Beliefs about Inflation and the Term Structure of Interest Rates[*]

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Abstract

We study how differences in beliefs about expected inflation affect the nominal term structure when investors have "catching up with the Joneses" preferences. In the model, "catching up with the Joneses" preferences help to match the level and slope of yields as well as the level of yield volatilities. Disagreement about expected inflation helps to match the dynamics of yields and yield volatilities. Expected inflation disagreement induces a spillover effect to the real side of the economy with a strong impact on the real yield curve. When investors share common preferences over consumption relative to the habit with a coefficient of relative risk aversion greater than one, real average yields across all maturities rise as disagreement increases. Real yield volatilities also rise with disagreement. To develop intuition concerning the role of different beliefs between investors, we consider a case where the real and nominal term structures can be computed as weighted-averages of quadratic Gaussian term structure models. We numerically find increased disagreement about expected inflation between the investors increases nominal yields and nominal yield volatilities at all maturities. We find empirical support for these predictions.

Keywords: Affine term structure, general equilibrium, time-varying term premiums, heterogeneous beliefs.

JEL Classification: D51, E43, E52, G12.

1 Introduction

Several sophisticated reduced-form term structure models exist that are successful in explaining empirical features of U.S. Treasury bonds.[1] However, the economic mechanisms driving these empirical regularities are not well understood.[2] A natural candidate for such an economic mechanism is perceived inflation uncertainty which we study by exploring how differences in beliefs about expected inflation influence the nominal term structure. We show that increased disagreement about expected inflation increases bond yields and bond yield volatilities at all maturities.

As Gürkaynak and Wright (2010) point out in their recent survey, inflation uncertainty seems to play an important role in potentially explaining the dynamics of nominal bond risk premiums.[3] This is the departure point for our work where we explore the role that differences in beliefs about inflation dynamics among investors plays in determining properties of real and nominal bond prices. While other heterogeneous beliefs works have explored specific features of bond prices that can be explained, for example predictability in Xiong and Yan (2010), our work focuses on exploring the tensions introduced by heterogeneous beliefs in not just explaining predictability, but also explaining other asset pricing properties such as the level and slope of nominal bond yield curves as well as spill-over effects of differences in inflation beliefs on real asset prices. In particular, we find that heterogeneous beliefs about inflation dynamics have strong implications for the level and volatilities of real and nominal yield curves.

As with any heterogeneous beliefs model, a common problem faced is linking investor beliefs to the true underlying economy. In fact, a common assumption is just to study the economy in a setting where one of the investors is assumed to have correct beliefs. Recent work by Piazzesi and Schneider (2011) argues that one potential source for predictability in long term bond excess

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[1]Examples of reduced-form no arbitrage models with latent state variables include Dai and Singleton (2000, 2002, 2003), Duffee (2002), and Ahn, Dittmar, and Gallant (2002). See Singleton (2006) for an overview of this literature.

[2]Building from the reduced-form no arbitrage models, an intermediate approach has been to introduce macroeconomic variables into these no-arbitrage settings as in Ang and Piazzesi (2003), Ang, Dong, and Piazzesi (2007b), and Ang, Bekaert, and Wei (2008). Other works have explored full structural models. Wachter (2006) and Piazzesi and Schneider (2007) study structural term structure models with exogenous inflation. Basak and Yan (2010) incorporate money illusion in a setting with exogenous inflation. Buraschi and Jiltsov (2005), Gallmeyer, Hollifield, and Zin (2005), Gallmeyer et al. (2007), Bekaert, Cho, and Moreno (2010), and Palomino (2010) study structural models that endogenize inflation. The recent surveys by Gürkaynak and Wright (2010) and Rudebusch (2010) summarize the implications of some of this structural work.

[3]Other works that explore the role of subjective beliefs or survey data on the term structure include Ang, Bekaert, and Wei (2007a), Chernov and Mueller (2008), Buraschi and Whelan (2010), and Chun (2011). See Adrian and Wu (2009) for complimentary work that extracts the term structure of inflation expectations by fitting an affine model of both real and nominal yield curves.

returns is that investors' actual predictions while trading in bond markets are different from the predictions derived from econometricians by using historical bond data. We also accommodate for this difference in beliefs between investors and the econometrician allowing us to explore the impact of heterogeneous beliefs about inflation along two dimensions. First, heterogeneity of beliefs about inflation between the investors in the economy allows us to capture the impact of speculative trade on bond properties. Second, heterogeneity of beliefs about inflation between the investors and the econometrician allows us to better understand the restrictions placed on investor beliefs to explain historical nominal bond price patterns.

To accomplish this, our work studies the equilibrium term structure of nominal interest rates in a pure exchange economy with heterogeneous external habit formation preferences and inflation uncertainty. Our benchmark common beliefs model is adapted from the habit-formation settings of Abel (1990) and Chan and Kogan (2002). Our innovation is to introduce differences in beliefs about expected inflation. In particular, investors observe the path of exogenous nominal consumption and the exogenous price level, but have different beliefs about expected inflation. Differences in beliefs about expected inflation alone impacts the equilibrium real pricing kernel generating a spill-over effect from the nominal to the real side of the economy not present in works such as Xiong and Yan (2010) who focus on a logarithmic utility model. This effect is generated by the investors having different expected inflation beliefs. The equilibrium real and nominal stochastic discount factors are determined in closed form and the effects of aggregate risk aversion, difference in beliefs, and inflation on the short rate and the market prices of risk are explored. In particular, the price system exhibits predictability when there is disagreement about the speed of mean reversion of expected inflation.

We find that heterogeneous beliefs about expected inflation have a strong impact on the level and volatility of the real yield curve. When both investors share common preferences over consumption relative to the habit with a coefficient of relative risk aversion greater than one, real average yields across all maturities rise as disagreement increases. When the coefficient of relative risk aversion is less than one, real average yields fall as disagreement increases. The direction of the shift in the real average yield curve is driven by the relative strength of income and substitution effects. Over both cases, yield volatilities increase with disagreement.

To develop additional intuition concerning the role of both different beliefs between the investors

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and the econometrician and different beliefs among the investors, we consider a simplifying case that provides more detailed closed-form equilibrium solutions, in particular closed-form term structures in the class of quadratic Gaussian term structure models. For this case, we numerically demonstrate the tension heterogeneous beliefs place on properties of the yield curve. When disagreement about expected inflation increases, average nominal yields rise, the nominal yield curve flattens, and nominal yield volatilities increase. These impacts can be large. For a reasonable increase in differences in beliefs, average nominal yields at short maturities can rise by as much as 200 basis points.

We also empirically explore how differences in beliefs impacts real and nominal term structure properties. We find support for increased inflation beliefs dispersion leading to higher nominal average yields and higher nominal yield volatilities. Turning to real yield curve properties, we also find that increased inflation beliefs dispersion leads to higher real yield volatilities.

The paper proceeds as follows. Section 2 describes the basic economic setup. The equilibrium is defined and characterized in Section 3. Section 4 studies the role of differences in beliefs about inflation expectations on real and nominal term structure properties. Section 5 studies the empirical relation between differences in beliefs about inflation expectations and nominal yields and the standard deviation of nominal yields, respectively. Section 6 concludes.

2   The Economy

To study the equilibrium impact of heterogeneous beliefs about inflation, our economic environment is a continuous-time pure exchange economy with heterogeneous investors. The economy is similar to Abel (1990) and Chan and Kogan (2002) modified to incorporate an exogenous inflation process as well as heterogeneous beliefs about expected inflation. The economy has a finite horizon equal to T with a single perishable consumption good. Real prices are measured in units of the consumption good and nominal prices are measured in dollars. Uncertainty is represented by the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$.

The exogenous real aggregate output $\epsilon(t)$ process follows a geometric Brownian motion with dynamics given by

$$d\epsilon(t) = \epsilon(t)\left[\mu_\epsilon \, dt + \sigma_\epsilon \, dz_\epsilon(t)\right], \qquad \epsilon(0) > 0, \tag{2.1}$$

where $z_\epsilon(t)$ is a one-dimensional Brownian motion that represents a real shock. The exogenous price level $\pi(t)$ in the economy is given by

$$d\pi(t) = \pi(t) \left[x(t)\, dt + \sigma_{\pi,\epsilon}\, dz_\epsilon(t) + \sigma_{\pi,\$}\, dz_\$(t) \right], \qquad \pi(0) = 1, \tag{2.2}$$

where $x(t)$ denotes expected inflation and $z_\$(t)$ is a one-dimensional Brownian motion that represents a nominal shock to the economy. The Brownian motions $z_\epsilon(t)$ and $z_\$(t)$ are locally uncorrelated. Since the price level also loads on $z_\epsilon(t)$, real aggregate output can be correlated with the price level.

Nominal aggregate output in the economy is denoted by $\epsilon_\$(t) \equiv \pi(t)\epsilon(t)$ with dynamics given by

$$d\epsilon_\$(t) = \epsilon_\$(t) \left[\mu_{\epsilon_\$}(t)\, dt + (\sigma_\epsilon + \sigma_{\pi,\epsilon})\, dz_\epsilon(t) + \sigma_{\pi,\$}\, dz_\$(t) \right], \qquad \epsilon_\$(0) > 0, \tag{2.3}$$

where $\mu_{\epsilon_\$}(t) \equiv \mu_\epsilon + x(t) + \sigma_\epsilon \sigma_{\pi,\epsilon}$.

To close the exogenous processes requires making an assumption about the unobserved expected inflation process. We assume that it follows an Ornstein-Uhlenbeck process; however, the analysis of Section 3 still holds for other dynamics such as finite-state Markov processes as in Veronesi (1999, 2000) for example. Specifically, we assume the dynamics of $x(t)$ are

$$dx(t) \;\; = \;\; \kappa\left(\bar{x} - x(t)\right)\, dt + \sigma_{\mu_\pi}\, dz_x(t), \quad x(0) \text{ given}, \tag{2.4}$$

where $x(0) \sim \mathrm{N}(\bar{x}(0), \sigma_x^2(0))$, $dz_x(t)\frac{d\epsilon(t)}{\sigma_\epsilon \epsilon(t)} = \rho_{x\epsilon}dt$, and $dz_x(t)\frac{d\pi(t)}{\sigma_\pi \pi(t)} = \rho_{x\pi}dt$.

2.1 Beliefs

Investors, as well as the econometrician, in the economy have heterogeneous beliefs about expected inflation because they only observe the continuous record of the price level $\pi(t)$. While both the econometrician and all investors agree that the evolution of the price level follows (2.2) respectively, they do not know the true expected inflation $x(t)$. Heterogeneous beliefs are modeled with investor-specific priors about these quantities as in for example Detemple and Murthy (1994), Basak (2000), and Basak (2005). Given $\pi(t)$ is observed continuously, all investors can perfectly estimate the volatilities in (2.2) by computing the quadratic variation of the process and the quadratic covariation

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with $\epsilon(t)$.

As with any economy when investors have different beliefs about economic fundamentals than the economy's objective beliefs, an important consideration is the linkage of the subjective beliefs of the investors to the objective beliefs of an econometrician as argued in works such as Cecchetti, Lam, and Mark (2000), Abel (2002), and Piazzesi and Schneider (2011). Here we explicitly model a set of beliefs for the econometrician distinct from our reference probabilities $\mathcal{P}$. This is driven by the desire to understand equilibrium properties through the lens of the econometrician's beliefs relative to the investors' beliefs. It is important to stress that the econometrician plays no optimizing role in the model. We only explicitly model the econometrician's beliefs to understand how they influence his interpretation of the economy.

For an econometrician denoted as $i = 0$ and investors denoted as $i = \{1, 2\}$, uncertainty in the economy is represented by the filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t^{\epsilon,\pi}\}, \mathcal{P}^i\right)$ where $\mathcal{F}_t^{\epsilon,\pi}$ denotes the filtration generated by real output and the price level. When referring to both the econometrician and the investors, we use *agents*. Agent i's best estimate for expected inflation is

$$x^i(t) = \mathrm{E}^i[x(t) \mid \mathcal{F}_t^{\epsilon,\pi}], \quad i \in \{0, 1, 2\}, \tag{2.5}$$

where $\mathrm{E}^i[\cdot]$ denotes the expectation with respect to agent i's belief $\mathcal{P}^i$.

As long as the agents in the economy start with different priors about expected inflation, they will continue to disagree about these quantities over the life of the economy. From standard filtering theory as in Liptser and Shiryaev (1974a,b), investor i's innovation processes for $z_\epsilon(t)$ and $z_\$(t)$ are related to the reference probability $\mathcal{P}$ and the econometrician's beliefs $\mathcal{P}^0$ via

$$dz_\$^i(t) = dz_\$(t) + \frac{x(t) - x^i(t)}{\sigma_{\pi,\$}} dt = dz_\$^0(t) + \frac{x^0(t) - x^i(t)}{\sigma_{\pi,\$}} dt. \tag{2.6}$$

Given the dynamics of real output $\epsilon(t)$ are known, $z_\epsilon(t)$ is known by all agents. This implies there are no feedback effects on the real pricing kernel from disagreement about real quantities. Instead, our framework isolates how disagreement about nominal quantities generates a feedback effect on the real pricing kernel.

From equation (2.6), investors' innovation processes are related to the econometrician and to

each other by

$$dz_\$^i(t) = dz_\$^0(t) + \Delta^i(t)dt, \quad i \in \{1,2\}, \tag{2.7}$$

$$dz_\$^2(t) = dz_\$^1(t) - \Delta(t)dt, \tag{2.8}$$

where the processes $\Delta^i(t)$ and $\Delta(t) = \Delta^1(t) - \Delta^2(t)$ represent disagreement processes with respect to the econometrician and across the two investors. They are defined as

$$\Delta^i(t) = \frac{x^0(t) - x^i(t)}{\sigma_{\pi,\$}} \quad \text{and} \quad \Delta(t) = \frac{x^2(t) - x^1(t)}{\sigma_{\pi,\$}}. \tag{2.9}$$

These processes summarize investors' differences in opinion about expected inflation. The disagreement is driven by initial priors and the paths of the realized real output and the price level.

We assume that the agents, including the econometrician, have different beliefs about expected inflation through two possible channels. Specifically, we focus on cases in which the econometrician and the investors differ with respect to (i) the long run mean of expected inflation $\bar{x}$, (ii) the speed of mean reversion of expected inflation κ, or (iii) both.

Since no participant in the economy, including the econometrician, perfectly observes expected inflation, they infer it by observing the path of the price level $\pi(t)$. Following standard filtering theory as in Liptser and Shiryaev (1974b), we can derive each market participant's dynamics for the estimator of expected inflation $x^i(t)$ where again we use $i = 1, 2$ to denote the two investors and $i = 0$ to denote the econometrician. This filtering problem is summarized as follows.

Proposition 1. *Under each agent's beliefs, the price level is*

$$d\pi(t) = \pi(t)\left[x^i(t)\,dt + \sigma_{\pi,\epsilon}\,dz_\epsilon(t) + \sigma_{\pi,\$}\,dz_\$^i(t)\right], \qquad \pi(0) = 1, \tag{2.10}$$

for $i \in \{0,1,2\}$.

Each agent's beliefs about the expected inflation rate follows the process:

$$dx^i(t) = \kappa^i\left(\bar{x}^i - x^i(t)\right)\,dt + \sigma_{x,\epsilon}^i\,dz_\epsilon(t) + \sigma_{x,\$}^i\,dz_\$^i(t), \quad x^i(0)\ given, \tag{2.11}$$

for $i \in \{0,1,2\}$ where $x^i(0) \sim N\left(\bar{x}^i(0), \sigma_{x^i(0)}^2\right)$. The volatility $\sigma_{x,\epsilon}^i$ is common across investors,

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$\sigma_{x,\epsilon}^i = \sigma_x \rho_{x\epsilon}$. The volatility $\sigma_{x,\$}^i$ for $i = 0, 1, 2$, given in the Appendix, differs across agents only when they disagree about the mean reversion of expected inflation, κ^i.

With Proposition 1, the dynamics of disagreement between the first and the second investor are

$$
\begin{aligned}
d\Delta(t) = & \left(\frac{\kappa^2 \bar{x}^2 - \kappa^1 \bar{x}^1}{\sigma_{\pi,\$}} + \frac{\sigma_{x,\$}^2 - \sigma_{x,\$}^1}{\sigma_{\pi,\$}^2} x^0(t) + \left[\frac{\kappa^1 - \kappa^2}{\sigma_{\pi,\$}} + \frac{\sigma_{x,\$}^1 - \sigma_{x,\$}^2}{\sigma_{\pi,\$}^2} \right] x^1(t) \right. \\
& \left. - \left(\kappa^2 + \frac{\sigma_{x,\$}^2}{\sigma_{\pi,\$}} \right) \Delta(t) \right) dt + \frac{\sigma_{x,\$}^2 - \sigma_{x,\$}^1}{\sigma_{\pi,\$}} dz_{\$}^0(t).
\end{aligned}
$$

(2.12)

Similarly, the dynamics of disagreement between the econometrician and investor i are

$$
d\Delta^i(t) = \left(\frac{\kappa^0 \bar{x}^0 - \kappa^i \bar{x}^i}{\sigma_{\pi,\$}} + \frac{\kappa^i - \kappa^0}{\sigma_{\pi,\$}} x^0(t) - \left(\kappa^i + \frac{\sigma_{x,\$}^i}{\sigma_{\pi,\$}} \right) \Delta^i(t) \right) dt + \frac{\sigma_{x,\$}^0 - \sigma_{x,\$}^i}{\sigma_{\pi,\$}} dz_{\$}^0(t).
$$

(2.13)

The expected inflation rate's instantaneous volatility $\sigma_{x,\$}^i$ on the $z_{\$}^i$ shock determines if $\Delta(t)$ and $\Delta^i(t)$ are stochastic or locally deterministic. These processes are locally deterministic when the agents only disagree about the long run mean of expected inflation $\bar{x}$. Disagreement about the mean reversion of expected inflation, κ^i, is the only channel that generates stochastic $\Delta(t)$ and $\Delta^i(t)$ dynamics.

2.2 Security Markets

Investors in the economy trade continuously in a real riskfree asset, a nominal money market account, and a security whose real return is locally perfectly correlated with real consumption growth. For simplicity, we will refer to this security as a "stock." This particular security structure is not crucial. We only require that the financial security structure be such that each investor can trade in a complete market.

The real risk-free asset is in zero-net supply and its real price is denoted by $B(t)$. The posited real price dynamics are

$$
dB(t) = B(t)\, r(t)\, dt, \qquad B(0) = 1,
$$

(2.14)

where $r(t)$ denotes the real riskfree rate to be determined in equilibrium. Investors observe the price of the real risk-free asset and hence know and agree on $r(t)$.

The nominal money market account is in zero-net supply and its nominal price is denoted by

$P_{\$}(t)$. The posited nominal price dynamics are

$$dP_{\$}(t) = P_{\$}(t)\, r_{\$}(t)\, dt, \qquad P_{\$}(0) = 1, \tag{2.15}$$

where $r_{\$}(t)$ denotes the nominal risk-free rate. Both investors agree on the nominal price of the nominal money market account and hence know and agree on the nominal risk-free rate $r_{\$}(t)$.

Applying Itô's lemma to the nominal price dynamics in equation (2.15) leads to the posited real price dynamics of the nominal money market account denoted by $P(t) \equiv \frac{P_{\$}(t)}{\pi(t)}$:

$$dP(t) = P(t)\left[\mu_P(t)\, dt - \sigma_{\pi,\epsilon}\, dz_\epsilon(t) - \sigma_{\pi,\$}\, dz_{\$}(t)\right], \qquad \mu_P(t) \equiv r_{\$}(t) - x(t) + \sigma_{\pi,\epsilon}^2 + \sigma_{\pi,\$}^2, \tag{2.16}$$

$$dP(t) = P(t)\left[\mu_P^i(t)\, dt - \sigma_{\pi,\epsilon}\, dz_\epsilon(t) - \sigma_{\pi,\$}\, dz_{\$}^i(t)\right], \qquad \mu_P^i(t) \equiv r_{\$}(t) - x^i(t) + \sigma_{\pi,\epsilon}^2 + \sigma_{\pi,\$}^2, \tag{2.17}$$

where equation (2.17) shows the real price dynamics of the nominal money market under each investor's beliefs. Since both investors agree on the real price of the nominal money market $P(t)$, the investor-specific expected returns are linked through

$$\mu_P^1(t) - \mu_P^2(t) = \sigma_{\pi,\$}\, \Delta(t) = x^2(t) - x^1(t). \tag{2.18}$$

This difference in expected returns is solely driven by the disagreement about expected inflation.

Let $S(t)$ denote the real price of a "stock" in zero net supply which is locally perfectly correlated with real consumption growth and has a strictly positive volatility $\sigma_{S,\epsilon}(t) > 0$.[4] The posited price dynamics are

$$dS(t) = S(t)\left[\mu_S(t)\, dt + \sigma_{S,\epsilon}(t)\, dz_\epsilon(t)\right], \qquad S(0) = 1, \tag{2.19}$$

where $\mu_S(t)$ denotes the expected real rate of return on the stock under all agents' beliefs given there is no disagreement about $z_\epsilon(t)$.

The endogenous price system $(r(t), r_{\$}(t), \mu_S(t))$ is determined in a dynamically complete equilibrium. It is convenient to summarize the price system in terms of investor-specific real state price densities, or stochastic discount factors, that capture the investor-specific beliefs, but common

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[4]By specifying this security to be non-dividend paying and in zero net supply, the volatility $\sigma_{S,\epsilon}(t)$ can be taken as exogenous allowing for a dynamically complete price system.

Arrow-Debreu prices across investors. Investor i's real state price density has dynamics

$$d\xi^i(t) = -\xi^i(t)\left[r(t)\,dt + \theta_\epsilon(t)\,dz_\epsilon(t) + \theta_\$^i(t)\,dz_\$^i(t)\right], \qquad \xi^i(0) = 1, \tag{2.20}$$

where

$$\theta_\epsilon(t) = \frac{\mu_S(t) - r(t)}{\sigma_{S,\epsilon}(t)} \tag{2.21}$$

and investor i's perceived market prices of risk to the nominal shock $\theta_\$^i(t)$ is

$$\theta_\$^i(t) = -\frac{\mu_P^i(t) - r(t)}{\sigma_{\pi,\$}} - \frac{\sigma_{\pi,\epsilon}}{\sigma_{\pi,\$}}\theta_\epsilon(t). \tag{2.22}$$

Given investor 1 and 2 agree on the security prices as well as the real interest rate, the investor-specific market prices of risk are linked through the disagreement process:

$$\theta_\$^2(t) - \theta_\$^1(t) = \Delta(t). \tag{2.23}$$

2.3 Investor Preferences and Consumption-Portfolio Choice Problem

Investors share endowment and inflation risks by continuously trading in the security market. Investors may differ with respect to endowments, beliefs, and preferences. Investors have "catching up with the Joneses" preferences as in Abel (1990) and Chan and Kogan (2002):

$$U^i = \mathrm{E}^i\left[\int_0^T e^{-\rho t}u^i\left(\frac{c^i(t)}{x(t)}\right)dt\right], \qquad i = \{1, 2\}, \tag{2.24}$$

where ρ denotes the common subjective discount factor and $x(t)$ denotes the standard of living process. While investors can have heterogeneous risk aversion, we mainly focus on a common CRRA-habit preferences assumption after Section 3. Using habit-based preferences, even with heterogeneous beliefs, is necessary to capture upward-sloping real term structures. In a standard CRRA economy, term structures are downward-sloping for plausible assumptions on consumption dynamics. See for example Campbell (1986) and Backus and Zin (1994).

The standard of living is measured as a weighted "geometric sum" of past realizations of aggre-

gate output

$$\log(x(t)) = \log(X(0))e^{-\delta t} + \delta \int_0^t e^{-\delta(t-a)} \log(\epsilon(a)) \, da, \qquad \delta > 0, \tag{2.25}$$

where δ describes the dependence of $x(t)$ on the history of aggregate output.[5] Defining relative log output as $\omega(t) = \log(\epsilon(t)/x(t))$, it follows a mean reverting process

$$d\omega(t) = \delta(\bar{\omega} - \omega(t)) \, dt + \sigma_\epsilon \, dz_\epsilon(t), \tag{2.26}$$

with $\bar{\omega} = (\mu_\epsilon - \sigma_\epsilon^2/2)/\delta$.

Investor i is endowed with a fraction of real aggregate output $\epsilon^i > 0$ where $\epsilon^1(t) + \epsilon^2(t) = \epsilon(t)$. The present value of investor's wealth is given by $W^i(0) = E^i \left[\int_0^T \xi^i(t)\epsilon^i(t)dt \right]$. He then chooses a nonnegative consumption process $c^i(t)$, and a portfolio process consisting of $\psi_B^i(t)$ shares in the real risk-free asset, $\psi_P^i(t)$ shares in the nominal money market account, and $\psi_S^i(t)$ shares in the stock.

Complete markets allow the use of standard martingale techniques (Karatzas et al. (1987) and Cox and Huang (1989)) to solve the consumption-portfolio problem of each investor. The optimal consumption process $\hat{c}^i(t)$ with supporting portfolio processes maximize the utility function given in equation (2.24) subject to the investor-specific static budget constraint $E^i \left[\int_0^T \xi^i(t)c^i(t) \, dt \right] \leq W^i(0)$. The optimal consumption process is $\hat{c}^i(t) = x(t)\mathcal{I} \left(y^i e^{\rho t} x(t)\xi^i(t) \right)$, where $\mathcal{I}(\cdot)$ denotes the inverse function of $\partial u(a)/\partial a$ and where the Lagrange multipliers y^i are determined from the investor-specific static budget constraints.

3 Equilibrium

Financial security prices and optimal allocations are characterized by appealing to general equilibrium restrictions when investors disagree on the expected inflation rate.

Definition 1. *Given preferences, endowments, and beliefs, an equilibrium is a collection of allocations* $((c^1(t), \psi_P^1(t), \psi_S^1(t)), (c^2(t), \psi_P^2(t), \psi_S^2(t))$ *and a price system* $(r(t), \mu_S(t), \mu_P^1(t), \mu_P^2(t))$ *such that* $(c^i(t), \psi_P^i(t), \psi_S^i(t))$ *is an optimal solution to investor i's consumption-portfolio problem given*

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[5] If δ is large, then shocks to relative output are transitory and hence the standard of living process resembles closely current output; i.e. $\omega(t) \approx 0$. If $\delta \approx 0$, then shocks to relative output are persistent and hence past aggregate output receives high weight in the standard of living process.

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his perceived price processes, security prices are consistent across investors, and all markets clear for $t \in [0, T]$. Specifically,

$$c^1(t) + c^2(t) = \epsilon(t), \qquad \psi_S^1(t) + \psi_S^2(t) = 0, \qquad \psi_P^1(t) + \psi_P^2(t) = 0.$$

The equilibrium can be constructed via a state-dependent representative agent as in Cuoco and He (1994) and Basak and Cuoco (1998) for example. The state-dependent representative agent at an arbitrary time t is constructed by

$$U(\epsilon(t), x(t), \lambda(t)) = \max_{\{c^1(t) + c^2(t) = \epsilon(t)\}} \left(u^1\left(\frac{c^1(t)}{x(t)}\right) + \lambda(t) u^2\left(\frac{c^2(t)}{x(t)}\right) \right). \tag{3.1}$$

Proposition 2 (Equilibrium). *If there exists an equilibrium, then the equilibrium consumption allocations are*

$$\begin{aligned}
\hat{c}^1(\epsilon(t), x(t), \lambda(t)) &= x(t)\mathcal{I}\left(x(t)U_\epsilon\left(\epsilon(t), x(t), \lambda(t)\right)\right) = f\left(\lambda(t)\right)\epsilon(t), \\
\hat{c}^2(\epsilon(t), x(t), \lambda(t)) &= x(t)\mathcal{I}\left(\frac{x(t)}{\lambda(t)}U_\epsilon\left(\epsilon(t), x(t), \lambda(t)\right)\right) = (1 - f\left(\lambda(t)\right))\epsilon(t),
\end{aligned} \tag{3.2}$$

in which $\mathcal{I}(\cdot)$ denotes the inverse of $(u)'(\cdot) = u_c$ and where the sharing rule $f(\cdot)$ solves $f\left(\lambda(t)\right) = \mathcal{I}\left(\lambda(t)\right)\left(1 - f\left(\lambda(t)\right)\right)$.

The investor's equilibrium state price densities are

$$\begin{aligned}
\xi^1\left(\epsilon(t), x(t), \lambda(t)\right) &= e^{-\rho t}\frac{U_\epsilon\left(\epsilon(t), x(t), \lambda(t)\right)}{U_\epsilon\left(\epsilon(0), X(0), \lambda(0)\right)} \\
&= e^{-\rho t + \omega(t) - \omega(0)}\frac{u_c\left(e^{\omega(t)}f\left(\lambda(t)\right)\right)\epsilon(0)}{u_c\left(e^{\omega(0)}f\left(\lambda(0)\right)\right)\epsilon(t)},
\end{aligned} \tag{3.3}$$

$$\xi^2\left(\epsilon(t), x(t), \lambda(t)\right) = \xi^1\left(\epsilon(t), x(t), \lambda(t)\right)\frac{\lambda(0)}{\lambda(t)}, \tag{3.4}$$

where the stochastic welfare weight $\lambda(t)$ has dynamics

$$d\lambda(t) = \lambda(t)\,\Delta(t)dz_\$^1(t), \tag{3.5}$$

and $\lambda(0)$ solves either investor's static budget constraint.

Given we are also interested in interpreting equilibrium quantities as viewed by the econome-

trician, the real state price density from the perspective of the econometrician can be expressed
as

$$\xi^0(t) = \lambda_i(t)\xi^i(t), \quad \text{where } \frac{d\lambda_{i(t)}}{\lambda_i(t)} = -\Delta^i(t)dz_\$^0(t). \quad (3.6)$$

In particular, $\lambda(t) = \frac{\lambda_2(t)}{\lambda_1(t)}$. The econometrician's perceived real state price density is the state price
density constructed from the path of observed security prices consistent with the econometrician's
beliefs about the dynamics of expected inflation $x^0(t)$.

From the equilibrium construction of the real state price density for each investor, the equi-
librium interest rate and market prices of risk can be computed highlighting the impact of belief
heterogeneity about expected inflation on real prices.

Proposition 3. *The market prices of risk for each investor $i \in \{1, 2\}$ are as follows:*

$$\theta_\epsilon(t) = \mathcal{A}(t)\epsilon(t)\sigma_\epsilon, \qquad \theta_\$^1(t) = -\frac{\mathcal{A}(t)}{\mathcal{A}_2(t)}\Delta(t), \qquad \theta_\$^2(t) = \frac{\mathcal{A}(t)}{\mathcal{A}_1(t)}\Delta(t), \quad (3.7)$$

*where $\mathcal{A}_i(t)$, $i \in \{1, 2\}$ is the absolute risk aversion coefficient for each investor and $\mathcal{A}(t)$ is
the absolution risk aversion coefficient for the state-dependent representative agent where $\mathcal{A}(t) \equiv \frac{1}{\frac{1}{\mathcal{A}_1(t)} + \frac{1}{\mathcal{A}_2(t)}}$.*

The real interest rate is given by

$$r(t) = \rho + \delta\omega(t) + \mathcal{A}(t)\epsilon(t)\left(\mu_\epsilon^1(t) - \delta\omega(t)\right) - \frac{1}{2}\mathcal{A}(t)^3\epsilon(t)^2\left[\frac{\mathcal{P}_1(t)}{\mathcal{A}_1(t)^2} + \frac{\mathcal{P}_2(t)}{\mathcal{A}_2(t)^2}\right]\sigma_\epsilon^2$$

$$+ \frac{\mathcal{A}(t)^2}{\mathcal{A}_1(t)\mathcal{A}_2(t)}\left[\frac{\mathcal{A}(t)}{\mathcal{A}_2(t)} + \frac{\mathcal{A}(t)}{\mathcal{A}_1(t)}\right]\Delta(t)^2 - \frac{1}{2}\frac{\mathcal{A}(t)^3}{\mathcal{A}_1(t)^2\mathcal{A}_2(t)^2}\left[\mathcal{P}_1(t) + \mathcal{P}_2(t)\right]\Delta(t)^2, \quad (3.8)$$

where $\mathcal{P}_i(t)$, $i \in \{1, 2\}$ is the absolute prudence coefficient for each investor.

Proposition 3 highlights the impact of speculative trade on real prices in the economy. Although
the investors in the economy do not disagree about any real quantities, disagreement about expected
inflation, a nominal quantity, induces a spillover effect on the real side of the economy as the
nominal shock $z_\$(t)$ is now priced through both the real interest rate $r(t)$ and the market price
of risk on the nominal shock $\theta^i(t)$. This mechanism, that heterogeneous beliefs about a nominal
quantity can induce nominal risks to be priced on the real side of the economy, is distinct from
New-Keynesian models such as Clarida, Galí, and Gertler (1999) where mechanisms such as sticky

prices are imposed so that the nominal side of the economy impacts the real side of the economy.

Using the state price densities determined in Proposition 2, real and nominal bond prices can be computed. All bonds considered in this paper are default-free zero-coupon bonds. A real bond pays one unit of the consumption good at its maturity and a nominal bond pays one unit of currency at its maturity. Real bonds and nominal bonds are in zero net supply.

Let $B(t; T')$ denote the real and $B_\$(t; T') = B(t; T')\pi(t)$ the nominal price of a real (inflation-protected) bond maturing at T'. The real price of a real bond with maturity T' is

$$B(t; T') = E_t^i \left[\frac{\xi^i(T')}{\xi^i(t)} \right]. \tag{3.9}$$

Let $P(t; T')$ denote the real and $P_\$(t; T') = P(t; T)\pi(t)$ the nominal price of a nominal bond maturing at T'. The nominal price of a nominal bond with maturity T' is

$$P_\$(t; T') = E_t^i \left[\frac{\xi^i(T')}{\xi^i(t)} \frac{\pi(t)}{\pi(T')} \right]. \tag{3.10}$$

Analogously, define the log-yields at time t of a real and nominal zero-coupon bond with τ years to maturity as $y_B^{(\tau)}(t) = -\frac{1}{\tau} \log \left(B(t; t + \tau) \right)$ and $y_{P_\$}^{(\tau)}(t) = -\frac{1}{\tau} \log \left(P_\$(t; t + \tau) \right)$ respectively. We provide closed-form solutions for real and nominal bond prices in the next sections.

4 The Impact of Different Beliefs Across Investors on Bond Prices

To gain additional insights on the impact of heterogeneous beliefs about expected inflation on equilibrium prices, it is useful to turn off the impact of heterogeneous preferences. To accomplish this, we assume that both habit-utility investors in the economy are endowed with identical CRRA preferences over consumption relative to the habit with a relative risk aversion coefficient γ.[6] Again, while the exogenous price level is observable, expected inflation $x(t)$, which follows an Ornstein-Uhlenbeck process from equation (2.4), is not. Here, the investors and econometrician have subjective beliefs about expected inflation in the economy. We assume that they have different beliefs about expected inflation through two possible channels. Specifically, we focus on cases in

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[6]When $\gamma < 1$, the standard of living process X_t and individual consumption are no longer complimentary goods as required to interpret investor preferences as a habit formation model. Instead, the preferences can be interpreted as state-dependent utility. We consider the $\gamma < 1$ case in our theoretical results only for completeness.

which the econometrician and the investors differ with respect to (i) the long run mean of expected inflation $\bar{x}$, (ii) the speed of mean reversion of expected inflation κ, or (iii) both.

Following the analysis from Sections 2 and 3, the consumption sharing rule is given by

$$f(t) = c_1(t)/\epsilon(t) = \frac{1}{1 + \lambda(t)^{\frac{1}{\gamma}}} \qquad \text{with} \qquad \lambda(t) = \frac{\lambda_2(t)}{\lambda_1(t)}, \tag{4.1}$$

leading to a real state price density as perceived by the econometrician given by

$$\xi^0(t) = \lambda_1(t) \left(1 + \lambda(t)^{\frac{1}{\gamma}}\right)^\gamma e^{-\rho t} \epsilon(t)^{-\gamma} e^{(1-\gamma)\omega(t)}. \tag{4.2}$$

Under the econometrician's beliefs, the equilibrium real interest rate and market prices of risk are summarized as follows.

Proposition 4. *The dynamics of the econometrician's real and nominal state price densities, $\xi^0(t)$ and $\xi_\$^0(t) = \frac{\xi^0(t)}{\pi(t)}$, are*

$$d\xi^0(t) = -\xi^0(t) \left[r(t)dt + \theta_\epsilon(t)dz_\epsilon(t) + \theta_\$^0(t)dz_\$^0(t)\right], \tag{4.3}$$

$$d\xi_\$^0(t) = -\xi_\$^0(t) \left[r_\$(t)dt + \theta_{\$,\epsilon}\, dz_\epsilon(t) + \theta_{\$,\$}^0(t)dz_\$^0(t)\right], \tag{4.4}$$

where the real and nominal market prices of risk for the econometrician are as follows:

$$\theta_\epsilon(t) = \gamma\, \sigma_\epsilon, \qquad\qquad \theta_\$^0(t) = \Delta_1(t) - (1 - f(t))\, \Delta(t), \tag{4.5}$$

$$\theta_{\$,\epsilon} = \gamma\sigma_\epsilon + \sigma_\pi\rho_{\epsilon\pi}, \qquad\qquad \theta_{\$,\$}^0(t) = \sigma_{\pi,\$} + \Delta_1(t) - (1 - f(t))\, \Delta(t), \tag{4.6}$$

and the real and nominal interest rates are given by

$$r(t) = \rho + \mu_\epsilon - \frac{1}{2}(\gamma^2 + 1)\sigma_\epsilon^2 + \delta(\gamma - 1)(\bar{\omega} - \omega(t))$$

$$+ \frac{1}{2}\left(1 - \frac{1}{\gamma}\right) f(t)(1 - f(t))(\Delta(t))^2, \tag{4.7}$$

$$r_\$(t) = r(t) + f(t)x_1(t) + (1 - f(t))x_2(t) - \gamma\sigma_\epsilon\sigma_\pi\rho_{\epsilon\pi} - \sigma_\pi^2. \tag{4.8}$$

The real equilibrium price system as viewed by the econometrician in Proposition 4 cleanly demonstrates a spillover effect that heterogeneous beliefs about nominal quantities through inflation

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dynamics can have on the real side of the economy. In particular, once heterogenous beliefs are introduced, the real equilibrium interest rate $r(t)$ is is driven by the difference in beliefs between the two investors captured through $\Delta(t)$ as long as the investors do not have logarithmic preferences as in Xiong and Yan (2010) for example.

When $\gamma > 1$ ($\gamma < 1$), the equilibrium real interest rate is increasing (decreasing) with the difference in beliefs. Given investors disagree about expected inflation, they choose to trade against each other. Both investors believe they will capture consumption from the other investor in the future. Hence, they believe they will consume more aggregate consumption in the future. Classical income and substitution effects then impact the demand for consumption today as discussed in Epstein (1988) and Gallmeyer and Hollifield (2008) for example. Given consumption today is fixed, the real interest rate must adjust to clear markets. If $\gamma = 1$, the income and substitution effects exactly offset implying no impact on the real interest rate. When $\gamma > 1$, the real interest rate rises to counterbalance increased demand for borrowing. When $\gamma < 1$, the real interest rate falls to counterbalance lowered demand for borrowing.

Additionally, the price level shock as perceived by the econometrician, $z_\$^0(t)$, is also priced as can be seen in the expression for $\theta_\$^0(t)$. This manifests itself through two channels. First, if the investors have different beliefs through $\Delta(t) \neq 0$, then $\theta_\$^0(t) \neq 0$. Here the pricing of the nominal shock is driven by speculative trade between the investors through their disagreement. This generates a feedback effect from the nominal to the real side of the economy. Second, the econometrician can perceive the nominal shock to be priced in the real pricing kernel even when both investors agree ($\Delta(t) = 0$). While there is no feedback effect from the nominal to the real side of the economy induced by the traders, the econometrician believes it exists by disagreeing with the common investor belief through $\Delta_1(t) \neq 0$. If disagreement across investors is driven by different views about the speed of mean reversion of expected inflation κ_i, $\Delta(t)$ is stochastic leading to predictability in real asset prices. Again, the econometrician can also perceive that predictability exists when it does not. This occurs when the econometrician's belief on κ^0 differs from the investors' common belief about κ. These results are summarized in the following corollary.

Corollary 1. *The price level shock is priced in the economy when the investors have different beliefs about expected inflation. When the econometrican has different beliefs about expected inflation than*

the common investor belief, then he perceives the price level shock to be priced.

The real price system exhibits predictability when the investors disagree about the speed of mean reversion of expected inflation. The econometrician can believe predictability exists when it does not if his belief about the speed of mean reversion of expected inflation differs from the common belief across investors.

Both Proposition 4 and Corollary 1 highlight the channels through which the price level shock can impact equilibrium quantities. These effects directly impact the real term structure of interest rates as summarized in the following proposition.

Proposition 5. *Suppose both habit-utility investors are endowed with a common relative risk aversion γ. Consider two economies at time t where $\lambda(t)$ is identical across the two economies implying time t consumption allocations are the same. Suppose one economy always exhibits more disagreement across the two investors than the other. Disagreement in the first economy, $\bar{\Delta}(s)$, then satisfies $\bar{\Delta}(s) \geq |\underline{\Delta}(s)|$ for $s \geq t$ where $\underline{\Delta}(s)$ denotes investor disagreement in the other economy.*

Then, real bond prices $\bar{B}(t, T')$ and $\underline{B}(t, T')$ of maturity T' in the higher and lower disagreement economies satisfy

$$\bar{B}(t, T') \begin{cases} > \underline{B}(t, T') & \text{if } \gamma < 1, \\ = \underline{B}(t, T') & \text{if } \gamma = 1, \\ < \underline{B}(t, T') & \text{if } \gamma > 1. \end{cases} \tag{4.9}$$

In particular, real bond yields increase with disagreement for $\gamma > 1$. Real bond volatility also rises with disagreement as compared to the no disagreement benchmark.

Proposition 5 highlights that the impact of increased disagreement on the real short rate propagates through the entire yield curve. When $\gamma > 1$, a case consistent with other observable equilibrium price properties, the real yield curve increases as inflation disagreement increases.

4.1 Closed-Form Bond Prices

The previous results have highlighted qualitative properties of the impact of expected inflation heterogeneous beliefs on equilibrium prices. To gain quantitative insights, it is useful to consider a setting where all bond prices can be computed in closed-form. To accomplish this, we assume that common relative risk aversion γ is an integer. This assumption allows us to construct exact

expansions of bond prices in artificial economies similar to equilibrium expansions computed in work such as Yan (2008), Dumas et al. (2009), Bhamra and Uppal (2010), and Cvitanić et al. (2011). To accomplish this, we decompose the real state price density through the following decomposition where economy k can be interpreted as a single investor habit formation economy with an aggregate endowment process given by $\frac{\epsilon(t)}{\lambda(t)^{\frac{k}{\gamma^2}}}$.

Proposition 6. *Assuming that γ is an integer, we can decompose the real state price density as*

$$\frac{\xi^0(t)}{\xi^0(0)} = \sum_{k=0}^{\gamma} w_k(0) \frac{\xi_k^0(t)}{\xi_k^0(0)}, \tag{4.10}$$

where $\xi_k^0(t)$ can be interpreted as a real state price density in a fictitious economy given by

$$\xi_k^0(t) = e^{-\rho t} \lambda_1(t) \lambda(t)^{\frac{k}{\gamma}} \epsilon(t)^{-\gamma} e^{(1-\gamma)\omega(t)}. \tag{4.11}$$

The dynamics of $\xi_k^0(t)$ are

$$\frac{d\xi_k^0(t)}{\xi_k^0(t)} = -r_k(t)dt - \theta_{k,\epsilon}(t)dz_\epsilon(t) - \theta_{k,\$}^0(t)dz_\$^0(t)(t), \tag{4.12}$$

where

$$\begin{aligned}
r_k(t) = {} & \rho + \gamma\mu_\epsilon - \frac{1}{2}\gamma(\gamma+1)\sigma_\epsilon^2 - \delta(\gamma-1)\omega(t) \\
& - \frac{1}{2}\frac{k}{\gamma}\left(\frac{k}{\gamma}-1\right)\frac{1}{\sigma_{\pi,\$}^2}(x^1(t) - x^2(t))^2,
\end{aligned} \tag{4.13}$$

and

$$\theta_{k,\epsilon}(t) = \gamma\sigma_\epsilon, \tag{4.14}$$

$$\begin{aligned}
\theta_{k,\$}^0(t) = {} & \frac{1}{\sigma_{\pi,\$}}\left(x^0(t) - x^1(t)\right) + \frac{k}{\gamma}\frac{1}{\sigma_{\pi,\$}}\left(x^1(t) - x^2(t)\right) \\
= {} & \frac{1}{\sigma_{\pi,\$}}x^0(t) - \frac{x^1(t)}{\sigma_{\pi,\$}}\left(1 - \frac{k}{\gamma}\right) - \frac{k}{\gamma}\frac{x^2(t)}{\sigma_{\pi,\$}}.
\end{aligned} \tag{4.15}$$

The quantity $w_k(t)$ denotes the weight placed on $\xi_k(t)$ and is driven by the sharing rule $f(t)$:

$$w_k(t) = \begin{pmatrix} \gamma \\ k \end{pmatrix} \frac{\lambda(t)^{\frac{k}{\gamma}}}{\left(1 + \lambda(t)^{\frac{1}{\gamma}}\right)^\gamma} = \begin{pmatrix} \gamma \\ k \end{pmatrix} f(t)^{\gamma-k}(1 - f(t))^k \tag{4.16}$$

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with $\sum_{k=0}^{\gamma} w_k(t) = 1$.

Likewise, we can also decompose the nominal state price density when γ is an integer as

$$\frac{\xi_\$^0(t)}{\xi_\$^0(0)} = \sum_{k=0}^{\gamma} w_k(0) \frac{\xi_{k\$}^0(t)}{\xi_{k\$}^0(0)}, \tag{4.17}$$

where $\xi_\$^0(t) = \frac{\xi^0(t)}{\pi(t)}$ and $\xi_{k\$}^0(t) = \frac{k\xi^0(t)}{\pi(t)}$. The dynamics of $\xi_{k\$}^0(t)$ are summarized in the following corollary.

Corollary 2. *The nominal stochastic discount factor in artificial economy k is defined as $\xi_{k\$}^0(t) = \xi_k^0(t)/\pi(t)$. We then have*

$$\frac{d\xi_{k\$}^0(t)}{\xi_{k\$}^0(t)} = -r_{k\$}(t)dt - \theta_{k\$,\epsilon}dz_\epsilon(t) - \theta_{k\$,\$}^0 dz_\$^0(t), \tag{4.18}$$

where

$$r_{k\$}(t) = r_k(t) + \left(1 - \frac{k}{\gamma}\right)x^1(t) + \frac{k}{\gamma}x^2(t) - \gamma\rho_{\epsilon\pi}\sigma_\pi\sigma_\epsilon - \sigma_\pi^2, \tag{4.19}$$

$$\theta_{k\$,\epsilon}(t) = \gamma\sigma_\epsilon + \rho_{\epsilon\pi}\sigma_\pi, \tag{4.20}$$

and

$$\theta_{k\$,\$}^0(t) = \frac{1}{\sigma_{\pi,\$}}x^0(t) - \frac{x^1(t)}{\sigma_{\pi,\$}}\left(1 - \frac{k}{\gamma}\right) - \frac{k}{\gamma}\frac{x^2(t)}{\sigma_{\pi,\$}} + \sigma_{\pi,\$}. \tag{4.21}$$

These state price decompositions allow us to interpret each fictitious economy k as a single investor economy where the difference in beliefs is captured through a fictitious aggregate endowment process. The weighting of each fictitious state price density to recover the actual state price density is solely driven by the sharing rule $f(t)$. By decoupling the sharing rule in the artificial economy, we can express bond prices as decompositions in the k fictitious economies.

The real price of a real zero-coupon bond is therefore

$$B(t; T') = \sum_{k=0}^{\gamma} w_k(t) B_k(t; T'), \tag{4.22}$$

where $B_k(t; T')$ denotes the real price of a real bond in artificial economy k given by

$$B_k(t; T') = E_t^0 \left[\frac{\xi_k^0(T')}{\xi_k^0(t)} \right] = E^0 \left[\frac{\xi_k^0(T')}{\xi_k^0(t)} \mid \omega(t) = \omega, x^1(t) = x^1, x^2(t) = x^2 \right]. \qquad (4.23)$$

Likewise, the nominal price of a nominal zero-coupon bond is therefore

$$P_\$(t; T') = \sum_{k=0}^{\gamma} w_k(t) P_{\$k}(t; T'), \qquad (4.24)$$

where $P_{k\$}(t; T')$ denotes the nominal price of a nominal bond in artificial economy k given by

$$P_{k\$}(t; T') = E_t^0 \left[\frac{\xi_{k\$}^0(T')}{\xi_{k\$}^0(t)} \right] = E^0 \left[\frac{\xi_{k\$}^0(T')}{\xi_{k\$}^0(t)} \mid \omega(t) = \omega, x^1(t) = x^1, x^2(t) = x^2 \right]. \qquad (4.25)$$

Given the structure of the artificial economies, we now show that the artificial real and nominal term structures are in the class of quadratic Gaussian term structure models as studied in Ahn, Dittmar, and Gallant (2002). To show this mapping, we adopt largely the same notation as Ahn, Dittmar, and Gallant (2002) for the state vector $Y(t)$ in the economy, where $Y(t) = (x^1(t), x^2(t), \omega(t))'$. Additional details are given in the Appendix.

The real bond prices in artificial economy k follow a quadratic Gaussian term structure model and are summarized in the following proposition.

Proposition 7. *The real and nominal bond prices, $B_k(t; T')$ and $P_{k\$}(t; T')$, in the artificial economy k are an exponential quadratic functions of the state vector given by*

$$B_k(t; T') = exp\left\{ \mathcal{A}_k(T' - t) + \mathcal{B}_k(\tau)'Y(T' - t) + Y(t)'\mathcal{C}_k(T' - t)Y(t) \right\}, \qquad (4.26)$$

$$P_{k\$}(t; T') = exp\left\{ \mathcal{A}_{k\$}(T' - t) + \mathcal{B}_{k\$}(\tau)'Y(T' - t) + Y(t)'\mathcal{C}_{k\$}(T' - t)Y(t) \right\}, \qquad (4.27)$$

where the coefficients are the solutions to ordinary differential equations summarized in the Appendix.

Summarizing, when both habit-utility investors are endowed with a common integer risk aversion γ, real and nominal bond prices can be expressed as expansions of artificial economics with quadratic Gaussian term structures. The weights in the expansions are driven by the sharing rule

$f(t)$ providing an additional channel to impact bond prices and their dynamics.

4.2 Quantitative Impact of Different Beliefs between Investors

We now turn to exploring how properties of the yield curve are quantitatively impacted by differences of beliefs in a numerical example. Here we use the closed-form bond prices when both habit-formation investors have a common coefficient of risk aversion γ as outlined in Section 4.1. Table 1 outlines the parameters used for the example. The consumption dynamics are taken from Chan and Kogan (2002), while the inflation dynamics are from Brennan and Xia (2002). While our two investors have a common risk aversion coefficient of $\gamma = 7$, they have different beliefs about the dynamics of inflation both through the long run mean $\bar{x}$ and the speed of mean reversion κ. Hence, the two investors agree to disagree about the dynamics of expected inflation. We assume that the investors are dogmatic in these beliefs. Introducing learning leads to a minor impact on the quantitative results. Throughout, we assume that the econometrician is endowed with the true beliefs.

Setting the difference in beliefs about the long run mean as $\Delta_{\bar{x}} = 0.01$ and the speed of mean reversion as $\Delta_{\kappa} = -0.3$, Figures 1 and 2 plot the average nominal yields and nominal yield volatilities as a function of maturity in the data and as predicted by the model.[7] In these plots, the sharing rule is set such that $f(t) = 0.5$ with belief dynamics computed in the steady state where $x^1(t) = 0.0279$ and $x^2(t) = 0.0309$.

In addition to plotting the data and the difference in beliefs economy (HE-Model), the two figures also plot three representative investor investor economies under either investor 1's, investor 2's, or the econometrician's beliefs. From Figure 1 with the introduction of the habit, the average yield curve is now upward sloping and roughly consistent with the data.[8] Relative to the representative investor model with the econometrician's beliefs, the difference in beliefs economy's average yield curve is always higher. Also, the difference in beliefs economy is not bracketed by the two representative investor economies with either investor 1's or investor 2's beliefs as the difference in beliefs economy's yield curve is higher than the investor 2 economy at longer maturities.

The yield volatilities for the data and the models are given in Figure 2. For the models, we plot

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[7]Based on several numerical trials, the numerical results presented appear insensitive to the signs of the disagreement over $\bar{x}$ and κ.

[8]Without the habit, the average yield curve is downward sloping. See for example Backus and Zin (1994).

the instantaneous volatilities, while the data plotted is for monthly frequency yield volatilities. Due to time aggregation, it is not surprising that the instantaneous volatilities are below the monthly volatilities from the data. Across the models, the difference in beliefs economy always has the higher yield volatilities relative to all of the representative investor models.

With this base case of the yield curve established, we now ask how heterogeneous beliefs impacts both the average yield curve and yield volatilities in Figure 3. In addition to the steady state case, we consider two other cases — high disagreement and no disagreement. In both these cases, the two investors still disagree about the underlying model driving expected inflation through a different $\bar{x}$ and a different κ. However, their current belief about expected inflation $x(t)$ is perturbed. In the high disagreement case, the spread between the two beliefs increases — $x^1(t) = 0.0253$ and $x^2(t) = 0.0336$. In the no disagreement case, the belief spread for expected inflation for the two investors collapses to zero — $x^1(t) = 0.03$ and $x^2(t) = 0.03$.

From these plots, the amount of disagreement has a strong impact on both the nominal and real yield curves. When disagreement goes down, yields go down, yield curves steepen, and yield volatilities fall. When disagreement goes up, nominal yields rise and the slopes of the yield curves flatten. In this particular example, the yield curves even invert. Additionally, yield volatilities increase as disagreement increases. So, increasing disagreement introduces a tension between average yields and volatilities. While disagreement increases bond yield volatility, it also flattens the average yield curve.[9]

Quantitatively, the example also highlights that the impact of expected inflation heterogeneous beliefs on yield curve properties can be large. For example, the nominal yield curve at a maturity of one year shifts by over 200 basis points when moving between the no disagreement and the high disagreement case. At longer maturities, the impact lessens. At a five year maturity, the shift is slightly more than 50 basis points.

Figures 4 and 5 show the impact of the sharing rule on the average real and nominal yield curves respectively under the state state disagreement. The real curve highlights the impact of the spillover effect. Given the two investors disagree about the expected inflation dynamics, their speculative trade spills over to the real side of the economy. In both plots, we see that the yield

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[9]Based on our numerical work, the feature that increased disagreement increases nominal average yields and volatilities as well flattens the yield curve seems robust across a large parameter space.

curve slopes are maximized when the consumption sharing between the two investors is roughly equal.

5   Empirical Evidence

We now empirically explore the relation between inflation disagreement and properties of the nominal and real yield curves. We construct our sample of nominal yields from the CRSP Risk-Free Rates File (1- and 3-month yields based on bid/ask average prices) and the Fama-Bliss Discount Bond File (yields with 1 to 5 years to maturity based on artificial discount bonds). Both data sets have a long time span and are widely used. To complement this data with yields with longer maturities, we use data from Gürkaynak et al. (2007) (yields with 6 to 30 years to maturity). This data, however, is derived from a factor model. The real yield sample merges the backed-out rates in Chernov and Mueller (2008) (Q3 1971 to Q4 2002) with the data in Gürkaynak et al. (2010).[10] From these yield series we estimate a GARCH(1,1) for yield volatilities.[11] We structure our yield and yield volatility data series at monthly, quarterly and semiannual frequency.

To build a measure of disagreement from expected inflation forecasts, we use three commonly used surveys of economic data — the Livingston Survey, the Michigan Surveys of Consumers, and the Survey of Professional Forecasters. From the raw series of forecasts, we compute the mean forecast ("Mean Inflation") as well as the standard deviation around the mean forecast, which we call "Dispersion," at each point in time. Dispersion is a proxy for inflation disagreement in our model. The Michigan Surveys of Consumers data is monthly (available since January 1978), the Survey of Professional Forecasters data is quarterly (available since September 1981), and the Livingston Survey data is semiannual (available since December 1946). We also obtain the mean inflation forecast and the inflation dispersion based on Blue Chip Financial Forecasts data (available since January 1988) from Chun (2011).[12] Below we focus on results obtained from employing the Michigan Surveys of Consumers data.[13]

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[10]The data of Gürkaynak et al. (2007, 2010) are at https://www.federalreserve.gov/econresdata/researchdata.htm.

[11]For some maturities there are missing observations between the backed-out data and the TIPS data. Estimating a GARCH(1,1) for yield volatilities on two separate series or on a merged series that ignores missing observations does not affect our regression results below.

[12]The correlation between the Dispersion series of the Livingston Survey and the Michigan Surveys of Consumers is 0.75. The other pair-wise correlations range from 0.40 to 0.56. Each pair-wise correlation is statistically significant with a p-value smaller than 0.006.

[13]Robustness checks, including checks that use the Livingston Survey, the Survey of Professional Forecasters, and

Figure 6 shows the term structure of nominal yields and nominal yield volatilities for the periods 1978 to 2010 and April 1981 to 2010. The second period April 1981 to 2010 excludes the high inflation years. Mean nominal yields increase with maturity but the term structure flattens out for yields between three and five year maturity. In contrast, average yield volatilities decrease with maturity. These features of the yield and yield volatility curves are standard and appear not to depend on the high inflation period.

Figure 7 shows the evolution of mean beliefs and dispersion for the Michigan Surveys of Consumers over time with NBER recessions as gray shaded areas. In the top panel, the monthly one year ahead CPI is plotted in addition to the mean survey forecast. We see from the figure that the mean of the survey forecast appears to contain valuable information regarding future realizations of inflation. In other words, the mean inflation forecast predicts realized inflation as in Ang et al. (2007a).[14] Yet, we also see from the figure that consumers at times are surprised by low realizations of inflation. From the bottom panel, we note the rather high inflation belief dispersion derived from the Michigan Surveys of Consumers. The high dispersion embedded in the data of the Michigan Surveys of Consumers, also relative to other surveys such as the Livingston Survey, the Survey of Professional Forecasters and the Blue Chip Financial Forecasts data, need not necessarily be surprising, considering that the Michigan Survey asks questions about price changes from the perspective of households. Since households have arguably different consumption bundles, this probably implies increased dispersion relative to a hypothetical survey that asks questions about the CPI instead. In addition, Malmendier and Nagel (2009) argue that dispersion in consumer forecast data is higher than in the professional forecaster data as older consumers consistently overestimate both inflation and its volatility based on past experience. Figure 7 supports this view as the path of the mean inflation forecast almost always lies above realized inflation after the high inflation period at the beginning of our sample.

Figure 8 shows the mean of nominal yields and nominal yield volatility curves that are sorted on belief dispersion. We compute curves for the following sorts: top 5% dispersion, 5% to 15% of the dispersion distribution, 15% to 30% of the dispersion distribution, 30% to 50% of the dispersion

the Blue Chip Financial Forecasts data, are available from the authors.

[14]In predictive regressions we find the following Newey-West corrected t-statistics on Dispersion 1.88, 18.90, 2.82, and 3.51 for Livingston Survey, the Michigan Surveys of Consumers, the Survey of Professional Forecasters, and Blue Chip Financial Forecasts, respectively. The adjusted R2 of these regressions are 11.6, 41.4, 3.9, and 3.9, respectively.

distribution, 50% to 70% of the dispersion distribution, and 70% to 100% of the dispersion distribution. Remarkably, none of these curves ever cross and higher dispersion portfolios always show higher yields and higher yield volatilities. These sorts strongly support our model predictions. We note that almost all of the differences in means are highly statistically significant and are always jointly significant with a p-value of 0.00. The difference in mean yields and mean yield volatilities is also significant with a p-value of 0.00 between the top 25% and the bottom 25% dispersion portfolios. The p-values of these tests are available from the authors.

Our next check is to test whether disagreement about expected inflation between investors, as measured by inflation forecast dispersion, shows a significantly positive relation with nominal bond yields in regression models. Panel A of Table 2 presents estimates of these dispersion regression coefficients for the following maturities: 3-month, 1-year, 2-year, 3-year, 5-year, and 10-year. For each maturity the table presents two models. Model 1 contains a constant and Dispersion as explanatory variables. We recognize that periods with high inflation mechanically imply high dispersion. At least for nominal yields, such a mechanical relation can lead to a rejection of the null hypothesis simply because high dispersion implies high inflation which obviously implies higher nominal bond yields. Model 2 addresses this concern by including Mean Inflation as an explanatory variable. All coefficients of Dispersion in Panel A of Table 2 have positive signs and are highly statistically significant as the smallest Newey-West corrected t-statistics is as high as 3.458. These results provide additional evidence that the model predictions are consistent with the data.

We then move on to test whether Dispersion shows a significantly positive relation with nominal bond yield volatilities. Panel B of Table 2 presents estimates of these dispersion regression coefficients for the following maturities: 3-month, 1-year, 2-year, 3-year, 5-year, and 10-year. Again, the table presents Model 1 and Model 2 for each maturity. As for yields, all coefficients of Dispersion in Panel B of Table 2 have positive signs and are highly statistically significant.

We now proceeded to test our main theoretical prediction by studying the relation between inflation belief dispersion and real yields and yield volatilities. Unfortunately, the TIPS time series it too short to be of use for our purpose. Nevertheless, we employ the backed out data from Chernov and Mueller (2008) together with the TIPS data to put the predictions to the test. Panel A and B of Table 3 presents coefficient estimates from the regressions. We see that in Model 1 of Panel A the sign of the coefficient is always positive consistent with our prediction. However,

the t-statistics are insignificant although increasing in the maturity. In Model 2, the coefficient is positive and statistically significant except for 3 month and 1 year maturities. All regressions with yield volatilities as dependent variable show the expected sign with highly significant coefficient estimates.

Overall, the empirical results are consistent with the model's theoretical predictions. The results are statistically significant and appear robust. Albeit, the significance of the coefficient estimates for real yields is mixed, in some sense it is expected given data limitations.

6 Conclusion

We study how differences in beliefs about expected inflation affect the real and nominal term structures when investors have "catching up with the Joneses" preferences in a pure exchange economy. The habit preferences and the differences in beliefs are both important ingredients of our model. Without "catching up with the Joneses" preferences it is difficult to match the level of yields as well as the level of yield volatilities. Our model shows that differences in beliefs about expected inflation impact the equilibrium real pricing kernel generating a spill-over effect from the nominal to the real side of the economy. We find that heterogeneous beliefs about expected inflation have a strong impact on the level and volatility of the real yield curve. When both investors share common preferences over consumption relative to the habit with a coefficient of relative risk aversion greater than one, real average yields across all maturities rise as disagreement increases. When the coefficient of relative risk aversion is less than one, real average yields fall as disagreement increases. Over both cases, yield volatilities increase with disagreement. To develop intuition concerning the role of different beliefs, we consider a simplifying case where the term structures can be computed in closed-form as a weighted-average of quadratic Gaussian term structure models. We demonstrate numerically how the nature of the difference in beliefs about inflation among investors is important in generating features of the real and nominal yield curves. From empirical work, we find a positive relation between expected inflation disagreement and both nominal bond yields and nominal bond yield volatilities.

Appendix

Proofs and Auxiliary Results

Proof of Proposition 1. The result follows from Theorem 12.1 of Liptser and Shiryaev (1974b). The volatilities $\sigma_{x,\epsilon}^i$ and $\sigma_{x,\$}^i$ for $i = 0, 1, 2$ are

$$\sigma_{x,\epsilon}^i = \sigma_{x,\epsilon} = \sigma_x \rho_{x\epsilon}, \tag{6.1}$$

$$\sigma_{x,\$}^i = \frac{\sigma_x}{\sqrt{1 - \rho_{\epsilon\pi}^2}} \left(\rho_{x\pi} - \rho_{\epsilon\pi} \rho_{x\epsilon} + \frac{1}{\sigma_\pi \sigma_x} v^i \right) = \frac{\sigma_x}{\sqrt{1 - \rho_{\epsilon\pi}^2}} (\rho_{x\pi} - \rho_{\epsilon\pi} \rho_{x\epsilon}) + \frac{v^i}{\sigma_{\pi,\$}}, \tag{6.2}$$

where v^i is agent i's estimation error.

Suppose the estimation error v^i is equal to its steady state value, i.e., it is a constant, given by

$$a \left(v^i \right)^2 + b^i v^i + c = 0, \tag{6.3}$$

with

$$a = -\frac{1}{\left(1 - \rho_{\epsilon\pi}^2 \right) \sigma_\pi^2}, \tag{6.4}$$

$$b^i = -2\kappa^i - \frac{2\sigma_x}{\sigma_\pi \left(1 - \rho_{\epsilon\pi}^2 \right)} (\rho_{\pi x} - \rho_{\epsilon\pi} \rho_{x\epsilon}), \tag{6.5}$$

$$c = \frac{\sigma_x^2}{1 - \rho_{\epsilon\pi}^2} \left(1 - \rho_{\pi\epsilon}^2 - \rho_{\pi x}^2 - \rho_{x\epsilon}^2 + 2\rho_{\pi\epsilon} \rho_{\pi x} \rho_{x\epsilon} \right). \tag{6.6}$$

$\square$

Proof of Proposition 2. The proof follows from Karatzas et al. (1990) with the appropriate modifications taken to accomodate for investors facing different state prices through heterogeneous beliefs.

$\square$

Proof of Propositions 3 and 4. The proof follows from applying Itô's lemma to each investor's first order conditions, imposing market clearing, and match coefficients in the dynamics of the real and nominal state price densities.

$\square$

Proof of Proposition 5. The real bond price written in terms of investor 1's beliefs is given by

$$
\begin{aligned}
B(t; T') &= \mathrm{E}_t^1 \left[\frac{\xi^i(T')}{\xi^i(t)} \right] \\
&= e^{-\rho(T-t)} \mathrm{E}_t^1 \left[\frac{\left(1 + \lambda(T)^{\frac{1}{\gamma}} \right)^\gamma}{\left(1 + \lambda(t)^{\frac{1}{\gamma}} \right)^\gamma} \left(\frac{\epsilon(T)}{\epsilon(t)(1 + \lambda(T)^{\frac{1}{\gamma}})^\gamma} \right)^{-\gamma} e^{(1-\gamma)(\omega(T) - \omega(t))} \right].
\end{aligned}
$$

Given we only focus on differences in beliefs about inflation, $\lambda(t)$ and $\epsilon(t)$ are uncorrelated

implying

$$B(t;T') = e^{-\rho(T-t)}\mathrm{E}_t^1\left[\left(\frac{\epsilon(T)}{\epsilon(t)}\right)^{-\gamma}e^{(1-\gamma)(\omega(T)-\omega(t))}\right] \times \mathrm{E}_t^1\left[\frac{\left(1+\lambda(T)^{\frac{1}{\gamma}}\right)^{\gamma}}{\left(1+\lambda(t)^{\frac{1}{\gamma}}\right)^{\gamma}}\right].$$

Increasing disagreement only impacts the last expectation. First, note that real bond prices are more volatile under disagreement as under the benchmark of no disagreement, $\lambda(t)$ is a constant.

To establish how increased disagreement impacts the expectation given by

$$\mathrm{E}_t^1\left[\frac{\left(1+\lambda(T)^{\frac{1}{\gamma}}\right)^{\gamma}}{\left(1+\lambda(t)^{\frac{1}{\gamma}}\right)^{\gamma}}\right],$$

we can apply the comparison theorem stated below due to Hajek (1985) where the weighting process $\lambda(t)$ is a martingale. $\qquad\square$

Theorem 1 (Mean Comparison Theorem Adapted from Hajek (1985))**.** *Let x be a continuous martingale with representation*

$$x(t) = x(0) + \int_0^t \sigma(s)dw(s)$$

such that for some Lipschitz continuous function ρ, $|\sigma(s)| \leq \rho(x(s))$ and let y be the unique solution to the stochastic differential equation

$$y(t) = x(0) + \int_0^t \rho(y(s))dw(s).$$

Then, for any convex function Φ and any $t \geq 0$,

$$E\left[\Phi(x(t))\right] \leq E\left[\Phi(y(t))\right].$$

Proof of Proposition 6. The proof follows by applying Proposition 4. $\qquad\square$

Proof of Proposition 7. The proof directly follows from applying Proposition 8 in the Appendix. Mapping into the Ahn et al. (2002) setting, the dynamics of $Y(t) = (x^0(t), x^1(t), x^2(t), \omega(t))'$ are

$$dY(t) = (\mu + \xi Y(t))\ dt + \Sigma dZ_2(t), \tag{6.7}$$

where

$$\mu = (\kappa_0 \bar{x}_0, \kappa_1 \bar{x}_1, \kappa_2 \bar{x}_2, \delta\bar{\omega})' \in \mathcal{R}^4, \tag{6.8}$$

$$\xi = \begin{pmatrix} -\kappa_0 & 0 & 0 & 0 \\ \frac{\sigma_{x,\$}^1}{\sigma_{\pi,\$}} & -\left(\kappa^1 + \frac{\sigma_{x,\$}^1}{\sigma_{\pi,\$}}\right) & 0 & 0 \\ \frac{\sigma_{x,\$}^2}{\sigma_{\pi,\$}} & 0 & -\left(\kappa^2 + \frac{\sigma_{x,\$}^2}{\sigma_{\pi,\$}}\right) & 0 \\ 0 & 0 & 0 & -\delta \end{pmatrix} \in \mathcal{R}^{4\times 4}, \tag{6.9}$$

$$\Sigma = \begin{pmatrix} \sigma_{x,\epsilon}^0 & \sigma_{x,\$}^0 \\ \sigma_{x,\epsilon}^1 & \sigma_{x,\$}^1 \\ \sigma_{x,\epsilon}^2 & \sigma_{x,\$}^2 \\ \sigma_\epsilon & 0 \end{pmatrix} \in \mathcal{R}^{4\times2}, \tag{6.10}$$

and

$$Z_2(t) = \left(z_{\epsilon,0}(t), z_\$^0(t)\right)' \in \mathcal{R}^2. \tag{6.11}$$

The volatilities $\sigma_{x,\epsilon}^i$ and $\sigma_{x,\$}^i$ for $i = 0, 1, 2$ are given in Proposition 1.

The coefficients for the real bond price, $\mathcal{A}_k(T'-t)$, $\mathcal{B}_k(\tau)$, and $\mathcal{C}_k(T'-t)$, are the solutions of the ordinary differential equations given in Proposition 8 where

$$\eta_{0,k} = -\left(\gamma\sigma_\epsilon, 0\right)' \tag{6.12}$$

$$\eta_{Y1,k} = 0_4 \tag{6.13}$$

$$\eta_{Y2,k} = -\frac{1}{\sigma_{\pi,\$}} \left(1, -\left(1 - \frac{k}{\gamma}\right), -\frac{k}{\gamma}, 0\right)' \tag{6.14}$$

$$\alpha_k = \rho + \gamma\mu_\epsilon - \frac{1}{2}\gamma(\gamma+1)\sigma_\epsilon^2 \tag{6.15}$$

$$\beta_k = (0, 0, 0, \delta(1-\gamma))' \tag{6.16}$$

$$\Psi_k = -\frac{1}{2}\frac{k}{\gamma}\left(\frac{k}{\gamma} - 1\right)\frac{1}{\sigma_{\pi,\$}^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6.17}$$

The matrix Ψ_k is positive semidefinite because $k/\gamma \leq 1$. Note Ψ_k is singular and that $\psi_K = 0_{4\times4}$ if $k = 0$ or $k = \gamma$.

The coefficients for the nominal bond price, $\mathcal{A}_{k\$}(T'-t)$, $\mathcal{B}_{k\$}(\tau)$, and $\mathcal{C}_{k\$}(T'-t)$, are the solutions of the ordinary differential equations given in Proposition 8 where

$$\eta_{0,k\$} = -\left(\gamma\sigma_\epsilon + \rho_{\epsilon\pi}\sigma_\pi, \sigma_{\pi,\$}\right)' \tag{6.18}$$

$$\eta_{Y1,k\$} = 0_4 \tag{6.19}$$

$$\eta_{Y2,k\$} = -\frac{1}{\sigma_{\pi,\$}} \left(1, -\left(1 - \frac{k}{\gamma}\right), -\frac{k}{\gamma}, 0\right)' \tag{6.20}$$

$$\alpha_{k\$} = \rho + \gamma\mu_\epsilon - \frac{1}{2}\gamma(\gamma+1)\sigma_\epsilon^2 - \gamma\rho_{\epsilon\pi}\sigma_\epsilon\sigma_\pi - \sigma_\pi^2 \tag{6.21}$$

$$\beta_{k\$} = \left(0, 1 - \frac{k}{\gamma}, \frac{k}{\gamma}, \delta(1-\gamma)\right)' \tag{6.22}$$

$$\Psi_{k\$} = -\frac{1}{2}\frac{k}{\gamma}\left(\frac{k}{\gamma} - 1\right)\frac{1}{\sigma_{\pi,\$}^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6.23}$$

The matrix $\Psi_{k\$}$ is positive semidefinite because $k/\gamma \leq 1$. Note that $\psi_{k\$}$ is singular and $\psi_{k\$} = 0_{4\times4}$ if $k = 0$ or $k = \gamma$.

$\square$

Quadratic Gaussian Term Structure Models

Here we use the same notation as Ahn, Dittmar, and Gallant (2002).[15] Let $Y(t)$ denote a $N-$dimensional vector of state variables and $Z_M(t)$ a $M-$dimensional vector of independent Brownian motions.

Assumption 1. *The dynamics of the stochastic discount factor $SDF(t)$ are*[16]

$$\frac{dSDF(t)}{SDF(t)} = -r(t)\,dt + 1'_M\,diag\left[\eta_{0m} + \eta'_{Ym}Y(t)\right]_M\,dZ_M(t) \tag{6.24}$$

with

$$\eta_0 = (\eta_{01}, \ldots, \eta_{0M})' \in \mathcal{R}^M \tag{6.25}$$

$$\eta_Y = (\eta_{Y1}, \ldots, \eta_{YM})' \in \mathcal{R}^{M \times N} \tag{6.26}$$

Hence, the market price of risk is an affine function of the state vector $Y(t)$.

Assumption 2. *The short rate is a quadratic function of the state variables:*

$$r(t) = \alpha + \beta'Y(t) + Y(t)'\Psi Y(t), \tag{6.27}$$

where α is a constant, β is an N-dimensional vector of constants, and Ψ is an $N \times N$ dimensional positive semidefinite matrix of constants.[17]

If the matrix Ψ is non singular, then $r(t) \geq \alpha - \frac{1}{2}\beta'\Psi^{-1}\beta\;\forall t$.

Assumption 3. *The state vector $Y(t)$ follows a multidimensional OU-process:*

$$dY(t) = (\mu + \xi Y(t))\,dt + \Sigma dZ_M(t), \tag{6.28}$$

where μ is an N-dimensional vector of constants, ξ is an N-dimensional square matrix of constants, and Σ is a $N \times M$-dimensional matrix of constants. We assume that ξ is diagonalizable and has negative real components of eigenvalues. Specifically, $\xi = U\Lambda U^{-1}$ in which U is the matrix of N eigenvectors and Λ is the diagonal matrix of eigenvalues.

Let $V(t, \tau)$ denote the price of a zero-coupon bond and $y(t, \tau)$ the corresponding yield. Specifically,

$$V(t, \tau) = E_t\left[\frac{SDF(t+\tau)}{SDF(t)}\right] \tag{6.29}$$

$$y(t, \tau) = -\frac{1}{\tau}\ln\left(V(t, \tau)\right). \tag{6.30}$$

The bond price and corresponding yield are given in the next proposition.

---

[15]In contrast to Ahn, Dittmar, and Gallant (2002): (i) we assume that the vector of Brownian motions driving the discount factor is identical to the vector of Brownian motions driving the state variables and thus Υ is the identify matrix, and (ii) we allow the vector of Brownian motions to have a dimension that is different from the number of state variables.

[16]An apostrophe denotes the transpose of a vector or matrix, $1'_M$ denotes a vector of ones, and diag $[Y'_m]_M$ denotes an M-dimensional matrix with diagonal elements $(Y_1, \ldots, Y_m)$.

[17]We don't impose an additional parameter restriction that guarantees non-negativity of the short rate.

Proposition 8 (Quadratic Gaussian Term Structure Model). *Let $\delta_0 = -\Sigma\Upsilon\eta_0 = -\Sigma\eta_0$ and $\delta_Y = -\Sigma\Upsilon\eta_Y = -\Sigma\eta_Y$. The bond price is an exponential quadratic function of the state vector*

$$V(t,\tau) = exp\left\{A(\tau) + B(\tau)'Y(t) + Y(t)'C(\tau)Y(t)\right\}, \tag{6.31}$$

where $A(\tau)$, $B(\tau)$, and $C(\tau)$ satisfy the ordinary differential equations,

$$\frac{dC(\tau)}{d\tau} = 2C(\tau)\Sigma\Sigma'C(\tau) + \left(C(\tau)(\xi - \delta_Y) + (\xi - \delta_Y)'C(\tau)\right) - \Psi \tag{6.32}$$

$$\frac{dB(\tau)}{d\tau} = 2C(\tau)\Sigma\Sigma'B(\tau) + (\xi - \delta_Y)'B(\tau) + 2C(\tau)(\mu - \delta_0) - \beta \tag{6.33}$$

$$\frac{dA(\tau)}{d\tau} = trace\left[\Sigma\Sigma'C(\tau)\right] + \frac{1}{2}B(\tau)'\Sigma\Sigma'B(\tau) + B(\tau)'(\mu - \delta_0) - \alpha, \tag{6.34}$$

in which $A(0) = 0$, $B(0) = 0_N$, and $C(0) = 0_{N\times N}$. Moreover, the yield is a quadratic function of the state vector $Y(t)$:

$$y(t,\tau) = A_y(\tau) + B_y(\tau)'Y(t) + Y(t)'C_y(\tau)Y(t) \tag{6.35}$$

with $A_y(\tau) = -A(\tau)/\tau$, $B_y(\tau) = -B(\tau)/\tau$, and $C_y(\tau) = -C(\tau)/\tau$.

Proof. See Ahn et al. (2002). $\qquad\square$

If the short rate is an affine function of the state vector $Y(t)$, then the bond price is an exponential affine function of the state vector $Y(t)$ because $\Psi = 0_{N\times N}$ implies $C(\tau) = 0_{N\times N}$ for all τ. The bond price in this case belongs to the class of essential affine term structure models (see Duffee (2002)) and is given in the next corollary.

Proposition 9 (Essential Affine Term Structure Model). *Let $\Psi = 0_{N\times N}$, $\delta_0 = -\Sigma\Upsilon\eta_0 = -\Sigma\eta_0$ and $\delta_Y = -\Sigma\Upsilon\eta_Y = -\Sigma\eta_Y$ and assume that $(\xi - \delta_Y)$ is invertible. The bond price is an exponential affine function of the state vector*

$$V(t,\tau) = exp\left\{A(\tau) + B(\tau)'Y(t)\right\}, \tag{6.36}$$

where

$$B(\tau) = -\left((\xi - \delta_Y)'\right)^{-1}\left(e^{(\xi-\delta_Y)'\tau} - I_{N\times N}\right)\beta, \tag{6.37}$$

$I_{N\times N}$ denotes the N dimensional identity matrix, and

$$
\begin{aligned}
A(\tau) = \; & \frac{1}{2}\beta'\left(\int_0^\tau \left(e^{(\xi-\delta_Y)'u}\right)' K\, e^{(\xi-\delta_Y)'u}\, du\right)\beta \\
& - \left(\beta'K + (\mu - \delta_0)'\left((\xi - \delta_Y)'\right)^{-1}\right)\left(\int_0^\tau e^{(\xi-\delta_Y)'u}\, du\right)\beta \\
& + \left(\frac{1}{2}\beta'K\,\beta + (\mu - \delta_0)'\left((\xi - \delta_Y)'\right)^{-1}\beta - \alpha\right)\tau
\end{aligned}
\tag{6.38}
$$

with

$$K = \left(\left((\xi - \delta_Y)'\right)^{-1}\right)'\Sigma\Sigma'\left((\xi - \delta_Y)'\right)^{-1}. \tag{6.39}$$

If $(\xi - \delta_Y)'$ is diagonalizable; i.e. $(\xi - \delta_Y)' = T\Lambda T^{-1}$ then[18]

$$B(\tau) = -T\,diag\left[\frac{1}{\lambda_i}\left(e^{\lambda_i\tau} - 1\right)\right]T^{-1}\beta, \tag{6.40}$$

$$\int_0^\tau e^{(\xi-\delta_Y)'u}\,du = T\,diag\left[\frac{1}{\lambda_i}\left(e^{\lambda_i\tau} - 1\right)\right]T^{-1}, \tag{6.41}$$

and

$$\int_0^\tau \left(e^{(\xi-\delta_Y)'u}\right)' K\,e^{(\xi-\delta_Y)'u}\,du = \left(T^{-1}\right)' G(\Lambda, t)T^{-1}, \tag{6.42}$$

where $G(\Lambda, t)$ is a $m \times m$-matrix with elements given by

$$G_{ij} = \frac{\omega_{ij}}{\lambda_i + \lambda_j}\left(e^{(\lambda_i+\lambda_j)t} - 1\right) \tag{6.43}$$

and ω_{ij} denotes the element of the matrix $\Omega = T'KT$ in the ith-row and j^{th}-column.
Moreover, the yield is an affine function of the state vector $Y(t)$:

$$y(t,\tau) = A_y(\tau) + B_y(\tau)'Y(t) \tag{6.44}$$

with $A_y(\tau) = -A(\tau)/\tau$, and $B_y(\tau) = -B(\tau)/\tau$.

Proof. where $A(\tau)$ and $B(\tau)$ satisfy the ordinary differential equations,

$$\frac{dB(\tau)}{d\tau} = (\xi - \delta_Y)'B(\tau) - \beta \tag{6.45}$$

$$\frac{dA(\tau)}{d\tau} = \frac{1}{2}B(\tau)'\Sigma\Sigma'B(\tau) + B(\tau)'(\mu - \delta_0) - \alpha, \tag{6.46}$$

in which $A(0) = 0$ and $B(0) = 0_N$. □

---

[18]The matrix $(\xi - \delta_Y)$ is invertible and thus all eigenvalues are nonzero.

Table 1: **Parameter Choice for Two Habit Investor Example.**

Parameter	Description	Value
Preferences		
ρ	Time Preference Parameter	2.5%
γ	Risk Aversion	7
δ	Habit Parameter	0.07
Consumption		
μ_ϵ	Expected Consumption Growth	1.72%
σ_ϵ	Volatility of Consumption	3.32%
Inflation		
σ_π	Inflation Volatility	1.3%
$\bar{x}$	Long Run Mean of Expected Inflation	3%
κ	Mean Reversion of Expected Inflation	0.35
σ_x	Volatility of Expected Inflation	1.4%
$\rho_{\epsilon\pi}$	ρ of Realized Inflation & Real Consumption Growth	-0.2
$\rho_{\pi x}$	ρ of Realized and Expected Inflation	0
$\rho_{x\epsilon}$	ρ of Expected Inflation & Real Consumption Growth	0
Disagreement		
$\bar{x}_1$	Long run mean of first investor	$\bar{x} - \frac{1}{2}\Delta_{\bar{x}}$
$\bar{x}_2$	Long run mean of second investor	$\bar{x} + \frac{1}{2}\Delta_{\bar{x}}$
κ_1	Mean reversion of first investor	$\kappa - \frac{1}{2}\Delta_\kappa$
κ_2	Mean reversion of first investor	$\kappa + \frac{1}{2}\Delta_\kappa$

Table 2: **Inflation Beliefs Dispersion and Nominal Yields** The table reports results from OLS regressions of the determinants of nominal yields (Panel A) and volatilities of nominal yields (Panel B). The dependent variables are from the Fama-Bliss Discount Bond Files (1, 2, 3, and 5 year yields) and from "The U.S. Treasury Yield Curve: 1961 to the Present," Gurkaynak, Sack, and Wright available at https://www.federalreserve.gov/econresdata/researchdata.htm (3 month and 10 year yields). Yield volatilities are estimated by a GARCH(1,1). Explanatory variables include inflation beliefs dispersion (Dispersion) and the mean of the inflation forecasts (Mean Inflation). The t-statistics are Newey-West corrected. The mean and dispersion of inflation forecasts are computed from raw data obtained from Thomson Reuters / University of Michigan. Sample: January 1978 - October 2010.

Panel A: Yields

	3 Month		1 Year		2 Year		3 Year		5 Year		10 Year	
	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2
Intercept	-0.018	-0.022	-0.014	-0.018	-0.011	-0.014	-0.007	-0.009	-0.002	-0.004	0.008	0.007
t-statistics	-1.919	-2.431	-1.623	-2.020	-1.331	-1.627	-0.944	-1.175	-0.293	-0.497	1.442	1.219
Dispersion	1.423	0.787	1.429	0.855	1.415	0.956	1.385	1.029	1.342	1.095	1.254	1.133
t-statistics	7.778	3.458	8.370	3.666	8.881	4.086	9.169	4.465	9.809	5.013	10.732	6.002
Mean Inflation		0.901		0.812		0.650		0.504		0.351		0.172
t-statistics		4.212		3.719		2.887		2.227		1.586		0.886
Adj.R2	0.435	0.544	0.442	0.531	0.456	0.516	0.470	0.508	0.497	0.517	0.545	0.550
N	394	394	394	394	394	394	394	394	394	394	394	394

Panel B: Yield Volatilities

	3 Month		1 Year		2 Year		3 Year		5 Year		10 Year	
	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2
Intercept	-0.016	-0.017	-0.012	-0.013	-0.016	-0.017	-0.015	-0.016	-0.019	-0.020	-0.003	-0.004
t-statistics	-2.051	-2.245	-1.712	-1.849	-2.732	-2.838	-2.572	-2.624	-3.976	-3.919	-0.718	-0.751
Dispersion	0.824	0.602	0.765	0.598	0.849	0.695	0.807	0.702	0.869	0.789	0.500	0.467
t-statistics	5.250	3.504	5.358	3.522	6.663	4.245	6.444	4.316	7.816	5.440	4.619	3.396
Mean Inflation		0.315		0.236		0.218		0.149		0.113		0.047
t-statistics		2.208		1.835		1.719		1.162		0.925		0.415
Adj.R2	0.313	0.341	0.296	0.312	0.377	0.391	0.361	0.367	0.439	0.442	0.232	0.231
N	394	394	394	394	394	394	394	394	394	394	394	394

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Table 3: **Inflation Beliefs Dispersion and Real Yields** The table reports results from OLS regressions of the determinants of real yields (Panel A) and volatilities of nominal yields (Panel B). The dependent variables are from "The Term Structure of Inflation Expectations," Chernov and Mueller (Q3 1971 to Q4 2002) and from "The TIPS Yield Curve and Inflation Compensation," Gurkaynak, Sack, and Wright available at https://www.federalreserve.gov/econresdata/researchdata.htm (3 month and 10 year yields). Yield volatilities are estimated by a GARCH(1,1). Explanatory variables include inflation beliefs dispersion (Dispersion) and the mean of the inflation forecasts (Mean Inflation). The t-statistics are Newey-West corrected. The mean and dispersion of inflation forecasts are computed from raw data obtained from Thomson Reuters / University of Michigan. Sample: Q1 1978 - Q3 2010.

Panel A: Yields

	3 Month		1 Year		2 Year		3 Year		5 Year		10 Year	
	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2
Intercept	0.015	0.015	0.017	0.016	0.009	0.010	0.012	0.013	0.013	0.014	0.019	0.020
t-statistics	1.174	1.170	1.524	1.542	1.058	1.234	1.430	1.710	1.761	2.215	3.383	4.517
Dispersion	0.221	0.327	0.187	0.321	0.280	0.423	0.251	0.413	0.244	0.432	0.173	0.354
t-statistics	0.909	0.993	0.877	1.148	1.550	2.128	1.485	2.309	1.590	2.792	1.399	2.972
Mean Inflation		-0.126		-0.160		-0.198		-0.224		-0.259		-0.251
t-statistics		-0.516		-0.766		-1.308		-1.577		-1.969		-2.257
Adj.R2	0.019	0.015	0.019	0.022	0.074	0.093	0.071	0.106	0.088	0.149	0.073	0.176
N	100	100	100	100	127	127	127	127	131	131	131	131

Panel B: Yield Volatilities

|	3 Month		1 Year		2 Year		3 Year		5 Year		10 Year	
	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2
Intercept	0.002	0.002	0.001	0.001	0.002	0.002	0.001	0.001	-0.001	-0.001	-0.002	-0.002
t-statistics	0.417	0.370	0.270	0.205	0.535	0.635	0.223	0.280	-0.338	-0.338	-1.693	-1.769
Dispersion	0.268	0.400	0.242	0.347	0.202	0.247	0.198	0.223	0.191	0.194	0.172	0.155
t-statistics	2.698	3.216	2.956	3.256	3.356	3.080	3.909	3.142	4.713	3.139	5.820	3.332
Mean Inflation		-0.158		-0.126		-0.062		-0.034		-0.004		0.024
t-statistics		-1.805		-1.669		-1.169		-0.710		-0.082		0.665
Adj.R2	0.150	0.177	0.166	0.188	0.168	0.173	0.197	0.195	0.231	0.225	0.283	0.281
N	100	100	100	100	127	127	127	127	131	131	131	131

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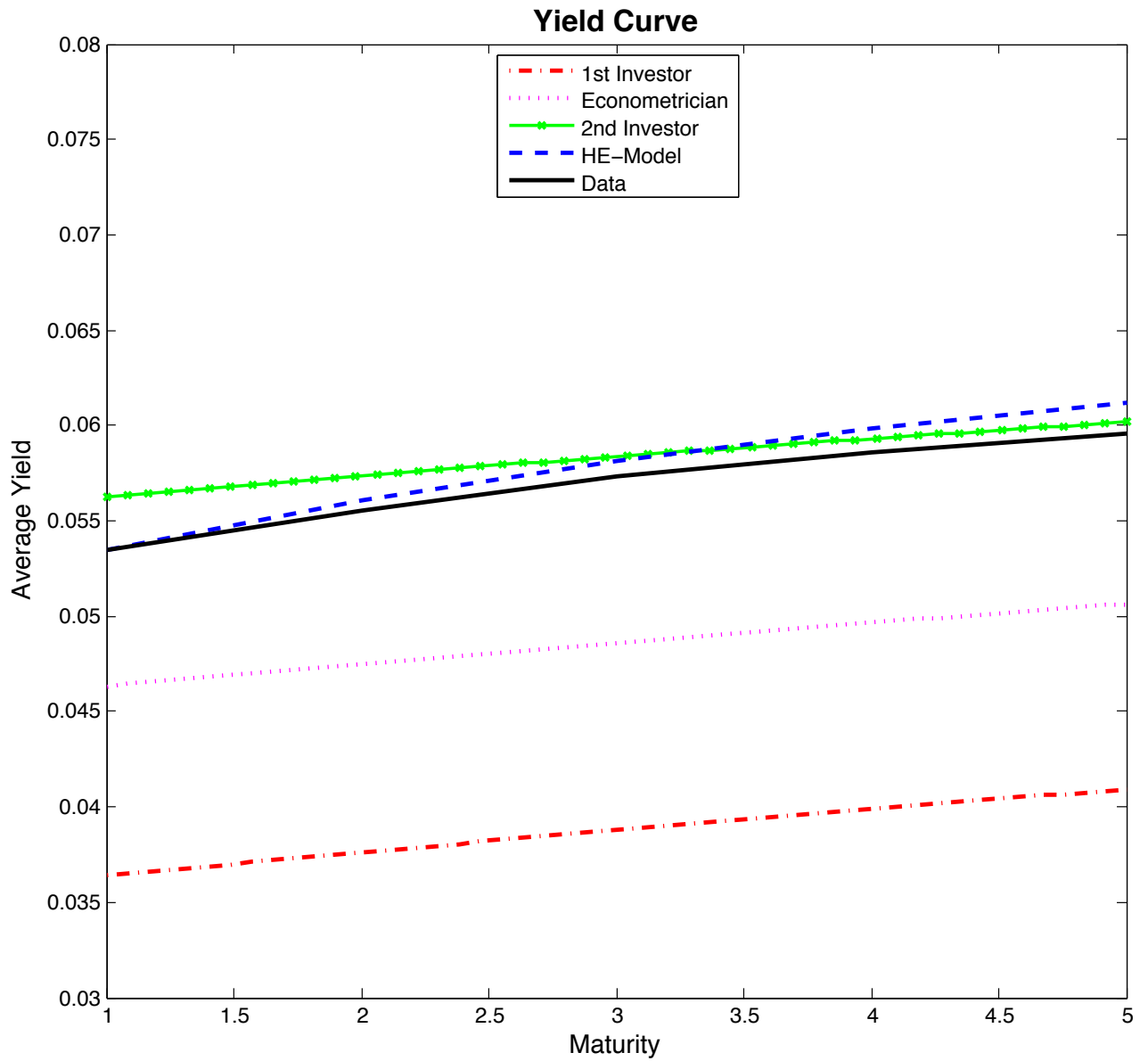
Figures

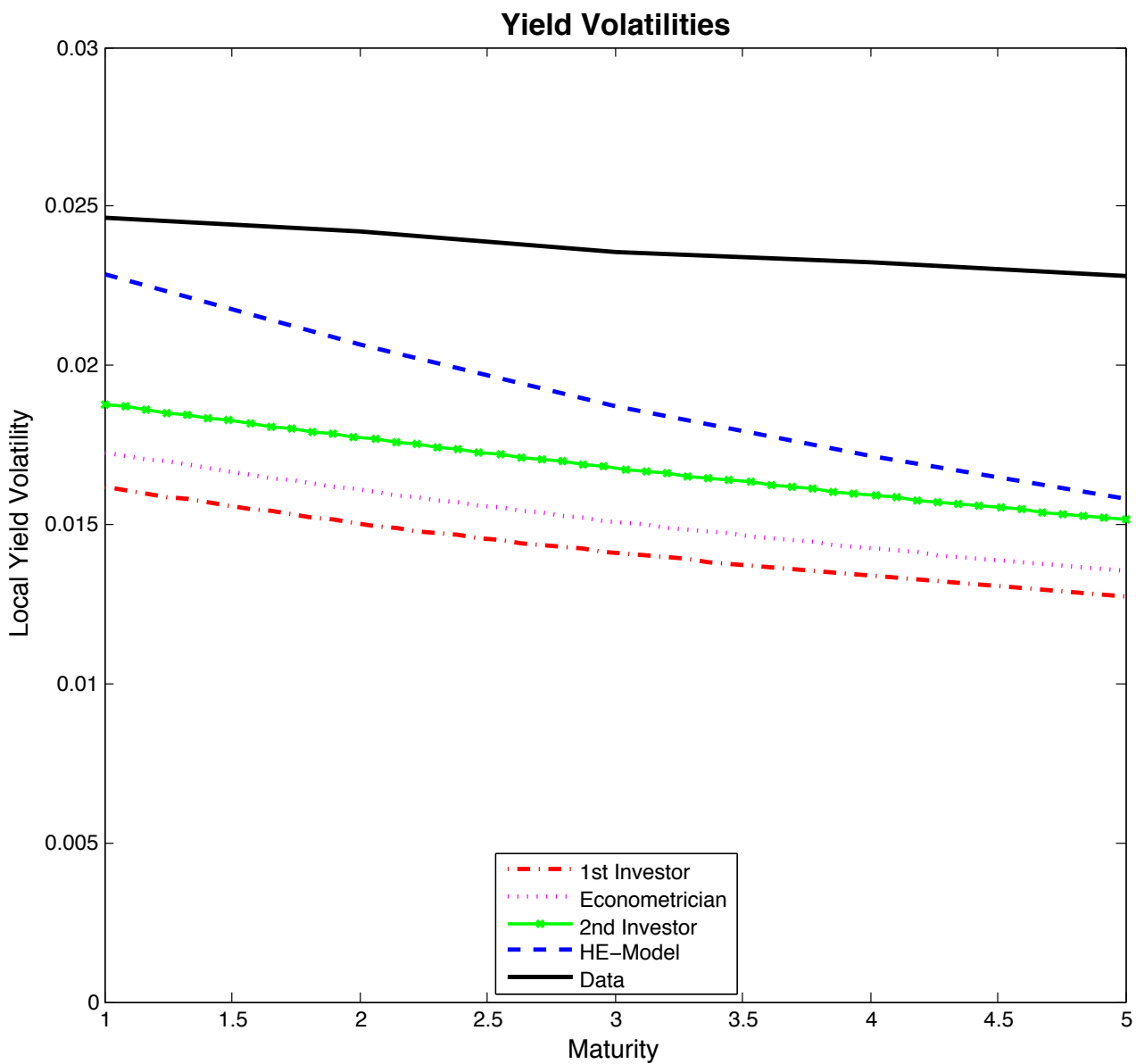Figure 1: Average Nominal Yields - Differences in Beliefs Example

Figure 2: Instantaneous Nominal Yield Volatilities - Differences in Beliefs Example
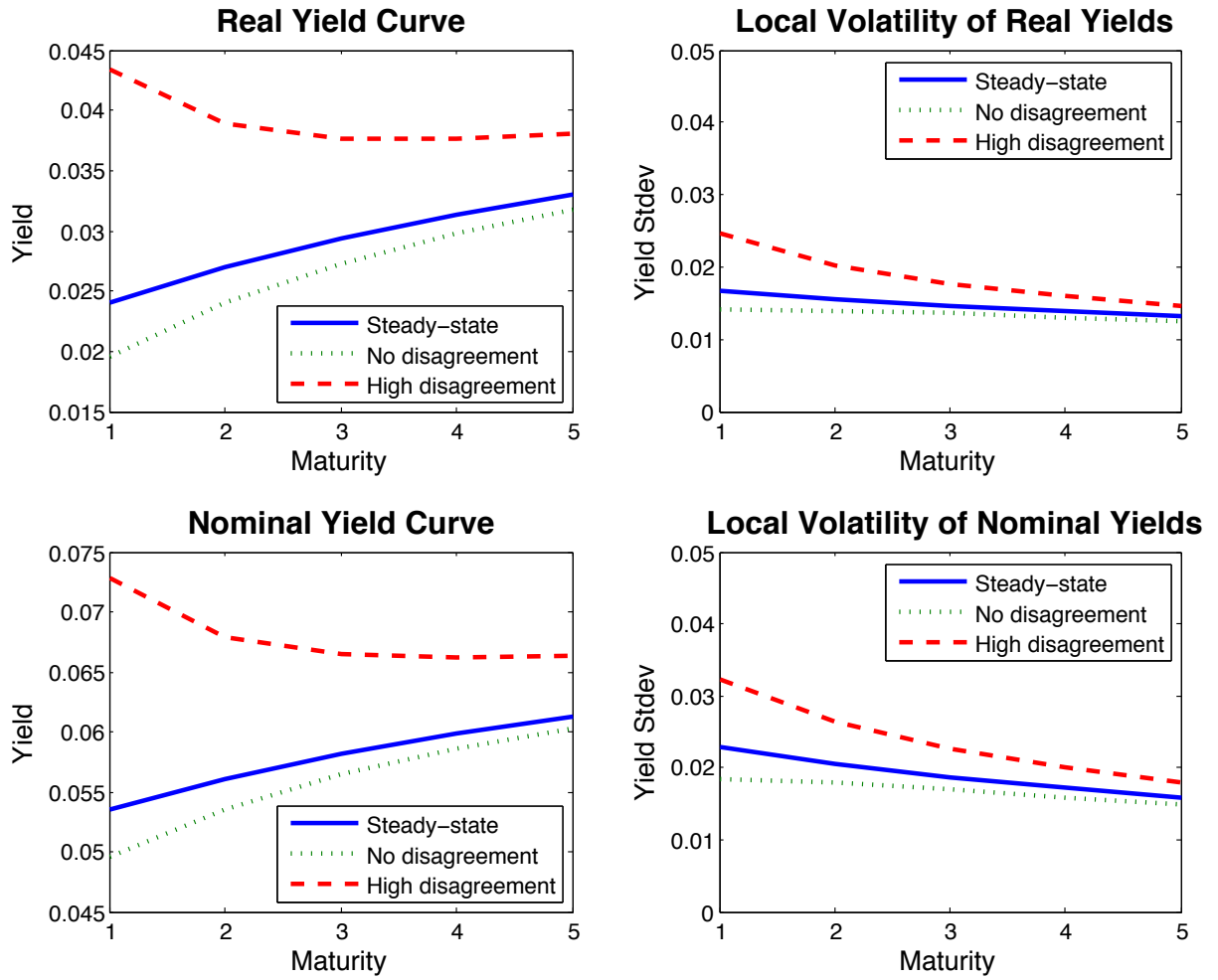
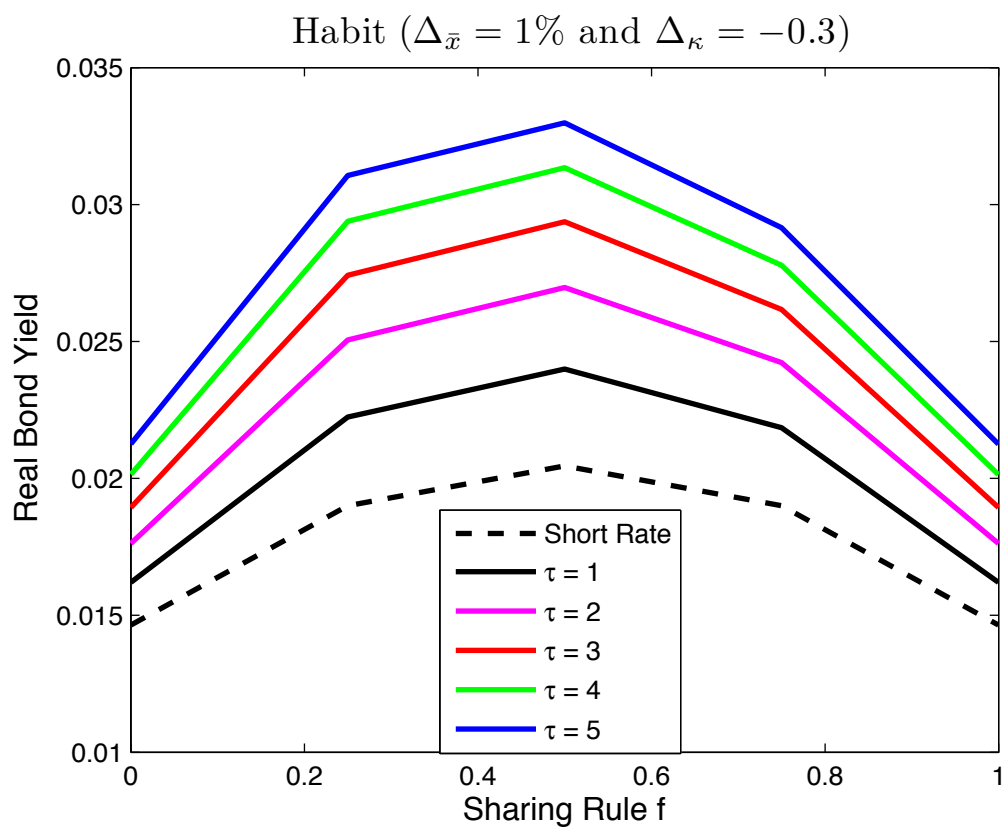Figure 3: Average Yields and Volatilities with Different Disagreement - Differences in Beliefs Example

Figure 4: Average Real Yields as a Function of the Sharing Rule - Differences in Beliefs Example
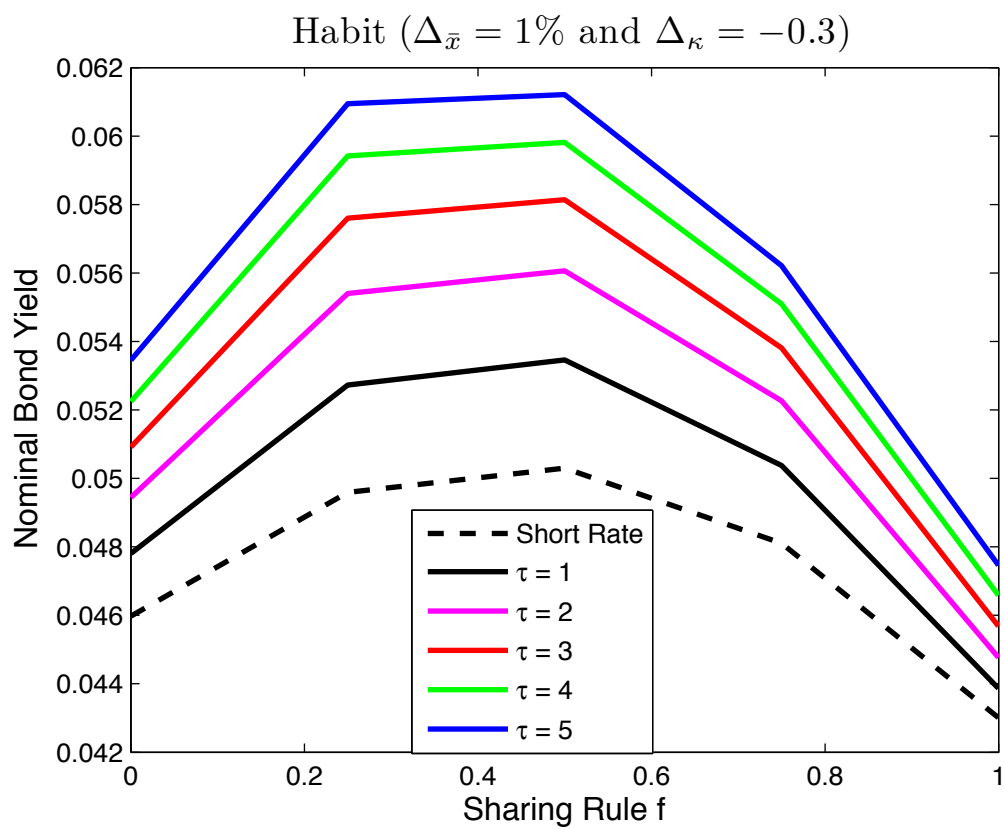
Figure 5: Average Nominal Yields as a Function of the Sharing Rule - Differences in Beliefs Example

Figure 6: **Mean Term Structure of Nominal Yields and Yield Volatilities.** This figure shows the mean of nominal yields and nominal yield volatilities for various maturities. Yield volatilities are computed from a GARCH(1,1). The figure shows the means for the entire sample and for a sample that excludes the high inflation period (with start date: April 1981). The yield data are from the Fama-Bliss Discount Bond Files (1, 2, 3, and 5 year yields) and from Gürkaynak et al. (2007) (3 month and 10 year yields). Sample: January 1978 - December 2010.
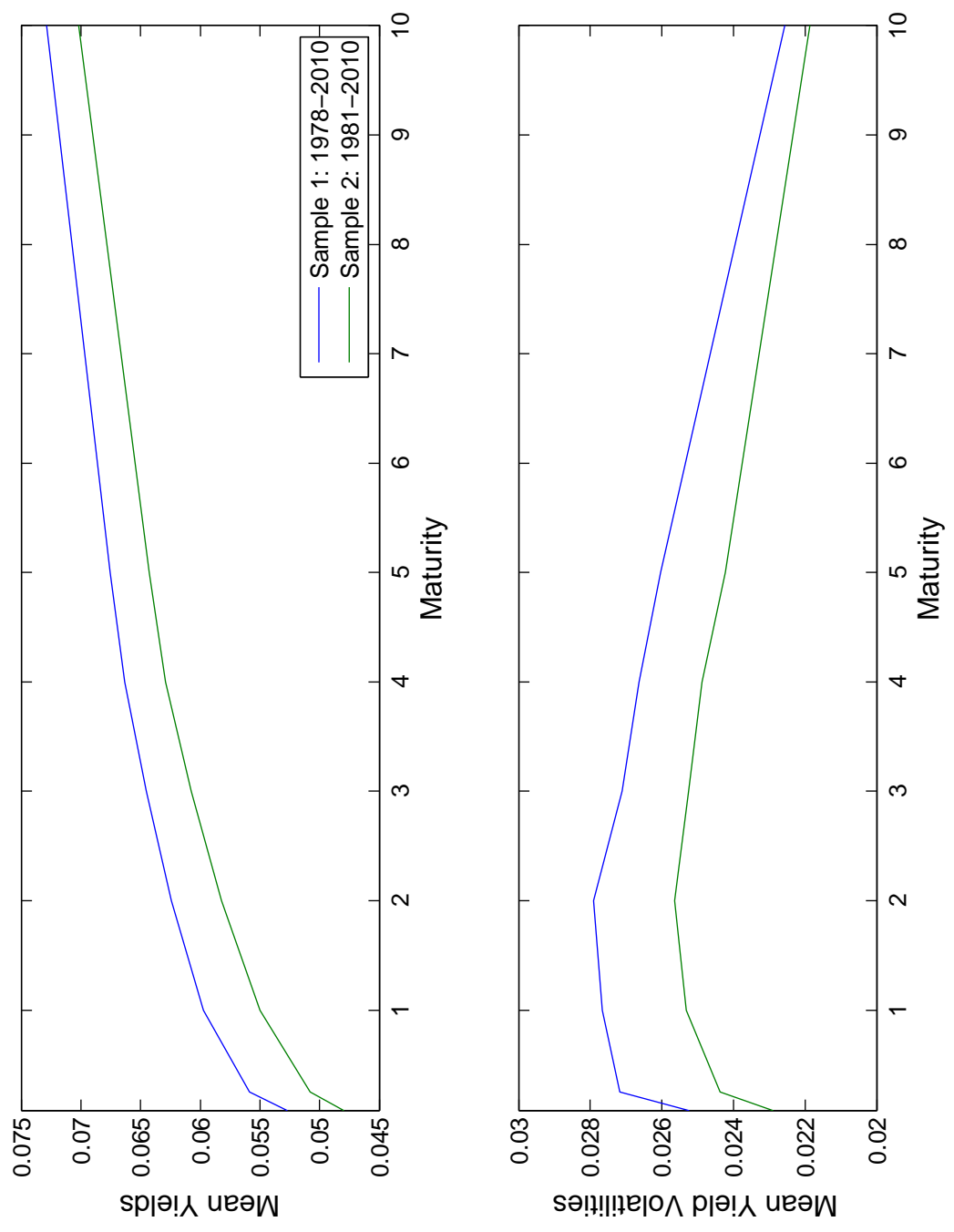
Figure 7: **Mean Inflation Beliefs and Inflation Beliefs Dispersion.** This figure shows mean inflation forecasts and inflation based on the CPI(top plot) and inflation forecast dispersions (bottom plot), the standard deviation of forecasts around the mean forecast —based on Michigan Surveys of Consumers— with NBER recessions as gray shaded areas. The monthly CPI is plotted one year ahead. Michigan Surveys of Consumers data are monthly surveys. The mean and dispersion of inflation forecasts are computed from raw data obtained from Thomson Reuters / University of Michigan. Sample: January 1978 - March 2011.
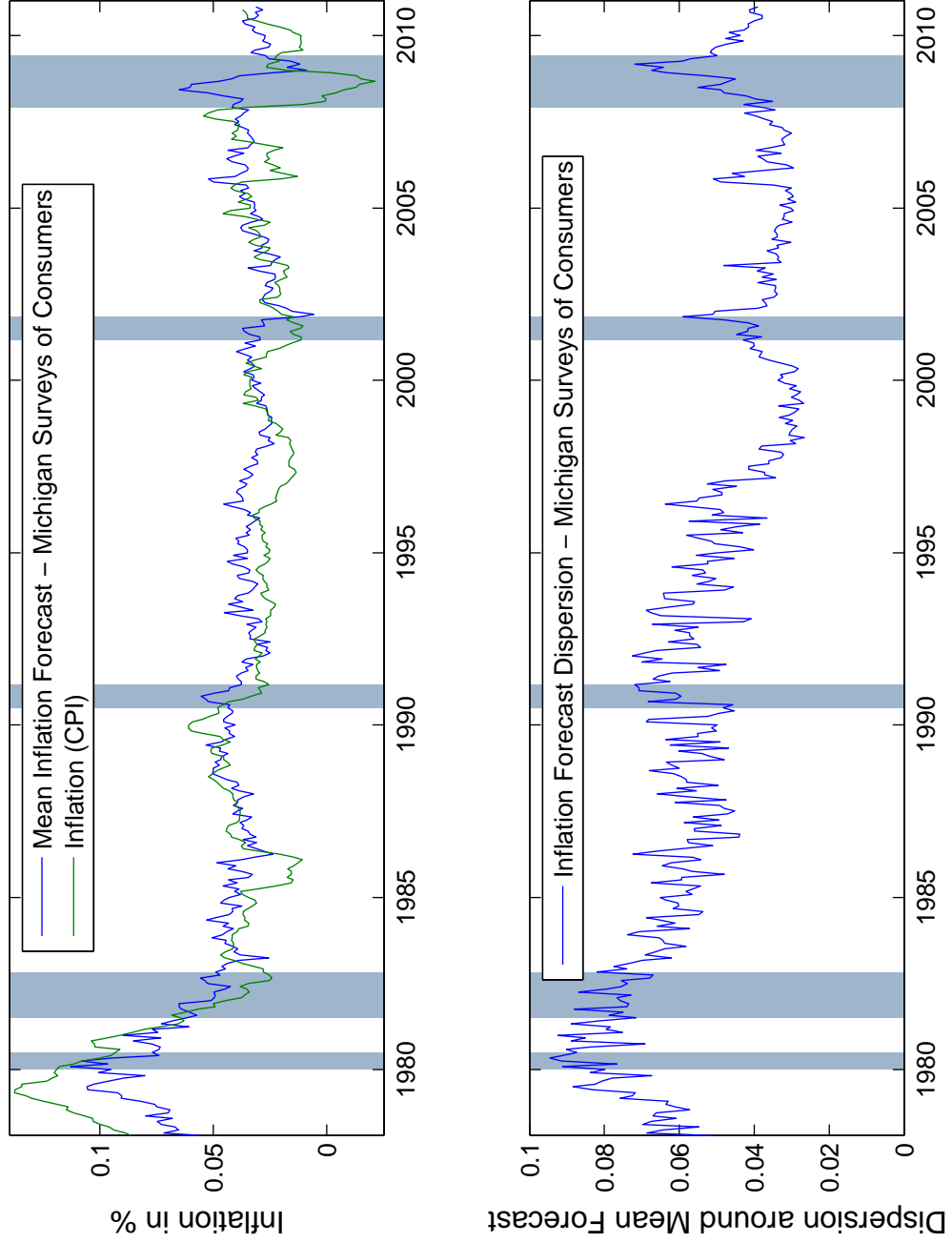
Figure 8: **Inflation Beliefs Dispersion and the Term Structure of Nominal Yields and Yield Volatilities.** This figure shows the mean of nominal yields and nominal yield volatilities for various maturities sorted on beliefs dispersion into six buckets. Yield volatilities are estimated by a GARCH(1,1). The yield data are from the Fama-Bliss Discount Bond Files (1, 2, 3, and 5 year yields) and from Gürkaynak et al. (2007) (3 month and 10 year yields). Inflation beliefs dispersion is based on Michigan Surveys of Consumers data. The mean and dispersion of inflation forecasts are computed from raw data obtained from Thomson Reuters / University of Michigan. Sample: January 1978 - December 2010.
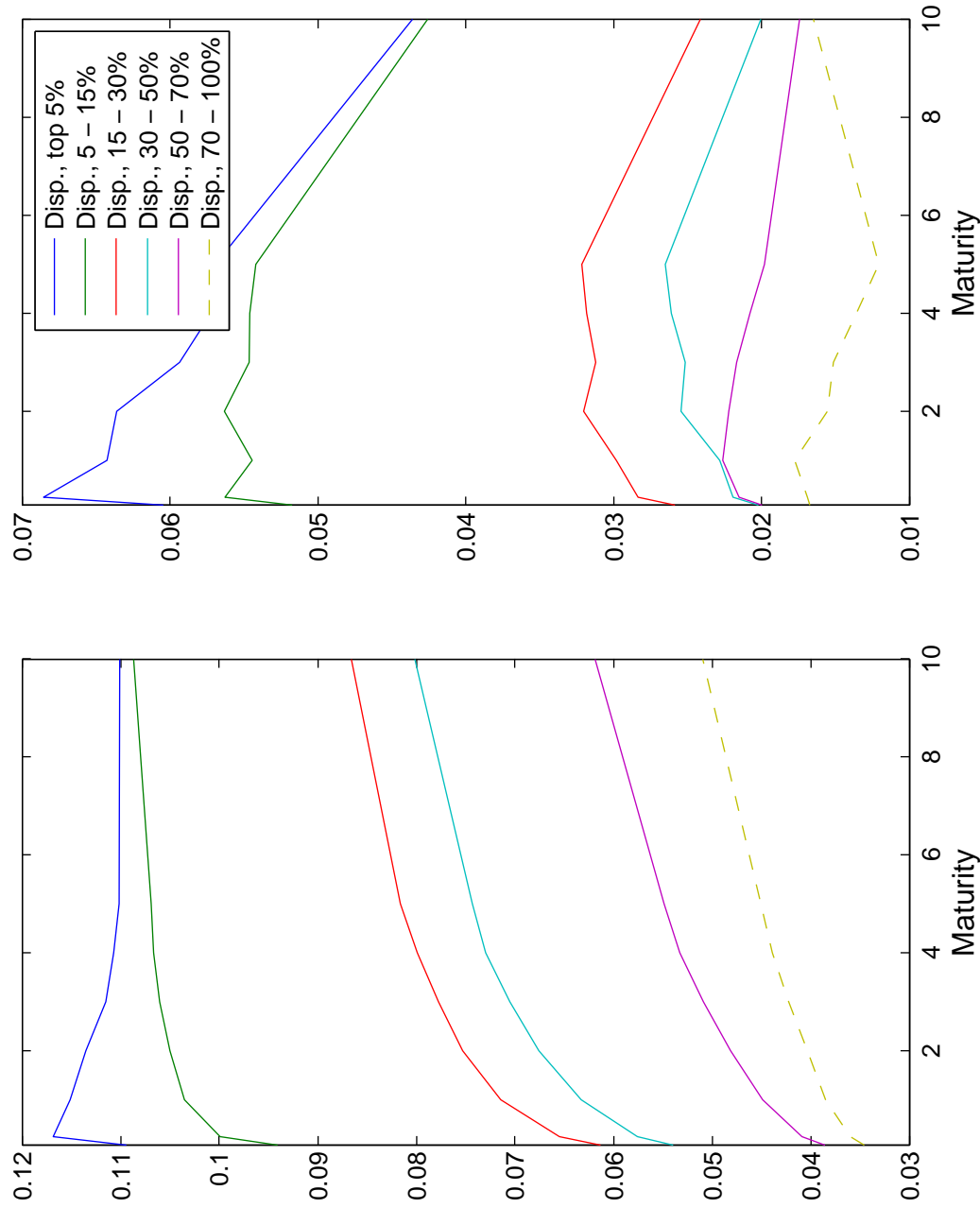
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