eq2 equilibria where both contestants chase the bonus, upside risk taking depends on the bonus threshold, r , and length of the rank plus bonus compensation region, $u_H - r$. Bonus compensation both makes competing for rank rewards at performance levels above the bonus threshold attractive relative to competing at performance levels below the threshold and makes competing for rank rewards less attractive than competing for bonus rewards. Given these insights the following proposition should not be too surprising.

Proposition 3 *If the equilibrium configuration is Eq1,*

- (i) *when strength asymmetry is not extreme, i.e., $\mu_S/\mu_W \leq 3 + 2\sqrt{2} \approx 5.83$, maximum contestant performance, r , weakly exceeds maximum contestant performance under pure rank competition, $2\mu_S$;*
- (ii) *when strength asymmetry is extreme, there exist bonus packages, (r, b) , under which maximum contestant performance, r , is less than maximum performance under pure rank competition, $2\mu_S$.*

If the equilibrium configuration is Eq2,

- (iii) *maximum contestant performance, u_H , is increasing in the bonus payment, b , for b sufficiently small, and decreasing for b sufficiently large. For all bonus payments $b > 0$,*

$$r \leq u_H(b) \leq \min \left[\mu_S + \sqrt{(r - \mu_S)^2 + \mu_S^2}, r + \frac{\mu_S}{b} \right]$$

Thus, if b is sufficiently large and $r < 2\mu_S$, i.e., maximum contestant performance is less than under pure rank competition.

The intuition for parts (i) and (ii) of the proposition is that, if the bonus threshold is less than maximum performance under pure rank competition, holding the strategy of the strong contestant fixed, chasing the bonus is attractive to the weak contestant. Thus, both the strong and weak contestant chase the bonus when their capacities are not too asymmetric.

If, instead, after the introduction of bonus competition, the adjustment in equilibrium strategies results in an Eq1 equilibrium, the payoff to the weak contestant from rank competition must increase sufficiently to dominate targeting a bonus with a threshold lower than the weak contestant's maximum performance under rank competition. This increase can be affected only if the strong contestant diverts significant capacity to bonus targeting

and the bonus associated with the threshold is not very large. Diverting significant capacity to targeting a small bonus is only optimal for the strong contestant if the strong contestant's marginal gain from rank competition is very small. The strong contestant's marginal gain is only very small when the weak contestant is very "easy to beat," i.e., strength asymmetry is extreme.

Now consider part (iii) of Proposition 3. Bonus compensation has two effects on contestants' incentives to compete at performance levels higher than the bonus threshold: a *subsidy effect* and an *opportunity-cost effect*. The bonus payment, b , provides a subsidy for rank competition above the bonus threshold but not for rank competition below the bonus threshold. This effect favors an increase in the length of the rank and bonus competition region relative to the pure rank competition region and, consequently, an increase in the upper bound of the rank and bonus competition region, u_H .

At the same time, bonus packages generate an opportunity cost for rank competition at performance levels above r : capacity devoted to rank competition could have been devoted to targeting the bonus. *Ceteris paribus*, the opportunity-cost effect leads contestants to place more weight on bonus targeting relative to rank competition and this effect favors a reduction of the length of the rank and bonus competition region.¹³

Figure 6 illustrates the interaction between the subsidy and opportunity-cost effects by plotting maximum performance, u_H , as a function of the level of bonus compensation, b . The subsidy effect dominates at low bonus levels while the opportunity cost effect dominates at high bonus levels.

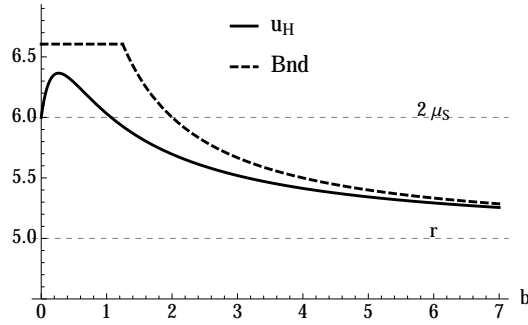


Figure 6: *Subsidy effect vs. opportunity-cost effect.* In the example, $\mu_S = 3$, $\mu_W = 2$, and $r = 5$. $2\mu_S$ represents the upper bound on performance under pure rank competition. $\text{Bnd} = \min \left[\mu_S + \sqrt{(r - \mu_S)^2 + \mu_S^2}, r + \mu_S/b \right]$ represents the upper bound on u_H defined in Proposition 3.iii.

¹³In general, the relation between the upper bound of performance in Eq2 equilibria, u_H , and the level of bonus compensation is not always quasiconcave (examples provided upon request) and, in general $b \mapsto u_H(b)$ can cross the upper bound of performance under pure rank competition, $2\mu_S$, more than once. However, unless the bonus threshold, r , is very close to $2\mu_S$ and strength asymmetry is also fairly close to the extreme asymmetry boundary, $3 + 2\sqrt{2}$, u_H will cross $2\mu_S$ once from above. The statement of the exact conditions required for a single crossing, which involves radical solutions to a cubic polynomial, is a bit cumbersome and thus is relegated to Lemma A.1 in Appendix A.

3.4 How low can downside risk taking go?

Figure 3 in Section 3.2 illustrates that bonus compensation can reduce downside risk taking dramatically. This example naturally raises the question of whether bonus compensations can eliminate downside risk taking entirely. As the next proposition shows, in fact, in all equilibria, the probability of downside risk taking is positive. However bonus packages exist which support equilibria in which the probability of downside risk taking is arbitrarily small. In the limit, these packages, can be very “expensive” in terms of the expected bonus compensation received by the strong contestant.

Proposition 4

- (i) *There exists no equilibrium in which downside risk taking is absent (i.e., $p_W^0 = 0$).*
- (ii) *However, there exists a sequence of bonus packages, $\{(r_n, b_n)\}$ which*
 - (a) *sustains Eq1 equilibria;*
 - (b) *satisfies $r_n \rightarrow \infty$, $b_n \rightarrow \infty$, $\frac{b_n}{r_n} \rightarrow \frac{1}{2\mu_W}$ as $n \rightarrow \infty$;*
 - (c) *yields a downside risk taking approaching 0, i.e., $p_W^0(n) \rightarrow 0$ as $n \rightarrow \infty$.*
 - (d) *In the limit, the distributions of contestant performance approach the equilibrium distribution in a pure rank competition in which both contestants’ capacities equal μ_W .*
- (iii) *The limiting expected value of bonus compensation under these bonus packages equals $\frac{1}{2} \frac{\mu_S - \mu_W}{\mu_W}$.*

Thus, a limiting sequence of bonus compensation packages, $\{(r_n, b_n)\}$ does exist which reduces downside risk taking to an arbitrarily large extent. Under this sequence of bonus packages, the expected bonus payment to the strong contestant, $p_S^r \times b$ converges to $(\mu_S - \mu_W)/\mu_W$. This payment can be quite large. For example if $\mu_W = 1/2$, the asymptotic bonus payment equals $\mu_S - \mu_W$, the entire difference in capacity between the strong and weak contestants.

3.5 Bonus compensation and overall risk?

The analysis thus far has shown that bonus compensation can reduce downside risk taking. However, bonus compensation has other effects on the distributions of contestants’ performance, e.g., it sometimes increases upside risk taking. In some cases, the weak contestant’s distribution under rank and bonus competition second-order stochastically dominates the weak contestant’s distribution under pure rank competition. In fact, this case is illustrated in Figure 3. However, in general, the performance distributions engendered by bonus packages are neither second-order stochastically dominant or dominated by the performance distributions engendered by pure rank competitions. Thus, the performance distributions engendered by bonus compensation will be preferred by some risk averse agents but not by others.

We aim to examine the question of whether the effects on risk taking produced by bonus compensation will be valued by a risk-averse agent endowed with a “reasonable” utility-of-consumption function. For this reason, we choose the utility specification most often used in field studies and laboratory experiments that calibrate risk aversion—constant relative risk aversion (CRRA). We parameterize the CRRA function by assuming a risk aversion coefficient of 3, a coefficient within the range of estimated risk aversion coefficient in a number of studies (Chiappori and Paiella, 2011; Garcia et al., 2003; Mankiw, 1985; Friend and Blume, 1975). In order to ensure that our results are not produced by unreasonably large, in expected value terms, bonus payments, we assume that the risk-averse agent’s consumption is reduced by bonus payments to the contestants. The utility of a risk-averse agent is thus determined by endowed wealth and the sum of two contestants’ performance less the bonus payments received by the contestants.

In our example $\mu_S = 3$ and $\mu_W = 1$, and the risk-averse agent’s consumption endowment equals 0.50.¹⁴ Using a numerical global optimization program, we determine the bonus package that maximizes the utility of the risk-averse agent given the equilibrium strategies of the contestants. The results of this exercise are reported in Table 1.

<i>Eq Config.</i>	<i>Optimal package</i>		<i>Contest outcome</i>			<i>Designer payoff</i>	
	r^*	b^*	u	p_W^0	p_S^r	Optimal package	Pure rank
Eq1	6.467	0.156	4.726	0.577	0.155	-32.475	-33.424

Table 1: *Optimal bonus compensation package.* In this example, the risk-averse agent’s utility of consumption is given by $u(w) = w^{1-\theta}/(1-\theta)$, $\theta = 3$. The capacities of the strong and weak contestants are given by $\mu_S = 3$ and $\mu_W = 1$, respectively. Consumption consists of an endowment equal to 0.5 and the sum of the performance of the two contestants less the cost of the bonus payment.

As we see from the table, the utility maximizing bonus package supports an Eq1 equilibrium configuration. At the maximizing bonus package, the weak contestant submits 0 performance with probability p_W^0 and otherwise randomizes uniformly over the pure rank competition region $(0, u]$. With probability $1 - p_W^0$, the strong contestant also randomizes uniformly over the pure rank competition region, and, with probability p_S^r chooses performance exactly equal to the bonus threshold, r .

Bonus compensation is positive, i.e., the risk-averse agent’s utility is increased by the provision of bonus compensation, even though bonus payments reduces the agent’s consumption. The expected cost of optimal bonus compensation, $p_S^r \times b \approx 0.024$, is quite modest relative to total expected performance, $\mu_S + \mu_W = 4$. Not surprisingly, the package is also far less costly than the limit of the downside risk elimination sequence, whose

¹⁴Given the equilibrium distribution of contestants’ strategies, a positive endowment is required for expected utility of performance to be defined for CRRA utility-of-consumption functions with conventional risk-aversion coefficients.

limiting expected value, as shown in Proposition 4, equals 1. Nevertheless, the reduction in downside risk taking is significant, from $0.6\bar{6}$, in the absence of bonus compensation (see Section 2.2), to 0.577, under the utility-maximizing bonus package. This example shows, for conventional specifications of risk aversion, offering bonus compensation can increase the certainty-equivalent value of contestant performance net of bonus payments. *A fortiori*, this implies that the certainty-equivalent value of performance gross of bonus compensation is higher when bonus compensation is offered.

4 Robustness of the Single-Period Baseline Model

The intuition we provided for our results was that, because strong contestants' marginal rewards from rank competition are smaller than weak contestants, they are more likely to be distracted by bonus compensation. Distraction makes strong contestants weaker rank competitors and thus reduces the amount of downside risk weak competitors need to assume to compete with strong competitors. Thus, our intuition rests on two conditions, whose satisfaction is verified in the baseline model: (a) In pure rank competitions, the strong contestants' marginal reward from capacity invested in rank competition is less than the weak contestants' marginal reward. (b) Diverting capacity to bonus chasing makes strong contestants weaker rank competitors.

Condition (a) relates to our specification of pure rank competition. The only obvious way to generalize our pure rank competition model within the risk-taking contest framework is to increase the number of contestant. In Section 4.1 we argue that condition (a) is robust to this generalization. In Section 4.2 we show that, under linear absolute performance rewards, e.g., pure equity compensation for the contestants, property (b) is not satisfied because contestant strategies are not affected by linear rewards. In Section 4.3 we show that when bonus packages are not challenging, property (b) is sometimes not satisfied because bonus chasing need not make the strong contestants weaker rank competitors.

4.1 Multiple contestants

In Appendix Section B, we show that in contests with more than two contestants, condition (a) is always satisfied when no two contestants have the same capacity. Thus, there is no reason to suspect that simply adding more contestants lead to different qualitative conclusions than the baseline model. Solving the general problem of characterizing equilibrium behavior in all asymmetric multi-contestant rank-and-bonus risk taking contests is a worthy objective. But is not a trivial task and not the objective of this paper. In fact, characterizing all possible equilibrium configurations in pure-rank, three contestant all-pay auctions which, as discussed in the introduction, are closely related to risk taking contests, is quite a challenging task that has only recently been accomplished (Xiao, 2018).

However, as we illustrate by an example in Appendix Section B, it is easy to numerically construct multi-contestant contest equilibria in which the Atalanta effect is verified.

4.2 Linear compensation

In order for condition (b) to hold, it clearly must be the case that compensation packages affect contestant behavior. However, contestants receiving linear compensation will play the same strategies as they would in a pure rank competition contest. To see this, let γ represent the linear compensation coefficient. Contestant payoffs in the linear compensation setting would still be characterized by the support line conditions provided by Remark 3. If we let α' , β' represent the multipliers for the support lines under linear compensation, the equilibrium best-reply conditions can be expressed as

$$\begin{aligned} \gamma x + F_W(x) &= \alpha'_S + \beta'_S x, \quad x \in \text{Supp}_S, & \text{for all } x \geq 0, \quad x + F_W(x) &\leq \alpha'_S + \beta'_S x; \\ \gamma x + F_S(x) &= \alpha'_W + \beta'_W x, \quad x \in \text{Supp}_W, & \text{for all } x \geq 0, \quad x + F_S(x) &\leq \alpha'_W + \beta'_W x. \end{aligned} \quad (4.1)$$

If we let $\alpha'_i = \alpha_i$ and $\beta'_i = \beta_i + \gamma$, $i = S, W$, where α_i and β_i represent the multipliers for the Eq0 equilibrium, we see that the distribution functions specified in Lemma 1 satisfy the equilibrium conditions given by equation (4.1). Thus, the Eq0 equilibrium distribution functions are equilibrium distribution functions under linear compensation.

Most other features of the compensation contract assumed in our analysis—e.g., a fixed bonus payment for reaching a specific target level of performance—are less important for verifying the Atalanta effect. However, the boundedness of bonus compensation does play an important role. The problem with unbounded compensation contracts is not that such contract reverse the Atalanta effect. Rather, when compensation is unbounded, equilibria may not exist. The non-existence problem could be surmounted by either imposing sufficient contestant risk aversion or by judiciously imposing a cap on performance. Resorting to these devices would simply result in a less tractable model without producing new insights.

4.3 Easy bonus compensation

When bonus packages are *easy*, i.e., not challenging, their attainment may not compel strong contestants to sacrifice rank dominance. Thus, condition (b) may not hold. Easy bonus packages are packages impose bonus targets under which at least one contestant is able to attain the target with certainty. Consider the case where only the strong contestant can attain the bonus with certainty, i.e., the case where the bonus threshold satisfies $\mu_W < r < \mu_S$. For the sake of brevity we consider specific examples, more general algebraic characterizations of these equilibrium configurations are provided in Section E of the online supplement.

Example 1 (Low bonus payment, b). In this example, $\mu_S = 4$, $\mu_W = 1$, $r = 2$, and $b = 0.25$. Because $\mu_S > r$, this case represents an easy bonus package. Using the same approach to computing equilibria as used for challenging bonus packages in previous sections yields the following equilibrium: the weak contestant randomizes between a performance of 0, chosen with probability $p_0^W = 23/30$ and a uniform performance distribution over $[r, u]$, where $u \approx 6.6$. The strong contestant places mass on the bonus threshold r , with probability $p_S^r \approx 0.13$, and selects the same uniform performance distribution with probability $1 - p_S^r$. The reward functions, Π , for the strong and weak contestant are graphed in Figure 7.

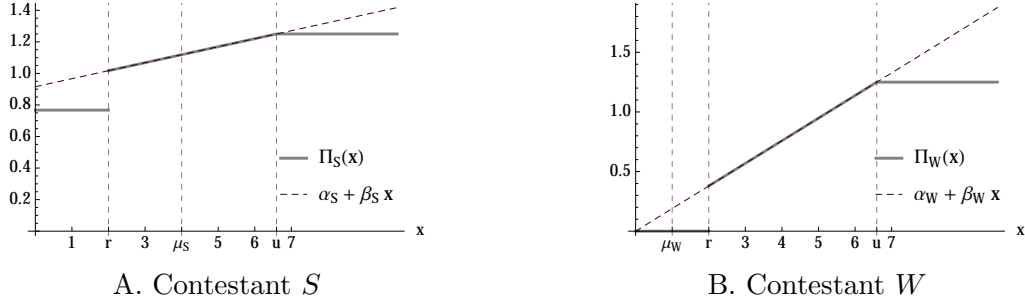


Figure 7: *Equilibrium reward functions under an easy bonus package: small bonus.* The figure illustrates the reward functions, Π_i and support lines, $\alpha_i + \beta_i x$ for contestant S (Panel A) and contestant W (Panel B). The horizontal axis represents performance, x . The parameters are $\mu_S = 4$, $\mu_W = 1$, $r = 2$ and $b = 0.25$.

In this equilibrium, the strong contestant, S , skips the pure rank competition region entirely and concentrates all performance at levels at or above the bonus threshold. This is possible because the bonus package is easy, i.e., features a bonus threshold lower than S 's capacity. For this reason, in contrast to the challenging bonus packages case, chasing the bonus does not make the strong contestant a weaker competitor over the pure rank competition region. In fact, because the strong contestant always submits performance higher than the bonus threshold, any performance in the pure rank competition region by the weak contestant is sure to result in losing the rank competition.

In this example, the bonus payment is fairly small relative to the rank reward, 1. Thus, the weak contestant W is still motivated to compete with S for rank rewards. Because, S does not compete over the pure rank competition region, to compete for rank, W must also submit performance in excess of the bonus threshold, r . Given W 's low capacity, this requires assuming considerable downside risk. In fact, the probability of downside risk taking in this equilibrium equals $p_W^0 = 23/30 > 3/4$, the probability of downside risk taking in a pure rank competition. Consequently, in this example, easy bonus compensation increases downside risk taking.

Now consider another example, fix all the parameters as in the first example except increase b from 0.25 to 2. This increase in the bonus level is sufficient to lead the weak contestant to give up on rank-competition entirely and reverses the conclusions of Example 1.

Example 2 (High bonus payment, b). Suppose that $\mu_S = 4$, $\mu_W = 1$, $r = 2$, and $b = 2$. The equilibrium in this example calls for W to place mass on 0 with probability $p_W^0 = 1/2$ and place mass on the bonus threshold, with probability $1/2$; S uniformly randomizes over $[r, u]$, where $u = 6$. Equilibrium reward functions for the two contestants are illustrated in Figure 8.

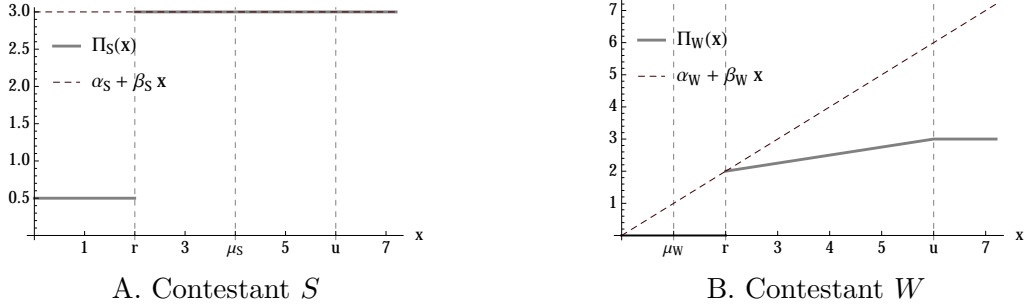


Figure 8: *Equilibrium reward functions under an easy bonus package: large bonus.* The figure illustrates the reward functions, Π_i and support lines, $\alpha_i + \beta_i x$ for contestant S (Panel A) and contestant W (Panel B). The horizontal axis represents performance, x . The parameters are $\mu_S = 4$, $\mu_W = 1$, $r = 2$ and $b = 2$.

In contrast to Example 1, the gains to W from attempting to compete for rank with a much stronger rival are not as attractive as chasing the very large bonus payment. Thus, W skips rank competition entirely, and adopts a pure bonus chasing strategy, conceding rank dominance to S . The fact that bonus targeting is more attractive to W than rank competition with S implies that the amount of downside risk that W must accept to compete with S for rank dominance is less than the amount of downside risk W must accept to target the bonus payment. Thus, the equilibrium probability of downside risk taking $p_W^0 = 1/2$ is less than the probability of downside risk taking under pure rank competition, $3/4$.

Examples 1 and 2 show that introducing easy bonus packages into rank competitions can either increase or decrease downside risk taking. Because the strong contestant eschews pure rank competition, chasing the bonus does not make the strong contestant a weaker competitor. Although bonus compensation can reduce downside risk taking (Example 2) the mechanism by which bonus compensation affects risk-taking is very different from the mechanism driving the Atalanta effect.¹⁵

As we can see, the effects of easy bonus packages on risk taking are quite different than the effect of challenging bonus packages. This raises the question of why we focus our

¹⁵By making bonus rewards even easier, i.e., setting $r < \mu_W$, and offering a sufficiently large bonus payment, it is also possible to support equilibria in which both S and W submit performance at or above the bonus threshold. In such equilibria, bonus payments, in equilibrium, are fixed non-random rewards conditioned on not submitting performance below the bonus threshold. The lower bound for net performance, performance less the fixed payment, is zero, and the probability of net performance equaling 0 is higher than the probability of performance equaling 0 under pure rank competition. See the online supplement (Section E) for details.

analysis on challenging packages. Our focus is motivated by three considerations. First “challenging,” at least as we define this term, appears to be a realistic description of most actual bonus packages. Even CEOs, whose bonus targets are notoriously unambitious, miss targets about 40% of the time (Kay et al., 2015). Unless the probability of meeting challenging targets is close to zero, this implies that between half and all of CEO bonus packages are challenging, i.e., there is some positive probability that bonus targets cannot be attained.

Second, easy bonus packages, do not, in the absence of rank competition, encourage risk-taking and are not the sort of compensation packages analyzed in the literature on bonus compensation and risk taking. Third, while the Atalanta effect might be viewed as being surprising—challenging bonus packages, which, in the absence of rank rewards, encourage risk taking, discourage downside risk taking by rank-motivated contestants—it is not particularly surprising that easy bonus packages, which do not encourage risk taking in the absence of rank competition, may not encourage risk-taking by rank-motivated contestants.

5 Dynamic Contests

Risk taking in rank competitions results from asymmetric competitor strength. Firms, banks, mutual funds, organizers of sporting contests, typically do not aim to set competitions between competitors with wildly differing abilities. Thus, in practice, risk taking appears to be the result of a contestant, say a trader, suffering an adverse shock, e.g., a large trading loss, and thereby “falling behind.” The trailing contestant, realizing that without taking big risks, he has little or no chance of besting his rivals, tries to catch up by ‘doubling down,’ i.e., through employing high risk strategies. The fact that risk taking in rank contests appears, in practice, to typically result from the evolution of dynamic contests naturally raises the question of whether the Atalanta effect persists in a dynamic contest, or, if the new incentives introduced by dynamic competition are sufficient to nullify the effect.

To address this question, we consider the Atalanta effect in the context of dynamic contests where contestants, *ex ante*, have equal capacity. We first develop, in Section 5.1, the general framework for our dynamic predation contest. Next, in Section 5.2 we parametrically specify the general formulation and solve for equilibrium strategies.

5.1 Dynamic contests—general formulation

We incorporate dynamics by assuming an infinite date world with time indexed by t . In each period, contestants compete for both rank and bonus-based period rewards. Performance in the current period affects capacity for performance in the next period. In this dynamic setting, the capacity of contestants evolves over time based on past performance.

Thus, we cannot follow the approach used in the previous section of labeling the contestants “strong” and “weak.” Instead, we label the contestants F and G , and apply the same labels to their performance distributions.

Initially, the two contestants have identical capacities equal to $\mu > 0$. If, in period t , the capacity of a given contestant equals μ_t , then, if the contestant wins the period- t rank competition, the contestant’s capacity in period $t + 1$ increases to $\mu_t + \delta$, where $\delta = \mu/m$, for some $m \in \{2, 3, 4 \dots\}$. If the contestant loses the rank competition in period t , the contestant’s capacity in period $t + 1$ falls to $\mu_t - \delta$. If a contestant’s capacity falls to zero, she becomes insolvent and is “knocked out,” i.e., cannot compete in future rounds. In this event, the surviving contestant has no competitor. Thus, surviving contestant’s capacity no longer changes and the survivor always wins the rank reward. Because the potential gains for the contestants are unbounded in this infinite date setting, we introduce a discount factor ϕ , where $\phi \in (0, 1)$, which is applied to continuation values.

We restrict attention to symmetric equilibria in which contestants play symmetric Markovian strategies. The assumption of Markovian strategies implies that strategies depend only on capacities at the start of the given period, μ_F and μ_G . Thus strategies are a function of the vector, (μ_1, μ_2) , where the first component of the vector equals the capacity of F and the second the capacity of G . These vectors are the states of the Markovian system, we represent state with ω and represent the set of all states by Ω .

The fact that contestants play symmetric strategies implies that the strategy played by G when $\mu_F = \mu_1$ and $\mu_G = \mu_2$ is the same as the strategy played by G when $\mu_F = \mu_2$ and $\mu_G = \mu_1$. Recalling that the first component of the state vector is the capacity of F , we see that the strategy played by G when the state is (μ_1, μ_2) equals the strategy played by F when the state is (μ_2, μ_1) . Thus, to simplify notation, define, for any state $\omega = (\mu_1, \mu_2)$, the *reversed state* $\tilde{\omega}$ by $\tilde{\omega} = \widetilde{(\mu_1, \mu_2)} = (\mu_2, \mu_1)$. Using this definition, the symmetry assumption can be expressed simply as the requirement that, the equilibrium strategy for contestant G in state ω equals the equilibrium strategy for contestant F in state $\tilde{\omega}$. Thus, we need only solve for the equilibrium strategy of contestant F .

A strategy for contestant F in state ω is a performance distribution. At each state, F ’s performance distribution is restricted by the capacity constraint, where capacity is determined by the state. Thus, at state ω , the set of feasible performance distribution for F is given by

$$\mathcal{F}_\omega = \mathcal{F}_{(\mu_F, \mu_G)} = \left\{ F : F[0-] = 0 \text{ and } \int x \, dF(x) \leq \mu_F \right\}. \quad (5.1)$$

Thus, the value function for F will, for each state vector ω map the current capacity vector, (μ_F, μ_G) into value, i.e., $V_F : \Omega \rightarrow \mathbb{R}$, where value is defined to equal the expected discounted rewards in the period and all subsequent periods. The dynamics of the value functions are described by a Markovian process in which the current state of the system is

ω . The states $(0, 2\mu)$ and $(2\mu, 0)$ are the absorbing states for the system. The set of these states is represented by \mathcal{A} . Because a best response for either F or G will never lead to a zero probability of winning the period rank contest, the set of $2m + 1$ recurrent states, represented by \mathcal{NA} , is given by

$$\mathcal{NA} = \left\{ \left(\mu \left(1 + \frac{i}{m} \right), \mu \left(1 - \frac{i}{m} \right) \right) : i \in \{1 - m, 2 - m, \dots, 0, \dots, m - 2, m - 1\} \right\}.$$

If, at a recurrent state, one contestant's capacity is greater than the other's, we will term the contestant with greater capacity the *leading* contestant and the contestant with less capacity the *trailing* contestant.

Computing values at the absorbing states, \mathcal{A} , is simple. If G 's capacity is 0, and thus G has been knocked out of the competition, F is sure to receive the rank reward, 1. Thus, F 's optimal performance distribution in the current period and thereafter maximizes the expected bonus reward. Given that capacity equals 2μ implies that F 's period payoff equals $1 + \frac{2\mu b}{r}$ in each period. If F has been knocked out, then clearly F 's payoff is 0 in the current and all subsequent periods. Discounting these period payoffs yields the value that F must receive at the absorbing states in any equilibrium:

$$V_F(2\mu, 0) = \frac{2b\mu + r}{r(1 - \phi)}, \quad V_F(0, 2\mu) = 0.$$

Now consider the payoff for contestant F in a recurrent state, $\omega \in \mathcal{NA}$. F will choose a random performance, $X \stackrel{d}{\sim} F$ in the current period. This will affect F 's current period reward, determined by the structure of bonus compensation and G 's performance distribution. In addition, performance in the current period will affect the probability of state transitions: if the contestant tops her rival in the current period, i.e., $X_F > X_G$, her capacity will increase by δ while her rival's capacity will decrease by δ . Similarly, if $X_F < X_G$ then her capacity will decrease by δ and her rival's will increase by δ . The effect of state transitions on F 's payoffs will depend on the value function, V_F , assumed.

To capture these effects, define, for $\omega = (\mu_F, \mu_G) \in \mathcal{NA}$, the function \mathcal{T} by

$$\begin{aligned} \mathcal{T}[V_F, F, G](\mu_F, \mu_G) = & \int (G(x) + b \mathbb{1}_r(x)) dF(x) + \\ & \phi \left(\mathbb{P}(X_F > X_G) V_F(\mu_F + \delta, \mu_G - \delta) + \mathbb{P}(X_F < X_G) V_F(\mu_F - \delta, \mu_G + \delta) \right). \end{aligned} \quad (5.2)$$

Note that \mathcal{T} , maps value functions, V_F , and current period contestant performance distributions, F and G , into a possibly different value function given by $\mathcal{T}[V_F, F, G](\mu_F, \mu_G)$. An equilibrium value function satisfies the consistency or “promise keeping” condition of dynamic programming—the value function produced by \mathcal{T} equals the value function input into \mathcal{T} . At the same time, the equilibrium performance distribution selected in each state must also be a best response for each contestant, given the equilibrium value function and

the performance distribution selected by the rival, G . These conditions, combined with symmetry condition (which implies that $G_\omega = F_{\tilde{\omega}}$), yields our definition of an equilibrium in the dynamic contest game:

Definition 1. An equilibrium is a collection of distributions $\{F_\omega : \omega \in \mathcal{NA}\}$, and a function $V^* : \Omega \rightarrow \mathbb{R}$ satisfying the following conditions:

1. For all $\omega \in \mathcal{NA}$,

$$V_F^*(\omega) = \max\{\mathcal{T}[V_F^*, F, F_{\tilde{\omega}}^*](\omega) : F \in \mathcal{F}_\omega\} \text{ and} \quad (\text{NA:PK})$$

$$\mathcal{T}[V_F^*, F_\omega, F_{\tilde{\omega}}^*](\omega) = \max\{\mathcal{T}[V_F^*, F, F_{\tilde{\omega}}^*](\omega) : F \in \mathcal{F}_\omega\}. \quad (\text{NA:BR})$$

2. For all $\omega \in \mathcal{A} = \{(2\mu, 0), (0, 2\mu)\}$,

$$V_F(2\mu, 0) = \frac{2\mu b + r}{r(1 - \phi)} \text{ and } V_F(0, 2\mu) = 0. \quad (\text{A:V})$$

We solve for equilibria numerically by using the standard iteration approach to solving dynamic programming problems. The procedure is detailed in Appendix Section C.2. It essentially involves the following steps: First, we develop a programming problem, whose solution for any fixed value function satisfying equation (A:V), yields performance distributions for each recurrent state, $\omega \in \mathcal{NA}$ which satisfy the best response condition given by equation (NA:BR). We then use the program to define inductively a sequence of value functions (V^n) , for $\omega \in \mathcal{NA}$ by

$$V^n(\omega) = \mathcal{T}[V_F^{n-1}, F_\omega^{n-1}, F_{\tilde{\omega}}^{n-1}](\omega), \quad \omega \in \mathcal{NA},$$

and, for the absorbing states $\omega \in \mathcal{A}$, we define $V^n(\omega)$ using equation (A:V). Using the contraction mapping theorem of dynamic programming, we show that $V^n \rightarrow V^*$ and that V^* satisfies the conditions of Definition 1 and thus is an equilibrium.

5.2 A dynamic predation game

We numerically solve the dynamic predation game in the case where $\mu = 1$ and $\delta = 1/2$. These assumptions imply that the system states, Ω , are given by $(1, 1)$, $(3/2, 1/2)$, $(1/2, 3/2)$, $(2, 0)$, and $(0, 2)$. The transition dynamics between the states is illustrated in Figure 9.

Because the state transition function depicted in Figure 9 implies that if a contestant's capacity reaches 0, the contestant is knocked out of the competition and the rival contestant thus always captures the rank reward in subsequent periods, the contest is a predation contest in which predation requires rank dominance. Thus, the continuation values for the two contestants, which will evolve solely based on success in rank competition, greatly augment the gains from rank competition versus bonus chasing. One might conjecture

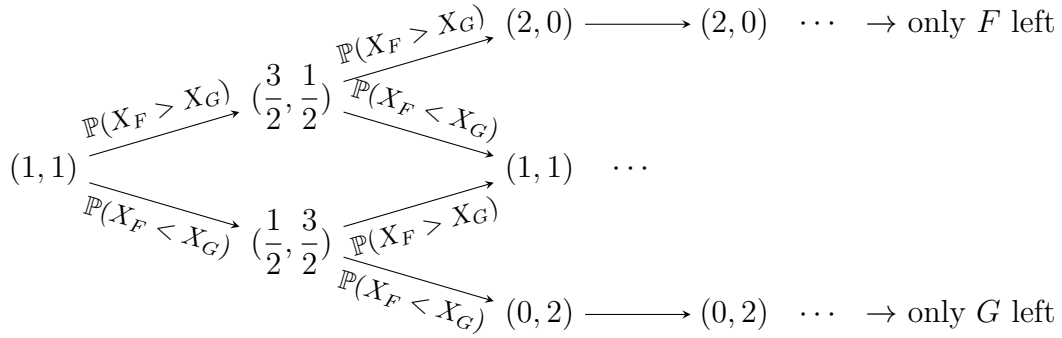


Figure 9: *State transition dynamics.*

that, in this predation setting, leading contestant should concentrate on finishing off the trailing contestant. However, the following result example shows this need not be the case. We confirm that these results are robust to local changes in the parameter values in Appendix Section H.

In this example, $b = 1$, $r = 3$, and $\phi = 0.9$. For the absorbing states, $(0, 2)$ and $(2, 0)$, equilibrium value function is determined by equation (A:V) of Definition 1. Table 2 presents the outcomes of the dynamic predation model (Panel A) and compares them with outcomes of the single-period model with the same bonus and bonus threshold (Panel B) and pure-rank dynamic predation contest (Panel C), i.e. a dynamic predation contest in which the bonus is set to 0. We only provide results at the states where contestant F is leading, as the values when G is leading are given by symmetry.

At state $(1, 1)$ neither contestant chases the bonus in the dynamic predation contest while both contestants chase the bonus in the single-period model. A greater focus on rank competition in the dynamic contest results from the rank-dependent effect of continuation values. Rank dominance increases capacity in the next period and thus increases F 's probability of winning at the continuation state $(\frac{3}{2}, \frac{1}{2})$. If F wins again at $(\frac{3}{2}, \frac{1}{2})$, G is knocked out of the competition and thus F will henceforth monopolize rank rewards. The large continuation reward associated with winning the rank competition at $(1, 1)$ is evidenced by the fact that the value of F at $(\frac{3}{2}, \frac{1}{2})$ is approximately 10 times higher than G 's value. Thus, the predation model clearly increases the focus of rank-based rewards.

At state $(\frac{3}{2}, \frac{1}{2})$, despite the fact that another ranking victory by F will knock G out of the competition, F diverts some capacity to bonus chasing, although much less than in the single-period model. The diversion of capacity produces the Atalanta effect—the probability that G 's performance equals 0 is 0.612 in the dynamic predation contest while under dynamic pure rank competition it equals 0.667. Thus bonus payments reduce downside risk-taking.

The reduction in G 's downside risk-taking is caused by F 's diversion of capacity to bonus chasing. Bonus chasing by F at $(\frac{3}{2}, \frac{1}{2})$, when F is leading, is motivated by two factors: First, at $(\frac{3}{2}, \frac{1}{2})$, for the same reasons as developed in Section 3.2, F 's capacity

	Regions				Values
	downside risk taking	pure rank competition	bonus targeting	rank+bonus competition	(V_L^*, V_T^*)
A: Dynamic rank+bonus					
$(\mu_L, \mu_T) = (1, 1)$ (Prob(L), Prob(T))	{0} (0, 0)	(0, 2] (1, 1)	{3} (0, 0)	\emptyset (0, 0)	(7.710, 7.710)
$(\mu_L, \mu_T) = (3/2, 1/2)$ (Prob(L), Prob(T))	{0} (0, 0.612)	(0, 2.296] (0.813, 0.360)	{3} (0.123, 0)	(3, 3.062] (0.064, 0.028)	(14.614, 1.406)
$(\mu_L, \mu_T) = (2, 0)$ (Prob(L), Prob(T))	{0} (0.333, n.a.)	\emptyset (0, n.a.)	{3} (0.667, n.a.)	\emptyset (0, n.a.)	(16.667, 0.0)
B: Single period rank+bonus					
$(\mu_S, \mu_W) = (1, 1)$ (Prob(S), Prob(W))	{0} (0, 0)	(0, 1.4] (0.875, 0.875)	{3} (0, 0)	(3, 3.2] (0.125, 0.125)	(0.625, 0.625)
$(\mu_S, \mu_W) = (3/2, 1/2)$ (Prob(S), Prob(W))	{0} (0, 0.445)	(0, 0.954] (0.599, 0.465)	{3} (0.284, 0)	(3, 3.184] (0.117, 0.090)	(1.176, 0.314)
C: Dynamic pure rank					
$(\mu_L, \mu_T) = (1, 1)$ (Prob(L), Prob(T))	{0} (0, 0)	(0, 2] (1, 1)	\emptyset (0, 0)	\emptyset (0, 0)	(5.0, 5.0)
$(\mu_L, \mu_T) = (3/2, 1/2)$ (Prob(L), Prob(T))	{0} (0, 0.667)	(0, 3] (1, 1)	\emptyset (0, 0)	\emptyset (0, 0)	(9.083, 0.917)

Table 2: *Equilibria at different states for dynamic, static, and pure rank competition.* The table presents the equilibrium strategies of the leading and trailing contestants, and compares the single-period vs. dynamic, and rank+bonus vs. pure rank cases. In the table, $r = 3$ and $b = 1$. “Prob” represents the probability that the performance of the contestants lies in a given region. (μ_L, μ_T) represent the capacities of leading and trailing contestants in the different states of the dynamic game. (μ_S, μ_W) represent the endowed capacities in the single-period game. Equilibrium values are the value pair (V_L^*, V_T^*) .

advantage reduces the marginal gain to F from rank competition with her weakened rival, G . Second, the value increase produced by winning at $(3/2, 1/2)$ is much less than from winning at $(1, 1)$: knocking out G by winning at $(3/2, 1/2)$ only increases F ’s value from 14.614 to 16.667. At $(3/2, 1/2)$, it is very likely that F will eventually knock out G and F ’s value at $(3/2, 1/2)$ already impounds this fact. If F loses at $(3/2, 1/2)$, F ’s value will fall to 7.710. Thus, the gap between the winning and losing values is less at $(3/2, 1/2)$ than it is at $(1, 1)$. Moreover, given F ’s capacity advantage at $(3/2, 1/2)$, F is much more likely to win at $(3/2, 1/2)$ even if F diverts significant capacity to chasing the bonus. Thus, F ’s value function weighs the small increase in value from winning more than the large decrease from losing.

Thus, when one contestant has a big lead in the dynamic contest, the trailing con-tes-

tant’s optimal strategy involves significant downside risk taking. At the same time, the big lead reduces the leading contestants marginal gain from focusing on rank competition. This reduced marginal gain encourages the strong contestant to chase the bonus. Bonus chasing in the dynamic contest, for the same reasons as advanced for the single-period contest, mitigates downside risk taking by the weak contestant.

6 Conclusions and Implications

In this paper, we analyzed the interaction between rank-based rewards, and rewards conditioned on attaining an absolute level of performance. We showed that introducing bonus compensation into rank-competitions always reduces downside risk taking and sometimes also lowers upside risk taking.

The extensive body of empirical research on tournament incentives and risk taking by financial professionals is motivated, to a very large extent, by theoretical models of rank-order contests. This theoretical literature models agents who do not receive any rewards for absolute performance, clearly not even a first-order approximation of the structure of incentives for financial professionals. Thus, our work has obvious implications for the development of hypotheses for this stream of empirical research.

The most obvious implication of our analysis is that tests of the effect of tournament incentives on risk taking should, at a minimum, control for the interaction between absolute rewards and rank-based rewards. With respect to downside risk taking, our analysis predicts that the interaction effect will be negative. With respect to upside risk taking, our analysis predicts that direction of the interaction effect will depend on the relative magnitude of absolute performance versus rank rewards and the degree of ability heterogeneity between the competitors.

The empirical literature has developed measures for pecuniary rank and absolute performance rewards (e.g., Kini and Williams, 2012; Coles et al., 2017; Ma et al., 2019). Measuring non-pecuniary rank rewards is more challenging. The theoretical literature frequently posits that non-pecuniary rewards for rank dominance are derived from the social status associated with high rankings. However, in a laboratory experiment, Kirchler et al. (2018) documents that the risk-taking behavior of financial professionals is profoundly influenced by rankings, even when rankings are anonymous, i.e., each subject is informed of her own ranking but not the rankings of the other subjects. In contrast, student risk-taking behavior is not affected by anonymous rankings. In another laboratory experiment, Lindner et al. (2019) find that student-subject behavior is affected by publicized rankings.

This evidence suggests that some agent populations have intrinsic rank-preferences, i.e., their welfare is increased by knowing that they “are the best,” even if their ranking is not communicated to other agents and is not associated with any pecuniary rewards. Other populations have rank preferences based on social status. Rank incentives based on social

status depend on the degree to which agents value social dominance. Research in the social psychology literature suggests that the valuation agents place on social dominance can be predicted by the personality sub-trait aggression (Lauriola and Levin, 2001).

Under the hypothesis that the agents competing in a given contest derive similar non-pecuniary rewards from rank dominance, cross sectional variation in their behavior will be determined by differences in their pecuniary incentives. Given the discussion above, this hypothesis appears to be reasonable. Agents competing for rankings compete with agents within the same occupational group. Both the sociology literature on homophily and group formation (Ruef et al., 2003), and the social psychology literature on personality/job matching (Holland, 1973), suggest that agents within a given occupational group have similar personalities and, thus, have similar valuations of rank dominance.

Thus, it seems possible to test the predictions of the risk-taking contest model by measuring the effects of the mix between pecuniary absolute and relative performance rewards on risk taking. Developing hypotheses for such tests would require extending our analysis to consider contests where contestants receive heterogeneous rewards for rank and absolute performance. Such an extension is certainly possible, but considering this paper's length, a task for future research.

References

- Admati, Anat R and Paul Pfleiderer**, "Robust financial contracting and the role of venture capitalists," *The Journal of Finance*, 1994, 49 (2), 371–402.
- Akerlof, George A and Janet L Yellen**, "The fair wage-effort hypothesis and unemployment," *The Quarterly Journal of Economics*, 1990, 105 (2), 255–283.
- Bates, Steve**, "Forced ranking," *HR magazine*, 2003, 48 (6), 62–62.
- Baye, Michael R, Dan Kovenock, and Casper G De Vries**, "The all-pay auction with complete information," *Economic Theory*, 1996, 8 (2), 291–305.
- Bebchuk, Lucian Arye, Jesse M Fried, and David I Walker**, "Executive compensation in America: Optimal contracting or extraction of rents?," Technical Report, National Bureau of Economic Research 2001.
- Becker, Gary S., Kevin M. Murphy, and Iván Werning**, "The equilibrium distribution of income and the market for status," *Journal of Political Economy*, 2005, 113 (2), 282–310.
- Carpenter, Jennifer N**, "Does option compensation increase managerial risk appetite?," *The Journal of Finance*, 2000, 55 (5), 2311–2331.
- Chen, Joseph, Eric Hughson, and Neal Stoughton**, "Strategic mutual fund tournaments," *Mimeo*, 2015.
- Chevalier, Judith A and Glenn D Ellison**, "Risk taking by mutual funds as a response to incentives," *Journal of Political Economy*, 1997, 105 (6), 1167–1200.

- Chiappori, Pierre-André and Monica Paiella**, “Relative risk aversion is constant: Evidence from panel data,” *Journal of the European Economic Association*, 2011, 9 (6), 1021–1052.
- Coles, Jeffrey L, Zhichuan Li, and Albert Y Wang**, “Industry tournament incentives,” *The Review of Financial Studies*, 2017, 31 (4), 1418–1459.
- DeMarzo, Peter M and Ron Kaniel**, “Relative pay for non-relative performance: Keeping up with the Joneses with optimal contracts,” Working paper SSRN 2755856 2017.
- Drechsler, Itamar**, “Risk choice under high-water marks,” *Review of Financial Studies*, 2014, 27 (7), 2052–2096.
- Ewerhart, Christian**, “Mixed equilibria in Tullock contests,” *Economic Theory*, 2015, 60 (1), 59–71.
- , “Contests with small noise and the robustness of the all-pay auction,” *Games and Economic Behavior*, 2017, 105, 195–211.
- Friend, Irwin and Marshall E Blume**, “The demand for risky assets,” *American Economic Review*, 1975, pp. 900–922.
- Fu, Qiang and Jingfeng Lu**, “Micro foundations of multi-prize lottery contests: A perspective of noisy performance ranking,” *Social Choice and Welfare*, 2012, 38 (3), 497–517.
- Garcia, Rene, Richard Luger, and Eric Renault**, “Empirical assessment of an intertemporal option pricing model with latent variables,” *Journal of Econometrics*, 2003, 116 (1-2), 49–83.
- Goel, Anand M. and Anjan V. Thakor**, “Overconfidence, CEO selection, and corporate governance,” *Journal of Finance*, 2008, 63 (6), 2737–2784.
- Green, Richard C**, “Investment incentives, debt, and warrants,” *Journal of Financial Economics*, 1984, 13 (1), 115–136.
- Guercio, Diane Del and Paula A Tkac**, “Star power: The effect of Morningstar ratings on mutual fund flow,” *Journal of Financial and Quantitative Analysis*, 2008, 43 (4), 907–936.
- Hillman, Arye L. and Dov Samet**, “Dissipation of contestable rents by small numbers of contenders,” *Public Choice*, 1987, 54, 63–82.
- Hillman, Arye L and John G Riley**, “Politically contestable rents and transfers,” *Economics & Politics*, 1989, 1 (1), 17–39.
- Holland, John L**, *Making Vocational Choices: A Theory of Careers*, Englewood Cliffs, 1973.
- Hölmstrom, Bengt and Paul Milgrom**, “Multitask principal-agent analyses: Incentive contracts, asset ownership, and job design,” *Journal of Law, Economics, & Organization*, 1991, 7, 24–52.
- Kale, Jayant R, Ebru Reis, and Anand Venkateswaran**, “Rank-order tournaments and incentive alignment: The effect on firm performance,” *Journal of Finance*, 2009, 64 (3), 1479–1512.
- Kay, Ira, Steve Friedman, Brian Lane, Blaine Martin, and Soren Meiseid**, “Are companies setting challenging target incentive goals?,” <https://corpgov.law.harvard.edu/2015/04/18/are-companies-setting-challenging-target-incentive-goals/> 2015.

- Kempf, Alexander and Stefan Ruenzi**, “Tournaments in mutual-fund families,” *Review of Financial Studies*, 2008, 21 (2), 1013–1036.
- Kini, Omesh and Ryan Williams**, “Tournament incentives, firm risk, and corporate policies,” *Journal of Financial Economics*, 2012, 103 (2), 350–376.
- Kirchler, Michael, Florian Lindner, and Utz Weitzel**, “Rankings and risk-taking in the finance industry,” *The Journal of Finance*, 2018, 73 (5), 2271–2302.
- Lauriola, Marco and Irwin P Levin**, “Personality traits and risky decision-making in a controlled experimental task: An exploratory study,” *Personality and Individual Differences*, 2001, 31 (2), 215–226.
- Lazear, Edward P**, “Pay equality and industrial politics,” *Journal of Political Economy*, 1989, 97 (3), 561–580.
- and **Sherwin Rosen**, “Rank-order tournaments as optimum labor contracts,” *Journal of Political Economy*, 1981, 89 (5), 841–864.
- Lee, Jung Hoon, Charles Trzcinka, and Shyam Venkatesan**, “Mutual fund risk-shifting and management contracts,” 2016.
- Levine, David I**, “Cohesiveness, productivity, and wage dispersion,” *Journal of Economic Behavior & Organization*, 1991, 15 (2), 237–255.
- Lindner, Florian, Michael Kirchler, Stephanie Rosenkranz, and Utz Weitzel**, “Social status and risk-taking in investment decisions,” 2019. MPI Collective Goods Discussion Paper.
- Ma, Linlin and Yuehua Tang**, “Portfolio manager ownership and mutual fund risk taking,” *Management Science*, Forthcoming 2019.
- , — , and **Juan pedro Gómez**, “Portfolio manager compensation in the US mutual fund industry,” *The Journal of Finance*, 2019, 74 (2), 587–638.
- Mankiw, NG**, “Consumer durables and the real interest rate,” *Review of Economics and Statistics*, 1985, 67 (3), 353–362.
- Martynova, Natalya and Enrico Perotti**, “Convertible bonds and bank risk-taking,” *Journal of Financial Intermediation*, 2018, 35, 61–80.
- Matějka, Michal and Korok Ray**, “Balancing difficulty of performance targets: Theory and evidence,” *Review of Accounting Studies*, 2017, 22 (4), 1666–1697.
- Murphy, Kevin J**, “Executive compensation,” *Handbook of Labor Economics*, 1999, 3, 2485–2563.
- Myerson, Roger B.**, “Incentives to cultivate favored minorities under alternative electoral systems,” *American Political Science Review*, 1993, 87 (04), 856–869.
- Noe, Thomas**, “Comparing the chosen: Selection bias when selection is competitive,” *Journal of Political Economy*, Forthcoming 2019.
- Olszewski, Wojciech and Ron Siegel**, “Large contests,” *Econometrica*, 2016, 84 (2), 835–854.

- Palomino, Frédéric and Andrea Prat**, “Risk taking and optimal contracts for money managers,” *RAND Journal of Economics*, 2003, *34* (1), 113–137.
- Ravid, S Abraham and Matthew Spiegel**, “Optimal financial contracts for a start-up with unlimited operating discretion,” *Journal of Financial and Quantitative Analysis*, 1997, *32* (3), 269–286.
- Ray, Debraj and Arthur Robson**, “Status, intertemporal choice, and risk-taking,” *Econometrica*, 2012, *80* (4), 1505–1531.
- Robson, Arthur J.**, “Status, the distribution of wealth, private and social attitudes to risk,” *Econometrica*, 1992, *60* (4), 837–857.
- Rose-Ackerman, Susan**, “Risk taking and ruin: Bankruptcy and investment choice,” *Journal of Legal Studies*, 1991, *20*, 277–310.
- Roussanov, Nikolai**, “Diversification and its discontents: Idiosyncratic and entrepreneurial risk in the quest for social status,” *Journal of Finance*, 2010, *65* (5), 1755–1788.
- Ruef, Martin, Howard E. Aldrich, and Nancy M. Carter**, “The structure of founding teams: Homophily, strong ties, and isolation among U.S. entrepreneurs,” *American Sociological Review*, 2003, *68* (2), 195–222.
- Schöttner, Anja**, “Relational contracts, multitasking, and job design,” *The Journal of Law, Economics, & Organization*, 2007, *24* (1), 138–162.
- Seel, Christian and Philipp Strack**, “Gambling in contests,” *Journal of Economic Theory*, 2013, *148* (5), 2033–2048.
- Siegel, Ron**, “Asymmetric contests with conditional investments,” *American Economic Review*, 2010, *100* (5), 2230–2260.
- Strack, Philipp**, “Risk-taking in contests: The impact of fund-manager compensation on investor welfare,” 2016. University of California, Berkeley: Working paper.
- Tullock, Gordon**, “Efficient rent seeking,” in Robert D. Tollison Buchanan, James M. and Gordon Tullock, eds., *Toward a Theory of the Rent-Seeking Society*, Texas A&M University Press, 1980, pp. 3–15.
- Van Wesep, Edward D and Sean Wang**, “The prevention of excess managerial risk taking,” *Journal of Corporate Finance*, 2014, *29*, 579–593.
- Xiao, Jun**, “Asymmetric all-pay contests with heterogeneous prizes,” *Journal of Economic Theory*, 2016, *163*, 178–221.
- , “Equilibrium analysis of the all-pay contest with two nonidentical prizes: Complete results,” *Journal of Mathematical Economics*, 2018, *74*, 21–34.

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Appendix: The Atalanta effect: How high-powered compensation reduces risk-taking

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A Proofs of results

Proof of Lemma 1 Lemma 1 is a consequence of Lemma 3, which is proved below. However, in order to permit the reader to follow the derivations of the results in the sequence in which they are presented in the manuscript, we provide a self-contained derivation below. The derivation is in the spirit of the derivation used in Hillman and Riley (1989), but their derivation is less detailed and their model is a bit different than ours. This derivation will exploit the fact that the pure rank competition game is a two-player constant sum game. Thus, each contestant's payoff is maximized by minimizing the payoff of the rival contestant. Consequently, the game can be solved using minimax approach.

A necessary (but not sufficient) condition for an equilibrium in which neither contestant chases the bonus is that the performance distributions of the contestants are equilibrium distributions for a pure rank competition contest, i.e., a contest where $b = 0$. To see this, note that the payoff for the contestants if neither chases the bonus, equals their payoff in a pure rank competition. If an alternative performance distribution produced a higher payoff, even ignoring whatever rank rewards it might garner, then a candidate equilibrium performance distribution, that candidate equilibrium distribution could not be a best reply.

We will show that the pure rank competition game has a unique equilibrium which yields the equilibrium performance distributions specified in Lemma 1. Thus, these performance distributions must characterize equilibrium performance when neither contestant chases the bonus. Let i be a given contestant and let j be the rival contestant. The definitions of the payoff function given by equation (3.1), show that, for a given performance distribution for i , F_i , j 's payoff equals

$$\max_{dF_j} \left\{ \int_0^\infty F_i(x) dF_j(x) : \int_0^\infty x dF_j(x) \leq \mu_j \text{ \& } F(0-) = 0 \right\} = \hat{F}_i(\mu_j),$$

where \hat{F}_i is the concave upper envelope of F_i , i.e., \hat{F}_i is the pointwise infimum of all affine functions that majorize F_i . This assertion simply follows from the definition of the concave upper envelope (cf. Aumann et al., 1995; Gentzkow and Kamenica, 2014). Because the total rank rewards received by i and j always sum to 1, the payoff to i equals $1 - \hat{F}_i(\mu_j)$. Thus, i 's payoff is maximized by minimizing $\hat{F}_i(\mu_j)$ over feasible performance distributions.

Because, by definition, the concave upper envelope of F_i is a concave distribution that is stochastically dominated by F_i , \hat{F}_i is feasible whenever F_i is feasible. If F_i is not concave, then F_i would strictly stochastically dominate \hat{F}_i . In which case, if F_i is feasible, \hat{F}_i would produce the same payoff as F_i and strictly satisfy the capacity constraint. Thus, F_i would not be optimal because a sufficiently small adjustment in \hat{F}_i would reduce the probability of the rival contest j winning and still satisfy the capacity constraint.

Thus, all optimal performance distributions for i are concave. Because the integral of the survival function, $S_i = 1 - F_i$, of a non-negative random variable equals its expectation, it is convenient to re-express these observations in terms of survival functions rather than distribution functions. Concavity of the distribution function implies convexity of the survival function. Expressed in terms of survival functions, i 's optimal survival functions are the survival functions which solve,

$$\text{PS: } \max_{S_i} \{ S_i(\mu_j) : S_i \text{ is a convex survival function, } S_i(0-) = 1 \text{ \& } \int_0^\infty S_i(x) dx \leq \mu_i \}.$$

Next we show, by a proof by contradiction, that all solutions to PS are affine over their support: suppose that there exists a solution, S_i^o , that is not affine over its support. Let ℓ be an affine function that is a lower support line for S_i^o at μ_j . Convexity assures the existence

of ℓ . Then, by definition, $\ell(\mu_j) = S_i^o(\mu_j)$, and $\ell \leq S_i^o$. Let $S_i'(x) = \max[\ell(x), 0]$. Then S_i' is a convex survival function and $S_i' \leq S_i^o$ and $S_i'(\mu_j) = S_i^o(\mu_j)$. Because S_i^o is not affine, $S_i' < S_i^o$ on a set of positive Lebesgue measure; thus,

$$\int_0^\infty x dF_i'(x) = \int_0^\infty S_i'(x) dx < \int_0^\infty S_i^o(x) dx = \int_0^\infty x dF_i(x) \leq \mu_i.$$

Thus, the capacity constraint is not binding at S_i' . Hence, it is possible to find a feasible survival function, S_i'' such that

$$S_i''(\mu_j) > S_i^o(\mu_j) \text{ and } \int_0^\infty S_i''(x) dx \leq \mu_i, \quad (\text{A-1})$$

through a small increase in the slope or intercept of ℓ . Equation (A-1) contradicts the optimality of the candidate non-affine solution, S^o . This argument is encapsulated by Figure A-1.

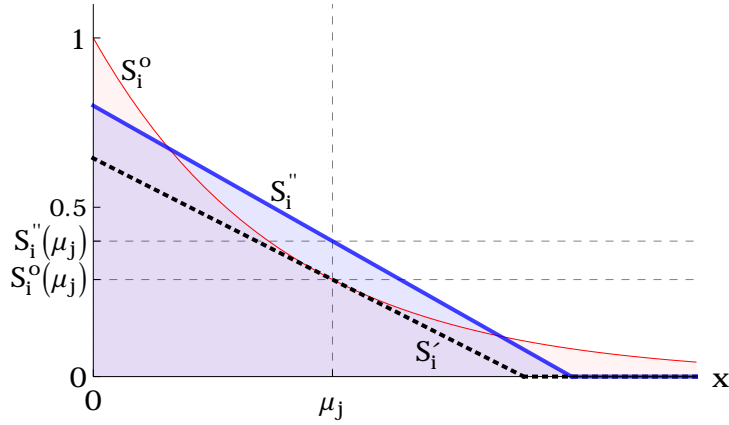


Figure A-1: *Optimal survival functions in pure rank competitions.*

Thus, optimal solutions to PS are affine over their supports and hence have the form, $S(x) = \max[a - bx, 0]$, $a \in [0, 1]$ and $b > 0$. Noting that

$$\int_0^\infty \max[a - bx, 0] dx = \frac{a^2}{2b},$$

we see that optimal survival functions solve the following simple programming problem:

$$\text{PSAff : } \max_{a \in [0, 1], b > 0} \{a - b\mu_j : \frac{a^2}{2b} \leq \mu_i\}.$$

The unique solution to PSAff is given by

$$S_i^*(x) = \min \left[\frac{\mu_i}{\mu_j}, 1 \right] - \frac{1}{2\mu_j} \min \left[\frac{\mu_i}{\mu_j}, 1 \right] x.$$

Thus, in any equilibrium in which neither S nor W chase the bonus, S^* characterizes optimal performance distributions. Considering the case where $i = S$ and $j = W$ as well as the case where $i = W$ and $j = S$, and translating the survival functions in to distribution functions, yields the characterizations of contestant strategies in Lemma 1.

Proof of Lemma 2 Suppose, to obtain a contradiction, that there exists an equilibrium in which only W chases the bonus. Thus, r is in the support of W 's performance distribution. By Remark 3, the reward from choosing $x = r$, $1 + b$, lies on W 's support line, i.e.,

$$\alpha_W + \beta_W r = 1 + b. \quad (\text{A-2})$$

Similarly, because not chasing the bonus is a best response for S , the payoff to S from not chasing the bonus lies weakly below S 's support line, i.e.,

$$\alpha_S + \beta_S r \geq 1 + b. \quad (\text{A-3})$$

Equations (A-2) and (A-3) imply that

$$\alpha_S + \beta_S r \geq \alpha_W + \beta_W r. \quad (\text{A-4})$$

Now let $\bar{x} = \max(\text{Supp}_S)$. The capacity constraint implies that $\bar{x} > 0$. The hypothesis that S is not chasing the bonus implies that $\bar{x} < r$. For $x < r$, the reward to S for performance x , $\Pi_S(x) = F_W(x)$, and, similarly, $\Pi_W(x) = F_S(x)$. Moreover, over $[0, r)$, the supports of F_S and F_W coincide (Remarks 2.a and 2.d) and, hence, $\bar{x} \in \text{Supp}_S \cap \text{Supp}_W$. This implies that \bar{x} lies on both S 's and W 's support lines. Because W is chasing the bonus and S is not, $F_W(\bar{x}) < 1 = F_S(\bar{x})$. These observations imply that

$$\alpha_S + \beta_S \bar{x} = F_W(\bar{x}) < 1 = F_S(\bar{x}) = \alpha_W + \beta_W \bar{x},$$

which, in turn, implies that

$$\alpha_S + \beta_S \bar{x} < \alpha_W + \beta_W \bar{x}, \quad \bar{x} < r. \quad (\text{A-5})$$

Now note that, because S 's capacity exceeds W 's, it cannot be the case that the performance distribution of W first-order stochastically dominates the performance distribution of S . Because S is not chasing the bonus, this implies that there must exist at least one point, say x_o such that $x_o \in [0, \bar{x})$ and $F_W(x_o) > F_S(x_o)$. Using the same argument as used in the previous case, we see that

$$\alpha_S + \beta_S x_o = F_W(x_o) > F_S(x_o) = \alpha_W + \beta_W x_o,$$

which implies that,

$$\alpha_S + \beta_S x_o > \alpha_W + \beta_W x_o, \quad x_o \in [0, \bar{x}). \quad (\text{A-6})$$

Inspection shows that equations (A-4), (A-5), and (A-6) cannot be simultaneously satisfied for any choice of multipliers, α_S , β_S , α_W , β_W , and thereby establishes the contradiction.

Proof of Lemma 3 First note that, in an Eq1 equilibrium, because, by hypothesis, S is chasing the bonus and W is not, if S submits performance r , S wins both the rank reward of 1 and bonus reward of b . Performance $x > r$ will not produce a larger reward and requires more capacity. Thus, r is in the support of S 's performance distribution, F_S , and r is not in the support of W 's performance distribution, F_W , and $x > r$ is not in the support of either contestants' distributions.

We claim that the support lines for the two contestants will have the following properties: $\alpha_S > 0$, $\alpha_W = 0$, $\beta_S > 0$, and $\beta_W > 0$. The fact that β_S and β_W are positive follows simply from the fact that the marginal value of capacity, which can always be applied to either increasing the probability of winning the rank reward or capturing the bonus, is positive. Now consider α_S , and α_W .

First, note that it cannot be the case that both α_S and α_W are positive. Over the rank competition region, the reward to each contestant from performance x equals the distribution function selected by the other contestant evaluated at x . Let $\ell_i(x) = \alpha_i + \beta_i x$, $i = S, W$ represent the support lines of the contestants. Because 0 is in the support of both contestants' performance distributions (Remark 2.d), at point x in the support of the performance distribution, the reward to a contestant from performance x lies on the contestant's support line, i.e., $\ell_i(0) = \Pi_i(0) = \alpha_i$, $i = S, W$ (Remark 3). Thus, if both α_S and α_W were positive, $\Pi_S(0) > 0$ and $\Pi_W(0) > 0$. Since $\Pi_S(0) = F_W(0)$ and $\Pi_W(0) = F_S(0)$, this would imply that there is a positive probability of tied performance, which is impossible (Remark 1). So, if it is not the case that $\alpha_S > 0$ and $\alpha_W = 0$, then $\alpha_S = 0$ and $\alpha_W \geq 0$.

So to obtain a contradiction, suppose that $\alpha_S = 0$ and $\alpha_W \geq 0$. By hypothesis, only S chases the bonus. Thus, the bonus threshold, r , must exceed $\bar{x} = \max(\text{Supp}_W)$. Performance by S equal to \bar{x} wins the rank competition with probability 1. But, because, by hypothesis, S is chasing the bonus, performance by W equal to \bar{x} does not win the rank competition with probability 1. Because \bar{x} is in the support of W 's performance distribution, \bar{x} is in the support of S 's performance distribution (Remark 2.a). Hence, \bar{x} lies on the contestant S 's support line, ℓ_S (Remark 3); thus, $\ell_S(\bar{x}) = \Pi_S(\bar{x}) = 1 > \ell_W(\bar{x}) = \Pi_W(\bar{x})$. Because $\ell_W(0) \geq \ell_S(0)$, and $\ell_W(\bar{x}) < \ell_S(\bar{x})$, it must be the case that the slope of ℓ_S exceeds the slope of ℓ_W . Thus, at all $x > \bar{x}$, $\ell_S(x) > \ell_W(x)$. This implies, because $r > \bar{x}$, it must be the case, that $\ell_S(r) > \ell_W(r)$. If only S chases the bonus, the reward from submitting performance r equals $1 + b$. Because, by hypothesis, S is chasing the bonus, r must lie on S 's support line (Remark 3). So, $\ell_S(r) = 1 + b$. Thus, $\ell_W(r) < 1 + b$. The reward to W from submitting any performance greater than r is $1 + b$. Thus,

$$1 + b = \Pi_W(x) > \ell_W(r), \quad x > r.$$

Because the reward function is right continuous and the support line is continuous, there exists $x > r$ such that $\Pi_W(x) > \ell_W(x)$, which is inconsistent with the support line ℓ_W , being an upper bound on the reward function, and thus is inconsistent with W not chasing the bonus.

Consequently, in any Eq1 equilibrium $\alpha_W = 0$ and $\alpha_S > 0$. Because $\alpha_S = F_W(0)$ and $\alpha_W = F_S(0)$ this implies that W , and only W , places some probability mass on 0. Let p_W^0 be the weight that W places on 0. Because a performance of r is sufficient to capture both the rank and bonus rewards, capacity has a positive shadow price, and, by hypothesis, S chases the bonus, the only point in the support of S performance distribution that is larger than the maximum performance of W is r . Thus, S places probability mass on r . Let p_S^r represent this weight. Let $u = \max(\text{Supp}_W)$ represent the upper bound on the performance of W . All $x \neq r$ are either in the supports of both contestants' distribution or in the supports of neither. Except perhaps at 0 and r , both equilibrium performance distributions are continuous (Remark 2.c). Moreover, Supp_W is connected (Remark 2.b) and the support of S 's distribution, is the same as W 's support except at $x = r$ (Remark 2.a). Thus, the support of W 's performance distribution equals $[0, u]$ and the support of S 's performance distribution equals $[0, u] \cup \{r\}$. All points in the support of the contestants' distributions lie on their support lines, which are affine. Thus, conditioned on choosing a performance level $x \in (0, u]$, both contestants submit uniformly distributed performance.

Thus, we know that in the equilibrium, contestant S puts point mass on r with probability p_S^r and plays a uniform distribution $\text{Unif}[0, u]$ with probability $1 - p_S^r$; whereas contestant W randomizes between 0 with probability p_W^0 and uniform $\text{Unif}[0, u]$ with probability $1 - p_W^0$. Our next task is to relate these parameters to the contestant's capacities and the structure of bonus compensation, defined by r and b .

For contestant S , her support line intersects $(r, 1 + b)$, i.e.,

$$\alpha_S + \beta_S r = 1 + b. \quad (\text{A-7})$$

Meanwhile if S submits u , she wins with probability one, i.e.,

$$\alpha_S + \beta_S u = 1. \quad (\text{A-8})$$

Combining equation (A-7) and (A-8) we have $\beta_S = b/(r - u)$, $\alpha_S = 1 - (b u)/(r - u)$, showing that

$$p_W^0 = \alpha_S = 1 - \frac{b u}{r - u}. \quad (\text{A-9})$$

For contestant W , we require that

$$(1 - p_S^r) \frac{x}{u} = \beta_W x, \quad x \in [0, u], \quad (\text{A-10})$$

Hence, $\beta_W = (1 - p_S^r)/u$.

Because both contestants use up their capacities in equilibrium, we have

$$(1 - p_S^r) \frac{u}{2} + r p_S^r = \mu_S, \quad (\text{A-11})$$

$$(1 - p_W^0) \frac{u}{2} = \mu_W. \quad (\text{A-12})$$

Combining three equations (A-9), (A-11) and (A-12), we are able to solve for the three unknowns $\{u, p_S^r, p_W^0\}$. Given nonnegativity, there is only one solution:

$$u = \frac{2 r \mu_W}{\mu_W + \sqrt{2 b r \mu_W + \mu_W^2}}, \quad p_S^r = \frac{2 \mu_S - u}{2 r - u}, \quad p_W^0 = 1 - \frac{2}{u} \mu_W.$$

Finally, from the results above it is not hard to calculate the contest reward functions for both contestants. The reward function for contestant S is given by

$$\Pi_S(x) = \begin{cases} p_W^0 + (1 - p_W^0) \frac{x}{u} & \text{for } x \leq u, \\ 1 & \text{for } x \in (u, r), \\ 1 + b & \text{for } x \geq r. \end{cases} \quad (\text{A-13})$$

Correspondingly, the reward function for contestant W is given by

$$\Pi_W(x) = \begin{cases} (1 - p_S^r) \frac{x}{u} & \text{for } x \leq u, \\ 1 - p_S^r & \text{for } x \in (u, r), \\ 1 + b & \text{for } x \geq r. \end{cases} \quad (\text{A-14})$$

Moreover, the parameters of support lines satisfy

$$\alpha_S = 1 - \frac{b u}{r - u}, \quad \beta_S = \frac{b}{r - u}, \quad \beta_W = \frac{1 - p_S^r}{u}.$$

Proof of Proposition 1 As shown in Lemma 1, in Eq0,

$$p_W^0(\text{Eq0}) = 1 - \frac{\mu_W}{\mu_S}.$$

As shown in Lemma 3, in Eq1,

$$p_W^0(\text{Eq1}) = 1 - \frac{2}{u} \mu_W.$$

Lemma 3 shows, $u < 2 \mu_S$, and thus $p_W^0(\text{Eq1}) < p_W^0(\text{Eq0})$.

Proof of Lemma 4 First note that, a proof analogous to the proof given for Eq1, shows that, in any Eq2 equilibrium, $\alpha_S > 0$ and $\alpha_W = 0$. Since, in an Eq2 equilibrium, both W and S chase the bonus, the supports of their performance distributions are identical. Let $\text{Supp} = \text{Supp}_W = \text{Supp}_S$. Let $u_H = \max(\text{Supp})$. Because u_H lies on the support of both contestants support lines, $\ell_S(u_H) = \Pi_H(u_H) = 1 + b$ and $\ell_W(u_H) = \Pi_W(u_H) = 1 + b$, so $\ell_S(u_H) = \ell_W(u_H)$. This implies, because the intercept of ℓ_S is positive and the intercept of ℓ_W is zero, and the slopes of both lines are positive, the slope of ℓ_W must be greater than that the slope of ℓ_S , i.e., $\beta_W > \beta_S$. Thus, for all $x \in \text{Supp}$, $x \neq u_H$ $\ell_S(x) > \ell_W(x)$. Next note that for all $x \in \text{Supp}$, Remark 3 and the definition of the reward functions imply that

$$\begin{aligned}\ell_W(x) &= \beta_W x = \Pi_W(x) = F_S(x) + \mathbb{1}_r(x), \\ \ell_S(x) &= \alpha_S + \beta_S x = \Pi_S(x) = F_W(x) + \mathbb{1}_r(x).\end{aligned}$$

Thus,

$$F_S(x) < F_W(x), \quad x \in \text{Supp} \setminus \{u_H\} \text{ and } F_S(u_H) = F_W(u_L) = 1,$$

i.e., F_S strictly first-order stochastically dominates F_W . Thus, at the upper bond of the rank competition region, u_L , $\Pi_W(u_L) < \Pi_S(u_L)$. For $x \in (u_L, r)$, Π_S and Π_W are constant. At r , the reward functions must jump up and each must meet their respective support lines in order for chasing the bonus to be a best response. The size of the jump will depend on the level of bonus compensation, b , and on mass placed on r . The larger the mass, the larger the jump from topping r . Since the slope of ℓ_W is greater than the slope of ℓ_S , Π_W is lower than Π_S for $x \in (u_L, r)$, and the component of the jump produced by the bonus is the same for both contestants, and the jump must be larger for W , S must place more mass on r . However, both contestants cannot place mass on r , as this would result in a tie with positive probability (Remark 1). Hence, the mass placed by W must be 0, the mass placed by S , which we denote by p_S^r , must be positive. By the same argument used in the proof of Lemma 3, W must place positive mass on 0, both contestants, conditioned on choosing performance levels in $(0, u_L]$, where $u_L < r$ must randomize uniformly; both contestants, conditioned on choosing performance levels in $(r, u_H]$ must randomize uniformly.

Let p_i^h , $i = S, W$, represent the probability that contestant i targets the rank and bonus competition region $(r, u_H]$, i.e., chooses performance levels in this region. The arguments above have shown that the performance distributions of the contestants can be described as follows:

- (a) Contestant S targets the rank and bonus competition region $(r, u_H]$ with probability p_S^h and, conditioned on targeting this region, randomizes using a uniform distribution $\text{Unif}[r, u_H]$; S puts point mass on r with probability p_S^r and targets the pure rank competition $(0, u_L]$ with probability $1 - p_S^h - p_S^r$ and, conditioned on targeting this region, randomizes using a $\text{Unif}[0, u_L]$ distribution;
- (b) Contestant W targets the rank and bonus competition region $(r, u_H]$ with probability p_W^h and, conditioned on targeting this region, randomizes using a uniform distribution $\text{Unif}[r, u_H]$; W puts point mass on r with probability p_W^0 and targets the pure rank competition $(0, u_L]$ with probability $1 - p_W^h - p_W^0$ and, conditioned on targeting this region, randomizes using a $\text{Unif}[0, u_L]$ distribution.

For contestant S , her reward at bonus threshold should meet her support line. As α_S is just the probability that contestant W plays zero, we have

$$\alpha_S + \beta_S r = p_W^0 + \beta_S r = (1 - p_W^r) + b. \quad (\text{A-15})$$

Meanwhile the slope over $[0, u_L]$ region should be the same as the slope over $[r, u_H]$ region,

which both equal to β_S , i.e.,

$$\beta_S = \frac{1 - p_W^h - p_W^0}{u_L} = \frac{p_W^h}{u_H - r}. \quad (\text{A-16})$$

Similarly, for contestant W , reward at r meets his support line, i.e.,

$$\beta_W r = (1 - p_S^h) + b. \quad (\text{A-17})$$

Meanwhile the slope over $[0, u_L]$ region should be the same as the slope over $[r, u_H]$ region, which both equal to β_W , i.e.,

$$\beta_W = \frac{1 - p_S^h - p_S^r}{u_L} = \frac{p_S^h}{u_H - r}. \quad (\text{A-18})$$

Thus, we have

$$\frac{1 - p_W^0 - p_W^h}{1 - p_S^r - p_S^h} = \frac{p_W^h}{p_S^h},$$

which implies that

$$\frac{p_W^h}{p_S^h} = \frac{1 - p_W^0}{1 - p_S^r}.$$

Because both contestants use up their capacities in equilibrium, we have

$$(1 - p_S^h - p_S^r) \frac{u_L}{2} + r p_S^r + p_S^h \frac{r + u_H}{2} = \mu_S, \quad (\text{A-19})$$

$$(1 - p_W^h - p_W^0) \frac{u_L}{2} + p_W^h \frac{r + u_H}{2} = \mu_W. \quad (\text{A-20})$$

Combining the six equations (A-15), (A-16), (A-17), (A-18), (A-19) and (A-20), we are able to solve for the six unknowns $\{u_L, u_H, p_S^h, p_W^h, p_S^r, p_W^0\}$. The solution can be expressed as follows,

$$p_S^r = \frac{b(\mu_S - \mu_W)}{br + \mu_W}, \quad p_S^h = \frac{(1+b)(u_H - r)}{u_H}, \quad (\text{A-21})$$

$$p_W^0 = \frac{(1+b)(\mu_S - \mu_W)}{br + \mu_S}, \quad p_W^h = \frac{b(u_H - r)}{r - u_L}, \quad (\text{A-22})$$

$$u_H = \frac{2(1+b)(br + \mu_S)(br + \mu_W)^2}{b^2(br + \mu_S)^2 + (1+b)^2(br + \mu_W)^2}, \quad (\text{A-23})$$

$$u_L = r - \frac{2b(br + \mu_S)^2(br + \mu_W)}{b^2(br + \mu_S)^2 + (1+b)^2(br + \mu_W)^2}. \quad (\text{A-24})$$

Finally, from the results above it is not hard to calculate the contest reward functions for both contestants. The reward function for contestant S from submitting performance x is as follows,

$$\Pi_S(x) = \begin{cases} p_W^0 + (1 - p_W^0 - p_W^h) \frac{x}{u_L} & \text{for } x \leq u_L, \\ 1 - p_W^h & \text{for } x \in (u_L, r), \\ b + (1 - p_W^h) + p_W^h \frac{x - r}{u_H - r} & \text{for } x \in [r, u_H], \\ 1 + b & \text{for } x > u_H. \end{cases} \quad (\text{A-25})$$

Correspondingly, the reward function for contestant W is

$$\Pi_W(x) = \begin{cases} (1 - p_S^r - p_S^h) \frac{x}{u_L} & \text{for } x \leq u_L, \\ 1 - p_S^r - p_S^h & \text{for } x \in (u_L, r), \\ b + (1 - p_S^h) + p_S^h \frac{x - r}{u_H - r} & \text{for } x \in [r, u_H], \\ 1 + b & \text{for } x > u_H. \end{cases} \quad (\text{A-26})$$

Moreover, the parameters of support lines satisfy

$$\alpha_S = p_W^0, \quad \beta_S = \frac{1 - p_W^h - p_W^0}{u_L}, \quad \beta_W = \frac{1 - p_S^h - p_S^r}{u_L}.$$

Proof of Proposition 2 Let $p_W^0(\text{Eq2})$ represent the probability that the weak contestant places point mass on zero in an Eq2 equilibrium. Let $p_W^0(\text{Eq0})$ represent the probability that the weak contestant places point mass on zero in an Eq0 equilibrium. Lemma 1 without bonus shows that $p_W^0(\text{Eq2}) = (\mu_S - \mu_W)/\mu_S$. Substituting the definitions of u_L , u_H and p_W^0 provided in Lemma 4 and equations (A-24) and (A-23) in this appendix shows that

$$p_W^0(\text{Eq2}) = p_W^0(\text{Eq0}) \left(1 - \frac{b(r - \mu_S)}{br + \mu_S} \right). \quad (\text{A-27})$$

It is always the case that

$$0 < \frac{b(r - \mu_S)}{br + \mu_S} < 1.$$

Thus, $p_W^0(\text{Eq2}) < p_W^0(\text{Eq0})$.

Proof of Lemma 5

The proof of this lemma is tedious and so we have broken the steps into a series of results. With this Lemma in hand, the proof of the characterizations in the next proposition, Proposition 3, is fairly straightforward.

To initiate the proof, for fixed contestant capacities, μ_S and μ_W , define the following functions:

$$\text{ConEq0}(r, b) = \left(1 - \frac{\mu_W}{\mu_S} \right) + \frac{\mu_W}{\mu_S} \frac{1}{2\mu_S} r - (1 + b), \quad (\text{A-28})$$

$$\text{ConEq1}(r, b) = \frac{2r(r - \mu_S)}{(2r - u)u} - (1 + b), \text{ where } u = \frac{2r\mu_W}{\mu_W + \sqrt{2br\mu_W + \mu_W^2}}, \quad (\text{A-29})$$

$$\text{ConEq2}(r, b) = u_H - r, \text{ where } u_H = \frac{2(1 + b)(br + \mu_S)(br + \mu_W)^2}{b^2(br + \mu_S)^2 + (1 + b)^2(br + \mu_W)^2}. \quad (\text{A-30})$$

Result A.1. For any bonus compensation package $(r, b) \in (\mu_S, \infty) \times (0, \infty)$,

- (i) An Eq0 equilibrium exists only if $\text{ConEq0}(r, b) \geq 0$.
- (ii) An Eq1 equilibrium exists only if $\text{ConEq1}(r, b) \geq 0$.
- (iii) An Eq2 equilibrium exists only if $\text{ConEq2}(r, b) > 0$.

PROOF: This result is fairly obvious. In an Eq0 equilibrium, $\alpha_S = 1 - \frac{\mu_W}{\mu_S}$, and $\beta_S = \frac{\mu_W}{\mu_S} \frac{1}{2\mu_S}$. Thus $\text{ConEq0}(r, b) \geq 0$ is equivalent to not chasing the bonus being a best response for S .

As shown in the proof of Lemma 3, in an Eq1 equilibrium, $\alpha_W = 0$ and $\beta_W = (1 - p_W^r)/u$, where u and p_W^r are defined in Lemma 3. Substituting, the definition of p_S^r also provided in that Lemma, shows that $\text{ConEq1} \geq 0$ is equivalent to the condition that $\beta_W r \geq (1 + b)$, the condition for not chasing the bonus to be a best reply for W .

By definition, in an Eq2 equilibrium, both contestants pursue rank competition at performance levels in excess of the reward threshold, r . Thus, the expression for the upper bound of the rank and bonus competition region, u_H , defined in equation (A-23), must exceed r . \square

Result A.2. For any bonus compensation package $(r, b) \in (\mu_S, \infty) \times (0, \infty)$, $\text{ConEq0}(r, b) \geq 0$ if and only if an Eq0 equilibrium exists.

PROOF: As stated above, $\text{ConEq0}(r, b) \geq 0$ implies that not chasing the bonus is a best reply for S , $\text{ConEq0}(r, b) \geq 0$ implies that $r > 2\mu_S$, the upper bound of performance in an Eq0 equilibrium. Thus, at $x = 2\mu_S$, the reward to both contestants is 1. Because the support lines cross at $x = 2\mu_S$ and because $\alpha_S > 0$ and $\alpha_W = 0$, for $x \geq 2\mu_S$, $\beta_W x \geq \alpha_S + \beta_S x$. Thus, $\alpha_S + \beta_S r \geq 1 + b$ implies that $\beta_W r \geq 1 + b$. Because, $\text{ConEq0}(r, b) \geq 0$ is equivalent to $\alpha_S + \beta_S r \geq 1 + b$, $\text{ConEq0}(r, b) \geq 0$ also implies that not chasing the bonus is a best reply for W . \square

Result A.3. For any bonus compensation package $(r, b) \in (\mu_S, \infty) \times (0, \infty)$, an Eq1 equilibrium exists if and only if $\text{ConEq1}(r, b) \geq 0$ and $\text{ConEq0}(r, b) < 0$.

PROOF: For an Eq1 equilibrium to exist it must be the case that $p_W^0 \in (0, 1)$, $p_S^r \in (0, 1]$, and $u < r$, where p_W^0 , p_S^r and u are defined in Lemma 3. Using the definition of p_S^r provided in Lemma 3, and the assumption that $r > \mu_S$, we see that $p_S^r < 1$. Inspecting the definition of p_W^0 provided in Lemma 3 shows that $p_W^0 < 1$. Using the definitions in Lemma 3, we see that

$$u = r \frac{2\mu_W}{\mu_W + \sqrt{2br\mu_W + \mu_W^2}} \text{ and } \frac{2\mu_W}{\mu_W + \sqrt{2br\mu_W + \mu_W^2}} < 1.$$

Hence, $u < r$. Inspection of the definition of u provided by Lemma 3 shows that $u > 0$.

Now consider p_W^0 . Algebraic simplification shows that ConEq1 can be expressed at follows:

$$\begin{aligned} \text{ConEq1}(r, b) &= \mathcal{S}(r, b) \frac{\mu_W (br + \mu_W)}{r \mu_W \sqrt{\mu_W (2br + \mu_W)}}, \text{ where} \\ \mathcal{S}(r, b) &= (r - \mu_S) - \frac{br + \mu_S}{br + \mu_W} \sqrt{\mu_W (2br + \mu_W)}. \end{aligned} \quad (\text{A-31})$$

Thus,

$$\text{sgn}(\text{ConEq1}(r, b)) = \text{sgn}(\mathcal{S}(r, b)). \quad (\text{A-32})$$

Thus, if $\text{ConEq1} \geq 0$, then using the definition of u in Lemma 3, and equations (A-31) and (A-32), we see that

$$\mathcal{S}(r, b) \geq 0 \Rightarrow u \geq \frac{2r\mu_W}{\mu_W + (r - \mu_S) \frac{br + \mu_W}{br + \mu_S}}.$$

The right-hand side of this expression is decreasing in b and

$$\lim_{b \rightarrow \infty} \frac{2r\mu_W}{\mu_W + (r - \mu_S) \frac{br + \mu_W}{br + \mu_S}} = 2\mu_W \frac{r}{r - (\mu_S - \mu_W)} > 2\mu_W.$$

Thus, $u > 2\mu_W$. This implies, using the definition of p_W^0 provided in Lemma 3, that $p_W^0 > 0$.

Now consider p_S^r . First note that because $u < r$, $p_S^r > 0$ if and only if $2\mu_S > u$. Using the quadratic formula, and the definition of u provided in Lemma 3 we see that

$$\text{ConEq0} < 0 \iff \mu_S > \mu_S^o = \frac{\sqrt{2br\mu_W + \mu_W^2} - \mu_W}{2b}. \quad (\text{A-33})$$

Next note, given the definition of u in Lemma 3,

$$2\mu_S - u = 2\mu_S - \frac{2r\mu_W}{\mu_W + \sqrt{2br\mu_W + \mu_W^2}}.$$

At $\mu_S = \mu_S^o$,

$$2\mu_S - \frac{2r\mu_W}{\mu_W + \sqrt{2br\mu_W + \mu_W^2}} = 0. \quad (\text{A-34})$$

Because, the left hand side of equation (A-34) is increasing in μ_S , equations (A-33) and (A-34) imply that

$$p_S^r > 0 \iff 2\mu_S - u > 0 \iff \mu_S > \mu_S^o \iff \text{ConEq0} < 0.$$

□

Result A.4. For any bonus compensation package $(r, b) \in (\mu_S, \infty) \times (0, \infty)$, an Eq2 equilibrium exists if and only if $\text{ConEq2}(r, b) > 0$.

PROOF: An Eq2 equilibrium exists if and only if all the parameters specified in Lemma 4 satisfy their relevant range restrictions. First note that u_L , as defined in the Proof of Lemma 4 by equation (A-24), can be shown by algebraic manipulation to be equal to

$$\frac{b^2 r (r - \mu_S) (r + 2br + \mu_S) + 2b (r - \mu_S) (r + 2br + \mu_S) \mu_W + (1 + b)^2 r \mu_W^2}{b^2 ((1 + 2b(1 + b)) r^2 + 2br\mu_S + \mu_S^2) + 2b(1 + b)^2 r \mu_W + (1 + b)^2 \mu_W^2} > 0.$$

Thus, $u_L > 0$. Inspecting the definition of u_L provided by equation (A-24) shows that $u_L < r$. Now consider p_W^0 . The definition of p_W^0 provided in Lemma 4 states that

$$p_W^0 = \frac{(1 + b) (\mu_S - \mu_W)}{br + \mu_S},$$

and

$$0 < \frac{(1 + b) (\mu_S - \mu_W)}{br + \mu_S} < \frac{(1 + b) (\mu_S - \mu_W)}{b\mu_S + \mu_S} = \frac{\mu_S - \mu_W}{\mu_S} < 1.$$

So, $p_W^0 \in (0, 1)$. Now consider p_S^r . The definition of p_S^r provided in Lemma 4 states that

$$p_S^r = \frac{b (\mu_S - \mu_W)}{br + \mu_W},$$

and

$$0 < \frac{b (\mu_S - \mu_W)}{b r + \mu_W} < \frac{b (\mu_S - \mu_W)}{b \mu_S + \mu_W} < 1.$$

Now suppose that $p_S^h \in (0, 1)$. The definitions p_S^h and p_W^h provided by equations (A-21) and (A-22) imply that

$$p_W^h = \frac{b r + \mu_W}{b r + \mu_S} p_S^h.$$

Thus if $p_S^h \in (0, 1)$ then $p_W^h \in (0, 1)$. So an Eq2 equilibrium will exist if $u_H > r$ and $p_S^h \in (0, 1)$.

First we show that it is always the case that $p_S^h < 1$. First, note that the definition of u_H provided by equation (A-23) implies that $p_S^h \geq 1$ if and only if

$$\frac{2 (b r + \mu_S) (b r + \mu_W)^2}{b^2 (b r + \mu_S)^2 + (1 + b)^2 (b r + \mu_W)^2} \geq \frac{r}{b}. \quad (\text{A-35})$$

Condition (A-35) can only be satisfied if

$$\begin{aligned} (2 (b r + \mu_S) (b r + \mu_W)^2) b - (b^2 (b r + \mu_S)^2 + (1 + b)^2 (b r + \mu_W)^2) r = \\ - r \mu_W^2 - 2 b^3 r^2 (r - \mu_W) - 2 b \mu_W (r^2 + \mu_W (r - \mu_S)) - \\ b^2 r (r^2 + (\mu_S^2 - \mu_W^2) + 4 \mu_W (r - \mu_S)) > 0, \end{aligned}$$

which is clearly impossible. Thus, $p_W^h < 1$.

Finally note that from the definition of p_W^h provided by equation (A-21), $p_W^h > 0$ if and only if $u_H > r$. Thus, an Eq1 equilibrium exists if and only if $u_H > r$, i.e., $\text{ConEq2} > 0$. \square

Result A.5. $\text{sgn}(\text{ConEq2}) = -\text{sgn}(\text{ConEq1})$.

PROOF: Let

$$\mathcal{S}_2(r, b) = (r - \mu_S)^2 - \mu_W (2 b r + \mu_W) \left(\frac{b r + \mu_S}{b r + \mu_W} \right)^2.$$

Algebraic simplification shows that

$\text{ConEq2}(r, b) = -K \mathcal{S}_2(r, b)$, where

$$K = \frac{(b r + \mu_W)^2}{r (((1 + b)^2 + b^2) (b r + \mu_W)^2 + (\mu_S - \mu_W) b^2 ((b r + \mu_W) + (b r + \mu_S)))}.$$

$K > 0$, so

$$\text{sgn}(\text{ConEq2}) = -\text{sgn}(\mathcal{S}_2(r, b)). \quad (\text{A-36})$$

Recall \mathcal{S} , defined in equation (A-31), and note that

$$\text{sgn}(\mathcal{S}_2(r, b)) = \text{sgn}(\mathcal{S}(r, b)). \quad (\text{A-37})$$

The result follows from (A-36), (A-37), and (A-32). \square

Result A.6. Define the parametric curves, with parameter $y > 0$ as follows:

$$\begin{aligned}\mathcal{B}_0^1(y) &= \frac{\sqrt{2y\mu_W + \mu_W^2} - \mu_W}{2\mu_S}, & \mathcal{R}_0^1(y) &= \frac{2y\mu_S}{\sqrt{2y\mu_W + \mu_W^2} - \mu_W}; \\ \mathcal{B}_1^2(y) &= y \left(\mu_S + \frac{y + \mu_S}{y + \mu_W} \sqrt{2y\mu_W + \mu_W^2} \right)^{-1}, & \mathcal{R}_1^2(y) &= \mu_S + \frac{y + \mu_S}{y + \mu_W} \sqrt{2y\mu_W + \mu_W^2}.\end{aligned}$$

These parametric curves have the following properties:

- (i) $\mathcal{B}_0^1(y)$ and $\mathcal{B}_1^2(y)$ are strictly increasing and map $(0, \infty)$ onto $(0, \infty)$.
- (ii) $\mathcal{B}_0^1(y) \mathcal{R}_0^1(y) = \mathcal{B}_1^2(y) \mathcal{R}_1^2(y) = y$.
- (iii) $\mathcal{B}_1^2(y) > \mathcal{B}_0^1(y)$.
- (iv) $\text{ConEq2}(r, b) = 0 \iff (r, b) = (\mathcal{R}_1^2(y), \mathcal{B}_1^2(y))$ for some $y > 0$.
- (v) $\text{ConEq1}(r, b) = 0 \iff (r, b) = (\mathcal{R}_1^2(y), \mathcal{B}_1^2(y))$ for some $y > 0$.
- (vi) $\text{ConEq0}(r, b) = 0 \iff (r, b) = (\mathcal{R}_0^1(y), \mathcal{B}_0^1(y))$ for some $y > 0$.

PROOF: Property (i) follows from differentiation. Property (ii) follows from inspection. To establish property (iii), first note that $\lim_{y \rightarrow 0} \mathcal{B}_1^2(y) - \mathcal{B}_0^1(y) = 0$ and $\lim_{y \rightarrow 0} (\mathcal{B}_1^2(y) - \mathcal{B}_0^1(y))' = 0$. Next note that differentiation shows that

$$\begin{aligned}\text{sgn}((\mathcal{B}_1^2(y) - \mathcal{B}_0^1(y))') &= \\ \text{sgn}\Big(y(\mu_S - \mu_W) \Big(2y^2\mu_W + 2\mu_S\mu_W \Big(\mu_W + \sqrt{\mu_W(2y + \mu_W)} \Big) + \\ &\quad y \Big(5\mu_S\mu_W + \mu_W^2 + 2\mu_S\sqrt{\mu_W(2y + \mu_W)} \Big) \Big) \Big). \quad (\text{A-38})\end{aligned}$$

Because, $\lim_{y \rightarrow 0} (\mathcal{B}_1^2(y) - \mathcal{B}_0^1(y))' = 0$ and $\lim_{y \rightarrow 0} \mathcal{B}_1^2(y) - \mathcal{B}_0^1(y) = 0$, to establish the result, it is sufficient to show that the derivative of the right-hand side of (A-38) is positive. This follows from differentiation.

To establish parts (iv), (v), first note that, by equations (A-32), (A-36), and (A-37), parts (iv), (v) are equivalent to the assertion that

$$\text{A1:} \quad \mathcal{S}(r, b) = 0 \iff \text{there exists } y > 0 \text{ such that } (r, b) = (\mathcal{B}_1^2(y), \mathcal{B}_1^2(y)).$$

Inspecting the definition of \mathcal{S} provided by equation (A-31), we see that $\mathcal{S}(r, b) = 0$ is equivalent to the assertion that

$$\text{A2:} \quad \text{there exists } y > 0 \text{ such that } r = \mu_S + \frac{y + \mu_S}{y + \mu_W} \sqrt{\mu_W(2y + \mu_W)} \text{ and } y = r b.$$

A1 and A2 are clearly equivalent.

Property (vi) follows from substitution of the definitions of \mathcal{B}_0^1 and \mathcal{B}_0^1 , provided by Result A.6, into ConEq0 (equation (A-28)). \square

Result A.7. If $(r_1, b) = (\mathcal{R}_0^1(y_1), \mathcal{B}_0^1(y_1))$ and $(r_2, b) = (\mathcal{R}_1^2(y_2), \mathcal{B}_1^2(y_2))$, then $r_1 > r_2$.

PROOF: By parts (i) and (iii) of Result A.6, if $b = \mathcal{B}_0^1(y_1) = \mathcal{B}_1^2(y_2)$, then $y_1 > y_2$. By part (ii) of Result A.6, $r_1 b = y_1$ and $r_2 b = y_2$. Thus $r_1 > r_2$. \square

Result A.8. For any fixed $b > 0$,

- (i) $\lim_{r \rightarrow \mu} \text{ConEq0}(r, b) < 0$ and $\lim_{r \rightarrow \infty} \text{ConEq0}(r, b) > 0$.
- (ii) $\lim_{r \rightarrow \mu} \text{ConEq1}(r, b) < 0$ and $\lim_{r \rightarrow \infty} \text{ConEq1}(r, b) > 0$.
- (iii) $\lim_{r \rightarrow \mu} \text{ConEq2}(r, b) > 0$ and $\lim_{r \rightarrow \infty} \text{ConEq2}(r, b) < 0$.

PROOF: These assertions follow from straightforward calculations. \square

Result A.9. For each $b > 0$,

- (i) there exists a unique $r_2 > \mu_S$, such that $\text{ConEq1}(r_2, b) = \text{ConEq2}(r_2, b) = 0$; if $r < r_2$, $\text{ConEq1}(r_2, b) < 0$ and $\text{ConEq2}(r_2, b) > 0$; if $r > r_2$, $\text{ConEq1}(r_2, b) > 0$ and $\text{ConEq2}(r_2, b) < 0$.
- (ii) There exists a unique $r_1 > \mu_S$, such that $\text{ConEq0}(r_1, b) = 0$; if $r < r_1$, $\text{ConEq0}(r_1, b) < 0$; if $r > r_1$, $\text{ConEq0}(r_1, b) > 0$.
- (iii) $r_1 > r_2$.

PROOF: The proofs of parts (i) and (ii) are identical. So we will only present the proof of (i). Suppose that $\text{ConEq1}(r'_2, b) = 0$ and $\text{ConEq1}(r''_2, b) = 0$. Then, by parts (v) and (iv) of Result A.6, $(r'_2, b) = (\mathcal{R}_1^2(y'), \mathcal{B}_1^2(y'))$ for some $y' > 0$ and $(r''_2, b) = (\mathcal{R}_1^2(y''), \mathcal{B}_1^2(y''))$ for some $y'' > 0$. By part (i) of Result A.6, $\mathcal{B}_1^2(y') = \mathcal{B}_1^2(y'')$ implies that $y' = y''$ and thus $r'_2 = r''_2$.

The fact that there is a unique $r = r_2$, such that $\text{ConEq1}(r_2, b) = 0$, combined with the continuity of ConEq1 implies that the sign of ConEq1 is constant for $r < r_2$ and is constant for $r > r_2$. Part (i) thus follows from part (ii) of Result A.9 and Result A.5.

Part (iii) follows from parts (v), (iv), (vi) of Result A.6, and Result A.7. \square

Result A.10. Define

$$\begin{aligned} E &= (\mu_S, \infty) \times (0, \infty), \\ E_{01} &= \{(r, b) : \exists y > 0 \text{ such that } r \geq \mathcal{R}_0^1(y) \text{ and } b = \mathcal{B}_0^1(y)\}, \\ E_{12} &= \{(r, b) : \exists y > 0 \text{ such that } r \geq \mathcal{R}_1^2(y) \text{ and } b = \mathcal{B}_1^2(y)\}. \end{aligned}$$

Then

- (i) $(r, b) \in E_{01} \iff \text{ConEq0}(r, b) \geq 0$,
- (ii) $(r, b) \in E_{12} \iff \text{ConEq1}(r, b) \geq 0$,
- (iii) $(r, b) \in E \setminus E_{12} \iff \text{ConEq2}(r, b) > 0$.

PROOF: The proofs of parts (i), (ii) are virtually identical. Thus we will only prove part (i). We first prove sufficiency. Note that if $(r, b) = (\mathcal{R}_0^1(y), \mathcal{B}_0^1(y))$, then by part (vi) of Result A.6, $\text{ConEq0}(r, b) = 0$. Next suppose that $b = \mathcal{R}_0^1(y)$ and $r > \mathcal{R}_0^1(y)$. Let $r_1 = \mathcal{R}_0^1(y)$. Part (ii) of Result A.9 shows that r_1 is the unique r satisfying $\text{ConEq0}(r, b) = 0$, and implies, because $r > r_1$, $\text{ConEq0}(r, b) > 0$.

To prove necessity, suppose that $(r, b) \notin E_{01}$. Let y' be the unique $y > 0$ such that $\mathcal{B}_0^1(y) = b$ (the existence and uniqueness of y' follows from property (i) in Result A.6). Thus, $\mathcal{B}_0^1(y') = b$. If $(r, b) \notin E_{01}$, then it must be the case that $r < \mathcal{R}_0^1(y')$, using the same argument as used above for sufficiency, we see that Part (ii) of Result A.9 implies that $\text{ConEq0}(r, b) < 0$.

Part (iii) follows from part (ii) and Result A.5. \square

Proof of Lemma 5 Result A.10, and Results A.2, A.3, and A.4 show that an Eq0 equilibrium can be sustained if and only if $(r, b) \in E_{01}$; an Eq1 equilibrium can be sustained if and only if $(r, b) \in E_{12} \setminus E_{01}$; an Eq2 equilibrium can be sustained if and only if $(r, b) \in E \setminus E_{12}$. We need only show that these three sets are disjoint. Clearly E_{01} and $E_{12} \setminus E_{01}$ are disjoint as are $E_{12} \setminus E_{01}$ and $E \setminus E_{12}$. Now consider E_{01} and $E \setminus E_{12}$. Result A.9 implies that $E_{01} \subset E_{12}$. Thus E_{01} and $E \setminus E_{12}$ are disjoint.

Result A.11. For $r_o > \mu_S$, $r_o > \inf_{y>0} \mathcal{R}_1^2(y)$ if and only if for some $b > 0$ there exists $r < r_o$ such that (r, b) sustains an Eq1 equilibrium.

PROOF: We start by proving sufficiency. Suppose that $r_o > \inf_{y>0} \mathcal{R}_1^2(y)$, then there exists some $y > 0$, say y_2 , such that $r_o > \mathcal{R}_1^2(y_2)$. Let $(r_2, b_o) = (\mathcal{R}_1^2(y_2), \mathcal{B}_1^2(y_2))$. By property (v) in Lemma A.6, $\text{ConEq1}(r_2, b_o) = 0$. The fact that $r_o > r_2$, implies, by part (i) of Result A.9, that $\text{ConEq1}(r_o, b_o) > 0$.

Parts (ii) and (iii) of Result A.9 show that there is a unique bonus threshold, r_1 , such that $\text{ConEq0}(r_1, b_o) = 0$ and that $r_1 > r_2$. Let $r^* = \min[r_1, r_o]$. Because $r_1 > r_2$ and $r_o > r_2$, $r^* > r_2$. By definition, $r^* \leq r_o$. Hence, if $r \in (r_2, r^*)$, $r < r_o$, $r > r_1$ and $r < r_2$. Parts (ii) and (i) of Result A.9 and Result A.3 thus establish that (r, b) sustains an Eq1 equilibrium.

To prove necessity, note simply that $r_o \leq \inf_{y>0} \mathcal{R}_1^2(y)$ implies that the set, $\{(r, b) : r < r_o\} \cap E_{12}$ is empty. And thus, by part (ii) of Result A.10, for all $b > 0$, if $r < r_o$ then $\text{ConEq1}(r_2, b_o) < 0$, and thus, by Result A.3, an Eq1 equilibrium cannot be sustained. \square

Proof of Proposition 3

Proof of Parts (i) and (ii) of Proposition 3 By definition provided in Result A.6, $\mathcal{R}_1^2(y') = 2\mu_S$, is equivalent to

$$2\mu_S = \mu_S + \frac{y + \mu_S}{y + \mu_W} \sqrt{2y\mu_W + \mu_W^2}.$$

The equation $\frac{y + \mu_S}{y + \mu_W} \sqrt{2y\mu_W + \mu_W^2} - \mu_S = 0$ has solution with $y > 0$ if and only if $y \leftrightarrow \left(\frac{y + \mu_S}{y + \mu_W}\right)^2 (2y\mu_W + \mu_W^2) - \mu_S^2$ has a positive real root. After algebraic simplification we see that

$$\left(\frac{y + \mu_S}{y + \mu_W}\right)^2 (2y\mu_W + \mu_W^2) - \mu_S^2 = \frac{y}{(y + \mu_W)^2} \text{Poly}(y), \quad \text{where}$$

$$\text{Poly}(y) = 2\mu_W y^2 + \mu_W^2 \left((\sqrt{5} - 2) + \frac{\mu_S}{\mu_W} \right) \left((2 + \sqrt{5}) - \frac{\mu_S}{\mu_W} \right) y + 2\mu_S \mu_W^2.$$

Taking the discriminant of Poly we obtain

$$\text{Disc} = (\mu_S - \mu_W)^2 \mu_W \mu_S \left(\frac{\mu_S}{\mu_W} + \frac{\mu_W}{\mu_S} - 6 \right). \quad (\text{A-39})$$

Let $a = \mu_S/\mu_W > 1$. $\text{Disc} \leq 0$ is equivalent to $a + 1/a - 6 \leq 0$. Thus, if $a \leq 3 + 2\sqrt{2}$, $\text{Disc} \leq 0$ and thus Poly(y) either has no roots or a double root at 0. Hence, $\mathcal{R}_1^2(y) - 2\mu_S \geq 0$, for all $y > 0$. Thus, $\inf_{y>0} \mathcal{R}_1^2(y) \geq 2\mu_S$. Thus, part (i) follows from Result A.11.

If $a > 3 + 2\sqrt{2}$ then Poly(y) has two real roots. Because Poly is convex and is positive at $y = 0$, both roots are either positive or negative. Because $3 + 2\sqrt{2} > 2 + \sqrt{5}$, $a > 3 + 2\sqrt{2}$

implies that $(2 + \sqrt{5}) - a < 0$. This implies that $\text{Poly}'(0) < 0$ which, in turn, implies that both roots are positive. Hence, there exists a y -interval over which $\text{Poly}(y) < 0$. This implies that, over this interval, $\mathcal{R}_1^2(y) - 2\mu_S < 0$. Hence, $\inf_{y>0} \mathcal{R}_1^2(y) < 2\mu_S$. Hence, part (ii) follows from Result A.11.

Proof of part (iii) of Proposition 3 First note that the upper bound of the rank plus bonus competition region in the Eq2 configuration, u_L provided by equation (A-23) in the proof of Lemma 4 is given by

$$u_H(b) = \frac{2(b r + \mu_S)(b r + \mu_W)^2(1+b)}{b^2(b r + \mu_S)^2 + (1+b)^2(b r + \mu_W)^2}. \quad (\text{A-40})$$

Next note that

$$\lim_{b \rightarrow \infty} u_H(b) = r.$$

Differentiating u_H and evaluating the derivative at 0 yields

$$u'_H(0) = 2(r - \mu_S) > 0. \quad (\text{A-41})$$

Thus u_H is increasing for b sufficiently small. To verify that u_H is asymptotically decreasing, define the function

$$u_H^o(q) = u_H(1/q), \quad q > 0. \quad (\text{A-42})$$

u_H will be asymptotically decreasing if and only if u_H^o is increasing in a neighborhood of 0. First note that

$$u_H^{o'}(q) = \frac{2(1+q)(r+q\mu_S)(r+q\mu_W)^2}{(2+q(2+q))r^2 + 2qr(\mu_S + (1+q)^2\mu_W) + q^2(\mu_S^2 + (1+q)^2\mu_W^2)}. \quad (\text{A-43})$$

Taking the limit of $u_H^{o'}$ as $q \rightarrow 0$ yields

$$\lim_{q \rightarrow 0} u_H^{o'}(q) = \mu_W > 0. \quad (\text{A-44})$$

Thus, u_H^o is increasing in a neighborhood of 0 and thus u_H is asymptotically decreasing.

To establish the bounds on u_H asserted in the theorem, first note from the proof of Proposition that $u_H \geq r$, where u_H is defined by equation (A-23), is a necessary and sufficient condition for the existence of an Eq2 equilibrium. This verifies the lower bound.

Using the definition of u_H provided by equation (A-23), and making the substitution, $y = br$ yields,

$$\frac{r}{u_H(y)} = \frac{1}{2} \left(\left(\frac{y}{r+y} \frac{y}{\mu_W+y} \frac{\mu_S+y}{\mu_W+y} \right) + \frac{r+y}{\mu_S+y} \right). \quad (\text{A-45})$$

The Arithmetic/Geometric mean inequality implies that

$$\frac{r}{u_H(y)} \geq \sqrt{\left(\frac{y}{r+y} \frac{y}{\mu_W+y} \frac{\mu_S+y}{\mu_W+y} \right) \frac{r+y}{\mu_S+y}} = \frac{y}{\mu_W+y}. \quad (\text{A-46})$$

Using the fact that $y = rb$ and simplifying shows that

$$u_H \leq r + \frac{\mu_W}{b}. \quad (\text{A-47})$$

Next note that we can also express r/u_H as

$$\frac{r}{u_H(y)} = \frac{y}{\mu_W + y} \frac{1}{2} \left(\frac{\mu_S + y}{r + y} \frac{y}{\mu_W + y} + \left(\frac{\mu_S + y}{r + y} \frac{y}{\mu_W + y} \right)^{-1} \right). \quad (\text{A-48})$$

Let

$$z = \frac{\mu_S + y}{r + y} \frac{y}{\mu_W + y}. \quad (\text{A-49})$$

Note that the function $y \rightarrow \frac{\mu_S + y}{r + y} \frac{y}{\mu_W + y}$ is strictly increasing and that the image of $[0, \infty]$ under this function is $[0, 1]$. Its inverse function is given by,

$$y = \frac{z(r + \mu_W) - \mu_S + \sqrt{4r(1-z)z\mu_W + (z(r + \mu_W) - \mu_S)^2}}{2(1-z)}. \quad (\text{A-50})$$

Using, equations (A-49) and (A-50) we can express (A-48) in terms of z as follows:

$$\frac{r}{u_H(z)} = \frac{\text{Num}(z)}{\text{Denom}(z)} = \frac{r(1+z^2)}{z(r - \mu_W) + \mu_S + \sqrt{4r(1-z)z\mu_W + (\mu_S - z(r + \mu_W))^2}}, \quad z \in [0, 1]. \quad (\text{A-51})$$

Because $\text{Denom}'' \geq 0$, Denom is convex and thus

$$\text{Denom}(z) = \text{Denom}((1-z)0 + z1) \leq (1-z)\text{Denom}(0) + z\text{Denom}(1) = 2((1-z)\mu_S + zr), \quad z \in [0, 1]. \quad (\text{A-52})$$

Equations (A-51) and (A-52) imply that

$$\frac{r}{u_H(z)} \geq \frac{r(1+z^2)}{2((1-z)\mu_S + zr)}, \quad z \in [0, 1]. \quad (\text{A-53})$$

Minimizing the right-hand side of the equation above shows that

$$\frac{r(1+z^2)}{2((1-z)\mu_S + zr)} \geq \min_{z \in [0, 1]} \frac{r(1+z^2)}{2((1-z)\mu_S + zr)} = \frac{r}{\mu_S + \sqrt{(r - \mu_S)^2 + \mu_S^2}}, \quad z \in [0, 1]. \quad (\text{A-54})$$

Equations (A-53) and (A-54) imply that

$$\frac{r}{u_H(z)} \geq \frac{r}{\mu_S + \sqrt{(r - \mu_S)^2 + \mu_S^2}}. \quad (\text{A-55})$$

Thus,

$$u_H \leq \mu_S + \sqrt{(r - \mu_S)^2 + \mu_S^2}. \quad (\text{A-56})$$

Combining (A-47) and (A-56) yields the result.

Condition for u_H crossing once the upper bound of pure rank competition risk taking, Lemma A.1

Lemma A.1 *Suppose that strength asymmetry between the two contestants is not extreme, i.e., $\mu_S/\mu_W < 3 + 2\sqrt{2}$ and that $r < 2\mu_S$. Define,*

$$\lambda^o = \frac{1}{6} \left(-\sqrt{2(18 - 7\sqrt{3})} - 3\sqrt{3} + 9 \right) \approx 0.063, \quad \rho^o = \frac{1}{2} \left(\sqrt{6(18 - 7\sqrt{3})} - 3\sqrt{3} + 9 \right) \approx 4.87.$$

If either of the following conditions is satisfied:

- (i) $\mu_S/\mu_W < \rho^o$, or*
- (ii) $r < (2 - \lambda^o)\mu_S$,*

then there exists a unique cut off bonus level $0 < b^o < \infty$ such that the upper bound for contestant performance under rank and bonus competition is greater (less) than the upper bound under pure rank competition if and only if $b < (>) b^o$.

PROOF: The upper bound of performance under pure rank competition is $2\mu_S$. We aim to provide conditions under which the upper bound under bonus competition is greater than the upper bound under rank and bonus competition. As shown by Proposition 3, the hypotheses of the theorem imply that, for all $b > 0$, all equilibria are Eq2 equilibria. Thus, the question we consider reduces to finding conditions under which u_H (defined in equation (A-23)) is greater (less) than $2\mu_S$, or equivalently, conditions under which

$$\frac{r}{u_H} < (>) \frac{r}{2\mu_S}. \quad (\text{A-57})$$

As in the proof of part (iii) of Proposition 3, we resolve this inequality by using the substitution, $b = y/r$ to replace b in the expression for u_H provided by equation (A-23), and then analyze the conditions on y required to verify

$$\frac{r}{u_H(y)} = \frac{1}{2} \left(\frac{y}{r+y} \frac{y}{\mu_W+y} \frac{\mu_S+y}{\mu_W+y} + \frac{r+y}{\mu_S+y} \right) < (>) \frac{2}{2\mu_S}, \quad y = rb. \quad (\text{A-58})$$

Algebraic simplification shows that

$$\text{sgn}(\Phi(y)) = \text{sgn} \left(\frac{1}{2} \left(\frac{y}{r+y} \frac{y}{\mu_W+y} \frac{\mu_S+y}{\mu_W+y} + \frac{r+y}{\mu_S+y} \right) - \frac{r}{2\mu_S} \right), \quad \text{where} \quad (\text{A-59})$$

$$\Phi(y) = (y + \mu_S)^2 y \mu_S - (y + \mu_W)^2 (r + y) (r - \mu_S). \quad (\text{A-60})$$

Remark A.1. Any result that implies that the discriminant of Φ is negative implies the conclusions of the lemma: Because, $\Phi(0) < 0$, and $\lim_{y \rightarrow \infty} \Phi = \infty$, the continuity of Φ implies that there exists $y^o > 0$ such that $\Phi(y^o) = 0$. Because Φ is a cubic polynomial, if Φ has a negative discriminant, Φ has one and only one real root (Chapt. 5.5, Theorem 8 Birkhoff and MacLane, 1977). Thus, a negative discriminant implies that there exists a unique $y^o > 0$ such that $\Phi(y) < (>) 0$ if and only if $y < (>) y^o$. Letting $b^o = y^o/r$, this implies that

$$\frac{r}{u_H} < (>) \frac{r}{2\mu_S} \iff b < (>) b^o,$$

which immediately implies that

$$u_H > (<) 2\mu_S \iff b < (>) b^o.$$

In the remainder of this proof we will use the notation Disc_x to represent the discriminant of a polynomial in x , where x represents a generic variable. Using this notation and performing some manipulations shows that

$$\begin{aligned} \text{sgn}(\text{Disc}_y[\Phi]) = \text{sgn} \Big(& -4r^4\mu_W + 4r^3\mu_W(\mu_S + 3\mu_W) - 4\mu_S(\mu_S^2 + \mu_W^2)^2 + \\ & r^2(\mu_S^3 + 18\mu_S^2\mu_W - 39\mu_S\mu_W^2 - 12\mu_W^3) + \\ & 4r\mu_W(-5\mu_S^3 + 9\mu_S^2\mu_W + 3\mu_S\mu_W^2 + \mu_W^3) \Big) \end{aligned} \quad (\text{A-61})$$

Next perform the variable substitutions,

$$\begin{aligned} r &= (1 - \lambda)2\mu_S + \lambda\mu_S \quad \text{and} \quad \mu_S = \rho\mu_W, \\ 0 &< \lambda < 1 \quad \text{and} \quad 1 < \rho < 3 + 2\sqrt{2}, \end{aligned} \quad (\text{A-62})$$

where the restrictions on the parameter range follow from the hypotheses of the lemma. After performing these substitutions and simplifying we see that the right hand side of equation (A-61) has the same sign as the polynomial, \mathcal{P} defined below:

$$\begin{aligned} \mathcal{P} = & \overbrace{((\lambda - 4)\lambda)}^{a_4} \rho^4 + \overbrace{(2\lambda(14 - 27\lambda + 14\lambda^2 - 2\lambda^3))}^{a_3} \rho^3 + \\ & \overbrace{(4 - 24\lambda + 33\lambda^2 - 12\lambda^3)}^{a_2} \rho^2 + \overbrace{(12(3\lambda - \lambda^2 - 2))}^{a_1} \rho + \overbrace{(4 - 4\lambda)}^{a_0}. \end{aligned} \quad (\text{A-63})$$

In equation (A-63) we expressed \mathcal{P} as a polynomial in ρ . We can also express \mathcal{P} as a polynomial in λ as follows

$$\begin{aligned} \mathcal{P} = & \overbrace{(-4\rho^3)}^{b_4} \lambda^4 + \overbrace{(4\rho^2(7\rho - 3))}^{b_3} \lambda^3 + \overbrace{(\rho(\rho(33 + (\rho - 54)\rho) - 12))}^{b_2} \lambda^2 + \\ & \overbrace{(-4(1 + \rho(-9 + (\rho - 6)(\rho - 1)\rho))}^{b_1} \lambda + \overbrace{(4(1 + (\rho - 6)\rho))}^{b_0}. \end{aligned} \quad (\text{A-64})$$

Remark A.2. These observations have established that the sign of $\text{Disc}_y(\Phi)$ equals the sign of \mathcal{P} .

Next we show that both condition (i) and condition (ii) are sufficient conditions for $\mathcal{P} < 0$. The key to establishing these results is signing the discriminants of \mathcal{P} when \mathcal{P} is viewed as a quartic polynomial in ρ and when it is viewed as a quartic polynomial in λ . These discriminants will provide information about the number real roots, which when combined with other characteristics of the polynomials, will identify the sign of \mathcal{P} .

Result A.12. If $\rho < \rho^o$, then $\text{Disc}_\lambda[\mathcal{P}] > 0$.

PROOF: First note that

$$\text{Disc}_\lambda[\mathcal{P}] = 256(\rho - 1)^2 \rho^8 (1 + \rho) (\rho^4 - 18\rho^3 + 54\rho^2 + 54\rho - 27)^3. \quad (\text{A-65})$$

Thus, because $\rho > 1$

$$\text{sgn}(\text{Disc}_\lambda[\mathcal{P}]) = \text{sgn}(\rho^4 - 18\rho^3 + 54\rho^2 + 54\rho - 27). \quad (\text{A-66})$$

The polynomial on the right hand side of equation (A-66) has four real roots, one negative and three positive. These roots are listed in order of magnitude below:

$$\overbrace{\frac{1}{2} \left(9 - 3\sqrt{3} - \sqrt{6(18 - 7\sqrt{3})} \right)}^{\approx -1.07}, \overbrace{\frac{1}{2} \left(9 + 3\sqrt{3} - \sqrt{6(18 + 7\sqrt{3})} \right)}^{\approx 0.376}, \overbrace{\frac{1}{2} \left(9 - 3\sqrt{3} + \sqrt{6(18 - 7\sqrt{3})} \right)}^{\rho^o}, \overbrace{\frac{1}{2} \left(9 + 3\sqrt{3} + \sqrt{6(18 + 7\sqrt{3})} \right)}^{\approx 13.8}.$$

Because this polynomial is monomial quartic, its limit is infinity as $\rho \rightarrow \pm\infty$ is infinity; hence, at the first root the polynomial crosses the x -axis from above, at the second from below, and at its third crossing, ρ^o , it crosses the x -axis from above. Because 1 is greater than the second root, this polynomial is positive between 1 and ρ^o , implying that $\text{Disc}_\lambda[\mathcal{P}] > 0$. \square

Result A.13. If $\lambda > \lambda^o$, then $\text{Disc}_\rho[\mathcal{P}] < 0$.

PROOF: First note that

$$\text{Disc}_\rho[\mathcal{P}] = 256(2 - \lambda)^2(1 - \lambda)\lambda(27\lambda^4 - 162\lambda^3 + 270\lambda^2 - 144\lambda + 8)^3. \quad (\text{A-67})$$

Thus, because $\lambda \in (0, 1)$,

$$\text{sgn}(\text{Disc}_\rho[\mathcal{P}]) = \text{sgn}(27\lambda^4 - 162\lambda^3 + 270\lambda^2 - 144\lambda + 8). \quad (\text{A-68})$$

The polynomial on the right hand side of equation (A-68), as four positive roots:

$$\overbrace{\frac{1}{6} \left(9 - 3\sqrt{3} - \sqrt{2(18 - 7\sqrt{3})} \right)}^{\lambda^o}, \overbrace{\frac{1}{6} \left(9 + 3\sqrt{3} - \sqrt{2(18 + 7\sqrt{3})} \right)}^{\approx 1.07}, \overbrace{\frac{1}{6} \left(9 - 3\sqrt{3} + \sqrt{2(18 - 7\sqrt{3})} \right)}^{\approx 1.21}, \overbrace{\frac{1}{6} \left(9 + 3\sqrt{3} + \sqrt{2(18 + 7\sqrt{3})} \right)}^{\approx 3.66}.$$

Its smallest root is λ_0 . The limit of the polynomial as $\lambda \rightarrow \pm\infty$ is infinity, thus, at λ_0 the polynomial crosses the x -axis from above. Because, the smallest root greater than λ^o is greater than 1, this implies for $\lambda \in (\lambda^o, 1)$ the polynomial is negative which implies that $\text{Disc}_\rho[\mathcal{P}] < 0$. \square

Result A.14. If $\rho < \rho^o$, $\mathcal{P} < 0$ for all $\lambda \in (0, 1)$.

PROOF: Transform the polynomial $\lambda \rightarrow \mathcal{P}$ defined by equation (A-64) into a monic polynomial by dividing the polynomial by the coefficient on its highest term, b_4 . This yields,

$$\lambda^4 + \left(-7 + \frac{3}{\rho}\right) \lambda^3 - \frac{-12 + \rho(33 + (-54 + \rho)\rho)}{4\rho^2} \lambda^2 + \left(-7 + \frac{1}{\rho^3} - \frac{9}{\rho^2} + \frac{6}{\rho} + \rho\right) \lambda - \frac{1 + (-6 + \rho)\rho}{\rho^3}. \quad (\text{A-69})$$

Next, perform the variable substitution (see Birkhoff and MacLane (Chapt 5.6, Theorem 9 1977)),

$$\lambda = z - \frac{1}{4} \left(-7 + \frac{3}{\rho} \right).$$

to yield the depressed monic polynomial

$$\begin{aligned}
& z^4 + q z^2 + r z + s, \text{ where} \\
& q = -\frac{3 - 60\rho + 39\rho^2 + 2\rho^3}{8\rho^2}, \\
& r = \frac{-1 - 78\rho + 96\rho^2 - 18\rho^3 + \rho^4}{8\rho^3}, \\
& s = \frac{-3 + 984\rho - 1746\rho^2 + 548\rho^3 + 221\rho^4 + 252\rho^5}{256\rho^4}.
\end{aligned} \tag{A-70}$$

Because the roots of the monic depressed quartic differ from the original quartic only by translation, the sign of the discriminant for the depressed monic polynomial as well as the number of real and complex roots are the same as for \mathcal{P} . Thus, Result A.12 ensures that $\text{Disc}_z[z^4 + q z^2 + r z + s] > 0$. This ensures that all roots of depressed monic polynomial are real or all roots are complex. If $s > q^2/4$ all roots of the depressed monic polynomial are complex (Rees, 1922). Next, note that

$$s - \frac{1}{4}q^2 = \frac{(\rho - 1)(3 - 333\rho + 1062\rho^2 - 242\rho^3 + 23\rho^4 - \rho^5)}{64\rho^4}. \tag{A-71}$$

Because $\rho > 1$, to show that $s - q^2/4 > 0$ we need only show that

$$p(\rho) = 3 - 333\rho + 1062\rho^2 - 242\rho^3 + 23\rho^4 - \rho^5 > 0, \quad \rho \in (1, \rho^o).$$

To show this, first note that at $\rho = 1$ and $\rho = \rho^o$, p is positive. So, if $p < 0$ at some point in $(1, \rho^o)$, it would have to be the case that the minimum of p over the interval is negative and attained at some point, say ρ^* , in $(1, \rho^o)$. Using Euclidean division we can divide p by p' to yield $p(\rho) = p_1(\rho)p'(\rho) + p_r(\rho)$, where p_r is the remainder after division. Because, $p'(\rho^*) = 0$, it must be the case that $p(\rho^*) = p_r(\rho^*)$. Computing the remainder polynomial, p_r , shows that

$$p_r(\rho) = \frac{16}{25} (-474 + 2637\rho - 48\rho^2 - 19\rho^3).$$

Consequently, if $p < 0$, at any point in $(1, \rho^o)$, there must exist $\rho^* \in (1, \rho^o)$, such that $p(\rho^*) = p_r(\rho^*) < 0$. However, in fact,

$$p_r(\rho) = \frac{16}{25} (-474 + 2637\rho - 48\rho^2 - 19\rho^3) > 0, \quad \rho \in [1, \rho^o].$$

To see this, first note that, because $\rho > 1$, $48\rho^2 + 19\rho^3 < (48 + 19)\rho^3 = 67\rho^3$. Thus,

$$\begin{aligned}
p_r(\rho) &= \frac{16}{25} (-474 + 2637\rho - 48\rho^2 - 19\rho^3) > \\
&\quad \frac{16}{25} (-474 + 2637\rho - 67\rho^3), \quad \rho \in [1, \rho^o].
\end{aligned} \tag{A-72}$$

The right-hand side expression in equation (A-72) is concave in ρ . Therefore, the expression's minimum is attained at one of the endpoints, 1 or ρ^o . Evaluating the expression at these endpoints shows that it is positive at both end points. Thus, the expression is positive over $\rho \in (1, \rho^o)$. Equation (A-72) shows that p_r is positive over $\rho \in (1, \rho^o)$. This contradicts p being negative at its minimum value and thus being negative at any point $\rho \in (1, \rho^o)$.

Thus, for $\rho \in (1, \rho^o)$, $s - q^2/4 > 0$. Therefore, all roots of \mathcal{P} are complex. Because, by hypothesis, $\rho < \rho^o < 3 + 2\sqrt{2}$, \mathcal{P} is negative at $\lambda = 0$ and, consequently, $\mathcal{P} < 0$ for all $\lambda \in (0, 1)$. \square

Result A.15. If $\lambda < \lambda^o$, $\mathcal{P} < 0$ for all $\rho \in (1, 3 + 2\sqrt{2})$.

PROOF: By Result A.13, the discriminant of $\rho \rightarrow \mathcal{P}(\rho)$ is negative. Because \mathcal{P} is a quartic polynomial, this implies that \mathcal{P} has two real and two complex roots (Rees, 1922). Next note that

$$\mathcal{P}(0) = 4(1 - \lambda) > 0, \mathcal{P}(1) = -4(2 + (-2 + \lambda)\lambda)^2 < 0, \text{ and } \lim_{\rho \rightarrow -\infty} \mathcal{P}(\rho) = -\infty.$$

Thus \mathcal{P} has two roots less than 1. Because \mathcal{P} has at most two roots, it must have no roots over $[1, 3 + 2\sqrt{2}]$. Thus, because $\mathcal{P}(1) < 0$, it must be the case that $\mathcal{P}(\rho) < 0$, for all $\rho \in [1, 3 + 2\sqrt{2}]$. \square

The lemma follows from Remarks A.1 and A.2, and Results A.14 and A.15. \square

Proof of Proposition 4 First, note that, as was shown in the proof of Result A.3, in all Eq1 equilibria, $p_W^0 > 0$. Thus all such equilibria feature some downside risk taking. As was shown in the proof of Result A.4, in all Eq2 equilibria, $p_W^0 > 0$. Thus, all such equilibria feature some downside risk taking. Lemma 1 shows that downside risk taking occurs in Eq0 equilibria. Thus, downside risk taking is never 0 in any equilibrium configuration.

In order to prove that a sequence of bonus thresholds and bonus payments (r_n, b_n) exist that support equilibria in which p_W^0 is arbitrarily small, we will employ the parametric curve, $y \mapsto (\mathcal{R}_1^2(y), \mathcal{B}_1^2(y))$, defined in Result A.6.

Define the sequence of bonus packages (r_n, b_n) as follows:

$$r_n = \mathcal{R}_1^2(n) = \mu_S + \frac{n + \mu_S}{n + \mu_W} \sqrt{\mu_W (2n + \mu_W)}, \quad (\text{A-73})$$

$$b_n = \mathcal{B}_1^2(n) = n \left(\mu_S + \frac{n + \mu_S}{n + \mu_W} \sqrt{\mu_W (2n + \mu_W)} \right)^{-1}. \quad (\text{A-74})$$

Result A.6 shows that for all n , (r_n, b_n) sustains an Eq1 equilibrium and only an Eq1 equilibrium. Using the definition of u , the upper bond of the rank plus bonus competition in Eq1 configurations provided in Lemma 3, we see that

$$u_n = 2\mu_W \left(\frac{\mu_S}{\mu_W + \sqrt{\mu_W (2n + \mu_W)}} + \frac{n + \mu_S}{n + \mu_W} \frac{\sqrt{\mu_W (2n + \mu_W)}}{\mu_W + \sqrt{\mu_W (2n + \mu_W)}} \right). \quad (\text{A-75})$$

The first term in the parentheses on the right-hand side of equation (A-75) converges to 0 as $n \rightarrow \infty$. The second term in the parentheses on the right-hand side of equation (A-75) converges to 1. Thus, $u_n \rightarrow 2\mu_W$. The definition of downside risk taking, p_W^0 in Eq1 configurations (see Lemma 3) is $p_W^0 = 1 - u/(2\mu_W)$. Because, $u_n \rightarrow 2\mu_W$, $1 - u_n/(2\mu_W) \rightarrow 0$.

Using the same arguments, it can also be shown that $p_S^r(n) \rightarrow 0$ as $n \rightarrow \infty$. Thus, both F_S^n and $F_W^n \rightarrow \text{Unif}[0, 2\mu_W]$ in distribution as $n \rightarrow \infty$.

B Multi-contestant equilibria

B.1 Marginal return from rank-competition when the number of contestants exceeds 2

In this section, we first show that a lower marginal return from rank competition for strong contestants—the foundation for the greater relative propensity of the strong contestant to chase the bonus in the manuscript—is a quite general feature of multi-contestant contests when contestants have varying capacities.

Consider a pure risk taking contest, with n contestants, assume that capacity is decreasing in the index of the contestant, i.e., $\mu_1 > \mu_2 > \mu_3 \dots \mu_{n-1} > \mu_n > 0$. Let $\boldsymbol{\mu} = (\mu_1, \mu_2 \dots \mu_n)$ represent the vector of capacities.

Assume, as in the manuscript, that contestants submit random bids subject to the capacity constraint—i.e., expected performance is no greater than capacity—and that the contestant with the highest performance receives a rank reward of 1, and all other contestants receive a rank reward of 0.¹ We call such a contest a $\boldsymbol{\mu}$ -contest. In a given equilibrium of the $\boldsymbol{\mu}$ -contest, let $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$ be the associated vectors of multipliers, and let \boldsymbol{F} represent the vector of contestant performance distributions.

Now consider an all-pay auction in which bidders bid x for an auctioned good. The highest bidder receives the good. All bidders, the winner as well as the losers, pay their bids to the auctioneer. Each of the n bidders has a positive valuation of the good, v_i , $i = 1 \dots n$. Let $\boldsymbol{v} = (v_1, v_2 \dots v_n)$ be the vector of such values. We call an all-pay auction where bidders' valuations are given by \boldsymbol{v} a \boldsymbol{v} -auction. Let \boldsymbol{G} represent the vector of contestant bid distributions in the auction and let $\boldsymbol{u}^* = (u_1^*, u_2^* \dots u_n^*)$ represent the vector of contestant payoffs given \boldsymbol{G} and \boldsymbol{v} .

Lemma B.1 *Suppose that \boldsymbol{F} is a $\boldsymbol{\mu}$ -contest equilibrium with associated multipliers $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. If*

$$v_i = \frac{1}{\beta_i} \text{ and } u_i^* = \frac{\alpha_i}{\beta_i} \implies \boldsymbol{F} \text{ is a } \boldsymbol{v}\text{-auction equilibrium,}$$

and, in this \boldsymbol{v} -auction equilibrium, $E[\tilde{x}_i] = \mu_i$.

PROOF: The condition for a best reply in the $\boldsymbol{\mu}$ -contest is that for all i ,

$$\begin{aligned} x \in \text{Supp}(F_i) &\Rightarrow \alpha_i + \beta_i x = \prod_{j \neq i} F_j(x), \\ \text{For all } x \geq 0, \alpha_i + \beta_i x &\geq \prod_{j \neq i} F_j(x), \end{aligned} \tag{B-1}$$

where $\prod_{j \neq i} F_j(x)$ simply represents the probability of x being the highest performance conditioned on the performance distributions of the other contestants.

In a \boldsymbol{v} -auction, the best reply condition is

$$\begin{aligned} x \in \text{Supp}(F_i) &\Rightarrow v_i \prod_{j \neq i} G_j(x) - x = u_i^*, \\ \text{For all } x \geq 0, v_i \prod_{j \neq i} G_j(x) - x &\leq u_i^*. \end{aligned} \tag{B-2}$$

¹Tie bids result in equal division of the rank reward between the tied contestants.

If $\mathbf{G} = \mathbf{F}$, then, after performing the transformation specified in the lemma, we see that the equilibrium conditions for the \mathbf{v} -auction are satisfied. Thus F is a \mathbf{v} -auction equilibrium. Because the distributions of contestant strategies are identical in the contest equilibrium and the corresponding auction equilibrium, it is obvious that the expected contestant performance in contest equilibrium and the bidder's expected bid in the corresponding auction equilibrium are equal to each other. \square

Our next lemma shows that, in a $\boldsymbol{\mu}$ -contest, the marginal gain from rank competition is always lowest for contestant 1, the contestant with the highest capacity.

Lemma B.2 *In any $\boldsymbol{\mu}$ -contest equilibrium, $\beta_1 < \min_{j \neq 1} \beta_j$.*

PROOF: Suppose that this is not the case, then $\beta_1 \geq \min_{j \neq 1} \beta_j$. In the corresponding auction equilibrium specified in Lemma B.1, $v_1 \leq \max_{j \neq 1} v_j$. Let $\mathcal{I} = \{i | v_i = \max_{j \neq 1} v_j\}$.

If $v_1 < \max_{j \neq 1} v_j$, then $1 \notin \mathcal{I}$. We show that this entails a contradiction by considering two cases. First suppose that \mathcal{I} contains only one element, say k , then $v_k > \max_{j \neq k} v_j$. In this case, by Lemma 11 in Baye et al. (1996), F_k strictly stochastically dominates F_j , $j \neq k$. Thus, (a) $\mathbb{E}[\tilde{x}_k] > \mathbb{E}[\tilde{x}_1]$. By Lemma B.1, (b) $\mathbb{E}[\tilde{x}_k] = \mu_k$ and $\mathbb{E}[\tilde{x}_1] = \mu_1$. (a) and (b) imply that $\mu_k > \mu_1$, contradicting the definition of the $\boldsymbol{\mu}$ vector, which entails $\mu_1 > \mu_k$.

Next, suppose that $v_1 < \max_{j \neq 1} v_j$ but \mathcal{I} contains more than one element, because, by hypothesis, $1 \notin \mathcal{I}$, Theorem 1 in Baye et al. (1996) shows that the 1's equilibrium auction bid is 0, which is absurd given Lemma B.1 and the fact that, by assumption, $\mu_1 > 0$.

Now consider the case where $v_1 = \max_{j \neq 1} v_j$, i.e., $1 \in \mathcal{I}$ but 1 is not the only element in \mathcal{I} . In this case, by Theorem 1 in Baye et al. (1996), at least two contestants in \mathcal{I} submit the same bid distribution. This implies that these two bidders' expected bids are the same. But, by assumption, no two components in the $\boldsymbol{\mu}$ vector are identical. And thus, given the correspondence specified in Lemma B.1, this is also impossible. \square

Lemma B.3 *In any $\boldsymbol{\mu}$ -contest equilibrium $\beta_2 = \beta_3 = \dots = \beta_n$, $\alpha_2 = \alpha_3 = \dots = \alpha_n = 0$, and $\alpha_1 > 0$.*

PROOF: Lemma B.2 and B.1 imply that, in the corresponding auction, $v_1 > \max_{j \neq 1} v_j$. Thus, we can partition the set of contestants as into three sets: $\{1\}$, $\mathcal{W} = \{i | v_i = \max_{j \neq 1} v_j\}$ and $\mathcal{O} = \{i | v_i < \max_{j \neq 1} v_j\}$. Theorem 2 in Baye et al. (1996) shows that if $i \in \mathcal{O}$, then $E[\tilde{x}_i] = 0$. The correspondence established in Lemma B.1 shows that $E[\tilde{x}_i] = \mu_i > 0$; thus \mathcal{O} is empty, which implies that $v_2 = v_3 = \dots = v_n$ and thus, by the correspondence, $\beta_2 = \beta_3 = \dots = \beta_n$. Lemma B.2 combined with this result, shows that in the corresponding auction, $v_1 > v_2 = v_3 = \dots = v_n$. Theorem 2 in Baye et al. (1996) shows that in this case $u_1^* > 0$ and $u_j^* = 0$, for $j \neq 1$. Thus because, under the correspondence, $u_i^* = \frac{\alpha_i}{\beta_i}$, $\alpha_1 > 0$ and $\alpha_i = 0$ for $i \neq 1$. \square

B.2 Example of Atalanta effect in a multi-contestant risk-taking contest

In this section, we present an example of the Atalanta effect in a three-contestant competition. In the competition, submitting the highest performance earns a rank reward of 1. Submitting less than the highest performance earns a rank reward of 0. There are three contestants, S , $W1$, and $W2$ and $\mu_S > \mu_{W1} > \mu_{W2} > 0$. As shown in Lemma B.1, the associated multipliers for the three contestants must satisfy the condition that $\alpha_S > 0$, and $\alpha_{W1} = \alpha_{W2} = 0$. Since Lemma B.1 also implies that the β 's of the two W contestants must be the same, we represent the β 's of two W contestants by β_W . Lemma B.2 implies that $\beta_S < \beta_W$.

Pure rank competition

The form of equilibrium distributions for the three contestants, (F_S, F_{W1}, F_{W2}) is provided by the following equation,

$$\begin{aligned} F_{W1} &= \begin{cases} \frac{\alpha_S + \beta_S x}{\sqrt{\alpha_S + \beta_S x_{W2}}} & x \in [0, x_{W2}), \\ \min[\sqrt{\alpha_S + \beta_S x}, 1] & x \geq x_{W2}; \end{cases} \\ F_{W2} &= \begin{cases} \sqrt{\alpha_S + \beta_S x_{W2}} & x \in [0, x_{W2}), \\ \min[\sqrt{\alpha_S + \beta_S x}, 1] & x \geq x_{W2}; \end{cases} \\ F_S(x) &= \begin{cases} \frac{x \beta_W}{\sqrt{\alpha_S + \beta_S x_{W2}}} & x \in [0, x_{W2}), \\ \min\left[\frac{x \beta_W}{\sqrt{\alpha_S + \beta_S x}}, 1\right] & x \geq x_{W2}. \end{cases} \end{aligned}$$

Because of the correspondence result, the results in Baye et al. (1996), and Lemma B.1, any pure rank competition equilibria must take this form for some parameters, β_S , β_W , x_{W2} , and α_S satisfying

$$0 < \beta_S < \beta_W, \quad x_{W2} < \frac{1 - \alpha_S}{\beta_S}, \quad \text{and } \alpha_S \in (0, 1). \quad (\text{B-3})$$

As inspection of Equation (B.2) shows that, in any equilibrium, both W contestants place positive mass on 0. The probability that both W contestants place mass on 0 equals α_S . S and $W1$ randomize over the interval $[0, u]$, where $u = \frac{1 - \alpha_S}{\beta_S}$. $W2$ randomizes over the interval $[x_{W2}, u]$. Over the interval $[x_{W2}, u]$, the distribution functions of $W1$ and $W2$ are identical.

Rank and bonus competition

Now consider rank and bonus competition. In addition to the rank reward, the contestants receive a bonus of b if their performance weakly exceeds the bonus threshold r . We aim to verify an equilibrium in which S chases the bonus but $W1$ and $W2$ do not. The form of the

candidate equilibrium strategies is given as follows.

$$\begin{aligned}
F_{W1} &= \begin{cases} \frac{\alpha_S + \beta_S x}{\sqrt{\alpha_S + \beta_S x_{W2}}} & x \in [0, x_{W2}), \\ \min [\sqrt{\alpha_S + \beta_S x}, 1] & x \geq x_{W2}; \end{cases} \\
F_{W2} &= \begin{cases} \sqrt{\alpha_S + \beta_S x_{W2}} & x \in [0, x_{W2}), \\ \min [\sqrt{\alpha_S + \beta_S x}, 1] & x \geq x_{W2}; \end{cases} \\
F_S(x) &= \begin{cases} \frac{x \beta_W}{\sqrt{\alpha_S + \beta_S x_{W2}}} & x \in [0, x_{W2}), \\ \min \left[\frac{x \beta_W}{\sqrt{\alpha_S + \beta_S x}}, 1 - p_S^r \right] & x \in [x_{W2}, r), \\ 1 & x \geq r. \end{cases}
\end{aligned}$$

In the candidate equilibrium, the parameters also satisfy the conditions of Equation (B-3) and the parameter, p_S^r , which represents the probability that S targets the bonus, satisfies $p_S^r \in (0, 1)$.

In order to illustrate the effects of introducing bonus compensation in a setting with more than two contestants, we verify equilibria in the pure rank and rank and bonus competition settings and show that qualitatively the behavior of the contestants is quite similar to contestant behavior in the two-contestant setting modeled in the manuscript. In both settings, the parameters—the multipliers and x_{W2} —and, in the case of rank and bonus competition, p_W^r , determine expected performance. However, with more than two contestants, it is not possible to analytically invert the map between the parameters and expected performance. So we proceed numerically and verify the equilibria for a specific parametric case described in the following table.

Assumed capacity			
$\mu_S = 0.50, \mu_{W1} = 0.2900, \mu_{W2} = 0.1584$			
Assumed bonus parameters			
$r = 2.4, b = 1.2477$			
Numerical solution			
Pure rank competition		Rank and bonus competition	
$p_{W1}^0 = 0.3726$	$p_{W2}^0 = 0.8159$	$p_{W1}^0 = 0.2362$	$p_{W2}^0 = 0.7766$
$u = 1.1446$	$\alpha_S = 0.3040$	$u = 0.9494$	$\alpha_S = 0.1834$
$\beta_S = 0.6081$	$\beta_W = 0.8737$	$\beta_S = 0.8601$	$\beta_W = 1.0006$
$x_{W2} = 0.5949$	—	$x_{W2} = 0.4880$	$p_S^r = 0.050$

Table 1: *Parameters for parametric example.*

Plots of the contestant reward functions, Π , and their support lines, ℓ , as well as the candidate equilibrium distributions, for the pure rank competition and rank and the rank and bonus competition are provided by Figures B-1 and B-2 respectively.

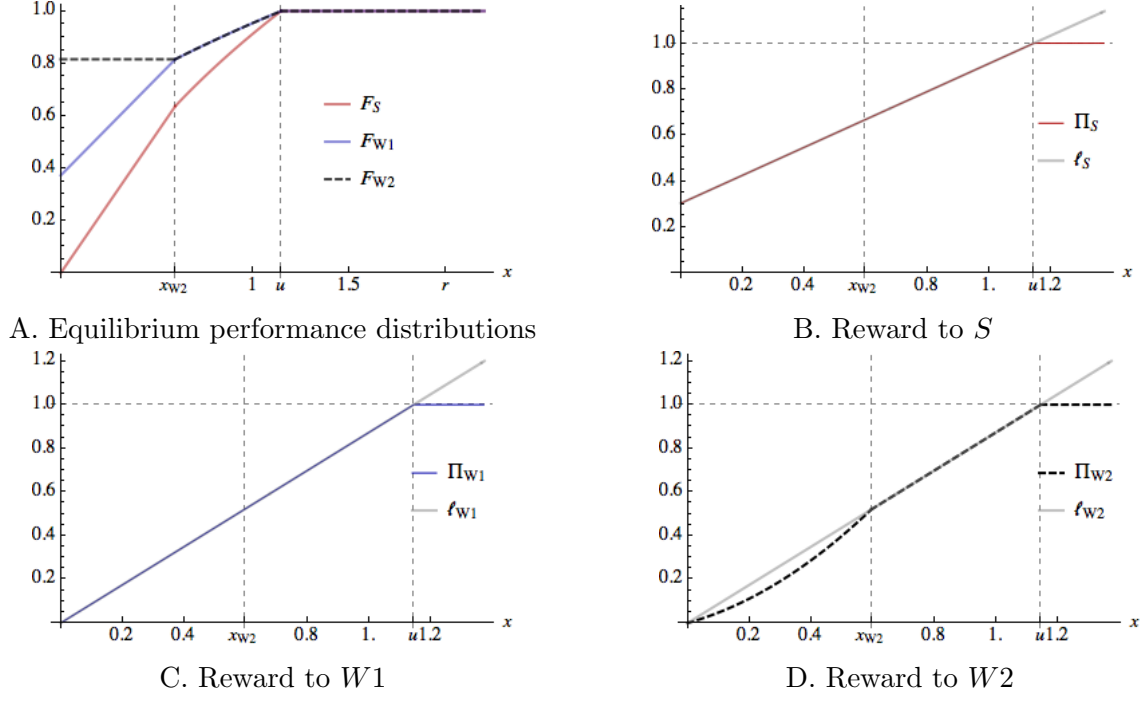


Figure B-1: *Pure rank competition: Equilibrium distributions and reward functions.*

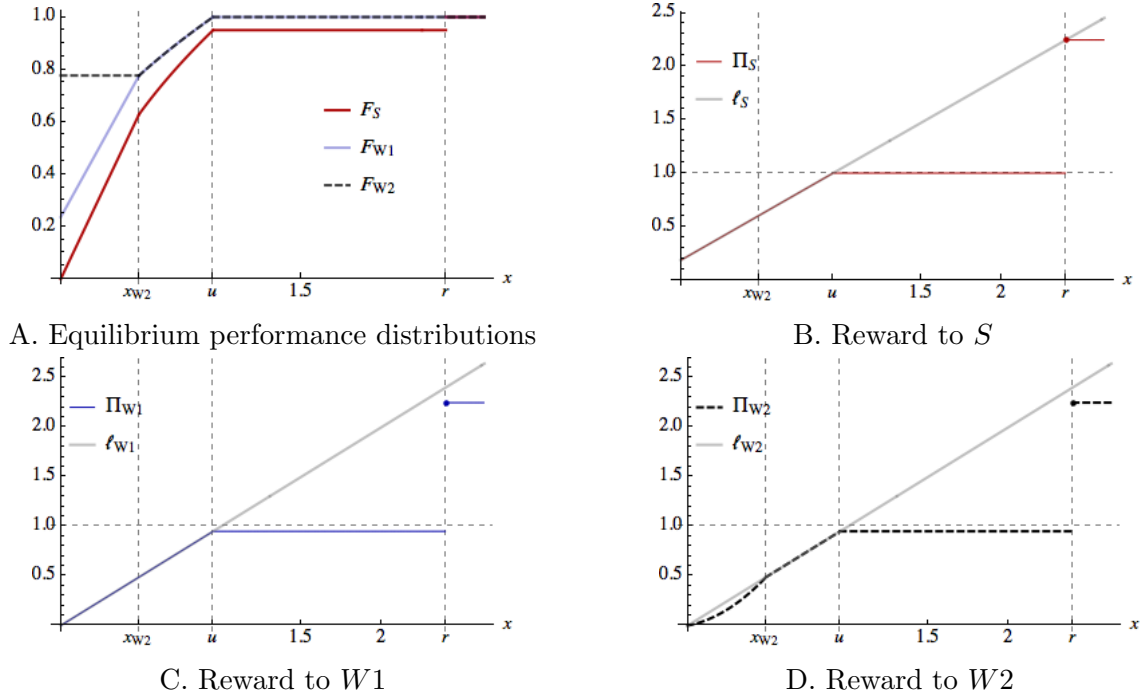


Figure B-2: *Rank and bonus competition: Equilibrium distributions and reward functions.*

As can be seen from the figures, the best response condition—that all performance levels in the supports of the contestants' distributions lie on their support lines and no performance levels lie above their support lines—are satisfied. Numerical integration shows that the capacity constraint is satisfied with equality for all contestants in the pure rank and rank and bonus competitions. Thus, the candidate equilibria are verified.

Turning to an analysis of the features of the equilibria, we see, from Table 1, that, qualitatively, equilibrium behavior is quite similar to behavior in the Eq0 configuration

developed in the paper: (a) The Atalanta effect is verified, i.e., the probability of downside risk taking for the two W contestants, represented by p_{W1}^0 and p_{W2}^0 , is reduced by the introduction of bonus compensation. (b) The length of the rank competition region, $(0, u]$, is reduced by the introduction of bonus compensation. Moreover, the logic underlying these two effects is the same as the logic developed in the manuscript: by diverting capacity to bonus chasing, S reduces capacity applied to the rank competition region. This reduction permits $W1$ and $W2$ to compete for rank dominance with S over a smaller region. The smaller region implies that they can satisfy their capacity constraints with performance distributions that place less weight on downside risk taking.

Thus, this example suggests that the qualitative effects of introducing bonus competition into rank competitions between multiple contestants are similar to the qualitative effects with two contestants. The analysis provides no reason to believe that the Atalanta effect depends on the two-contestant setting utilized in the manuscript.

C Dynamic contests

C.1 Dynamic optimization problem

Consider two contestants, F and G , who, as in Section 5, compete for rank and bonus rewards in the dynamic contest game. Their capacities are μ_F and μ_G respectively. The transition dynamics between the states are provided by the following tree structure, where the first component of the state vector represents the capacity of F and the second represents the capacity of G . By each competition on a time node, the winner will gain a capacity of δ and her rival will lose the same amount, with the sum of capacities constant due to the limited market size. We aim to first specify the contestant value functions and equilibrium distributions on each possible state. This will be the basis for solving the dynamic contest model.

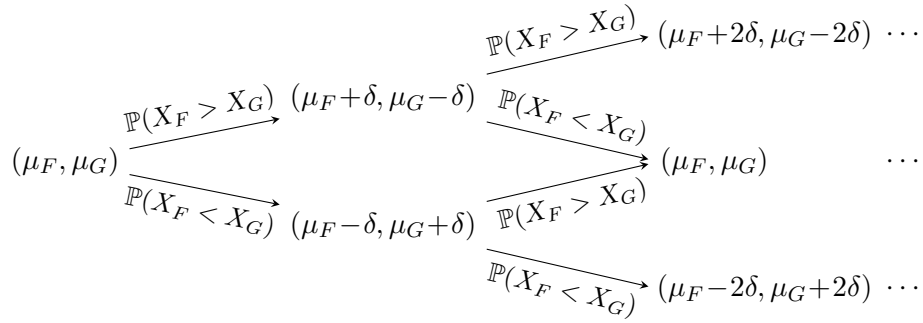


Figure C-1: *State transition dynamics.*

We can divide the support of performance into $[0, r)$ where the bonus is not captured, and $[r, \infty)$ where bonus is obtained with certainty. The key insight we use in the formulation of the programming problem, is that picking a performance distribution is equivalent to making the following two decisions: (i) Fix the probability of submitting performance in a specific region and, conditioned on submitting performance in this region, determine how much capacity to allocate to performance in this region. (ii) Conditioned on the allocated capacity and probabilities, determine an equilibrium performance distribution over each region. Using the equilibrium payoffs provided by (ii) we compute the payoffs associated with each allocation and then choose the allocation that maximizes the payoff of a contestant given the other contestant's allocation. We will perform these calculations for contestant F , the payoff calculations for G follow by symmetry.

First consider the payoff to F over $[0, r)$. With probability p_F , F devotes capacity μ_F^L to this region, and her rival, G , targets this region with probability p_G and devotes capacity μ_G^L to this region. Let $\mu_S^L = \max[\mu_F^L, \mu_G^L]$. The leading contestant, i.e., the contestant who possesses more capacity in the current state, will play a uniform distribution over $[0, u_L] = [0, 2\mu_S^L]$; whereas the trailing contestant will randomize between copying his rival and putting a point mass on zero. In order to define the payoff effects of allocations of capacity, the function, N , defined below is quite useful.

$$N(x, y) = \begin{cases} 1 - \frac{1}{2} \frac{y}{x} & \text{if } x > y, \\ \frac{1}{2} \frac{x}{y} & \text{if } x \leq y. \end{cases} \quad (\text{C-1})$$

Despite the piecewise definition of N function, we show its nice properties in the following lemma.

Lemma C.1 (Properties of N). *Function N is not only continuous but is also differentiable everywhere, both quasiconvex and quasiconcave (i.e., quasilinear), bounded between 0 and 1, and concave in each of its individual arguments.*

PROOF: We define a new variable, to be the ratio $r(x, y) := x/y$. With the kink point happening at $x = y$ (or equivalently, $r = 1$), the piecewise function (C-1) can be rewritten as

$$N(x, y) = \Psi(r) = \begin{cases} 1 - \frac{1}{2r} & \text{if } r \geq 1, \\ \frac{1}{2}r & \text{if } r < 1. \end{cases} \quad (\text{C-2})$$

Intuitively, Figure C-2 shows the contouring plot of N with respect to (x, y) and the shape of Ψ with respect to r .

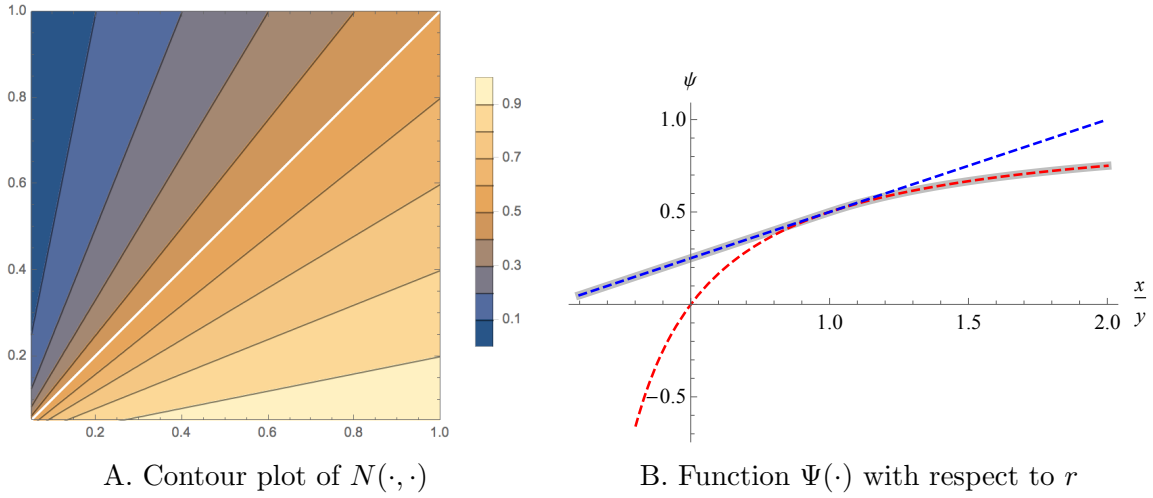


Figure C-2: *Contouring plot of N with respect to (x, y) and Ψ with respect to x/y .*

By definition, we have

$$\Psi(r) = \frac{1}{2} \int_0^r \min \left[1, \frac{1}{s^2} \right] ds,$$

where Ψ is differentiable everywhere because $s \mapsto \min[1, 1/s^2]$ is continuous, and $\Psi' > 0$. Thus Ψ is strictly increasing and $\Psi \in [0, 1]$.

The function $r(x, y) = x/y$ is semi-strictly quasilinear. Because Ψ is strictly increasing, $\Psi(x/y) = N(x, y)$ is semi-strictly quasilinear as well.

To put it in detail, let $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$. Then

$$r(\lambda v_1 + (1 - \lambda) v_2) = \frac{\lambda x_1 + (1 - \lambda) x_2}{\lambda y_1 + (1 - \lambda) y_2}.$$

If viewed as a function of $\lambda \in [0, 1]$, this function is strictly monotone provided $x_1/y_1 \neq x_2/y_2$. Hence for $\lambda \in (0, 1)$, $r(v_1) \neq r(v_2)$, we have

$$\min [r(v_1), r(v_2)] < r(\lambda v_1 + (1 - \lambda) v_2) < \max [r(v_1), r(v_2)].$$

Thus same results apply to the function $N(\cdot, \cdot)$. □

Using N , we can express the probability that contestant F wins in region $[0, r)$ as $N(\mu_F^L, \mu_G^L)$. Winning the rank competition contest only ensures the rank prize if and only

if the other contestant, G , also plays in this region. Thus, F will win the rank reward with probability $p_G N(\mu_F^L, \mu_G^L)$. Consequently, the expected payoff produced by targeting this region for F will equal the probability that F targets the region, p_F , times the reward above:

$$p_F p_G N(\mu_F^L, \mu_G^L). \quad (\text{C-3})$$

Now consider performance levels in $[r, \infty)$ that aim for both rank and bonus rewards. Assume the probability weights allocated to $[r, \infty)$ are $1 - p_F$ and $1 - p_G$, respectively. In this region, the contestant receives the bonus for sure and must always choose performance weakly in excess of r . Thus, we can think of competition in this region involving first paying a participation fee of r and then submitting performance in excess of r . If the conditional capacities allocated to $[r, \infty)$ are μ_F^H and μ_G^H , then essentially the two contestants bid $\mu_F^H - r$ and $\mu_G^H - r$, respectively, for the rank prize. Thus, if we let $\mu_S^H = \max[\mu_F^H, \mu_G^H]$, the support of this region will be $[r, u_H] = [r, r + 2(\mu_S^H - r)] = [r, 2\mu_S^H - r]$. The payoff for performance targeted at this region consists of a bonus reward and a rank reward. The bonus will always be received. For the rank reward, it is either (i) always attained whenever the rival targets $[0, r)$, or (ii) realized when the rival targets $[r, \infty)$ and is topped. Therefore, again using the N function, the payoff of a contestant, F , by targeting $[r, \infty)$, can be expressed as

$$(1 - p_F) [b + (p_G + (1 - p_G) N(\mu_F^H - r, \mu_G^H - r))]. \quad (\text{C-4})$$

Thus, the current reward Π_F to F conditioned on the allocations of capacity by F and G is the following sum

$$\Pi_F := p_F p_G N(\mu_F^L, \mu_G^L) + (1 - p_F) [b + (p_G + (1 - p_G) N(\mu_F^H - r, \mu_G^H - r))].$$

$\mathbb{P}(X_F > X_G)$ denotes the probability that contestant F wins in the game. When F targets $[0, r)$, the winning probability is given by $N(\mu_F^L, \mu_G^L)$. When F targets $[r, \infty)$, the winning probability consists of the sum of two parts: (i) when G targets $[0, r)$ then F wins for sure; and (ii) when both targets $[r, \infty)$ then F wins with probability $N(\mu_F^H - r, \mu_G^H - r)$. Thus, $\mathbb{P}(X_F > X_G) = p_F N(\mu_F^L, \mu_G^L) p_G + (1 - p_F) (p_G + (1 - p_G) N(\mu_F^H - r, \mu_G^H - r))$. We have $\mathbb{P}(X_F < X_G) = 1 - \mathbb{P}(X_F > X_G)$ by the no-ties property (Remark 1).

On a given state, each contestant's policy choice can be characterized by a vector $\Theta_i := \{p_i, \mu_i^L, \mu_i^H\} \in [0, 1] \times [0, \infty) \times [r, \infty)$, $i = F$ or G . Focusing on a single state, we term the exogenous value function of F as V_F . Based on the above analysis, we can see that the contest can be characterized by two programming problems, one for each contestant. Contestant F solves

$$\begin{aligned} \max_{\Theta_F} \quad & \Pi_F + \phi \left(\mathbb{P}(X_F > X_G) V_F(\mu_F + \delta, \mu_G - \delta) + \mathbb{P}(X_F < X_G) V_F(\mu_F - \delta, \mu_G + \delta) \right); \\ \text{s.t.} \quad & p_F \mu_F^L + (1 - p_F) \mu_F^H \leq \mu_F, \\ & \mu_F^H \geq r, \quad p_F \in [0, 1]. \end{aligned} \quad (\text{C-5})$$

The programming problem for G is completely symmetric. Each contestant solves her programming problem based on the conjectured allocations of the other contestant. Equilibrium results of the conjectured actions of the rival contestant are best responses for the rival.

Reviewing the problem (C-5), we find that the capacity constraint is given by the lower contour set of a quasi-convex function. Thus, the capacity constraint set is convex; the other constraint is linear. Hence, the overall constraint set is convex. Although objective function is not quasiconcave, it is differentiable and thus interior maxima for the problem are characterized by the Karush-Kuhn-Tucker (KKT) conditions. We solve the KKT problem analytically for the stationary points and compare these points and payoffs from the

boundary solutions. By comparing these local maxima and the boundary solutions we can identify global maximizers for both F and G 's optimization problems.

C.2 Dynamic equilibrium

We first develop a useful characterization, Lemma C.2, of the solution: the ratios of expected “available capacity” allocated to $[0, r)$ and $[r, \infty)$ are the same for both the leading and trailing contestants.

Lemma C.2 (Equilibrium ratio property). *In the dynamic contest, a contestant's equilibrium allocation at each state satisfies the following property:*

$$\frac{p_F}{1 - p_F} \frac{\mu_F^L}{\mu_F^H - r} = \frac{p_G}{1 - p_G} \frac{\mu_G^L}{\mu_G^H - r}, \quad (\text{C-6})$$

as long as $p_i \neq 1$, $i = F, G$.

We will prove Lemma C.2 after characterizing the equilibrium for exogenous continuation values. The function $N(\cdot, \cdot)$ defined in equation (C-1), is a smooth, concave, increasing function of its first argument and a smooth, convex, decreasing function of its second argument, with

$$\partial_1 N = \frac{1}{2} \min \left(\frac{1}{y}, \frac{y}{x^2} \right), \quad \partial_2 N = -\frac{1}{2} \min \left(\frac{1}{x}, \frac{x}{y^2} \right).$$

Because we add in a discounted expected future value, the objective function (NA:PK) changes compared to the static model. Nevertheless the rationale for the results is exactly the same as in Section 3. The equilibrium for any fixed continuation values can be characterized by the equation systems provided in the following lemma.

Lemma C.3 (Equilibrium for exogenous continuation values). *The equilibrium performance at each state of the dynamic game is characterized by the solution to the following equations:*

$$\mu_F = p_F \mu_F^L + (1 - p_F) \mu_F^H, \quad (\text{C-7})$$

$$p_G \partial_1 N(\mu_F^L, \mu_G^L) = (1 - p_G) \partial_1 N(\mu_F^H - r, \mu_G^H - r), \quad (\text{C-8})$$

$$p_G \partial_1 N(\mu_F^L, \mu_G^L) = \frac{(1 + A_F) p_G N(\mu_F^L, \mu_G^L) - [b + (1 + A_F) (p_G + (1 - p_G) N(\mu_F^H - r, \mu_G^H - r))]}{(1 + A_F) (\mu_F^L - \mu_F^H)}; \quad (\text{C-9})$$

$$\mu_G = p_G \mu_G^L + (1 - p_G) \mu_G^H, \quad (\text{C-10})$$

$$p_F \partial_1 N(\mu_G^L, \mu_F^L) = (1 - p_F) \partial_1 N(\mu_G^H - r, \mu_F^H - r), \quad (\text{C-11})$$

$$p_F \partial_1 N(\mu_G^L, \mu_F^L) = \frac{(1 + A_G) p_F N(\mu_G^L, \mu_F^L) - [b + (1 + A_G) (p_F + (1 - p_F) N(\mu_G^H - r, \mu_F^H - r))]}{(1 + A_G) (\mu_G^L - \mu_G^H)}; \quad (\text{C-12})$$

where we denote the terms related to the discounted future payoff as $A_F := \phi(V_F(\mu_F + \delta, \mu_G - \delta) - V_F(\mu_F - \delta, \mu_G + \delta))$ and $A_G := \phi(V_G(\mu_F - \delta, \mu_G + \delta) - V_G(\mu_F + \delta, \mu_G - \delta))$ for simplification.

PROOF: Denote the Lagrangian multipliers of contestant F and G by λ_F and λ_G , respectively. The Lagrangian function for contestant F is formulated as follows (Symmetry

implies an analogous expressions for G):

$$\mathcal{L} = p_F (1 + A_F) p_G N(\mu_F^L, \mu_G^L) + (1 - p_F) [b + (1 + A_F) (p_G + (1 - p_G) N(\mu_F^H - r, \mu_G^H - r))] - \lambda_F (p_F \mu_F^L + (1 - p_F) \mu_F^H - \mu_F). \quad (\text{C-13})$$

Differentiating the Lagrangian function with respect to F 's control variables, we must have the following three equalities at a stationary point:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial p_F} &= (1 + A_F) p_G N(\mu_F^L, \mu_G^L) - [b + (1 + A_F) (p_G + (1 - p_G) N(\mu_F^H - r, \mu_G^H - r))] + \lambda_F (\mu_F^H - \mu_F^L) = 0, \\ \frac{\partial \mathcal{L}}{\partial \mu_F^L} &= p_F ((1 + A_F) p_G \partial_1 N(\mu_F^L, \mu_G^L) - \lambda_F) = 0, \\ \frac{\partial \mathcal{L}}{\partial \mu_F^H} &= (1 - p_F) ((1 + A_F) (1 - p_G) \partial_1 N(\mu_F^H - r, \mu_G^H - r) - \lambda_F) = 0. \end{aligned}$$

So, at a stationary point,

$$\lambda_F = \frac{(1 + A_F) p_G N(\mu_F^L, \mu_G^L) - [b + (1 + A_F) (p_G + (1 - p_G) N(\mu_F^H - r, \mu_G^H - r))]}{\mu_F^L - \mu_F^H}, \quad (\text{C-14})$$

$$\lambda_F = (1 + A_F) p_G \partial_1 N(\mu_F^L, \mu_G^L), \quad (\text{C-15})$$

$$\lambda_F = (1 + A_F) (1 - p_G) \partial_1 N(\mu_F^H - r, \mu_G^H - r). \quad (\text{C-16})$$

Equations (C-15) and (C-16) imply that

$$p_G \partial_1 N(\mu_F^L, \mu_G^L) = (1 - p_G) \partial_1 N(\mu_F^H - r, \mu_G^H - r). \quad (\text{C-17})$$

Equations (C-14) and (C-15) imply that

$$\frac{(1 + A_F) p_G N(\mu_F^L, \mu_G^L) - [b + (1 + A_F) (p_G + (1 - p_G) N(\mu_F^H - r, \mu_G^H - r))]}{\mu_F^L - \mu_F^H} = (1 + A_F) p_G \partial_1 N(\mu_F^L, \mu_G^L). \quad (\text{C-18})$$

The binding capacity constraint is given by

$$p_F \mu_F^L + (1 - p_F) \mu_F^H = \mu_F. \quad (\text{C-19})$$

Same set of equations characterizes G 's performance with the required transpositions. Therefore, the above six equations fully characterize the stationary points.

Because of the piecewise definition of function $N(\cdot, \cdot)$, determining the solutions to these stationary points involves analyzing four cases. However, we can focus on just two cases as the other two are obtained through symmetry by exchanging the positions of the two contestants.

(i) $\mu_F^L \leq \mu_G^L$, and $\mu_F^H \geq \mu_G^H$: In this case we have

$$N(\mu_F^L, \mu_G^L) = \frac{\mu_F^L}{2\mu_G^L}, \quad N(\mu_F^H - r, \mu_G^H - r) = 1 - \frac{\mu_G^H - r}{2(\mu_F^H - r)}.$$

Plugging the expressions for $N(\cdot, \cdot)$ into the equation system (C-14)–(C-16) as well as into the analogous three equations for contestant G , we can first solve the stationary

point for μ_F^{H*} and μ_G^{L*} , and the other variables follow:

$$\begin{aligned}\mu_F^{H*} &= \frac{b + (1 + A_F) + \lambda_F r}{2 \lambda_F}, & \mu_G^{L*} &= \frac{\lambda_G r - b}{2 \lambda_G}, \\ p_F^* &= 1 - \frac{2 \lambda_G (\mu_F^{H*} - r)}{1 + A_G}, & p_G^* &= \frac{2 \lambda_F \mu_G^{L*}}{1 + A_F}, \\ \mu_F^{L*} &= \frac{2 \lambda_G (\mu_G^{L*})^2}{p_F^* (1 + A_G)}, & \mu_G^{H*} &= \frac{2 \lambda_F (\mu_F^{H*} - r)^2}{(1 + A_F) (1 - p_G^*)} + r;\end{aligned}$$

where λ_F and λ_G take the values that bind the two capacity constraints.

- (ii) $\mu_F^L \leq \mu_G^L$, and $\mu_F^H \leq \mu_G^H$:
In this case we have

$$N(\mu_F^L, \mu_G^L) = \frac{\mu_F^L}{2 \mu_G^L}, \quad N(\mu_F^H - r, \mu_G^H - r) = \frac{\mu_F^H - r}{2 (\mu_G^H - r)}.$$

Plugging the expressions of $N(\cdot, \cdot)$ into the equation system (C-14)–(C-16) as well as the other three analogous equations of contestant G , we can first find the stationary points for the three control variables of G , and then for F , as follows:

$$\begin{aligned}p_G^* &= \frac{\lambda_F r - b}{1 + A_F}, & p_F^* &= 1 - \frac{2 \lambda_G (\mu_G^{H*} - \mu_G^{L*}) - b - \lambda_G r}{1 + A_G}, \\ \mu_G^{L*} &= \frac{\lambda_F r - b}{2 \lambda_F}, & \mu_F^{L*} &= \frac{2 \lambda_G (\mu_G^{L*})^2}{(1 + A_G) p_F^*}, \\ \mu_G^{H*} &= \frac{1 + A_F + b}{2 \lambda_F} + \frac{r}{2}, & \mu_F^{H*} &= \frac{2 \lambda_G (\mu_G^{H*} - r)^2}{(1 + A_G) (1 - p_F^*)} + r;\end{aligned}$$

where λ_F and λ_G take the values that bind the two capacity constraints.

The derivation is purely symmetric for the other two cases, (iii) $\mu_F^L \geq \mu_G^L$, and $\mu_F^H \leq \mu_G^H$ and (iv) $\mu_F^L \geq \mu_G^L$, and $\mu_F^H \geq \mu_G^H$. These analytical solutions for the stationary points are indeed the local maximizer because the Hessian matrix is semi-negative definite on the hyperplane orthogonal to the constraint set. By comparing these local maxima and the boundary solutions we can identify the unique global maximizer for both F and G when, for example, $\mu_F \leq \mu_G$, the stationary point for case (i) turns out to produce the highest value. \square

Now we give the formal proof of Lemma C.2.

PROOF: Plugging in the first derivative of $N(\cdot, \cdot)$, equation (C-8) gives:

$$p_G \min \left(\frac{1}{\mu_G^L}, \frac{\mu_F^L}{(\mu_F^L)^2} \right) = (1 - p_G) \min \left(\frac{1}{\mu_G^H - r}, \frac{\mu_F^H - r}{(\mu_F^H - r)^2} \right).$$

Similarly, from (C-11) we have

$$p_F \min \left(\frac{1}{\mu_F^L}, \frac{\mu_F^L}{(\mu_F^L)^2} \right) = (1 - p_F) \min \left(\frac{1}{\mu_F^H - r}, \frac{\mu_F^H - r}{(\mu_F^H - r)^2} \right).$$

Hence, we have the following four cases:

(i) $\mu_F^L \leq \mu_G^L$ and $\mu_F^H \leq \mu_G^H$:

$$\frac{p_G}{\mu_G^L} = \frac{1 - p_G}{\mu_G^H - r}, \quad \frac{p_F \mu_F^L}{(\mu_G^L)^2} = \frac{(1 - p_F)(\mu_F^H - r)}{(\mu_G^H - r)^2},$$

(ii) $\mu_F^L \leq \mu_G^L$ and $\mu_F^H > \mu_G^H$:

$$\frac{p_G}{\mu_G^L} = \frac{(1 - p_G)(\mu_G^H - r)}{(\mu_F^H - r)^2}, \quad \frac{p_F \mu_F^L}{(\mu_G^L)^2} = \frac{1 - p_F}{\mu_F^H - r},$$

(iii) $\mu_F^L > \mu_G^L$ and $\mu_F^H \leq \mu_G^H$:

$$\frac{p_G \mu_G^L}{(\mu_F^L)^2} = \frac{1 - p_G}{\mu_G^H - r}, \quad \frac{p_F}{\mu_F^L} = \frac{(1 - p_F)(\mu_F^H - r)}{(\mu_G^H - r)^2},$$

(iv) $\mu_F^L > \mu_G^L$ and $\mu_F^H > \mu_G^H$:

$$\frac{p_G \mu_G^L}{(\mu_F^L)^2} = \frac{(1 - p_G)(\mu_G^H - r)}{(\mu_F^H - r)^2}, \quad \frac{p_F}{\mu_F^L} = \frac{1 - p_F}{\mu_F^H - r}.$$

Algebraic simplification shows that these equalities are satisfied. \square

Now we numerically characterize the equilibrium of the dynamic predation game. The equilibrium result at the initial state can be found using *dynamic programming*. To formalize these notions in detail, denote each state by $\omega \in \Omega$. Define a transition function, Γ , mapping the current state into the state in the following period, consisting of the following two sub-functions:

$$\Gamma^+(\omega) = \begin{cases} \omega + (\delta, -\delta) & \text{if } X_F > X_G \text{ and } \omega \in \mathbb{R}_+^2, \\ \omega & \text{otherwise;} \end{cases}$$

and

$$\Gamma^-(\omega) = \begin{cases} \omega - (\delta, -\delta) & \text{if } X_F < X_G \text{ and } \omega \in \mathbb{R}_+^2, \\ \omega & \text{otherwise.} \end{cases}$$

Then the Bellman equation for contestant F can be represented by

$$T[V_F](\omega) = \max_{\Theta_F} \Pi_F(\Theta_F, \Theta_G|\omega) + \phi \left[\mathbb{P}(X_F > X_G | \Theta_F, \Theta_G) V_F(\Gamma^+(\omega)) + \mathbb{P}(X_F < X_G | \Theta_F, \Theta_G) V_F(\Gamma^-(\omega)) \right], \quad (\text{C-20})$$

where Π_F is the function of F 's current rewards and ϕ is a discount rate. Operator T is a map from functions to functions. Similarly, for contestant G we have

$$T[V_G](\omega) = \max_{\Theta_G} \Pi_G(\Theta_F, \Theta_G|\omega) + \phi \left[\mathbb{P}(X_G > X_F | \Theta_F, \Theta_G) V_G(\Gamma^+(\omega)) + \mathbb{P}(X_G < X_F | \Theta_F, \Theta_G) V_G(\Gamma^-(\omega)) \right]. \quad (\text{C-21})$$

Thus, the equilibrium is a fixed point, $V^* = (V_F^*, V_G^*)$, of the Bellman equations (C-20) and (C-21).

The solution for the fixed point is based on the following iterative procedure: assume a value v_0 at state $(1, 1)$ for both F and G . This assumed value will determine the value at all other states, for instance, in the example in the main body of the paper, as follows: once state $(2, 0)$ is reached, G is eliminated and thus F receives the rank reward in all subsequent periods. Thus, the value at $(2, 0)$ is easy to compute. Symmetry implies that these observations also apply at $(0, 2)$ with the roles of the contestants reversed. Values at

$(3/2, 1/2)$ and $(1/2, 3/2)$ are given by the discounted expectations of the value at $(1, 1)$, and the fixed values at $(0, 2)$, and $(2, 0)$. Conditioned on these values, we can determine equilibrium performance strategies at each state using Lemma C.3. The result will imply a new value v_1 at $(1, 1)$, and so forth. Using these iterations, we apply the *contraction mapping* principle to solve for the fixed point solution of the Bellman equation, $V^*(1, 1)$, as follows: (i) We start by assigning some value v_0 to state $(1, 1)$: We use this value to calculate the values at the other states. (ii) Given these values we solve optimal strategies of the contestants using the conditions in Lemma C.3. (iii) These strategies generate a new value at $(1, 1)$, $v_1 = T(v_0)$. (iv) We keep iterating steps one and two to get new values $v_{n+1} = T(v_n)$. (v) We continue iteration until we reach the fixed point $v^* = T(v^*) = V^*(1, 1)$.

The following figure illustrates the convergence of $\{v_n\}$ to $V^*(1, 1)$ in this example for a starting values $v_0 = 0$ or $v_0 = 8$. As implied by the contraction mapping principle, the limit of the sequence is robust to different choices of initial value. For the asymmetric dynamic game, the logic for convergence is similar.

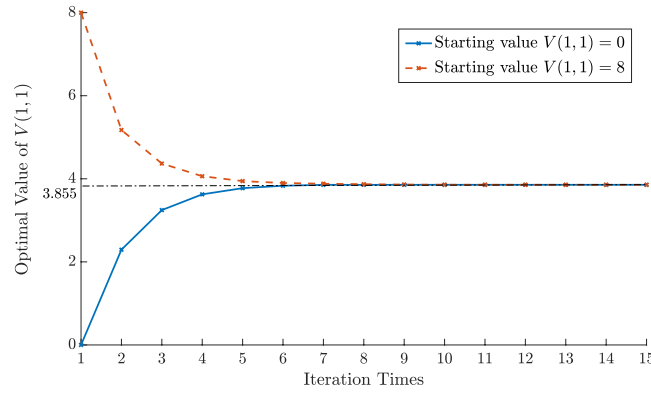


Figure C-3: *Convergent sequences of iterates under different starting values, v_0 .*

D Formal derivations of the generic properties of contest equilibria

The derivations in this appendix are novel in the sense that they have not been established in a contest game with exactly the same structure as our contest game. However, the arguments and characterizations closely track characterizations for other all pay and risk taking contest games (e.g., Siegel, 2009; Hillman and Riley, 1989). More generally, the approach taken to deriving the results—bounding the support of the payoff distribution with an envelope of affine functions that it majorizes—is equivalent to the concave envelope approach frequently used to analyze all-pay auction, risk-taking, and Bayesian persuasion games (e.g., Aumann et al., 1995; Kamenica and Gentzkow, 2011). We deal with the problem of tied performance, which generates discontinuities in the reward function, by replacing the natural reward function for the game with a more tractable reward function and then establish an equivalence between the equilibria under the natural and tractable functions. This is a very standard approach in games with discontinuous best reply correspondences (cf. Simon and Zame, 1990; Siegel, 2009).

D.1 Reward functions, best responses, ties, and support lines

Overall this section is to deal with the problem of tied performance and formally justify the support-line arguments informally developed in the paper. These results will confirm the assertions in Remarks 2 and 3 as well as confirming that the reward functions used in our definition of equilibrium produces the same set of equilibria as a reward function that splits the rank reward in the event of tied performance.

D.1.1 Payoffs, best replies, and reward functions

Let $-S$ represent W and $-W$ represent S . Consider the following reward functions, Π_i^T and Π_i^N , both defined over \mathbb{R}_+ :

$$\Pi_i^T(x) = F_{-i}(x-) + \frac{1}{2}(F_{-i}(x) - F_{-i}(x-)) + \mathbb{1}_r(x), \quad i = S, W, \quad (\text{D-1})$$

$$\Pi_i^N(x) = F_{-i}(x) + \mathbb{1}_r(x), \quad i = S, W. \quad (\text{D-2})$$

Π_i^T accounts for the possibility of tied performance and, in the event of tied performance, splits the rank reward of 1 equally between the two contestants. As an inspection of the following derivations will show, an equal split is not essential to the arguments, any division that does not assign the entire reward to one of the contestants suffices to establish the subsequent results. An equal division is used here simply to avoid introducing more notation.

In contrast, Π_i^N ignores for the possibility of tied performance and, in the event of tied performance, provides a reward of 1 to both contestants. This is the reward function used in the body of the paper.

Let \mathcal{P}^+ be the set of all probability distribution functions supported by $[0, \infty)$ and define \mathcal{F}_i by

$$\mathcal{F}_i = \left\{ F \in \mathcal{P}^+ : \int_{0-}^{\infty} x \, dF(x) \leq \mu_i \right\}, \quad i = S, W.$$

A best reply for contestant i to F_{-i} under Π^k , $k = T, N$ is a probability distribution F^* that satisfies

$$\int_{0-}^{\infty} \Pi_i^k(x) \, dF^*(x) = \sup \left\{ \int_{0-}^{\infty} \Pi_i^k(x) \, dF(x) : F \in \mathcal{F}_i \right\}, \quad k = T, N, \quad i = S, W.$$

Both Π_i^T and Π_i^N are bounded, nondecreasing functions defined over \mathbb{R}_+ that only differ with respect to how they treat tied performance. Tied performance occurs with positive probability if and only if both contestants choose discontinuous performance distributions that both put positive mass on some performance level. To formalize this notion let

$$\mathcal{M}_i = \{x \geq 0 : F_i(x) - F_i(x-) > 0\}, \quad i = S, W. \quad (\text{D-3})$$

Note that because distributions are non decreasing, \mathcal{M}_i is at most countable. We will say that tied performance does not occur if $\mathcal{M}_S \cap \mathcal{M}_W = \emptyset$. When ties do not occur, $\Pi^T = \Pi^N$.

Π^N is right-continuous, and this implies, because it is also non-decreasing, that it is upper semicontinuous. This implies that the map

$$F \mapsto \int_{0-}^{\infty} \Pi^N(x) dF(x)$$

is upper semicontinuous (Ash, 1972, Theorem 4.5.1.b). Thus, whenever the payoff from sequence of distribution converges to the supremum of a contestant's payoff, and the sequence of distributions converges to a distribution function F_o , F_o is a best reply.

In contrast, in general, Π^T is not upper semicontinuous. Thus, the limit of a convergent sequences of distributions producing payoffs that converge to the supremum, need not be a best reply. For this reason, Π^T is not a very convenient reward function. However, it does accurately reflect the fact that the total rank reward is constant, while Π^N does not.

Fortunately, as we will show below, the set of Nash equilibria given Π^T equals the set of Nash equilibria under Π^N which satisfy the condition of no-tied performance. This equivalence results simply because, under Π^T , contestants will never choose in equilibrium to submit performance distributions that result in performance ties with positive probability. This general approach is used in Siegel (2009) to apply Simon and Zame (1990) existence result for games with discontinuous best reply correspondences to all-pay auctions.

D.1.2 Support lines and optimal performance distributions

To establish this equivalence, and to characterize the general properties of equilibrium performance distributions, we require a simple means of characterizing best replies. This desideratum will be supplied by a support line characterization of best replies developed in this section.

Because, in this section, we are only concerned with characterizing the properties of optimal performance distribution for an individual contestant under a given reward function, we will simplify notation by suppressing the subscript representing agent type. Because the results in this section hold for both Π^N and Π^T we will simply represent the contest reward function with Π . The contestant's problem is to maximize her payoff under the contest reward function, Π , over distribution functions in \mathcal{P}^+ subject to the capacity constraint, let ν^* represent the supremum of this problem, i.e.,

$$\nu^* = \sup_{F \in \mathcal{P}^+} \left\{ \int_{0-}^{\infty} \Pi(x) dF(x) : \int_{0-}^{\infty} x dF(x) - \mu \leq 0 \right\}. \quad (\text{D-4})$$

Because, For all $x \geq 0$, $\Pi(x) \in [0, 1 + b]$, $\nu^* < \infty$. Define the associated Lagrange function, $\mathcal{L} : \mathcal{P}^+ \times [0, \infty) \rightarrow \mathbb{R}$:

$$\mathcal{L}(F, \lambda) = \int_{0-}^{\infty} (\Pi(x) - \lambda x) dF(x) + \lambda \mu. \quad (\text{D-5})$$

The objective function,

$$dF \hookrightarrow \int_{0-}^{\infty} \Pi(x) dF(x)$$

is linear and hence concave, the constraint set is convex, and clearly, there exists F such that the constraint is strictly satisfied. Thus, there exists a multiplier, $\lambda^* \geq 0$, such that

$$\nu^* = \sup_{F \in \mathcal{P}^+} L(F, \lambda^*), \quad (\text{D-6})$$

and, if the supremum in (D-4) is attained at F^* , then F^* attains the supremum in equation (D-6) (Chapt 1., Theorem 1 Luenberger, 1969). Let

$$\beta = \lambda^*, \quad \text{and } \alpha = \sup_{x \geq 0} \Pi(x) - \beta x. \quad (\text{D-7})$$

Note that because $\Pi(0) \geq 0$, $\alpha \geq 0$.

Thus, if F^* is an optimal policy, using equations (D-5) and (D-7), we can express the Lagrange function, evaluated at F^* and λ^* , in terms of α and β as follows:

$$\nu^* = \sup_{F \in \mathcal{P}^+} \mathcal{L}(F, \lambda^*) = \mathcal{L}(F^*, \lambda^*) = \int_{0-}^{\infty} \left(\Pi(x) - (\alpha + \beta x) \right) dF^*(x) + \alpha + \beta \mu. \quad (\text{D-8})$$

Inspection of Equation (D-8) implies that the optimal performance distribution, F^* , satisfies

$$\begin{aligned} \text{Supp}(F^*) &\subseteq \{x \geq 0 : \Pi(x) - (\alpha + \beta x) = 0\}, \\ &\quad \forall x \geq 0, \Pi(x) \leq \alpha + \beta x, \\ 0 &= \int_{0-}^{\infty} \left(\Pi(x) - (\alpha + \beta x) \right) dF^*(x). \end{aligned} \quad (\text{D-9})$$

Clearly $\beta > 0$. If $\beta = 0$ then $\alpha = 1 + b$, which implies, by (D-8) and (D-9), that $\nu^* = 1 + b$, which is impossible by our assumption that $\mu < r$, and thus the bonus b cannot be captured with probability 1.

Lemma D-1 (Multipliers and support lines) *If $\Pi = \Pi^T$ or Π^N and F^* is a best response to Π , there exists $\alpha \geq 0$ and $\beta > 0$ and support line, $\ell(x) = \alpha + \beta x$, such that F^* satisfies*

$$\text{Supp}(F^*) \subseteq \{x \geq 0 : \Pi(x) = \ell(x)\}, \quad (\text{D-10})$$

$$\forall x \geq 0, \Pi(x) \leq \ell(x). \quad (\text{D-11})$$

Lemma D-1 confirms Remark 3. Note that Lemma D-1 establishes necessary conditions for optimal performance distributions under both Π^T and Π^N . These conditions do not speak to the question of whether an optimal performance distribution exists, i.e. whether the supremum of the contestants' optimization problems is attained. In this respect, Π^T and Π^N can differ substantially, as pointed out earlier.

D.1.3 Ties

The next result confirms Remark 1 by showing that, that under Π^T , best responses never produce tied performance. Intuitively, this is obvious, at a tie point, a contestant can divert infinitesimal to slightly increasing performance at the tie point and to “just top” her rival, breaking the tie and increasing her payoff by a non infinitesimal amount.

Lemma D-2 *If F_i^* is best response to Π^T , then $\mathcal{M}_{-i} \cap \text{Supp}_i = \emptyset$, i.e., a best response by i to $-i$ never produces tied performance.*

PROOF: Suppose not. Then there exists $x_o \geq 0$ to which both contestants assign positive probability mass. In the event of a tie, the rank reward is divided between the two contestants, with each receiving a rank-based reward of $1/2$. Let $\{x_n\}$ be a decreasing sequence converging to x_o .

Consider a contestant's, say S , reward function. The rank based-reward to W if W plays F_W is

$$\Pi_S^T(x_o) = \mathbb{P}[X_W < x_o] + \frac{1}{2} \mathbb{P}[X_W = x_o], \quad \mathbb{P}[X_W = x_o] > 0. \quad (\text{D-12})$$

The rank based reward to S from x_n equals

$$\Pi_S^T(x_n) = \mathbb{P}[X_W < x_n] + \frac{1}{2} \mathbb{P}[X_W = x_n] \geq \mathbb{P}[X_W \leq x_o] = \mathbb{P}[X_W < x_o] + \mathbb{P}[X_W = x_o]. \quad (\text{D-13})$$

The bonus reward at x_n is no less than the bonus reward at x_o . Thus,

$$\Pi_S^T(x_n) \geq \Pi_S^T(x_o) + \frac{1}{2} \mathbb{P}[X_W = x_o]. \quad (\text{D-14})$$

Because $x_o \in \text{Supp}_S$ by hypothesis, condition (D-10) implies that $\ell_S(x_o) = \Pi_S(x_o)$. In order for condition (D-11) to be satisfied, it must be the case that

$$\forall n \in \mathbb{N}, \quad \ell_S(x_n) \geq \Pi_S^T(x_n). \quad (\text{D-15})$$

Equations (D-14) and (D-15) imply that

$$\forall n \in \mathbb{N}, \quad \ell_S(x_n) \geq \Pi_S^T(x_o) + \frac{1}{2} \mathbb{P}[X_W = x_o].$$

Thus

$$\Pi_S^T(x_o) = \ell_S(x_o) = \lim_{n \rightarrow \infty} \ell_S(x_n) \geq \Pi_S^T(x_o) + \frac{1}{2} \mathbb{P}[X_W = x_o] > \Pi^T(x_o).$$

This contradiction establishes the lemma. \square

Lemma D-2 provides an equivalence relation between equilibria under Π^T and Π^N which rationalizes the reward function used in the body of the paper (equation (3.1)).

Lemma D-3 *The following statements are equivalent:*

- (i) For $i = S, W$, F_i is a best response to F_{-i} under Π_i^T ,
- (ii) For $i = S, W$, F_i is a best response to F_{-i} under Π_i^N and $\mathcal{M}_S \cap \mathcal{M}_W = \emptyset$.

PROOF: (i) \Rightarrow (ii): If (i) holds, then Lemma D-2 implies that $\mathcal{M}_S \cap \mathcal{M}_W = \emptyset$, which implies that $\Pi_i^T = \Pi_i^N$ and thus, (ii) holds.

(ii) \Rightarrow (i): If (ii) holds then $\Pi_i^T = \Pi_i^N$, and thus (i) holds. \square

This result formally shows that the payoff function used in the body of the paper, which in the body of the paper is simply called Π and in this appendix, thus far, has been termed Π^N , characterizes equilibrium behavior even when the game specifies a division of the rank reward in the event of ties.

D.2 Properties of equilibrium performance strategies

Henceforth, making a slight abuse of notation we represent Π_i^N , $i = S, W$, simply by Π_i . Note that this definition coincides with the definition of Π_i in the main body of the paper.

Lemma D-4 *If (F_W, F_S) are equilibrium performance distributions,*

(i)

$$(0, r) \cap \text{Supp}_S = (0, r) \cap \text{Supp}_W,$$

(ii)

$$(r, \infty) \cap \text{Supp}_S = (r, \infty) \cap \text{Supp}_W.$$

PROOF: We prove (i), the proof of (ii) is omitted because it is virtually identical. Suppose to obtain a contradiction, that there exists x_o such that $x_o \in (0, r) \cap \text{Supp}_W$ but $x_o \notin (0, r) \cap \text{Supp}_S$ (the argument with the roles of the contestants reversed is identical up to transpositions of the type names).

Because, by definition, Supp_S is closed, there exists an open neighborhood N of x_o in $(0, r)$, such that, for all $x \in N$, $x \notin \text{Supp}_S$. Over $(0, r)$, $\Pi_W = F_S$ and thus because F_S is constant on N , Π_W is constant on N .

By hypothesis, $x_o \in N \cap \text{Supp}_W$, thus by condition (D-10), $\ell_W(x_o) = \Pi_W(x_o)$. Because ℓ_W is increasing, for $x \in N$ and $x < x_o$, $\ell_W(x) < \ell_W(x_o)$ and, because Π_W is constant over N , $\Pi_W(x) = \Pi_W(x_o)$. Thus, for $x \in N$ and $x < x_o$, $\ell(x) < \Pi_W(x)$, contradicting condition (D-11). \square

Lemma D-4 confirms Remark 2.a in the main body of the paper. The next result shows that, in all equilibria, both contestants assign some probability weight to the pure rank competition region.

Lemma D-5 *If (F_W, F_S) are equilibrium performance distributions,*

$$(0, r) \cap \text{Supp}_i \neq \emptyset, \quad i = S, W. \quad (\text{D-16})$$

PROOF: If $(0, r) \cap \text{Supp}_i = \emptyset$, then, by Lemma D-4, $(0, r) \cap \text{Supp}_j = \emptyset$, $j \neq i$. Performance distributions that are supported by $[r, \infty)$ are inconsistent with the capacity constraint given the assumption that $\mu_i < r$, $i = S, W$. Thus, both S and W must place positive mass on 0, i.e., the probability of tied performance must be positive. But this is impossible by Lemma D-2. \square

Lemma D-5 will now be used to derive the next lemma, Lemma D-6, which confirms Remark 2.d.

Lemma D-6 *If (F_W, F_S) are equilibrium performance distributions,*

(i)

$$0 \in \text{Supp}_i, \quad i = S, W.$$

(ii)

$$\text{If } \text{Supp}_i \cap [r, \infty) \neq \emptyset, \quad r \in \text{Supp}_i, \quad i = S, W.$$

PROOF: Again we prove only (i), because the proof of (ii) is virtually identical. Suppose, to obtain a contradiction, that 0 is not in the support of one of the contestant's performance distributions, say S . Then there would exist a neighborhood of 0, open in $[0, \infty)$ such that for all $x \in N$, $x \notin \text{Supp}_S$. This implies that on N , F_S is constant. Let $x_o = \min\{x \geq 0 : x \in \text{Supp}_S\}$. By Lemma D-5, $x_o < r$. For $x \in [0, x_o)$, F_S is constant and thus Π_W is constant. Thus, for all $x \in [0, x_o)$, $\Pi_W(x) = \Pi(0)$, which implies that $\Pi(x_o-) = \Pi(0)$. If Π_W is continuous at x_o (i.e., F_S is continuous at x_o) then $\Pi_W(x_o) = \Pi_W(0)$. Condition (D-11) implies that $\ell_W(0) \geq \Pi_W(0)$. Because ℓ_W is increasing, and $\Pi(x_o-) = \Pi(0)$, $\ell(x_o) > \Pi_W(x_o)$, which, by condition (D-10), implies that $x_o \notin \text{Supp}_W$. By Lemma D-4 this implies that $x_o \notin \text{Supp}_S$, contradicting the definition of x_o .

Thus, F_S must jump at x_o . This implies by Lemma D-2, that F_W does not jump at x_o . On N , F_W is constant, and does not jump at x_o . Hence, $\Pi_S(x_o) = \Pi_S(0)$. Because, by construction, $x_o \in \text{Supp}_S$, $\ell_S(x_o) = \Pi_S(x_o)$ by condition (D-10). Because Π_S is constant on N and ℓ_S is increasing, for $x \in [0, x_o)$, $\ell_S(x) < \Pi_S(x)$, contradicting condition (D-11). \square

In addition to confirming Remark 2.d, Lemma D-6 shows that when the rank plus bonus competition region is not empty, its greatest lower bound always equals the bonus threshold. The following lemma shows that the rank and rank plus bonus competition regions are in fact intervals confirming Remark 2.b.

Lemma D-7 *If (F_W, F_S) are equilibrium performance distributions,*

(i)

$\text{Supp}_i \cap [0, r)$ *is connected.*

(ii)

If $\text{Supp}_i \cap [r, \infty) \neq \emptyset \Rightarrow \text{Supp}_i \cap [r, \infty)$ is connected.

PROOF: We shall prove (i). The proof of (ii) is virtually identical save for adding the bonus compensation reward b to the contest reward function. To obtain a contradiction suppose that $\text{Supp}_i \cap [0, r)$ is not connected. Without loss of generality, suppose that $i = S$. For $\tau, \nu > 0$, let

$$G = \bigcup_{\tau, \nu > 0} \{(x_o - \tau, x_o + \nu) : (x_o - \tau, x_o + \nu) \cap \text{Supp}_S = \emptyset\}. \quad (\text{D-17})$$

Let

$$\underline{x} = \inf G, \quad \bar{x} = \sup G.$$

Then $\underline{x}, \bar{x} \in \text{Supp}_S$ and thus, by Lemma D-4 and D-6, $\underline{x}, \bar{x} \in \text{Supp}_W$. Thus, condition (D-10) implies that $\ell_S(\underline{x}) = \Pi_S(\underline{x})$, $\ell_S(\bar{x}) = \Pi_S(\bar{x})$, $\ell_W(\underline{x}) = \Pi_W(\underline{x})$, and $\ell_W(\bar{x}) = \Pi_W(\bar{x})$. Because, for $x < r$, $\Pi_S = F_W$ and $\Pi_W = F_S$ and because G does not meet the supports of S and W , Π_S and Π_W are constant on G . Thus because ℓ_S and ℓ_W are increasing, it must be the case that both F_S and F_W jump up at \bar{x} , this implies tied performance with positive probability at \bar{x} , contradicting Lemma D-2. \square

The next two results, Lemma D-8 and Lemma D-9, are fairly obvious technical results that will be used to establish our final characterization, continuity, in Lemma D-10.

Lemma D-8 *If (F_W, F_S) are equilibrium performance distributions,*

$$\sup\{\text{Supp}_i \cap [0, r)\} < r.$$

PROOF: $\Pi_i(r) > \Pi_i(r-) + b$. Because $b > 0$, condition (D-11) implies that $\ell_i(r) \geq \Pi_i(r)$, $\ell_i(r) = \ell_i(r-)$, thus $\ell_i(r) > \Pi_i(r-)$. Because ℓ is continuous, this implies that for all x in a sufficiently small lower neighborhood of r , $\ell_i(r) > \Pi_i(x)$. Thus by condition (D-10), such x are not in Supp_i . \square

Lemma D-9 *If (F_W, F_S) are equilibrium performance distributions, $\sup(\text{Supp}_i) < \infty$, $i = S, W$.*

PROOF: We establish this result for S , the proof for W is identical save for transpositions of S and W . For $x \geq r$, $\Pi_S(x) = b + F_S(x) \leq 1 + b$ and $\lim_{x \rightarrow \infty} \ell_S(x) = \infty$. So for x sufficiently large, $\ell_S > \Pi_S$, implying, by condition (D-11), that for x sufficiently large, $x \notin \text{Supp}_S$. \square

Finally, we confirm Remark 2.c.

Lemma D-10 *If (F_W, F_S) are equilibrium performance distributions, then F_S and F_W are continuous except perhaps at 0 and r .*

PROOF: Suppose that one of the distributions, say F_S , has a jump at $x_o \neq r$ or 0. Suppose that $x_o < r$. The proof when $x_o > r$ is the same except that the bonus reward is added to the payoffs. In this case, obviously $x_o \in \text{Supp}_S$ which implies, by Lemma D-4, that $x_o \in \text{Supp}_W$. For $x < x_o < r$, $\Pi_W = F_S$, thus, by condition (D-10), $\ell_W(x_o) = \Pi_W(x_o)$. Lemma D-6 shows that $0 \in \text{Supp}_W$, so, again by condition (D-10), $\ell_W(0) = \Pi_W(0)$. Because Π_W jumps up at x_o and ℓ is continuous, for a sufficiently small lower neighborhood of x_o , $\Pi_W(x) < \ell_W(x)$. By condition (D-10) this implies that points in this neighborhood are not in Supp_W . Thus, there exists $0 < x' < x_o$ such that $x' \notin \text{Supp}_W$ but $0 \in \text{Supp}_W$. $x_o \in \text{Supp}_W$, i.e., Supp_W is not connected, contradicting Lemma D-7. \square

This result confirms Remark 2.c in the body of the paper. The following lemma simply summarizes the implications of the previous lemmas for equilibrium contestant reward functions.

Lemma D-11 *If (F_W, F_S) are equilibrium performance distributions, then there exist constants, $\alpha_S, \beta_S, \alpha_W, \beta_W, u_H, u_L$, such that (a) $\beta_S, \beta_W > 0$ and $\alpha_S, \alpha_W \geq 0$, (b) $u_L \in (0, r)$, (c) $u_H \in [r, \infty)$, and*

(i) *If $x \in [0, r)$,*

$$\Pi_i(x) = \begin{cases} \alpha_i + \beta_i x & x \in [0, u_L], \\ \Pi_i(u_L) & x \in (u_L, r). \end{cases}$$

(ii) If $x \in [r, \infty)$ and $u_H = r$,

$$\begin{aligned} r \in \text{Supp}_S &\Leftrightarrow r \notin \text{Supp}_W, \\ r \in \text{Supp}_i &\Rightarrow \alpha_i + \beta_i r = 1 + b, \quad r \notin \text{Supp}_i \Rightarrow \alpha_i + \beta_i r \leq 1 + b, \text{ and} \\ \Pi_i(x) &= 1 + b, \quad i = S, W. \end{aligned}$$

(iii) If $x \in [r, \infty)$ and $u_H > r$,

$$\Pi_i(x) = \begin{cases} \alpha_i + \beta_i x & x \in [r, u_H], \\ 1 + b & x > u_H. \end{cases}$$

E Equilibrium configurations when bonus packages are easy

In this section, we characterize equilibria when bonus packages are easy, i.e., $\mu_S > r$. Thus, in this section (and only this section), we assume that $\mu_S > r$. Our first result is that when Eq2 equilibrium configurations can sometimes be sustained when $\mu_S > r$.

The necessary and sufficient condition for configuration Eq2 to hold when $\mu_S > r$ is provided by Lemma E-1.

Lemma E-1 *When $\mu_S > r$, Eq2 equilibria exist if and only if*

$$\mu_S \leq r \left((1+b) \sqrt{1 + \frac{\mu_W^2}{br(br + 2\mu_W)}} - b \right). \quad (\text{E-1})$$

PROOF: We first check that when $\mu_S > r$, no Eq1 configuration exists. This follows easily from showing that

$$1 - p_S^r = 1 - \frac{2\mu_S - u}{2r - u} = \frac{(r - \mu_S)(\mu_W + \sqrt{\mu_W(2br + \mu_W)})}{r\sqrt{\mu_W(2br + \mu_W)}} < 0.$$

Thus no Eq1 equilibrium exists.

First, we prove the necessity of condition (E-1). For Eq2 equilibrium to be verified, we must have $u_L \geq 0$. By the definition in equation (A-24) from Lemma 4, u_L has the same sign as

$$b^2 r (r - \mu_S) (r + 2br + \mu_S) + 2b (r - \mu_S) (r + 2br + \mu_S) \mu_W + (1+b)^2 r \mu_W^2.$$

Define $x := \mu_S - r$. The above expression becomes quadratic in x :

$$-b(br + 2\mu_W)x^2 - 2b(1+b)r(br + 2\mu_W)x + (1+b)^2 r \mu_W^2.$$

The positive root of this expression is given by

$$x = \mu_S - r \leq (1+b)r \left(\sqrt{1 + \frac{\mu_W^2}{br(br + 2\mu_W)}} - 1 \right),$$

i.e., inequality (E-1) in the lemma.

Next, we prove the sufficiency, it is sufficient to show that if condition (E-1) is satisfied, the constraints on other parameters hold.

Start with p_S^r . From equation (A-21), for $p_S^r < 1$ we need to show that $\mu_S < r + \mu_W + \mu_W/b$. Condition (E-1) is sufficient for this inequality to hold because

$$r + \mu_W + \frac{\mu_W}{b} - r \left((1+b) \sqrt{1 + \frac{\mu_W^2}{br(br + 2\mu_W)}} - b \right) = \frac{(1+b)(br + \mu_W) \left(1 - \frac{br}{\sqrt{br(br + 2\mu_W)}} \right)}{b} > 0$$

guarantees that $r + \mu_W + \mu_W/b - \mu_S > 0$.

Next, we show that $u_H > r$. From definition (A-23), $u_H - r$ has the same sign as

$$(r - \mu_S)^2(br + \mu_W)^2 - \mu_W(2br + \mu_W)(br + \mu_S)^2, \quad (\text{E-2})$$

which is a convex function of μ_S defined over $\mu_S \in [r, \bar{\mu}_S]$, where

$$\bar{\mu}_S := r \left((1+b) \sqrt{1 + \frac{\mu_W^2}{br(br+2\mu_W)}} - b \right).$$

Expression (E-2) is negative at $\mu_S = r$. Letting $\mu_S = \bar{\mu}_S$, we see that the expression has the same sign as $1 + \frac{br}{br+2\mu_W} - 2 \sqrt{\frac{(br+\mu)^2}{br(br+2\mu_W)}}$, which has the same sign as

$$\left(1 + \frac{br}{br+2\mu_W}\right)^2 - \left(2 \sqrt{\frac{(br+\mu)^2}{br(br+2\mu_W)}}\right)^2 = -\frac{8\mu_W(br+\mu_W)^2}{br(br+2\mu_W)^2} < 0.$$

Thus expression (E-2) is negative at $\mu_S = \bar{\mu}_S$ as well. Because the maxima of a convex function are attained at its extreme points, $u_H > r$. The constraints for p_S^r , p_W^0 , p_S^h and p_W^h follow in like fashion using the definitions provided by (A-21) and (A-22). \square

When an upper bound on μ_S , defined by equation (E), is not satisfied, new equilibrium configurations can be verified. Three new configurations can be realized for some choice of model parameter values.

1. Eq0B: Contestant W randomizes between 0 and uniformly randomizing over the rank plus bonus competition region; contestant S randomizes between r and uniformly randomizing over rank plus bonus competition region.
2. Eq0r: Contestant W randomizes between 0 and r , and contestant S captures both the rank and bonus reward with probability 1. In the Eq0r configuration, S 's strategies are not uniformly determined and the capacity constraint for S need not bind; but, if, for any given parameter choice, any performance strategy verifies an Eq0r equilibrium, uniform randomization by S over the rank and bonus competition region verifies an Eq0r equilibrium.
3. EqBB: Contestant W places point mass at r and uniformly randomizes over the rank plus bonus competition region; contestant S uniformly randomizes over rank plus bonus competition region.

Lemma E-2 (Eq0B, Eq0r and EqBB)

(i) When $\mu_W \leq r < \mu_S$,

(a) Eq0B equilibria exist if and only if $r \left((1+b) \sqrt{1 + \frac{\mu_W^2}{br(br+2\mu_W)}} - b \right) < \mu_S < r + \frac{r}{2b}$, and can be characterized by the following distributions:

$$F_S^* = p_S^r \mathbb{1}_r + (1 - p_S^r) \text{Unif}[r, u], \quad F_W^* = p_W^0 \mathbb{1}_0 + (1 - p_W^0) \text{Unif}[r, u].$$

(b) Eq0r equilibria exist if and only if $\mu_S \geq r + \frac{r}{2b}$, and can be characterized by the following distributions:

$$F_S^* = \text{Unif}[r, 2\mu_S - r], \quad F_W^* = p_W^0 \mathbb{1}_0 + (1 - p_W^0) \mathbb{1}_r.$$

(ii) When $r < \mu_W$, EqBB equilibria exist if and only if $\mu_S \geq r + \frac{r}{2b}$, and can be characterized by the following performance distributions:

$$F_S^* = \text{Unif}[r, 2\mu_S - r], \quad F_W^* = p_W^r \mathbb{1}_r + (1 - p_W^r) \text{Unif}[r, 2\mu_S - r].$$

All the parameters are specified in the proof. In all the equilibria, W 's capacity constraint binds; In all equilibria, except perhaps Eq0r equilibria, S 's capacity constraint binds.

PROOF: For Eq0B, the capacity constraint for S is $p_S^r r + (1 - p_S^r)(r/2 + u/2) = \mu_S$. The slope constraint for W , produced by S 's distribution, is $\frac{1-p_S^r}{u-r} = \frac{1+b}{u}$. The two constraints jointly determine the value pair

$$p_S^r = \frac{\mu_S - \sqrt{(\mu_S - r)(r + 2br + \mu_S)}}{r}, \quad u = \frac{br + \mu_S + \sqrt{(\mu_S - r)(r + 2br + \mu_S)}}{1+b} > r.$$

To show that $p_S^r < 1$ is equivalent to show that $(\mu_S - r)^2 < (\mu_S - r)(r + 2br + \mu_S)$, which holds for sure. For $p_S^r > 0$, we need $\mu_S^2 - (\mu_S - r)(r + 2br + \mu_S) = r(r + 2br - 2b\mu_S)$ to be positive. Thus we need the following condition:

$$\mu_S \leq r + \frac{r}{2b}.$$

The capacity constraint of W is $(1 - p_W^0)(r/2 + u/2) = \mu_W$, which implies that

$$p_W^0 = \frac{r + u - 2\mu_W}{r + u}. \quad (\text{E-3})$$

$p_W^0 \in [0, 1]$ because $u > r > \mu_W$. We also need $p_W^0 \leq (1 - b) - (1 - p_W^0)\frac{u}{u-r}$ to insure that S 's support line lies above $(0, p_W^0)$. Thus we check

$$(1 - b) - (1 - p_W^0)\frac{u}{u-r} - p_W^0 = \frac{b(br + \mu_S + \sqrt{(\mu_S - r)(r + 2br + \mu_S)})^2}{(1+b)^2} - r(br + 2\mu_W),$$

which is nonnegative if

$$\mu_S \geq r \left((1+b) \sqrt{1 + \frac{\mu_W^2}{br(br + 2\mu_W)}} - b \right).$$

Now we discuss the Atlanta effect under Eq0B. We have the downside risk taking under pure ranking competition as $p_W^{\text{rank}} = \frac{\mu_S - \mu_W}{\mu_S}$. Thus we compare the downside risk in Eq0B given by (E-3) to the one under pure ranking competition,

$$p_W^{\text{rank}} - p_W^0 = \frac{\mu_S - \mu_W}{\mu_S} - \frac{r + u - 2\mu_W}{r + u} = -\frac{(r + u - 2\mu_S)\mu_W}{(r + u)\mu_S},$$

which has the same sign as (if we define $x := \mu_S - r$)

$$-(r + u - 2\mu_S) = \frac{x + 2bx - \sqrt{x(2r + 2br + x)}}{1+b},$$

which has the same sign as

$$(x + 2bx)^2 - x(2r + 2br + x) = -2(1+b)x(r - 2bx),$$

which is negative when $\mu_S < r + r/(2b)$. Thus, downside risk is larger under Eq0B configuration compared to the pure ranking game. Thus, the Atlanta effect is reversed.

(ib) For Eq0r, if S randomizes uniformly over a rank plus bonus competition region $[r, u]$, then her capacity constraint implies that $u = 2\mu_S - r$. Thus the slope of W 's payoff function over $[r, u]$ equals $\frac{1}{2(\mu_S - r)}$. The slope of W 's support line if W simply chases the bonus equals, $\frac{b}{r}$. For not pursuing rank competition to be a best reply for W we need

$$\frac{b}{r} \geq \frac{1}{2(\mu_S - r)}.$$

This holds if $\mu_S \geq r + r/(2b)$.

Now we discuss the Atalanta effect under Eq0r. As W plays a two-point discrete strategy, his capacity constraint implies that $p_W^r = \mu_W/r$ and $p_W^0 = 1 - \mu_W/r$. By the condition on μ_S , we have

$$p_W^{\text{rank}} = 1 - \frac{\mu_W}{\mu_S} \geq 1 - \frac{\mu_W}{r + \frac{r}{2b}} = \frac{r - \mu_W + \frac{r}{2b}}{r + \frac{r}{2b}} > \frac{r - \mu_W}{r} = p_W^0.$$

Thus, the Atalanta effect persists under Eq0r.

(ii) For EqBB, similarly, the rank plus bonus competition region $[r, u]$ has a support $u = 2\mu_S - r$. Contestant W 's capacity constraint $p_W^r r + (1 - p_W^r)\mu_S = \mu_W$ uniquely determines the point mass on bonus threshold, p_W^r . From the assumption $\mu_S > r$, we conclude that $\mu_W > r$.

The support line for S is

$$(1 + b) - (1 - p_W^r) \left(\frac{x - r}{u - r} \right);$$

and for W is

$$(1 + b) - \frac{1}{u - r} (u - x).$$

Because the support lines must lie above the origin, condition $\mu_S \geq r + r/(2b)$ must hold to sustain the EqBB equilibrium.

As there is no point mass at 0 for EqBB configuration, the Atlanta effect persists in this configuration, albeit for the completely obvious reason that bonus compensation is so large that and so easy to capture that it lifts both contestants minimum performance to the bonus threshold. The bonus compensation threshold is also so low that bonus compensation, in the absence of rank incentives, does not motivate risk taking. \square

We plot the graph of equilibria for easy bonus ($\mu_S > r$) case in Figure E-1. For each bonus package (b, r) , one and only one configuration can be sustained.

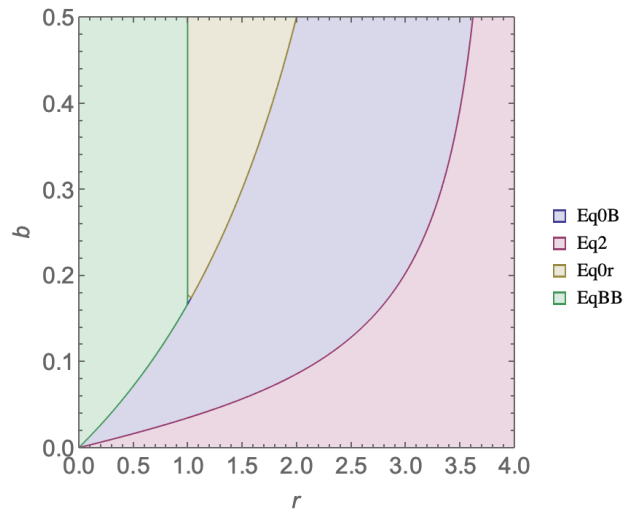


Figure E-1: *Equilibrium configurations and parametric curves when $\mu_S > r$. In this figure, $\mu_S = 4$ and $\mu_W = 1$.*

F Endogenous bonus compensation

In this section we assume that contestant capacity is endogenous. More specifically, we assume that each contestant buys a random variable whose cost equals $\gamma_i \mu$, $i = S, W$, and simultaneously selects a distribution satisfying the capacity constraint implied by μ . We assume that the marginal cost of capacity for S , γ_S , is less than the marginal cost of capacity for W , μ_W .

We assume that the welfare function of the designer is linear in contestant capacity and expected bonus payments, i.e.,

$$\alpha \left(\frac{1}{2} \mu_S + \frac{1}{2} \mu_W \right) - \mathbb{E}[\text{BonusPmts}],$$

where $\alpha > 0$ is a scale parameter. Note that designer welfare is only affected by bonus payments and capacity. Because capacity equals expected performance, the designer is indifferent to the effect of bonus compensation on the riskiness of contestant performance.

The timing of events is as follows: First the designer selects a bonus compensation package, (r, b) , where $r \geq 0$ and $b \geq 0$. Note that we do not restrict the designer to challenging bonus packages, i.e. packages satisfying $r > \mu_S$. Conditioned on the package selected, the two contestants select random performance strategies with mean μ_i , paying capacity cost, $\mu_i \gamma_i$. The structure of the game is identical to the risk-taking game analyzed in the main body of the paper except for the capacity choice.

Note that for a support line $\alpha + \beta x$, β measures the gain from relaxing the capacity constraint. Because, in equilibrium, the marginal cost of capacity equals its marginal benefit, and the payoff to each contestant, holding the other contestant's strategies fixed, is a concave function of capacity constraint, optimal capacity choices will satisfy, $\gamma_i = \beta$, $i = S, W$. Thus, for each equilibrium configuration, we can determine, using $\gamma = \beta$ relation, the equilibrium mean performance μ_S and μ_W as a function of r , b , γ_S and γ_W . The marginal benefit of capacity, β depends on the equilibrium configuration. The results in Lemma 5 and in Section E, identify the set constraints on the parameters r, b, μ_S , and μ_W required to support each equilibrium configuration. Thus, for each configuration, we can determine an optimal bonus package by solving a constrained maximization problem. Once this is accomplished, we compare welfare under the different configurations. Specifying these constrained optimization problems requires computing the marginal gain from capacity and the expected cost of bonus compensation in the configurations.

F.1 Slopes, rewards, and expected bonus payments

In this section, we characterize the slopes of support line (β_i), contestant payoffs evaluating at μ_i , i.e., ($\pi_i := \Pi_i(\mu_i)$), and the expected bonus payments to both contestants.

The computations for all configurations follow the same pattern. We use the Eq2 configuration to illustrate our approach. In the Eq2 configuration, the slopes for the contestants' support lines are

$$\beta_W = \frac{1+b}{u_H} = \frac{b^2 (br + \mu_S)^2 + (1+b)^2 (br + \mu_W)^2}{2 (br + \mu_S) (br + \mu_W)^2},$$

and

$$\beta_S = \frac{1+b-p_W^0}{u_H} = \frac{b^2 (br + \mu_S)^2 + (1+b)^2 (br + \mu_W)^2}{2 (br + \mu_W) (br + \mu_S)^2}.$$

From the solutions in Lemma 4, we can compute the expected bonus payments to contestant S as follows:

$$b(p_S^r + p_S^h) = (br + \mu_S) b^2 \frac{br + 2\mu_W}{2(br + \mu_W)^2} + b \frac{2\mu_S - r(1 + b^2)}{2(br + \mu_S)},$$

and the expected bonus payments to contestant W as follows:

$$bp_W^h = b \frac{br + \mu_W}{br + \mu_S} \left(\frac{2\mu_S(1 + b) - (1 - b^2)r}{2(br + \mu_S)} - \frac{b^2 r (br + \mu_S)}{2(br + \mu_W)^2} \right).$$

Using the same approach, we determine β 's and expected bonus payments, using the characterizations of contestant strategies in the main body of the paper for the Eq0, Eq1, and Eq2 configurations and the characterizations of Section E for the other configurations. Configuration Eq0r is not included in this list because the marginal benefit to S from

Equilibria	Eq0B	Eq0r	EqBB	Eq1
β_W	$\frac{(1+b)^2}{\mu_S + br + \sqrt{(\mu_S - r)(r + 2br + \mu_S)}}$	$\frac{b}{r}$	$\frac{1}{2(\mu_S - r)}$	$\frac{(r - \mu_S)(\mu_W + \sqrt{\mu_W(2br + \mu_W)})^2}{2r^2 \mu_W \sqrt{\mu_W(2br + \mu_W)}}$
β_S	$\frac{\mu_W \beta_W}{\sqrt{(\mu_S - r)(r + 2br + \mu_S)}}$	0	$\frac{\mu_W - r}{2(\mu_S - r)^2}$	$\frac{br + \mu_W + \sqrt{\mu_W(2br + \mu_W)}}{r^2}$
BonusPmt $_W$	$\frac{2b(1+b)\mu_W}{r + 2br + \mu_S + \sqrt{(\mu_S - r)(r + 2br + \mu_S)}}$	$\frac{b}{r} \mu_W$	b	0
BonusPmt $_S$	b	b	b	$\frac{b\mu_S(\mu_W + \sqrt{\mu_W(2br + \mu_W)}) - br\mu_W}{r\sqrt{\mu_W(2br + \mu_W)}}$

Table 2: List of support line slopes, β , and expected bonus payments in different equilibrium configurations.

capacity in this configuration is 0 (S wins both the rank and bonus reward in this equilibrium with probability 1). Thus, it is not possible to equate marginal cost and benefit under this configuration.

F.2 Equilibrium expected performance

Using $\gamma = \beta$ relation, the equilibrium mean performance μ_S and μ_W as a function of r , b , γ_S and γ_W , we can now compute expected capacity, μ . Again, taking the Eq2 configuration as our example, the equilibrium mean performance pair (μ_S, μ_W) for the two contestants, S and W , is determined as follows:

$$\begin{cases} \beta_W = \frac{b^2 (br + \mu_S)^2 + (1 + b)^2 (br + \mu_W)^2}{2 (br + \mu_S) (br + \mu_W)^2} = \gamma_W, \\ \beta_S = \frac{b^2 (br + \mu_S)^2 + (1 + b)^2 (br + \mu_W)^2}{2 (br + \mu_W) (br + \mu_S)^2} = \gamma_S. \end{cases}$$

Thus we have

$$\mu_S = \frac{1}{2} \left(\frac{(1 + b)^2}{\gamma_W} + \frac{b^2 \gamma_W}{\gamma_S^2} - 2br \right), \quad \mu_W = \frac{(1 + b)^2 \gamma_S^2 + b(b - 2r \gamma_S) \gamma_W^2}{2 \gamma_S \gamma_W^2}.$$

Similarly, we can determine the equilibrium mean performance for the other configurations. These expected performance levels are provided in the Table 3:

Equilibria	Eq0	EqBB	Eq0B	Eq1
μ_W	$\frac{\gamma_S}{2\gamma_W^2}$	$r + \frac{\gamma_S}{2\gamma_W^2}$	$\frac{\gamma_S((1+b)^2 - r^2\gamma_W^2)}{2\gamma_W^2}$	$\frac{(b-r\gamma_S)^2}{2\gamma_S}$
μ_S	$\frac{1}{2\gamma_W}$	$r + \frac{1}{2\gamma_W}$	$\frac{(1+b)^2 - 2br\gamma_W + r^2\gamma_W^2}{2\gamma_W}$	$r + \frac{\gamma_W(b+r\gamma_S)}{\gamma_S(b-r\gamma_S)}$

Table 3: *List of equilibrium mean performance in different configurations.*

F.3 Maximizing designer welfare

We are not able to obtain a closed form solution to the optimal design problem. So we take a numerical approach and provide an example in which Eq1 configuration maximizes the designer welfare. The parameters assumed for this example are $\gamma_W = 0.45$, $\gamma_S = 0.40$, $\alpha = 0.75$. Numerical solutions for each configuration are provided in Table 4.

Equilibria	Eq0	Eq2	Eq0B	EqBB	Eq1
Designer Welfare	0.787	0.787	0.442	0.768	0.868

Table 4: *Designer welfare for different configurations.*

From the table, we see that designer welfare is maximized by a bonus package supporting an Eq1 equilibrium. The optimal bonus package in this configuration is $r = 10.45$, $b = 3.24$. Under the optimal package, the endogenous expected performance levels are $\mu_S = 1.59$ and $\mu_W = 1.11$, and downside risk taking is given by $p_W^0 = 0.045$.

The key to the optimality of Eq1 packages in this example is the relatively low value of α . Numerical analysis of the effect of varying the scale parameter, α , reveals that the fight for optimality between the configurations boils down to a fight between Eq1 and EqBB. The tradeoff characterizing this fight is quite standard: maximizing total output vs. limiting agency rents. EqBB, where bonus compensation ensures that both contestants always submit performance at or above the bonus threshold, requires paying both contestants W 's marginal cost of capacity, γ_W . This provides S , who has a lower marginal capacity cost, with significant rents and requires very high expected bonus payments relative to Eq1 optimal solution. However, EqBB solutions allow for unlimited scaling upwards of μ . Thus, if α is sufficiently large, designer welfare is unbounded over EqBB solutions.

The advantage of Eq1 over EqBB configuration is that it pushes the marginal reward to S lower, closer to S 's marginal cost of capacity. Thus, the bonus package can be fixed so that so that S 's rents are small. However, the Eq1 solutions do not permit an unlimited scaling up of capacity because Eq1 configurations cannot be supported when bonus rewards are too generous.

This example shows that (a) it is quite easy to construct examples in which challenging bonus packages arise endogenously, as optimal solutions to a designer's welfare maximization problem (b) the welfare function of the designer which engenders challenging bonus package need not depend on the effects of bonus compensation on risk taking.

G Computation of risk averse agent welfare

This section explains how the numerical results in Section 3.5 were generated. We optimized the utility of a risk-averse agent, who owns the performance of the two contestants, and pays any bonus compensation they receive. The preferences of this agent are defined in Section 3.5. Our solution was produced by solving two constrained maximization problems, one for Eq1 configurations and one for Eq2 configurations, over bonus compensation packages. The constraints imposed on these problems were the model parameter assumptions and the necessary and sufficient conditions for the two equilibrium configurations (See the proof of Lemma 5 in this appendix). For the sake of comparison, we also computed the payoff from pure rank competition and asymptotic limit payoff as downside risk goes to 0.

First, we used Mathematica's "Differential Evolution" algorithm, a global maximization algorithm based on genetic evolution, to identify global maximum points (r^*, b^*) for each configuration, Eq1 and Eq2. Next, using Mathematica's local optimization algorithm, we tested the robustness of our solutions by solving local optimization problems, setting the starting values for these problems equal to the solutions produced by global optimization algorithm. The local maxima identified were equal to the global solutions identified by the global optimization algorithm.

Both optimization exercises produced the same solution, a solution on boundary between the Eq1 and Eq2 regions. The optimal bonus package identified by this optimization exercise yields greater utility for the risk-averse agent than either the limiting downside risk-elimination sequence or pure rank competition. Under the optimal bonus package, downside risk taking is reduced substantially but not eliminated. This exercise shows that the risk profile of contestant behavior produced by introducing bonus compensation can be preferred by a risk averse agent to the risk profile produced by pure rank competition, even though bonus compensation packages are challenging and thus, in isolation, encourage risk taking.

H Numerical comparative statics for dynamic contests

In this section, we consider the effect of changing the compensation parameters, b and r , on contestant risk taking.

Figure H-1 illustrates the effect of varying b from 0 to 10. It shows that, increasing the bonus decreases downside risk taking (p_T^0 decreases) by the trailing contestant, on the other hand, increases the incentive of the leading contestant to chase bonus (p_L^b increases). When one of the contestants is leading ($\omega = (3/2, 1/2)$ or $(1/2, 3/2)$), we find the length of the rank and bonus competition region increases quickly initially and then appears to approach a limit.

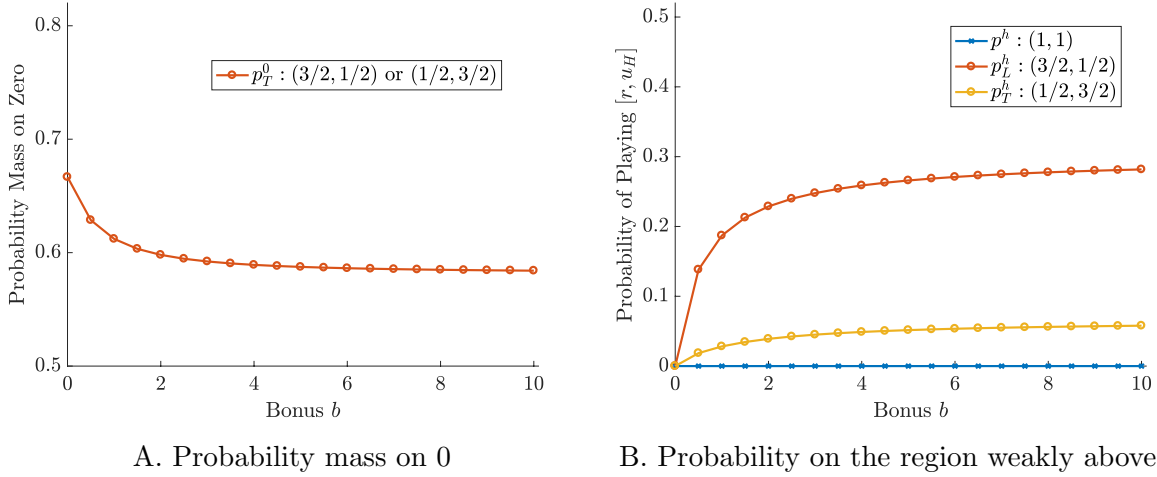


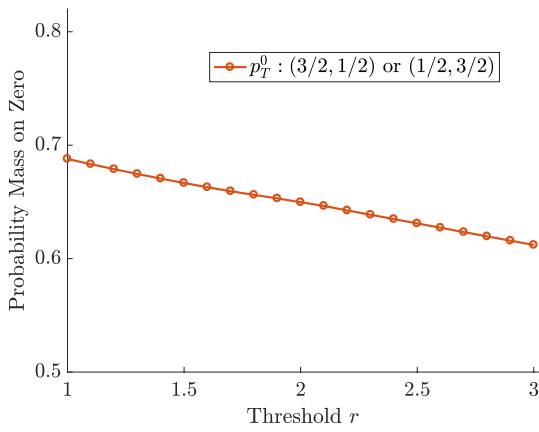
Figure H-1: *Effect on outcomes from varying the bonus reward.* We vary the bonus b over $[0, 10]$. The figure plots the probability of downside risk taking, p_T^0 by the trailing contestant (Panel A), and the probability of targeting the region weakly above r (Panel B, we term it p^b for the state with symmetric capacity pair, p_L^b and p_T^b for the leading and trailing contestant in the rest states, respectively). Note that, because the bonus threshold r is large relative to the bonus b , at state $(1, 1)$, both F and G ignore the bonus and exclusively target the rank competition region.

Next, we consider the effect of varying the bonus threshold, r . The baseline value for the bonus threshold for the symmetric dynamic predation game is 3. This is too high to induce the contestants to chase the bonus at $(1, 1)$. Consequently, in Figure H-2, we vary the bonus threshold above 1, from 1 to 3, to examine cases where bonus chasing occurs in state $(1, 1)$.

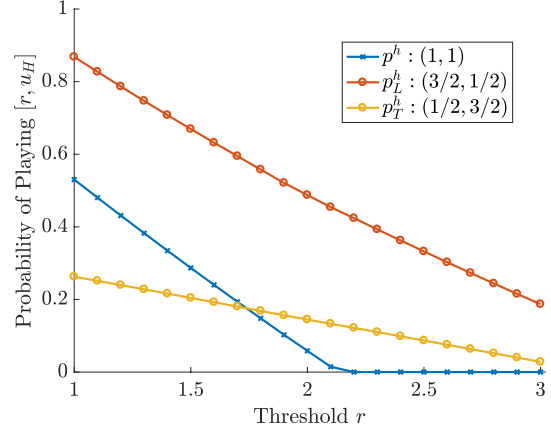
Panel A shows that the downside risk taking decreases. As shown in Panel B, the probability for the leading contestant of targeting the rank and bonus competition region decreases when chasing the bonus becomes more costly. The length of the rank and bonus competition region also shrinks as the threshold increases.

References

- Ash, Robert B., *Real Analysis and Probability*, Academic Press, 1972.
- Aumann, Robert J, Michael Maschler, and Richard E Stearns, *Repeated Games with Incomplete Information*, MIT Press, 1995.



A. Probability mass on 0



B. Probability on the region weakly above r

Figure H-2: *Effect on outcomes from varying the bonus threshold.* We vary the bonus threshold r over $[1, 3]$. The figure plots the probability of downside risk taking, p_T^0 by the trailing contestant (Panel A), and the probability of targeting the region weakly above r (Panel B, we term it p^h for the state with symmetric capacity pair, p_L^h and p_T^h for the leading and trailing contestant in the rest states, respectively).

Baye, Michael R, Dan Kovenock, and Casper G De Vries, “The all-pay auction with complete information,” *Economic Theory*, 1996, 8 (2), 291–305.

Birkhoff, Garret and Saunders MacLane, *A Survey of Modern Algebra*, 4th ed., Macmillan, 1977.

Gentzkow, Matthew and Emir Kamenica, “Costly persuasion,” *American Economic Review*, 2014, 104, 457–462.

Hillman, Arye L and John G Riley, “Politically contestable rents and transfers,” *Economics & Politics*, 1989, 1 (1), 17–39.

Kamenica, Emir and Matthew Gentzkow, “Bayesian persuasion,” *American Economic Review*, 2011, 101 (6), 2590–2615.

Luenberger, David, *Optimization by Vector Space Methods*, Wiley, 1969.

Rees, E. L., “Graphical discussion of the roots of a quartic equation,” *American Mathematical Monthly*, 1922, 29 (2), 51–55.

Siegel, Ron, “All-pay contests,” *Econometrica*, 2009, 77 (1), 71–92.

Simon, Leo K and William R Zame, “Discontinuous games and endogenous sharing rules,” *Econometrica*, 1990, 58 (4), 861–872.