Rotation of a Rigid Object About a Fixed Axis

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ANGULAR POSITION, VELOCITY, AND ACCELERATION

The **angular position** is described by

$$s = r\theta$$

Where s is the arc length, r is the radius, and θ is the angle.

The **average angular speed** ω_{avg} is

$$\omega_{\text{avg}} = \frac{\Delta\theta}{\Delta t}$$

The **instantaneous angular speed** ω is

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

Furthermore, the **average angular acceleration** is

$$\alpha_{\text{avg}} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_f - \omega_i}{t_f - t_i}$$

The **instantaneous angular acceleration** is

$$\alpha = \lim_{t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

Rotational Kinetmatic Equations

Through integration, one can derive that the final angular speed is

$$\omega_f = \omega_i + \alpha t \quad \text{(for constant } \alpha\text{)}$$

And likewise, the angular position is

$$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2 \quad \text{(for constant } \alpha\text{)}$$

where θ_i is the angular position at time $t = 0$.t can also be derived that

$$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i) \quad \text{(for constant } \alpha\text{)}$$

And if you were to determine angular position in terms of average angular speed

$$\begin{aligned}\theta_f &= \theta_i + \omega_{\text{avg}} t \\ &= \theta_i + \frac{1}{2}(\omega_i + \omega_i)t\end{aligned}$$

Angular and Translational Quantities

The tangential velocity is measured as $v = \frac{ds}{dt}$, where s is the distance traveled by the point measured along the circular path. Since $s = r\theta$

$$\begin{aligned} v &= \frac{ds}{dt} \\ &= r\frac{d\theta}{dt} \\ &= r\omega \end{aligned}$$

Similarly, for tangential acceleration

$$\begin{aligned} a_t &= \frac{dv}{dt} \\ &= r\frac{d\omega}{dt} \\ &= r\alpha \end{aligned}$$

In the case of centripetal acceleration, since we now have an equation for tangential velocity

$$\begin{aligned} a_c &= \frac{v^2}{r} \\ &= \frac{r^2\omega^2}{r} \\ &= r\omega^2 \end{aligned}$$

Note that the total acceleration is $\vec{a} = \vec{a_t} + \vec{a_c}$, hence the magnitude is

$$a = \sqrt{a_t^2 + a_c^2} = \sqrt{r^2\alpha^2 + r^2\omega^4} = r\sqrt{\alpha^2 + \omega^4}$$

TORQUE

The *change in rotational motion* depends on both the *force applied* and *location* that this force was applied from. The measure of this change in rotational motion is measured by the quantity **torque**, denoted by $\vec{\tau}$. Torque is defined to be

$$\vec{\tau} = \vec{r} \times \vec{F}$$

And by definition of cross product, the magnitude of the torque is

$$\tau = rFsin\theta = Fd$$

Where d is the perpendicular distance from the orientation axis to the line of action of $\vec{F}$, called the **moment arm** (or *lever arm*) of $\vec{F}$. See Figure 1.

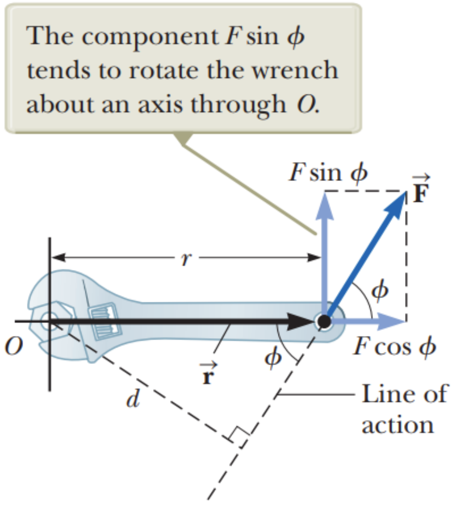

Figure 1

Rigid Object Under a Net Torque and Moment of Inertia

For a particle with mass m rotating in a circle of radius r

$$\sum F_t = ma_t$$

The net torque is

$$\sum \tau = \sum F_t r = (ma_t)r$$

But recalling that $a_t = r\alpha$

$$\begin{aligned}\sum \tau &= (mr\alpha)r \\ &= (mr^2)\alpha \\ \sum \tau &= I\alpha\end{aligned}$$

Where we let $I = mr^2$. The quantity I is called the **moment of inertia**. For a given rigid body that is made up of a collection of particles, even though they may have different translational accelerations a_i, they all have the *same* angular acceleration α. Therefore

$$\begin{aligned}\sum \tau_{\text{ext}} &= \sum \tau_i \\ &= \sum mr^2\alpha \\ &= (\sum_i m_i r_i^2)\alpha \\ &= I\alpha\end{aligned}$$

Where

$$I = \sum_i m_i r_i^2$$

Notice that this equation has the same form as Newton's second law for a system of particles

$$\sum \vec{F}_{\text{ext}} = M\vec{a}_{\text{CM}}$$

Consequently, the moment of inertia I must lay the same role in rotational motion as the role that mass plays in translational motion: the moment of inertia is the resistance to changes in rotational motion. See Figure 2.

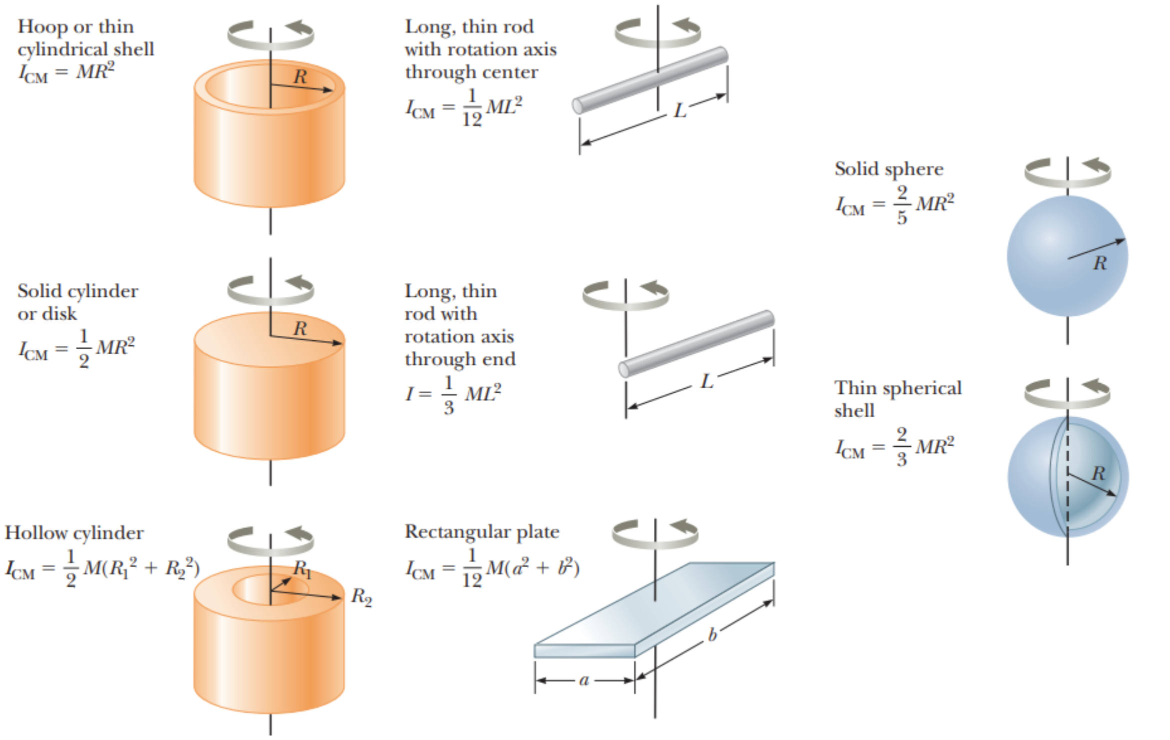

Figure 2: Moments of inertia of homogeneous rigid objects with different geometries

Deriving Moments of Inertia for Geometries

Because the moment of inertia for a rigid body is defined to be

$$I = \sum_i m_i r_i^2$$

If we were to consider for an infinitesimal number of particles that make up that rigid body, the summation simplifies to become

$$I = \int r^2\, dm$$

Deriving Moment of Inertia Formula for Uniform Thin Rod with Axis Through the Center

Suppose we had a rod that was oriented along the x-axis and it were to be rotated about the z-axis (Figure 3). But since $I = \int r^2 dm$ and it is impractical to integrate over mass, we will perform u-substition. We will do this using the linear mass density λ of the object, which is

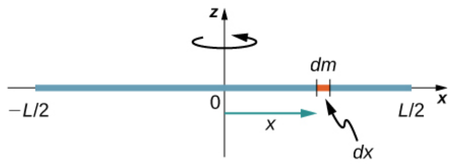

Figure 3

the mass per unit length. Since the mass density of this object is uniform

$$\begin{aligned}\lambda &= \frac{m}{x} \\ m &= \lambda x \\ dm &= d(\lambda x) \\ &= \lambda dx\end{aligned}$$

So performing this substitution yields

$$\begin{aligned}I &= \int x^2\,dm \\ &= \int x^2\lambda\,dx \\ &= \lambda\int x^2\,dx\end{aligned}$$

For the limits of integration, we are integrating from $x = -\frac{L}{2}$ to $x = \frac{L}{2}$

$$\begin{aligned}I &= \lambda\int_{-\frac{L}{2}}^{\frac{L}{2}} x^2\,dx \\ &= \lambda\left[\frac{1}{3}x^3\right]_{-\frac{L}{2}}^{\frac{L}{2}} \\ &= \frac{\lambda}{3}\left[\frac{L^3}{8}+\frac{L^3}{8}\right] \\ &= \frac{1}{12}\lambda L^3\end{aligned}$$

But noting that $\lambda = \frac{M}{L}$ yields the final equation

$$I = \frac{1}{12}ML^2$$

Deriving Moment of Inertia for a Uniform Thin Rod with Axis at the End

This is a simple setup which has the moment of inertia evaluating to one simple integral

$$\begin{aligned} I &= \lambda \int_0^L x^2 dx \\ &= \lambda \left[\frac{1}{3} x^3 \right]_0^L \\ &= \frac{1}{3} \lambda L^3 \\ &= \frac{1}{3} M L^2 \end{aligned}$$

Deriving Moment of Inertia for a Disk Through the Center

We can utilize the disk method for integration (Figure 4). We start with the relationship for

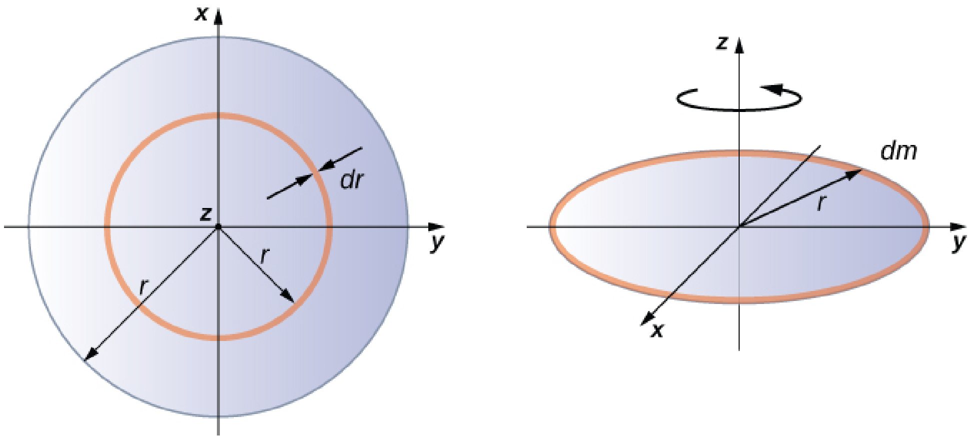

Figure 4

the surface mass density, which is the mass per unit surface area. Since it is uniform, the surface mass density σ is constant

$$\begin{aligned} \sigma &= \frac{m}{A} \\ m &= \sigma A \\ dm &= \sigma(dA) \end{aligned}$$

Since we have the relationship $A = \pi r^2$, we can calculate for dA

$$\begin{aligned} dA &= d(\pi r^2) \\ &= 2\pi r dr \end{aligned}$$

So integrating from $r = 0$ to $r = R$ yields

$$\begin{aligned} I &= \int_0^R r^2[\sigma(2\pi r)]\,dr \\ &= 2\pi\sigma \int_0^R r^3\,dr \\ &= 2\pi\sigma \left[\frac{1}{4}r^4\right]_0^R \\ &= \frac{\pi\sigma}{2}R^4 \end{aligned}$$

But rectall that $\sigma = \frac{m}{A} = \frac{m}{\pi R^2}$ so

$$\begin{aligned} I &= \frac{m}{\pi R^2}\frac{\pi}{2}R^4 \\ &= \frac{1}{2}mR^2 \end{aligned}$$

Parallel-Axis Theorem

The similarity between the process of finding the moment of inertia of a rod about an axis through its middle and about an axis through its end is striking, and suggests that there might be a simpler method for determining the moment of inertia for a rod about any axis parallel to the axis through the center of mass. Such an axis is called a **parallel axis**, and the **parallel-axis theorem** is stated as

$$I_{\text{parallel-axis}} = I_{\text{center of mass}} + md^2$$

Where m is the mass of an object and d is the distance from an axis through the object's center of mass to a new axis.

Derivation of Parallel-Axis Theorem

We may assume, without loss of generality, that in a Cartesian coordinate system the distance between the axes lies along the x-axis and that the center of mass lies at the origin (Figure 5). The moment of inertia relative to the z-axis is then

$$I_{\text{CM}} = \int (x^2 + y^2)\,dm$$

The moment of inertia relative to the axis z', which is at a distance D from the center of mass along the x-axis, is

$$I = \int ((x - D)^2 + y^2)\,dm$$

Expanding the bracket yields

$$I = \int (x^2 + y^2)\,dm + D^2 \int dm - 2D \int x\,dm$$

And notice that the first term is I_{CM}, the second term becomes MD^2, while the third term is a multiple of the x-coordinate of the center of mass which is 0 since the center of mass lies at the origin. So the equation becomes

$$I = I_{\text{CM}} + MD^2$$

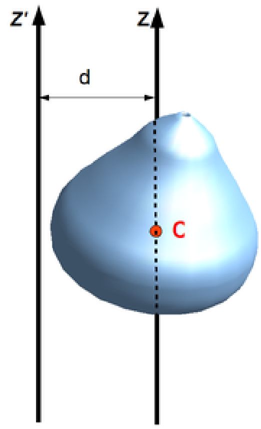

Figure 5