Lec 1.

① Basic ODE theory

② Existence and uniqueness

③ Lipschitz continuity

④ Linear multistep method

$\mathbb{R}^n$: n-tuple real numbers

$\mathbb{C}^n$: " " complex "

Length of a Vector (vector 2-norm)

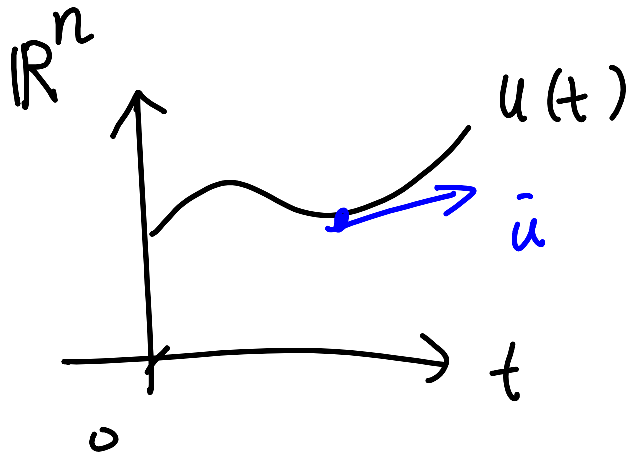
$$\|x\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}$$

$$x(t) : \mathbb{R} \longrightarrow \mathbb{R}^n$$

derivative $\dot{x}(t) \equiv x'(t) \equiv \frac{d}{dt} x(t)$

Initial value problem (IVP)

$$\begin{cases} u'(t) = f(u(t), t), \quad 0 \le t \le T. \\ u(0) = u_0 \in \mathbb{R}^n \end{cases}$$



Matrix $\quad A \in \mathbb{R}^{m \times n}$

Operator norm (matrix 2-norm)

$$\|A\|_2 := \sup_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{\|Ax\|_2}{\|x\|_2}$$

2-norm $\mathbb{R}^m$

2-norm $\mathbb{R}^n$

Ex. $\|Ax\|_2 \leq \|A\|_2 \|x\|_2$

other norms.

vector 1-norm. $\|x\|_1 = |x_1| + \cdots + |x_n|$

special case.

$$f(u(t), t) = f(u(t))$$ autonomous eq.

"autonomization" $$\xi(t) = \begin{pmatrix} u(t) \\ t \end{pmatrix} \in \mathbb{R}^{n+1}$$

$$\frac{d}{dt}\xi(t) = \begin{pmatrix} \dot{u}(t) \\ \dot{t} \end{pmatrix} = \begin{pmatrix} f(\xi(t)) \\ 1 \end{pmatrix} = F(\xi(t))$$

High order ODES.

Ex. $\ddot{x}(t) = f(x(t), t)$. $x(t) \in \mathbb{R}^n$

Introduce $v(t) = \dot{x}(t)$

$$\begin{cases} \dot{x}(t) = v(t) \\ \dot{v}(t) = f(x(t), t) \end{cases}$$

. sys. ODES. in $\mathbb{R}^{2n}$

In general.

$$x^{(n)}(t) + \alpha_{n-1}(t)\, x^{(n-1)}(t) + \cdots + \alpha_0(t)\, x(t)$$

$$= f(x(t), \dot{x}(t), \cdots, x^{(n-1)}(t), t), \quad x(t) \in \mathbb{R}^m$$

Define

$$\begin{cases} u_1(t) = x(t) \\ u_2(t) = \dot{x}(t) \\ \vdots \\ u_n(t) = x^{(n-1)}(t) \end{cases} \qquad u(t) \in \mathbb{R}^{mn}$$

$$\frac{d}{dt} u(t) = A\, u(t) + \tilde{f}(u(t), t)$$

$$A = \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 \\ 0 & 0 & I_m & & 0 \\ 0 & \cdots & \cdots & \cdots & I_m \\ -\alpha_0(t) I_m & & \cdots & & -\alpha_{n-1}(t) I_m \end{bmatrix}$$

$$\tilde{f} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ f(u(t),t) \end{bmatrix}$$

$$f(u(t),t) : \underset{\text{space}}{\underset{\uparrow}{\mathbb{R}^n}} \times \underset{\text{time.}}{\underset{\uparrow}{\mathbb{R}}} \longrightarrow \mathbb{R}^n$$

f continuous w.r.t. u, t.

<u>Thm</u>. (Cauchy-Peano)

f continuous $\Rightarrow$ sol of IVP exists.

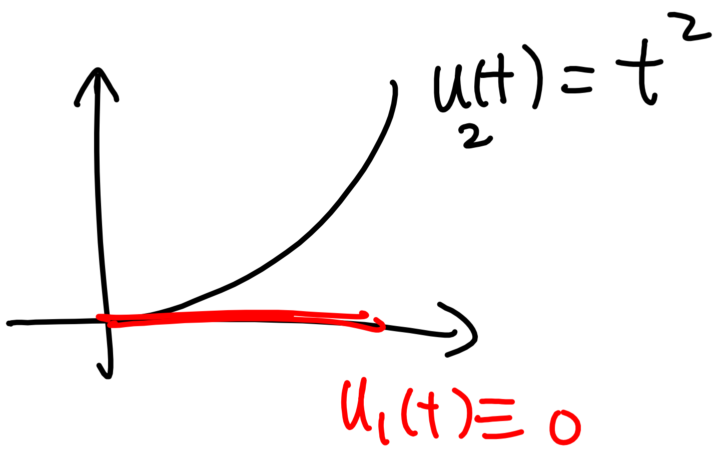
Ref: [Hai] I. 7.

Uniqueness.

Ex. $\begin{cases} \dot{u}(t) = 2\sqrt{u(t)} \\ u(0) = 0. \end{cases}$

sol.1. $u(t) \equiv 0$.

sol.2. $u(t) = t^2$



$f(u,t) = 2\sqrt{u}$ cont. w.r.t. u & t.

$$\frac{\partial f}{\partial u} = u^{-1/2}, \quad u \to 0 \quad \frac{\partial f}{\partial u} \to \infty.$$

Lipschitz continuity

Def $f: \mathbb{R}^n \to \mathbb{R}^m$ is L.C.

if $\exists\, L > 0$, s.t. $\forall\, x, y \in \mathbb{R}^n$

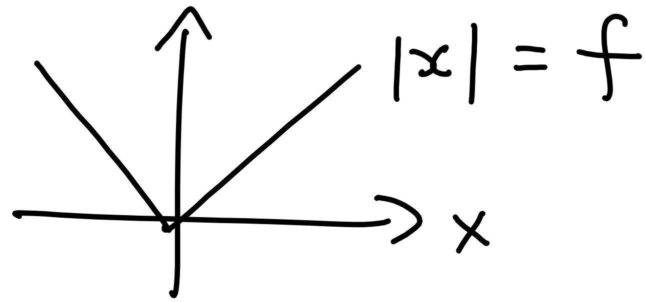
$$\underbrace{\|f(x) - f(y)\|_2}_{\text{in } \mathbb{R}^m} \le L \underbrace{\|x - y\|_2}_{\mathbb{R}^n}.$$

Ex. $f(x) = 2\sqrt{x}$.

$$\frac{|f(y) - f(0)|}{|y - 0|} = \frac{2\sqrt{y}}{y} = \frac{2}{\sqrt{y}} \quad \text{unbounded.}$$

$2\sqrt{x}$ is NOT L.C.

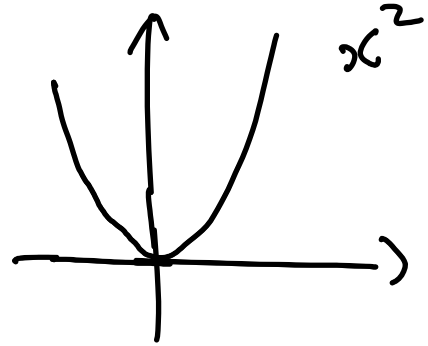
Ex. $f(x) = |x|$



$$\frac{|f(x)-f(y)|}{|x-y|} = \frac{\big||x|-|y|\big|}{|x-y|} \le \frac{|x-y|}{|x-y|} = 1$$

$|x|$ is L.C. w. $L=1$.

Ex. $f(x) = x^2$



$$\frac{|f(x)-f(y)|}{|x-y|} = \frac{|x^2-y^2|}{|x-y|} = |x+y|$$

① $x, y \in [-A, A]$. $L = 2A$.

② on $\mathbb{R}$ NOT L.C.