Lec 19.

Collocation based RK.

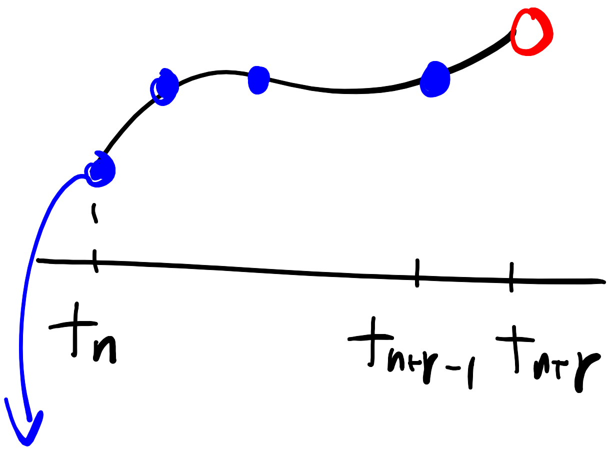
"co-locate"

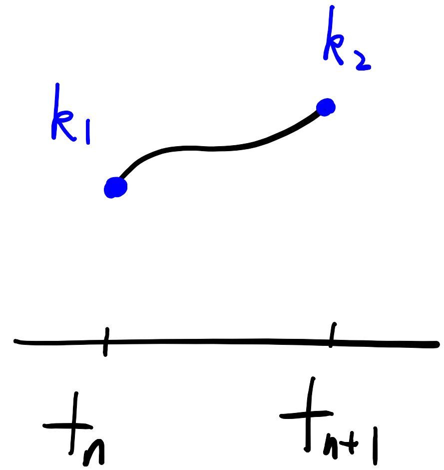
$$\begin{cases} \vec{k} = f \circ (u_n \vec{e} + hA\vec{k},\ t_n \vec{e} + \vec{c}h) \\ u_{n+1} = u_n + h\,\vec{b}^T \vec{k} \end{cases}$$

Ex. trapezoidal rule.

$$\begin{cases} k_1 = f(u_n) \\ k_2 = f\left(u_n + \frac{h}{2}(k_1 + k_2)\right) \\ u_{n+1} = u_n + \frac{h}{2}(k_1 + k_2) \end{cases}$$

Recall LMM





already know

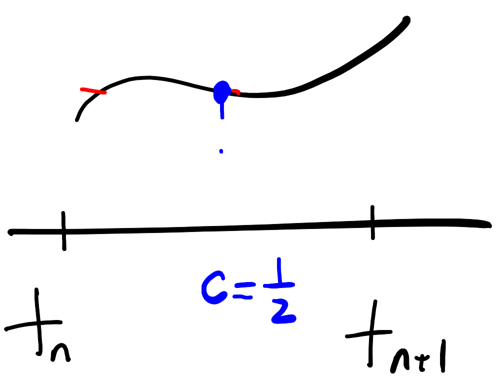
Ex. $\begin{cases} k = f\left(u_n + \frac{h}{2} k\right) \\ u_{n+1} = u_n + h\,k \end{cases}$

unknown. $\tilde{u}$

$$u(t_n + ch)$$

$$= u(t_n) + \int_{t_n}^{t_n+ch} f(u(s))\,ds$$

$$\approx u(t_n) + \frac{h}{2} \underbrace{f\big(u(t_n+ch)\big)}_{k}$$

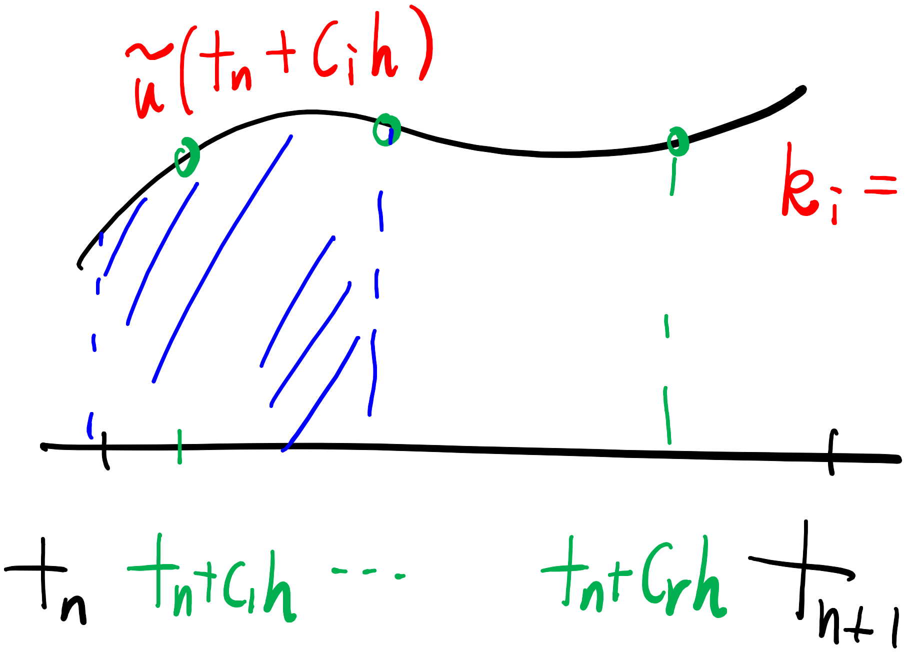


alternatively

$$\tilde{u} = u_n + \frac{h}{2} \underbrace{f(\tilde{u})}_{k}$$

Collocation:

"co-locate" all functions on the same set of points.



$$k_i = f(\tilde{u}(t_n+c_ih))$$

$$0 \le c_1 < c_2 < \cdots < c_r \le 1.$$

Lagrange interpolation for f

$$\tilde{u}(t_n+c_ih) = \underset{\substack{\parallel \\ u_n}}{\tilde{u}(t_n)} + \int_{t_n}^{t_n+c_ih} f(\tilde{u}(s))\,ds$$

Lag-

$\approx$

use ALL

r points

$$u_n + \sum_{j=1}^{r} \int_{t_n}^{t_n+c_i h} P_j(s)\, k_j$$

$$\left(P_j(t_n+c_i h) = \delta_{ij} . \quad P_j \in \mathbb{P}_{r-1} \right)$$

$$= u_n + h \sum_{j=1}^{r} A_{ij} k_j$$

$$\Rightarrow \quad k_i = f\left(u_n + h \sum_{j=1}^{r} A_{ij} k_j\right)$$

$$u_{n+1} = u_n + \int_{t_n}^{t_{n+1}} f\left(\tilde{u}(s)\right) ds$$

$$\rightarrow u_n + h \sum_{j=1}^{r} b_j k_j$$

$$\begin{cases} b_j = \frac{1}{h} \int_{t_n}^{t_{n+1}} p_j(s)\, ds \\ A_{ij} = \frac{1}{h} \int_{t_n}^{t_n + c_i h} p_j(s)\, ds \end{cases} \Rightarrow$$

RK COMPLETELY determined by $\vec{c}$

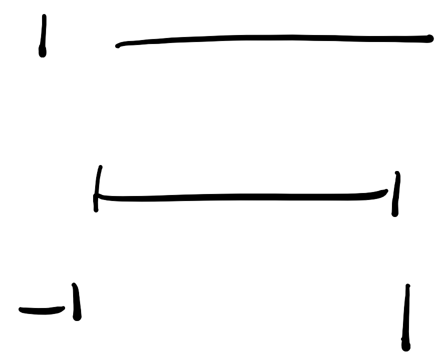
(exer) Kutta's condition is always satisfied.

$$\sum_j A_{ij} = c_i$$

Gauss quadrature.

r pts. $2r$ order.

Gauss-Legendre.

Orthogonal poly.  $\omega(x) \equiv 1$.

$$\int_{-1}^{1} f(x)\,dx$$

$$\int_{-1}^{1} P_l(x) P_k(x)\,dx = 0 \text{ . if } l \neq k.$$

$$P_l(1) = 1.$$

Gram - Schmit $\longrightarrow P_\ell$

$$\begin{cases} P_0(x) = 1 \\ P_1(x) = x \\ P_2(x) = \frac{3x^2}{2} - \frac{1}{2} \\ \dots \end{cases}$$

Quadrature pts. roots of $P_r(x)$

$\{x_i\}_{i=1}^{r}$ $\quad P_r(x_i) = 0.$

Weights Lagrange interpolation on $\{x_i\}_{i=1}^{r}$

$$w_i = \int_{-1}^{1} p_i(x)\,dx.$$