



Second quantization

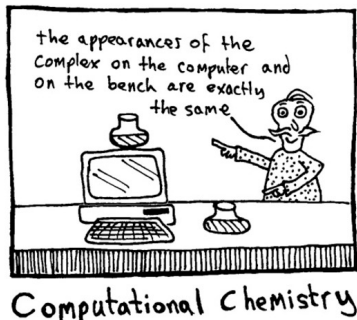
Basics

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Asking Nature ... and the computer



To learn about the world

- the **experimentalist** asks Nature using his experimental apparatus
- the **theoretician** asks the wave function Ψ using mathematical operators $\hat{\Omega}$
- The most important operator is the **Hamiltonian**

The molecular problem



- The time-independent molecular Schrödinger equation

$$\hat{H}^{mol}\psi^{mol} = E^{tot}\psi^{mol}$$

- The molecular Hamiltonian

$$\hat{H}^{mol} = \hat{T}_N + \hat{T}_e + V_{en} + V_{ee} + V_{nn}$$

- ▶ $\hat{T}_N(\mathbf{R})$ - kinetic energy of nuclei
- ▶ $\hat{T}_e(\mathbf{r})$ - kinetic energy of electrons
- ▶ $V_{en}(\mathbf{r}, \mathbf{R})$ - electron-nucleus interaction
- ▶ $V_{ee}(\mathbf{r})$ - electron-electron interaction
- ▶ $V_{nn}(\mathbf{R})$ - nucleus-nucleus interaction

- The Born-Oppenheimer approximation leads to a separation of the
 - ▶ electronic problem

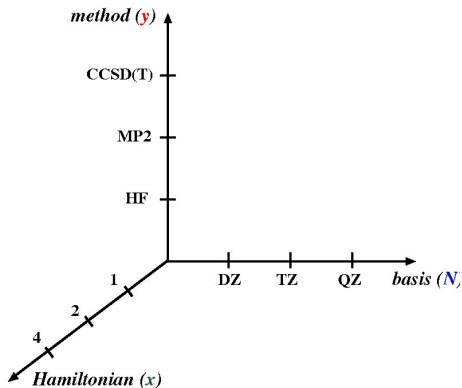
$$H^{el}\Psi^{el}(\mathbf{r}; \mathbf{R}) = E^{el}(\mathbf{R})\Psi^{el}(\mathbf{r}; \mathbf{R}); \quad H^{el} = \hat{T}_e + V_{en} + V_{ee} + V_{nn}$$

- ▶ ... from the nuclear problem

$$\left[\hat{T}_N + E^{el}(\mathbf{R}) \right] \chi(\mathbf{R}) = E^{tot} \chi(\mathbf{R})$$

- ... although many of us stop after the electronic part.

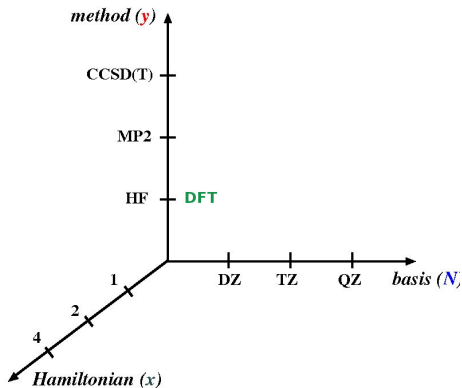
Theoretical model chemistries



Electronic Hamiltonian:
$$\hat{H} = \sum_{i=1}^N \hat{h}(i) + \frac{1}{2} \sum_{i \neq j}^N \hat{g}(i, j) + V_{nn}$$

Computational cost: $\propto N^y$

Theoretical model chemistries



Electronic Hamiltonian:
$$\hat{H} = \sum_{i=1}^N \hat{h}(i) + \frac{1}{2} \sum_{i \neq j}^N \hat{g}(i, j) + V_{nn}$$

Computational cost: $\propto N^y$

The electronic energy

- The electronic wave function

$$\Psi(1, 2, \dots, N)$$

is an extraordinarily complicated mathematical beast and generally not available in exact form.

- The expectation value of the electronic Hamiltonian is

$$E = \langle \Psi | \hat{H} | \Psi \rangle = \sum_{i=1}^N \langle \Psi | \hat{h}(i) | \Psi \rangle + \frac{1}{2} \sum_{i \neq j}^N \langle \Psi | \hat{g}(i, j) | \Psi \rangle + \langle \Psi | V_{nn} | \Psi \rangle$$

- ...and can be simplified.
- The constant term is

$$E_0 = \langle \Psi | V_{nn} | \Psi \rangle = V_{nn} \langle \Psi | \Psi \rangle = V_{nn}$$

which follows from the normalization of the wave function.

The electronic energy

One-electron energy

- We can simplify the one-electron energy

$$\begin{aligned} E_1 &= \sum_{i=1}^N \langle \Psi | \hat{h}(i) | \Psi \rangle \\ &= \sum_{i=1}^N \int \Psi^\dagger(1, 2, \dots, N) \hat{h}(i) \Psi(1, 2, \dots, N) d1 d2 \dots dN \end{aligned}$$

- ... by noting that since electrons are **indistinguishable**,
all one-electron integrals have the same value

$$\langle \Psi | \hat{h}(1) | \Psi \rangle = \langle \Psi | \hat{h}(2) | \Psi \rangle = \dots = \langle \Psi | \hat{h}(N) | \Psi \rangle$$

- We therefore pick one and multiply with the number N of electrons

$$E_1 = N \langle \Psi | \hat{h}(1) | \Psi \rangle$$

The electronic energy

Two-electron energy

- For the two-electron energy

$$\begin{aligned} E_2 &= \frac{1}{2} \sum_{i \neq j}^N \langle \Psi | \hat{g}(i, j) | \Psi \rangle \\ &= \frac{1}{2} \sum_{i \neq j}^N \int \Psi^\dagger(1, 2, \dots, N) \hat{g}(i, j) \Psi(1, 2, \dots, N) d1 d2 \dots dN \end{aligned}$$

we can proceed in similar fashion.

- Since electrons are **indistinguishable**,
all two-electron integrals have the same value

$$\langle \Psi | \hat{g}(1, 2) | \Psi \rangle = \langle \Psi | \hat{g}(1, 3) | \Psi \rangle = \dots = \langle \Psi | \hat{g}(N-1, N) | \Psi \rangle$$

- We can therefore write

$$E_2 = \frac{1}{2} N(N-1) \langle \Psi | \hat{g}(1, 2) | \Psi \rangle$$

where $\frac{1}{2} N(N-1)$ is the number of electron pairs.

The electronic Hamiltonian

1-electron density matrix

- The one-electron Hamiltonian can be split into a free-electron part (kinetic energy) and a term describing the electron-nucleus interaction

$$\hat{h}(1) = \hat{h}_0(1) + \hat{v}_{en}(1)$$

- The interaction operators $\hat{v}_{en}(i)$ and $\hat{g}(i,j)$ are **multiplicative operators**, that is, they do not contain derivatives and can be moved around inside integrals, e.g.

$$\begin{aligned}\langle \hat{V}_{eN} \rangle &= N \int \Psi^\dagger(1, 2, \dots, N) \hat{v}_{eN}(1) \Psi(1, 2, \dots, N) d1 d2 \dots dN \\ &= N \sum_{i=1}^N \int \hat{v}_{eN}(1) \Psi^\dagger(1, 2, \dots, N) \Psi(1, 2, \dots, N) d1 d2 \dots dN \\ &= \int \hat{v}_{eN}(1) n_1(1; 1) d1\end{aligned}$$

- where we have introduced the **one-electron density matrix**

$$n_1(1; 1') = N \int \Psi^\dagger(1, 2, \dots, N) \Psi(1', 2, \dots, N) d2 \dots dN$$

The electronic Hamiltonian

2-electron density matrix

- The kinetic part \hat{h}_0 is in general not multiplicative, but we can write the expectation value of kinetic energy in terms of the one-electron density by a trick

$$\begin{aligned}\langle \hat{T}_e \rangle &= N \int \Psi^\dagger(1, 2, \dots, N) \hat{h}_0(1) \Psi(1, 2, \dots, N) d1 d2 \dots dN \\ &= \int [\hat{h}_0(1') n(1; 1')]_{1' \rightarrow 1} d1\end{aligned}$$

- The expectation value of the two-electron interaction

$$\begin{aligned}\langle \hat{V}_{ee} \rangle &= \frac{1}{2} N(N-1) \int \Psi^\dagger(1, 2, \dots, N) \hat{g}(1, 2) \Psi(1, 2, \dots, N) d1 d2 \dots dN \\ &= \frac{1}{2} \int \hat{g}(1, 2) n_2(1, 2; 1, 2) d1 d2\end{aligned}$$

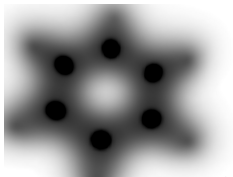
may be expressed in terms of the **two-electron density matrix**

$$n_2(1, 2; 1', 2') = N(N-1) \int \Psi^\dagger(1, 2, \dots, N) \Psi(1', 2', \dots, N) d3 \dots dN$$

The electronic energy

What is needed to calculate the energy?

- This exercise has shown that in order to calculate the electronic energy we do not need the full wave function in terms of N electron coordinates
- It suffices to have the near-diagonal elements of the one-electron density matrix and the diagonal elements of the two-electron density matrix.
- Density functional theory goes a big step further and proposes that we only need the electron density $\rho(\mathbf{r}) = -e \sum_{\text{spin}} n(1; 1)$



The electronic problem

- The generic form of the electronic Hamiltonian, relativistic or not, is

$$\hat{H} = \sum_{i=1}^n \hat{h}(i) + \frac{1}{2} \sum_{i \neq j}^n \hat{g}(i, j) + V_{NN}$$

and is supposed to specify our system.

- The problematic term is the two-electron interaction $\hat{g}(i, j)$.
- Let us for a moment drop this term, as well as V_{NN} (a number), and consider a two-electron system

$$\left[\hat{h}(1) + \hat{h}(2) \right] \Psi(1, 2) = E \Psi(1, 2)$$

The electronic problem

Separation of variables

- We write the two-electron wave function as

$$\Psi(1,2) = \varphi_a(1)\varphi_b(2)$$

- Insertion into the wave equation gives

$$\{h(1)\varphi_a(1)\}\varphi_b(2) + \varphi_a(1)\{h(2)\varphi_b(2)\} = E\varphi_a(1)\varphi_b(2)$$

- Division by $\Psi(1,2)$ gives

$$\frac{\{h(1)\varphi_a(1)\}}{\varphi_a(1)} + \frac{\{h(2)\varphi_b(2)\}}{\varphi_b(2)} = E$$

The electronic problem

Separation of variables

- In order for this relation to hold for any choice of electron coordinates 1 and 2, we must have

$$\underbrace{\frac{\{h(1)\varphi_a(1)\}}{\varphi_a(1)}}_{\varepsilon_a} + \underbrace{\frac{\{h(2)\varphi_b(2)\}}{\varphi_b(2)}}_{\varepsilon_b} = E$$

- A single wave equation for two electrons

$$\left[\hat{h}(1) + \hat{h}(2) \right] \Psi(1, 2) = E\Psi(1, 2)$$

- ... is thereby converted into two wave equations for single electrons

$$h(1)\varphi_a(1) = \varepsilon_a\varphi_a(1); \quad h(2)\varphi_b(2) = \varepsilon_b\varphi_b(2)$$

- The situation is even simpler ...

The electronic problem

Indistinguishability

- Electrons can not be distinguished,
so it suffices to solve a single wave equation

$$h(1)\varphi_x(1) = \varepsilon_x\varphi_x(1); \quad x = a, b, c, \dots$$

- However, the form

$$\Psi(1, 2) = \varphi_a(1)\varphi_b(2)$$

is not an acceptable wave function:

- ▶ electrons are identical particles
- ▶ electrons are fermions:
the wave function must be antisymmetric under particle exchange

- This leads to the form

$$\Psi(1, 2) = \frac{1}{\sqrt{2}} \{ \varphi_a(1)\varphi_b(2) - \varphi_b(1)\varphi_a(2) \}$$

The electronic problem

Slater determinants

- This result is readily generalized:
The exact wave function for a system of N non-interacting electrons is an antisymmetrized product of one-electron wave functions (orbitals).

$$\Psi(1, 2, \dots, N) = \frac{1}{\sqrt{N!}} \hat{A} \varphi_1(1) \varphi_2(2) \dots \varphi_N(N) = |\varphi_1(1) \varphi_2(2) \dots \varphi_N(N)|$$

where \hat{A} is the **anti-symmetrization operator**.

- The wave function for a system of N interacting electrons is typically expanded in an N -electron basis of Slater determinants.
- The fermionic nature of electrons is not built into the electronic Hamiltonian.
- This is achieved with **second quantization** !

First quantization

- The quantum-mechanical Hamiltonian \hat{H} is obtained from its classical counterpart, the Hamiltonian function $H \equiv H(\mathbf{r}, \mathbf{p})$, by replacing the dynamical variables (position \mathbf{r} and momentum \mathbf{p}) by operators:
 - ▶ in the coordinate representation:

$$\mathbf{r} \rightarrow \hat{\mathbf{r}} = \mathbf{r}; \quad \mathbf{p} \rightarrow \hat{\mathbf{p}} = -i\hbar \frac{\partial}{\partial \mathbf{r}} = -i\hbar \nabla$$

- ▶ in the momentum representation:

$$\mathbf{r} \rightarrow \hat{\mathbf{r}} = i\hbar \frac{\partial}{\partial \mathbf{p}}; \quad \mathbf{p} \rightarrow \hat{\mathbf{p}} = \mathbf{p}$$

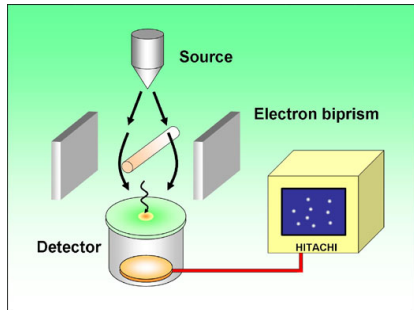
- ...in order to obey the fundamental commutator relation

$$[r_i, p_j] = i\hbar \delta_{ij}$$

- Quantization leads to discrete values of the energy E (as well as angular momentum etc.)

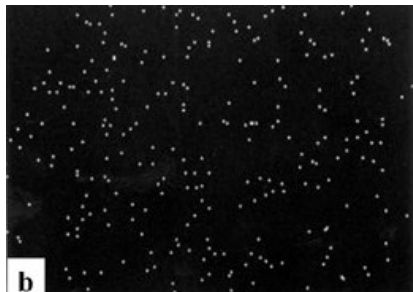
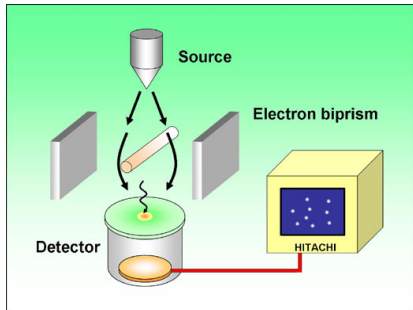
Intermission: the double slit experiment

Researchers of Hitachi has reproduced the famous double-slit experiment using an electron microscope as electron source, an “electron biprism” as double slit and a very sensitive electron detector.



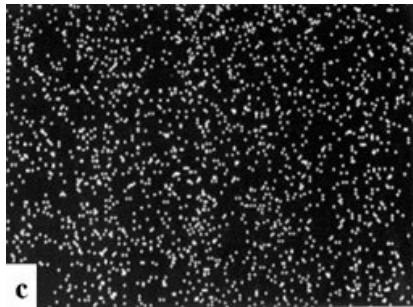
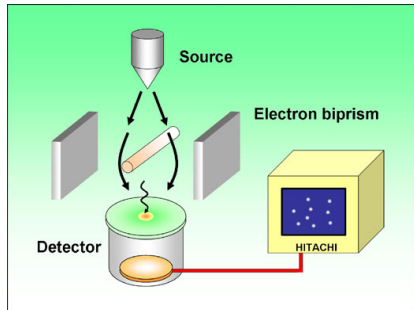
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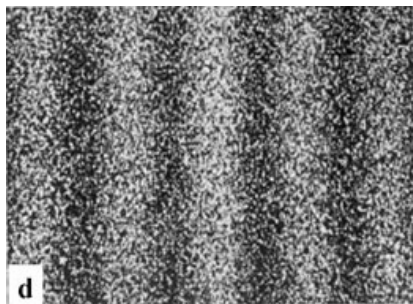
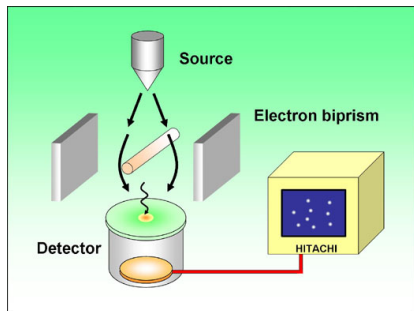
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This information is contained in the wave function.

Interpretation of the wave function

- The wave function is interpreted as a probability amplitude.
- For a system of N electrons the quantity

$$\psi^\dagger(1, 2, \dots, N) \psi(1, 2, \dots, N) d1 d2 \dots dN$$

represents the probability of finding the electrons in the infinitesimal volumes $d1, d2, \dots dN$ about the space/spin coordinates $1, 2, \dots N$

- This leads to the normalization condition

$$\int \psi^\dagger(1, 2, \dots, N) \psi(1, 2, \dots, N) d1 d2 \dots dN = 1$$

Second quantization

- The electron density can be obtained from the quantity

$$\rho(1) = N \int \Psi^\dagger(1, 2, \dots, N) \Psi(1, 2, \dots, N) d2 \dots dN$$

- The electron density integrates to the number of electrons

$$\int \rho(1) d1 = N.$$

- We now introduce an **operator**

$$\hat{N} = \int \hat{\psi}^\dagger(1) \hat{\psi}(1) d1$$

- ... in terms of operators $\hat{\psi}^\dagger(1)$ and $\hat{\psi}(1)$, creating and annihilating electron density amplitude at position 1, respectively.
- We want the total operator \hat{N} to return the particle number N , when acting on an object representing an N -electron system.

Field operators

- In order to represent electrons (fermions) field operators must obey the following anti-commutation relations

$$\left[\hat{\psi}^\dagger(1), \hat{\psi}^\dagger(2) \right]_+ = \hat{\psi}^\dagger(1)\hat{\psi}^\dagger(2) + \hat{\psi}^\dagger(2)\hat{\psi}^\dagger(1) = 0$$

$$\left[\hat{\psi}(1), \hat{\psi}(2) \right]_+ = \hat{\psi}(1)\hat{\psi}(2) + \hat{\psi}(2)\hat{\psi}(1) = 0$$

$$\left[\hat{\psi}(1), \hat{\psi}^\dagger(2) \right]_+ = \hat{\psi}(1)\hat{\psi}^\dagger(2) + \hat{\psi}^\dagger(2)\hat{\psi}(1) = \delta(1 - 2)$$

- Bosons obey corresponding commutator relations.

Expansion of field operators

- Suppose that we have some orthonormal orbital basis $\{\varphi_p(1)\}_{p=1}^M$

$$\int \varphi_p^\dagger(1)\varphi_q(1)d1 = \langle \varphi_p | \varphi_q \rangle = S_{pq} = \delta_{pq}$$

- We now expand the field operators in this basis

$$\hat{\psi}(1) = \sum_q \varphi_q(1)\hat{a}_q; \quad \hat{\psi}^\dagger(1) = \sum_q \varphi_q^\dagger(1)\hat{a}_q^\dagger$$

- We find the expansion coefficients \hat{a}_p and \hat{a}_p^\dagger by

$$\hat{a}_p = \int \varphi_p^\dagger(1)\hat{\psi}(1)d1; \quad \hat{a}_p^\dagger = \int \hat{\psi}^\dagger(1)\varphi_p(1)d1$$

Expansion of field operators

- This is perhaps easier seen using bracket notation, for instance

$$\hat{\psi}(1) = \sum_q \varphi_q(1) \hat{a}_p \rightarrow |\hat{\psi}\rangle = \sum_q |\varphi_q\rangle \hat{a}_p$$

- ... such that

$$\langle \varphi_p | \hat{\psi} \rangle = \sum_q \langle \varphi_p | \varphi_q \rangle \hat{a}_q = \sum_q \delta_{pq} \hat{a}_q = \hat{a}_p$$

- The expansion coefficients \hat{a}_p and \hat{a}_p^\dagger are operators as well:
 - ▶ \hat{a}_p is denoted an annihilation operator
 - ▶ \hat{a}_p^\dagger is denoted a creation operator and is the conjugate of \hat{a}_p
 - ▶ which also means that $(\hat{a}_p^\dagger)^\dagger = \hat{a}_p$

Annihilation and creation operators

- The algebra of the annihilation and creation operators follows from the algebra of the field operators. We have

$$\left[\hat{\psi}(1), \hat{\psi}^\dagger(2) \right]_+ = \delta(1 - 2)$$

- .. from which we deduce

$$\left[\hat{a}_p, \hat{a}_q^\dagger \right]_+ = \left[\int \varphi_p^\dagger(1) \hat{\psi}(1) d1, \int \hat{\psi}^\dagger(2) \varphi_q(2) d2 \right]_+$$

- Remembering that the integral signs are like summation signs we obtain

$$\begin{aligned} \left[\hat{a}_p, \hat{a}_q^\dagger \right]_+ &= \int \int \varphi_p^\dagger(1) \varphi_q(2) \left[\hat{\psi}(1), \hat{\psi}^\dagger(2) \right]_+ d1 d2 \\ &= \int \int \varphi_p^\dagger(1) \varphi_q(2) \delta(1 - 2) d1 d2 \\ &= \int \varphi_p^\dagger(1) \varphi_q(1) d1 = \delta_{pq} \end{aligned}$$

Algebra of annihilation and creation operators

- We just found that (using an orthonormal basis)

$$\left[\hat{\psi}(1), \hat{\psi}^\dagger(2)\right]_+ = \delta(1-2) \Rightarrow \left[\hat{a}_p, \hat{a}_q^\dagger\right]_+ = \delta_{pq}$$

- In a similar manner we find that

$$\left[\hat{\psi}^\dagger(1), \hat{\psi}^\dagger(2)\right]_+ = 0 \Rightarrow \left[\hat{a}_p^\dagger, \hat{a}_q^\dagger\right]_+ = 0$$

$$\left[\hat{\psi}(1), \hat{\psi}(2)\right]_+ = 0 \Rightarrow \left[\hat{a}_p, \hat{a}_q\right]_+ = 0$$

Occupation-number vectors

- Let us consider a simple example:

We have 4 orbitals $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ ($M=4$).

- With two electrons ($N=2$) we can build $\binom{4}{2} = 6$ determinants.

One example is

$$\Phi(1, 2) = \frac{1}{\sqrt{2!}} \begin{vmatrix} \varphi_1(1) & \varphi_3(1) \\ \varphi_1(2) & \varphi_3(2) \end{vmatrix}$$

- or, in short-hand notation

$$\Phi(1, 2) = |\varphi_1 \varphi_3|$$

- We can map this into an **occupation-number vector**

$$\Phi_k(1, 2) = |\varphi_1 \varphi_3| \rightarrow |\mathbf{k}\rangle = |k_1, k_2, k_3, k_4\rangle = |1, 0, 1, 0\rangle$$

- ... where *occupation numbers* k_p are either 0 or 1,
since electrons are fermions.

Occupation-number vectors

Vacuum state and annihilation

- Another example is

$$\Phi_m(1, 2, 3) = \frac{1}{\sqrt{3!}} \begin{vmatrix} \varphi_1(1) & \varphi_2(1) & \varphi_4(1) \\ \varphi_1(2) & \varphi_2(2) & \varphi_4(2) \\ \varphi_1(3) & \varphi_2(3) & \varphi_4(3) \end{vmatrix} = |\varphi_1 \varphi_2 \varphi_4| \rightarrow |\mathbf{m}\rangle = |1, 1, 0, 1\rangle$$

- A special occupation-number vector is the **vacuum state**

$$|vac\rangle = |0, 0, 0, 0\rangle$$

- Annihilation operators reduce occupation numbers by one and therefore all give zero when acting on $|vac\rangle$

$$\hat{a}_p |vac\rangle = \mathbf{0}; \quad \forall \hat{a}_p$$

- This even serves as a definition of the vacuum state.

Occupation-number vectors

Creation operators

- Creation operators increase occupation numbers by one

$$\hat{a}_1^\dagger |vac\rangle = |1, 0, 0, 0\rangle$$

$$\hat{a}_2^\dagger |vac\rangle = |0, 1, 0, 0\rangle$$

$$\hat{a}_3^\dagger |vac\rangle = |0, 0, 1, 0\rangle$$

$$\hat{a}_4^\dagger |vac\rangle = |0, 0, 0, 1\rangle$$

- ..but, since they refer to fermions,
occupation numbers can not be greater than one

$$\hat{a}_1^\dagger |1, 0, 0, 0\rangle = \hat{a}_1^\dagger \hat{a}_1^\dagger |vac\rangle = \mathbf{0}$$

- This follows directly from the special case

$$[\hat{a}_p^\dagger, \hat{a}_p^\dagger]_+ = \hat{a}_p^\dagger \hat{a}_p^\dagger + \hat{a}_p^\dagger \hat{a}_p^\dagger = 2\hat{a}_p^\dagger \hat{a}_p^\dagger = 0$$

Occupation-number vectors

More creation

- We can build ONVs corresponding to $N = 2$

$$\hat{a}_1^\dagger |0, 1, 0, 0\rangle = |1, 1, 0, 0\rangle = \hat{a}_1^\dagger \hat{a}_2^\dagger |vac\rangle$$

- Using the algebra of creation operators we find

$$\hat{a}_2^\dagger |1, 0, 0, 0\rangle = \hat{a}_2^\dagger \hat{a}_1^\dagger |vac\rangle = -\hat{a}_1^\dagger \hat{a}_2^\dagger |vac\rangle = -|1, 1, 0, 0\rangle$$

- ..showing how the fermion antisymmetry is built into the operators.

Fock space

..or occupation-number space

- Occupation number vectors (ONVs) have the general form

$$|\mathbf{k}\rangle = |k_1, k_2, \dots, k_M\rangle; \quad k_p = 0 \text{ or } 1$$

and reside in a Fock space of dimension 2^M .

- Any ONV can be generated from the vacuum state

$$|\mathbf{k}\rangle = \prod_{p=1}^M (\hat{a}_p^\dagger)^{k_p} |vac\rangle$$

- An inner-product in Fock space is defined by

$$\langle \mathbf{k} | \mathbf{m} \rangle = \delta_{\mathbf{k}, \mathbf{m}} = \prod_{p=1}^M \delta_{k_p, m_p}$$

and is one if all occupation numbers are identical, zero otherwise.

- A special case

$$\langle vac | vac \rangle = 1$$

Fock space

..or occupation-number space

- The dual vector $\langle \mathbf{k} |$ is obtained by conjugation, e.g. starting from

$$|\mathbf{k}\rangle = |1, 0, 1, 1\rangle = \hat{a}_1^\dagger \hat{a}_3^\dagger \hat{a}_4^\dagger |vac\rangle$$

- ... we have

$$\langle \mathbf{k} | = \langle 1, 0, 1, 1 | = \langle vac | \left(\hat{a}_1^\dagger \hat{a}_3^\dagger \hat{a}_4^\dagger \right)^\dagger = \langle vac | \hat{a}_4 \hat{a}_3 \hat{a}_1$$

- ▶ notice the change of operator order under conjugation
- ▶ annihilation operators become creators when operating to the left
- ▶ the dual vacuum state can therefore be defined by

$$\langle vac | \hat{a}_p^\dagger = \mathbf{0}; \quad \forall \hat{a}_p^\dagger$$

The number operator

- Notice that in Fock space there is no restriction on particle number N , except $N \leq M$.
- We now return to the operator

$$\hat{N} = \int \hat{\psi}^\dagger(1) \hat{\psi}(1) d1$$

- Expanding the field operators in some orthonormal orbital basis $\{\varphi_p(1)\}_{p=1}^M$ we obtain

$$\hat{N} = \int \hat{\psi}^\dagger(1) \hat{\psi}(1) d1 = \sum_{pq} \left\{ \int \varphi_p^\dagger(1) \varphi_q(1) d1 \right\} \hat{a}_p^\dagger \hat{a}_q = \sum_{pq} \delta_{pq} \hat{a}_p^\dagger \hat{a}_q = \sum_p \hat{a}_p^\dagger \hat{a}_p$$

- ... which defines the **number operator** \hat{N} . For instance

$$\hat{N} |1, 0, 1, 1\rangle = 3 |1, 0, 1, 1\rangle$$

- The occupation number vectors are eigenvectors of the number operator.

Counting electrons

- The field operators do not relate to specific electrons; rather, they sample contributions to the **electron quantum field** in space



- Quantum field theory explains why electrons are the same everywhere; they all belong to the same field !

The number operator

and commutator algebra

- Suppose that

$$\hat{N} |\mathbf{k}\rangle = N |\mathbf{k}\rangle$$

- What about $\hat{N} \hat{a}_p^\dagger |\mathbf{k}\rangle$?
- We can rewrite this as

$$\hat{N} \hat{a}_p^\dagger |\mathbf{k}\rangle = \left(\hat{a}_p^\dagger \hat{N} + [\hat{N}, \hat{a}_p^\dagger] \right) |\mathbf{k}\rangle$$

- We need to solve the commutator

$$[\hat{N}, \hat{a}_p^\dagger] = \sum_q [\hat{a}_q^\dagger \hat{a}_q, \hat{a}_p^\dagger]$$

- We may use a commutator rule such as

$$[\hat{A}\hat{B}, \hat{C}] = \hat{A}\hat{B}\hat{C} - \hat{C}\hat{A}\hat{B} = \hat{A}\hat{B}\hat{C} - \hat{A}\hat{C}\hat{B} + \hat{A}\hat{C}\hat{B} - \hat{C}\hat{A}\hat{B} = \hat{A} [\hat{B}, \hat{C}] + [\hat{A}, \hat{C}] \hat{B}$$

The number operator

and commutator algebra

- The algebra of creation and annihilation operators is, however, expressed in terms of anti-commutators

$$[\hat{a}_p^\dagger, \hat{a}_q^\dagger]_+ = 0; \quad [\hat{a}_p, \hat{a}_q]_+ = 0; \quad [\hat{a}_p, \hat{a}_q^\dagger]_+ = \delta_{pq}$$

- We therefore rather form the rule

$$[\hat{A}\hat{B}, \hat{C}] = \hat{A}\hat{B}\hat{C} - \hat{C}\hat{A}\hat{B} = \hat{A}\hat{B}\hat{C} + \hat{A}\hat{C}\hat{B} - \hat{A}\hat{C}\hat{B} - \hat{C}\hat{A}\hat{B} = \hat{A} [\hat{B}, \hat{C}]_+ - [\hat{A}, \hat{C}]_+ \hat{B}$$

- ... which gives

$$[\hat{N}, \hat{a}_p^\dagger] = \sum_q [\hat{a}_q^\dagger \hat{a}_q, \hat{a}_p^\dagger] = \sum_q \left(\hat{a}_q^\dagger \underbrace{[\hat{a}_q, \hat{a}_p^\dagger]_+}_{=\delta_{pq}} - \underbrace{[\hat{a}_q^\dagger, \hat{a}_p^\dagger]_+}_{=0} \hat{a}_q \right) = \hat{a}_p^\dagger$$

- Our final result is thereby

$$\hat{N}\hat{a}_p^\dagger |\mathbf{k}\rangle = \left(\hat{a}_p^\dagger \hat{N} + [\hat{N}, \hat{a}_p^\dagger] \right) |\mathbf{k}\rangle = \hat{a}_p^\dagger (\hat{N} + 1) |\mathbf{k}\rangle = (N + 1) \hat{a}_p^\dagger |\mathbf{k}\rangle$$

The number operator

and commutator algebra

- What about $\hat{N}a_p |\mathbf{k}\rangle$?
- We can write this as

$$\hat{N}a_p |\mathbf{k}\rangle = \left(\hat{a}_p \hat{N} + [\hat{N}, \hat{a}_p] \right) |\mathbf{k}\rangle$$

- We can proceed as before, but instead we note that

- ▶ $[\hat{A}, \hat{B}]^\dagger = (\hat{A}\hat{B} - \hat{B}\hat{A})^\dagger = \hat{B}^\dagger \hat{A}^\dagger - \hat{A}^\dagger \hat{B}^\dagger = -[\hat{A}^\dagger, \hat{B}^\dagger]$
- ▶ $\hat{N}^\dagger = \sum_p (\hat{a}_p^\dagger \hat{a}_p)^\dagger = \hat{N}$ (hermitian operator)

- ... so that

$$\hat{N}a_p |\mathbf{k}\rangle = \left(\hat{a}_p \hat{N} - [\hat{N}, \hat{a}_p^\dagger]^\dagger \right) |\mathbf{k}\rangle = \hat{a}_p (\hat{N} - 1) |\mathbf{k}\rangle = (N - 1) \hat{a}_p |\mathbf{k}\rangle$$

Counting electron pairs

- Let us have a look at the second quantization operator

$$\begin{aligned}\hat{N}_{pair} &= \frac{1}{2} \int \hat{\psi}^\dagger(1) \hat{\psi}^\dagger(2) \hat{\psi}(2) \hat{\psi}(1) d1 d2 \\&= \frac{1}{2} \sum_{pqrs} \left\{ \int \varphi_p^\dagger(1) \varphi_q^\dagger(2) \varphi_r(2) \varphi_s(1) d1 d2 \right\} \hat{a}_p^\dagger \hat{a}_q^\dagger \hat{a}_r \hat{a}_s \\&= \frac{1}{2} \sum_{pqrs} \left\{ \int \varphi_p^\dagger(1) \varphi_s(1) d1 \int \varphi_q^\dagger(2) \varphi_r(2) d2 \right\} \hat{a}_p^\dagger \hat{a}_q^\dagger \hat{a}_r \hat{a}_s \\&= \frac{1}{2} \sum_{pqrs} \{ \delta_{ps} \delta_{qr} \} \hat{a}_p^\dagger \hat{a}_q^\dagger \hat{a}_r \hat{a}_s = \frac{1}{2} \sum_{pq} \hat{a}_p^\dagger \hat{a}_q^\dagger \hat{a}_q \hat{a}_p\end{aligned}$$

- Operator algebra

$$\hat{a}_p^\dagger \hat{a}_q^\dagger \hat{a}_q \hat{a}_p = -\hat{a}_p^\dagger \hat{a}_q^\dagger \hat{a}_p \hat{a}_q = -\hat{a}_p^\dagger (\delta_{pq} - \hat{a}_p \hat{a}_q^\dagger) \hat{a}_q = \hat{a}_p^\dagger \hat{a}_p \hat{a}_q^\dagger \hat{a}_q - \delta_{pq} \hat{a}_p^\dagger \hat{a}_q$$

- ...shows that it counts electron pairs

$$\hat{N}_{pair} = \frac{1}{2} \hat{N} (\hat{N} - 1)$$

What does the second-quantized electronic Hamiltonian look like ?

- The first-quantized form

$$\hat{H} = \sum_{i=1}^N \hat{h}(i) + \frac{1}{2} \sum_{i \neq j}^N \hat{g}(i, j) + V_{NN}$$

- The second-quantized form

$$\hat{H} = \int \hat{\psi}^\dagger(1) \hat{h}(1) \hat{\psi}(1) d1 + \frac{1}{2} \int \hat{\psi}^\dagger(1) \hat{\psi}^\dagger(2) \hat{g}(1, 2) \hat{\psi}(2) \hat{\psi}(1) d1 d2 + V_{NN}$$

(notice the order of electron coordinates in the two-electron operator)

- This gives a formula for finding the second-quantized form of any one- and two-electron operator.

What does the second-quantized electronic Hamiltonian look like ?

- The one-electron part

$$\begin{aligned}\hat{H}_1 &= \int \hat{\psi}^\dagger(1) \hat{h}(1) \hat{\psi}(1) d1 \\ &= \sum_p \sum_q \left\{ \int \varphi_p^\dagger(1) h(1) \varphi_q(1) d1 \right\} \hat{a}_p^\dagger \hat{a}_q = \sum_{p,q} h_{pq} \hat{a}_p^\dagger \hat{a}_q\end{aligned}$$

- Proceeding in the same way with the two-electron part we obtain

$$\begin{aligned}\hat{H}_2 &= \frac{1}{2} \int \hat{\psi}^\dagger(1) \hat{\psi}^\dagger(2) \hat{g}(1,2) \hat{\psi}(2) \hat{\psi}(1) d1 d2 \\ &= \frac{1}{2} \sum_{pq,rs} V_{pq,rs} \hat{a}_p^\dagger \hat{a}_q^\dagger \hat{a}_s \hat{a}_r\end{aligned}$$

- Dirac notation: $V_{pq,rs} = \langle \varphi_p \varphi_q | \varphi_r \varphi_s \rangle = \int \varphi_p^\dagger(1) \varphi_q^\dagger(2) \hat{g}(1,2) \varphi_r(1) \varphi_s(2) d1 d2$
- Mulliken notation: $g_{pq,rs} = (\varphi_p \varphi_q | \varphi_r \varphi_s) = \int \varphi_p^\dagger(1) \varphi_r^\dagger(2) \hat{g}(1,2) \varphi_q(1) \varphi_s(2) d1 d2$

What does the second-quantized electronic Hamiltonian look like ?

- The final form is

$$\hat{H} = \sum_{pq} h_{pq} \hat{a}_p^\dagger \hat{a}_q + \frac{1}{2} \sum_{pq,rs} V_{pq,rs} \hat{a}_p^\dagger \hat{a}_q^\dagger \hat{a}_s \hat{a}_r + V_{NN}$$

- This is a very convenient operator form:
 - ▶ The fermion antisymmetry is built into the operator
 - ▶ The operator is expressed in terms of one- and two-electron integrals, which are the basic ingredients of quantum chemistry codes
 - ▶ The form is universal; there is no reference to the number of electrons !
- ..but note that it is a *projected* operator:
- it “lives” in the space defined by the orbital set $\{\varphi_p\}_{p=1}^M$.

The electronic energy in second quantization

- The electronic Hamiltonian is

$$\hat{H} = \sum_{pq} h_{pq} \hat{a}_p^\dagger \hat{a}_q + \frac{1}{2} \sum_{pq,rs} V_{pq,rs} \hat{a}_p^\dagger \hat{a}_q^\dagger \hat{a}_s \hat{a}_r + V_{NN}$$

- The wave function is now expressed as a linear combination of occupation number vectors (limited to occupation N)

$$|0\rangle = \sum_k C_k |\mathbf{k}\rangle$$

- The energy is given as the expectation value

$$E = \langle 0 | \hat{H} | 0 \rangle = \sum_{pq} h_{pq} D_{pq} + \frac{1}{2} \sum_{pq,rs} V_{pq,rs} d_{pq,rs} + V_{NN}$$

- ▶ Matrix elements h_{pq} and $V_{pq,rs}$ depends on the operator, but are independent of wave function
- ▶ **Orbital density matrices** D_{pq} and $d_{pq,rs}$ are independent of operator, but depend on wave function

Orbital density matrices

- **One-electron orbital density matrix**

$$D_{pq} = \langle 0 | \hat{a}_p^\dagger \hat{a}_q | 0 \rangle$$

- ▶ dimension: M^2
- ▶ contains all information needed to calculate expectation values of one-electron operators
- ▶ diagonalization gives **natural orbitals**

- **Two-electron orbital density matrix**

$$d_{pq,rs} = \langle 0 | \hat{a}_p^\dagger \hat{a}_q^\dagger \hat{a}_s \hat{a}_r | 0 \rangle$$

- ▶ dimension: M^4
- ▶ contains all information needed to calculate expectation values of two-electron operators

- **Data reduction:** $\mathbf{C} : \begin{pmatrix} M \\ N \end{pmatrix} \rightarrow \mathbf{D}/\mathbf{d} : M^2/M^4 \quad !!!$

What about spin ?

- By convention, the z-axis is chosen as spin-axis such that the electron spin functions $|s, m_s\rangle$ are eigenfunctions of \hat{s}^2 and \hat{s}_z

$$\hat{s}^2 |s, m_s\rangle = s(s+1) |s, m_s\rangle; \quad \hat{s}_z |s, m_s\rangle = m_s |s, m_s\rangle$$

- It is also convenient to introduce **step operators**

$$\hat{s}_+ = \hat{s}_x + i\hat{s}_y \text{ and } \hat{s}_- = \hat{s}_x - i\hat{s}_y$$

$$\hat{s}_\pm |s, m_s\rangle = \sqrt{s(s+1) - m_s(m_s \pm 1)} |s, m_s \pm 1\rangle$$

- Electrons are spin- $\frac{1}{2}$ particles with spin functions denoted $|\alpha\rangle = |\frac{1}{2}, \frac{1}{2}\rangle$ and $|\beta\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle$. The action of the spin operators is summarized by

	\hat{s}^2	\hat{s}_z	\hat{s}_+	\hat{s}_-
$ \alpha\rangle$	$\frac{3}{4} \alpha\rangle$	$\frac{1}{2} \alpha\rangle$	0	$ \beta\rangle$
$ \beta\rangle$	$\frac{3}{4} \beta\rangle$	$-\frac{1}{2} \beta\rangle$	$ \alpha\rangle$	0

Spin in second quantization

- We may separate out spin from spatial parts of the creation- and annihilation operators, giving

$$\left[\hat{a}_{p\sigma}^\dagger, \hat{a}_{q\sigma'}^\dagger\right]_+ = 0; \quad \left[\hat{a}_{p\sigma}, \hat{a}_{q\sigma'}\right]_+ = 0; \quad \left[\hat{a}_{p\sigma}, \hat{a}_{q\sigma'}^\dagger\right]_+ = \delta_{pq}\delta_{\sigma\sigma'}; \quad \sigma, \sigma' = \alpha \text{ or } \beta$$

- We may also separate out spin in the electronic Hamiltonian. For the (non-relativistic) one-electron part we obtain

$$\begin{aligned}\hat{H}_1 &= \sum_{pq} \sum_{\sigma, \sigma'} \langle \varphi_p \sigma | \hat{h} | \varphi_q \sigma' \rangle \hat{a}_{p\sigma}^\dagger \hat{a}_{q\sigma'} \\ &= \sum_{pq} \sum_{\sigma, \sigma'} \langle \varphi_p | \hat{h} | \varphi_q \rangle \langle \sigma | \sigma' \rangle \hat{a}_{p\sigma}^\dagger \hat{a}_{q\sigma'} \\ &= \sum_{pq} \sum_{\sigma, \sigma'} \langle \varphi_p | \hat{h} | \varphi_q \rangle \delta_{\sigma\sigma'} \hat{a}_{p\sigma}^\dagger \hat{a}_{q\sigma'} \\ &= \sum_{pq} \langle \varphi_p | \hat{h} | \varphi_q \rangle E_{pq}; \quad E_{pq} = \sum_{\sigma} \hat{a}_{p\sigma}^\dagger \hat{a}_{q\sigma}\end{aligned}$$

Spin in second quantization

- For the (non-relativistic) two-electron part we obtain

$$\begin{aligned}
 \hat{H}_2 &= \frac{1}{2} \sum_{pqrs} \sum_{\sigma\tau\sigma'\tau'} \langle \varphi_p \sigma \varphi_q \tau | \hat{g} | \varphi_r \sigma' \varphi_s \tau' \rangle a_{p\sigma}^\dagger a_{q\tau}^\dagger a_{s\tau'} a_{r\sigma'} \\
 &= \frac{1}{2} \sum_{pqrs} \sum_{\sigma\tau\sigma'\tau'} \langle \varphi_p \varphi_q | \hat{g} | \varphi_r \varphi_s \rangle \langle \sigma | \sigma' \rangle \langle \tau | \tau' \rangle a_{p\sigma}^\dagger a_{q\tau}^\dagger a_{s\tau'} a_{r\sigma'} \\
 &= \frac{1}{2} \sum_{pqrs} \sum_{\sigma\tau\sigma'\tau'} \langle \varphi_p \varphi_q | \hat{g} | \varphi_r \varphi_s \rangle \delta_{\sigma\sigma'} \delta_{\tau\tau'} a_{p\sigma}^\dagger a_{q\tau}^\dagger a_{s\tau'} a_{r\sigma'} \\
 &= \frac{1}{2} \sum_{pqrs} \sum_{\sigma\tau\sigma'\tau'} \langle \varphi_p \varphi_q | \hat{g} | \varphi_r \varphi_s \rangle e_{pq,rs}; \quad e_{pq,rs} = \sum_{\sigma\tau} a_{p\sigma}^\dagger a_{q\tau}^\dagger a_{s\tau} a_{r\sigma}
 \end{aligned}$$

- Operator algebra

$$a_{p\sigma}^\dagger a_{q\tau}^\dagger a_{s\tau'} a_{r\sigma'} = -a_{p\sigma}^\dagger a_{q\tau}^\dagger a_{r\sigma} a_{s\tau} = a_{p\sigma}^\dagger a_{r\sigma} a_{q\tau}^\dagger a_{s\tau} - \delta_{qr} \delta_{\sigma\tau} a_{p\sigma}^\dagger a_{s\tau}$$

- .. shows that

$$e_{pq,rs} = E_{pr} E_{qs} - \delta_{qr} E_{ps}$$

Summary

- Second quantization starts from **field operators** $\psi^\dagger(1)$, $\psi(1)$ sampling the electron field in space. It provides a very convenient language for the formulation and implementation of quantum chemical methods.
- **Occupation number vectors** (ONVs) are defined with respect to some (orthonormal) orbital set $\{\varphi_p(\mathbf{r})\}_{p=1}^M$
- Their occupation numbers are manipulated using **creation- and annihilation operators**, \hat{a}_p^\dagger and \hat{a}_p , which are conjugates of each other.
- The algebra of these operators is summarized by **anti-commutator relations**

$$\left[\hat{a}_p^\dagger, \hat{a}_q^\dagger\right]_+ = 0; \quad \left[\hat{a}_p, \hat{a}_q\right]_+ = 0; \quad \left[\hat{a}_p, \hat{a}_q^\dagger\right]_+ = \delta_{pq}$$

and reflects the fermionic nature of electrons.

Summary

- **One-electron operators** are translated into their second quantized form by

$$\sum_{i=1}^N \hat{f}(i) \rightarrow \int \psi^\dagger(1) f(1) \psi(1) d1 = \sum_{pq} \langle \varphi_p | \hat{f} | \varphi_q \rangle \hat{a}_p^\dagger \hat{a}_q$$

- **Two-electron operators** are translated into their second quantized form by

$$\begin{aligned} \frac{1}{2} \sum_{i,j=1}^N g(i,j) &\rightarrow \frac{1}{2} \int \hat{\psi}^\dagger(1) \hat{\psi}^\dagger(2) \hat{g}(1,2) \hat{\psi}(2) \hat{\psi}(1) d1 d2 \\ &= \frac{1}{2} \sum_{pq,rs} \langle \varphi_p \varphi_q | \hat{g} | \varphi_r \varphi_s \rangle \hat{a}_p^\dagger \hat{a}_q^\dagger \hat{a}_s \hat{a}_r \end{aligned}$$

- Nice features is that:
 - ▶ **Antisymmetry** is automatically built into the operators
 - ▶ They are **expressed in terms of integrals**, building blocks of quantum chemistry codes

- The **second-quantized electronic Hamiltonian** is expressed as

$$\hat{H} = \sum_{pq} h_{pq} \hat{a}_p^\dagger \hat{a}_q + \frac{1}{2} \sum_{pq,rs} V_{pq,rs} \hat{a}_p^\dagger \hat{a}_q^\dagger \hat{a}_s \hat{a}_r + V_{nn}$$

- The **electronic energy** becomes

$$E = \langle 0 | \hat{H} | 0 \rangle = \sum_{pq} h_{pq} D_{pq} + \frac{1}{2} \sum_{pq,rs} V_{pq,rs} d_{pq,rs} + V_{nn}$$

- which nicely separates
 - ▶ **operator content**,
in terms of integrals h_{pq} and $V_{pq,rs}$, and
 - ▶ **wave function content**,
in terms of orbital density matrices D_{pq} and $d_{pq,rs}$
(data compression)