Part 1: q Theory and Irreversible Investment

Goal: Endogenize firm characteristics and risk.

- Value/growth
- Size
- Leverage
- New issues, ...

This lecture:

- q theory of investment
- Irreversible investment and real options
- Value and risk under perfect competition

Notation

$$
\begin{aligned}
r &= \text{interest rate} \\
\delta &= \text{depreciation rate} \\
K_t &= \text{capital stock} \\
k_0 &= \text{initial capital stock} \\
I_t &= \text{investment rate} \\
\theta(i,k) &= \text{investment cost} \\
B &= \text{BM under risk-neutral probability} \\
\mathrm{d}X &= \mu(X)\,\mathrm{d}t + \sigma(X)\,\mathrm{d}B \\
\pi(x,k) &= \text{operating cash flow} \\
J(x,k) &= \text{market value of firm}
\end{aligned}
$$

Value of Firm

$$J(k,x) = \sup_I \mathsf{E} \int_0^\infty \mathrm{e}^{-rt}[\pi(X_t, K_t) - \theta(I_t, K_t)]\,\mathrm{d}t$$

subject to

$$\begin{aligned} \mathrm{d}K &= I\,\mathrm{d}t - \delta K\,\mathrm{d}t\,, \\ K_0 &= k\,, \\ X_0 &= x\,. \end{aligned}$$

Examples of investment cost θ

- Costless adjustment: $\theta(i,k) = ai$ for constant a.
- Quadratic (and linearly homogeneous): $\theta(i,k) = ai + bi^2/k$.
- Purchase price of capital different from resale price: $\theta(i,k) = ai^+ - bi^-$.
- Irreversible investment (zero resale price): $\theta(i,k) = ai^+$.
- Irreversible investment with fixed costs: $\theta(i,k) = ai^+ + f1_{\{i>o\}}$.

Operating Cash Flow

Usual model: Assume there is a production function $y = k^\alpha \ell^\beta$ for $\alpha, \beta > 0$ with $\alpha + \beta \leq 1$. Labor is hired in a perfectly competitive market at wage rate w. The industry demand curve has constant elasticity: $p = Xy^{-1/\gamma}$, where X is a GBM. Different versions are obtained from

- Constant ($\alpha + \beta = 1$) or decreasing ($\alpha + \beta < 1$) returns to scale .
- Perfect competition, monopoly, or Cournot oligopoly.

We end up with something like $\pi(x, k) = xk^\lambda$.

Linearity/Concavity

- Operating cash flow is linear in capital ($\lambda = 1$) if there are constant returns to scale and perfect competition.
- Operating cash flow is strictly concave in capital ($\lambda < 1$) otherwise.

Basic q Theory

Assume θ is linearly homogeneous, so $\theta(i,k) = k\phi(i/k)$.

- If ϕ is differentiable and strictly convex, then the optimal investment-to-capital ratio is a function of the marginal value of capital (marginal q).
- In the quadratic case, the optimal investment-to-capital ratio is an affine function of the marginal value of capital.
- If π is linear in k, then the marginal value of capital equals the average value of capital (marginal q equals average q). In other words, $J_k(x,k) = J(x,k)/k$.

Proof

HJB Equation:

$$0 = \sup_i \left[\pi(k,x) - \theta(i,k) - rJ(k,x) + J_x\mu + J_k(i - \delta k) + \frac{1}{2}J_{xx}\sigma^2\right].$$

First-order condition: $\theta_i = J_k \;\Leftrightarrow\; \phi'(i/k) = J_k$.

- Strict convexity implies ϕ is strictly decreasing, hence invertible, so $i/k = (\phi')^{-1}(J_k)$.
- If $\phi(y) = ay + by^2$, then $\phi'(i/k) = J_k \;\Leftrightarrow\; i/k = (J_k - a)/(2b)$.
- If $\pi(k,x) = f(x)k$, guess $J(k,x) = g(x)k$ and verify. The HJB equation simplifies to

$$0 = \sup_{i/k} \left[f(x) - \phi(i/k) - rg(x) + \mu g'(x) + g(x)(i/k - \delta) + \frac{1}{2}g''(x)\sigma^2\right].$$

This equation is independent of k. Solve it for g (given boundary conditions).

Nonlinear π

Suppose $\pi(x, k) = xk^\lambda$ with $\lambda < 1$. Then average q is larger than marginal q.

Suppose $\pi(x, k) = xk - c$ for a constant c. Then average q is less than marginal q. This is called operating leverage.

Irreversible Investment

Assume $\theta(i, k) = i^+$ and π is monotone in k. Let I now denote cumulative investment instead of the investment rate. Then

$$J(x, k) = \sup_I \mathsf{E} \int_0^\infty \mathrm{e}^{-rt}[\pi(K_t, X_t)\,\mathrm{d}t - \mathrm{d}I_t]$$

subject to

$$\begin{aligned} \mathrm{d}K &= \mathrm{d}I - \delta K\,\mathrm{d}t\,, \\ K_0 &= k\,, \\ X_0 &= x\,, \end{aligned}$$

and subject to I being an increasing process.

Zero Depreciation

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$\delta = 0 \;\Rightarrow\; K_t = k + I_t$. Some references in which π depends directly on the control process:

- Karatzas, I. and S. E. Shreve, 1984, "Connections between Optimal Stopping and Singular Stochastic Control I. Monotone Follower problems," *SIAM J. Contr. Opt.* 6, 856–877.
- Scheinkman, J. A. and T. Zariphopoulou, 2001, "Optimal Environmental Management in the Presence of Irreversibilities," *J. Econ. Theory* 96, 180–207.
- Bank, P., 2005, "Optimal Control under a Dynamic Fuel Constraint," *SIAM J. Contr. Opt.* 44, 1529–1541.
- Back, K. and D. Paulsen, 2009, "Open-Loop Equilibria and Perfect Competition in Option Exercise Games," *Rev. Fin. Stud.* 22, 4531–4552.
- Aguerrevere, F., 2009, "Real Options, Product Market Competition, and Asset Returns, "*J. Fin.* 64, 957–983.
- Steg, J.-H., 2012, "Irreversible Investment in Oligopoly," *Finance Stoch.* 16, 207–224.

Assets in Place and Growth Options

Write $\pi' = \pi_k$ and

$$\pi(x,k) = \pi(x,k_0) + \int_{k_0}^{k} \pi'(X_t,\ell)\,\mathrm{d}\ell\,,$$

so the objective function is

$$\mathsf{E}\int_0^\infty \mathrm{e}^{-rt}\pi(X_t,k_0)\,\mathrm{d}t + \mathsf{E}\int_0^\infty \mathrm{e}^{-rt}\left[\int_{k_0}^{K_t}\pi'(X_t,k)\,\mathrm{d}k\,\mathrm{d}t - \mathrm{d}I_t\right].$$

The first term is the value of assets in place. The maximized value of the second term is the value of growth options.

First-Order Condition

- Alternative to dynamic programming when $\delta = 0$.
- See Bank and Riedel (AAP, 2001), Bank (SIAM J. Contr. Opt., 2005), Steg(FS, 2012).
- Given K, define a "gradient" D by

$$D_\tau = \mathsf{E}_\tau \int_\tau^\infty \mathrm{e}^{-rt}\pi'(X_t, K_t)\,\mathrm{d}t - \mathrm{e}^{-r\tau}$$

  for all stopping times τ.
- Given some technical conditions, a necessary and sufficient condition for K to be optimal is that $D \le 0$ and

$$\int_0^\infty D_t\,\mathrm{d}K_t = 0\,.$$

Real Options

Rather than choosing the optimal capital stock K_t at each date t, we can equivalently choose the optimal date t at which to invest the unit of capital k for each $k \ge k_0$.

In this formulation, the value of growth options is the integral of a continuum of call option values, indexed by the level $k \ge k_0$ of the capital stock.

The equivalence is based on changing the order of integration and using

$$\tau_k = \inf\{t \mid K_t > k\}\,,$$

which is the right-continuous inverse of the path $t \mapsto K_t$.

From the optimal investment times τ_k, we can recover the optimal capital stock process as $K_t = \inf\{k \mid \tau_k > t\}$.

Real Options cont.

The underlying asset for option k has price

$$S(x,k) = \mathsf{E}\left[\int_t^\infty \mathrm{e}^{-r(u-t)}\pi'(X_u,k)\,\mathrm{d}u \,\middle|\, X_t = x\right].$$

The benefit of investment is that you earn the marginal profit π' in perpetuity after investment, which has value $S(X_t,k)$ at date t.

The options are perpetual. The strikes equal 1 (the price of capital).

The value of growth options is

$$\int_{k_0}^\infty \sup_{\tau_k} \mathsf{E}\left[\mathrm{e}^{-r\tau_k}\{S(X_{\tau_k},k)-1\}\right]\,\mathrm{d}k.$$

Value Matching and Optimal Capital

- Define the value of option k:

$$V(x,k) = \sup_\tau \mathsf{E}\left[\mathrm{e}^{-r\tau}\{S(X_\tau,k)-1\} \,\middle|\, X_0 = x\right].$$

- When it is optimal to invest, we must have value matching: $V(X_t,k) = S(X_t,k) - 1$.
  - In other words, the optimal exercise boundary is $\{(k,x) \mid V(X_t,k) = S(X_t,k) - 1\}$.
- Given x, the largest capital stock such that it would be optimal to invest is

$$\kappa(x) \stackrel{\text{def}}{=} \sup\{k \mid V(x,k) = S(x,k) - 1\}.$$

- Hysteresis: The optimal capital stock process is

$$K_t = k_0 \vee \sup_{0\le s\le t} \kappa(X_s).$$

Real Options and q Theory

The value-matching condition can be expressed in the q language.

The marginal value of investment (marginal q) is $S(y,x) - V(y,x)$.

- Investing earns the marginal cash flow π' in perpetuity but extinguishes the option.
- Hence, the marginal value of investing is $S - V$.
- In fact, a direct calculation shows that $J_k(x,k) = S(x,k) - V(x,k)$.

So, value matching can be expressed as: invest when marginal q equals 1.

Perfect Competition

Assume constant returns to scale and perfect competition, so $\pi(x,y,k) = h(x,y)k$ for some h, where y denotes industry capital (which is exogenous from the point of view of any firm).

The equilibrium condition for perfect competition is that investment occurs as soon as any investment option reaches the money (no barriers to entry). Because the options are never strictly in the money, growth options have zero value.

The value of any firm is the value of its assets in place:

$$J(x,y,k) = k\mathsf{E}\left[\int_0^\infty \mathrm{e}^{-rt} f(X_t, Y_t)\,\mathrm{d}t \mid X_t = x, Y_t = y\right] \stackrel{\text{def}}{=} kq(x,y)\,.$$

Risk

The return on the firm is its dividend yield plus capital gain:

$$\frac{\pi\,\mathrm{d}t - \mathrm{d}K_t + \mathrm{d}J}{J}\,.$$

Because $J = Kq$ and K is continuous with finite variation,

$$\frac{\mathrm{d}J}{J} = \frac{\mathrm{d}K}{K} + \frac{\mathrm{d}q}{q}\,.$$

Because the firm only invests when $q = 1 \;\Leftrightarrow J = K$, we have $\mathrm{d}K/K = \mathrm{d}K/J$, so the return is

$$\frac{\pi\,\mathrm{d}t}{J} + \frac{\mathrm{d}q}{q}\,.$$

If investment were perfectly reversible, industry capital would adjust to maintain $q = 1$, and we would have $\pi/J = r$. With irreversibility, fluctuations in q (the market-to-book ratio) add risk.

Example

The production function is $f(k) = k$. The industry output price is $P_t = X_t Y_t^{-1/\gamma}$, where X is a GBM with coefficients μ and σ, where $\mu < r$.

Let β denote the positive root (guaranteed to be larger than one) of the quadratic equation

$$\frac{1}{2}\sigma^2\beta^2 + \left(\mu - \frac{1}{2}\sigma^2\right)\beta = r\,.$$

The investment options reach the money when P_t reaches

$$p_c^* \stackrel{\text{def}}{=} (r-\mu)\left(\frac{\beta}{\beta-1}\right)\,.$$

In equilibrium, P is a GBM reflected at p_c^*.

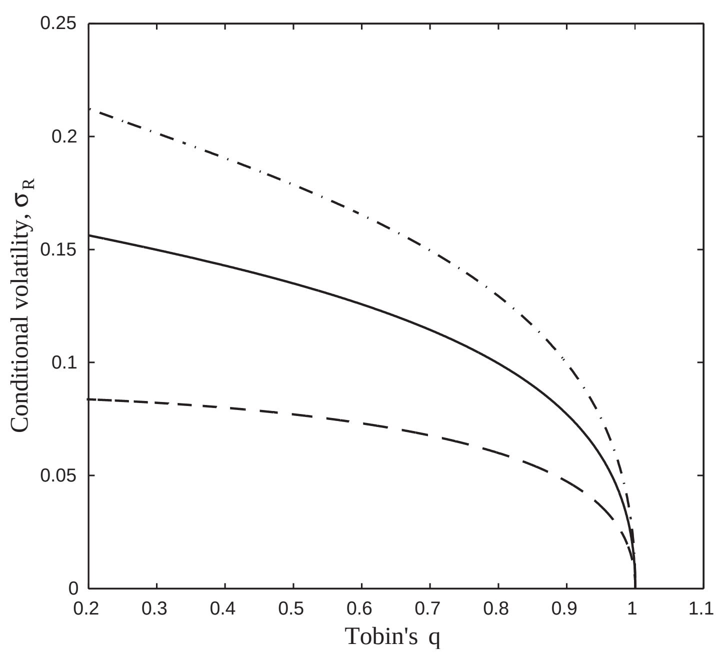
Example cont.

$$\text{Marginal } q = \text{Average } q = \frac{\beta}{\beta-1}\frac{P_t}{p_c^*} - \frac{1}{\beta-1}\left(\frac{P_t}{p_c^*}\right)^\beta .$$

The stochastic part of $\mathrm{d}q/q$ is

$$\left(\frac{1-(P_t/p_c^*)^{\beta-1}}{\beta-(P_t/p_c^*)^{\beta-1}}\right)\beta\sigma\,\mathrm{d}B\,.$$

Note that risk decreases as P_t increases towards p_c^*, vanishing at $P_t = p_c^*$.



Kogan, L., 2004, “Asset Prices and Real Investment,” *JFE* 73, 411–431. The three lines correspond to different levels of risk aversion of the representative investor.

Explanation

The risk comes from the dependence of q on X. When X changes, the demand for capital changes.

Suppose the demand for capital increases.

- If the supply of capital were perfectly elastic, then the quantity supplied would increase with no change in q.
- If the supply of capital were perfectly inelastic, then q would increase with no change in the quantity supplied.

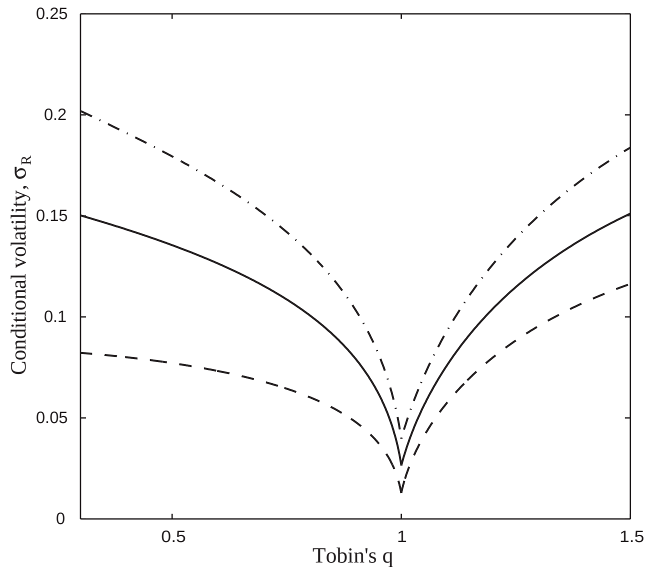
Perfectly reversible investment implies a perfectly elastic supply of capital. Irreversibility implies that supply is elastic only at $q = 1$.

Bounded Investment Rate

A simple example of convex adjustment costs is

$$\theta(i, k) = \begin{cases} 0 & \text{if } i \leq 0\,, \\ i & \text{if } 0 \leq i \leq i^{\max}\,, \\ \infty & \text{otherwise}\,. \end{cases}$$

This is equivalent to the constraint $0 \leq \mathrm{d}I_t/\mathrm{d}t \leq i^{\max}$. At the optimum, the firm will invest at the maximum rate whenever $q > 1$.



Kogan, L., 2004, “Asset Prices and Real Investment,” *JFE* 73, 411–431. We expect similar figures for more general convex adjustment costs: risks are high when the adjustment costs are particularly constraining.