

Lec 19.

Collocation based RK.

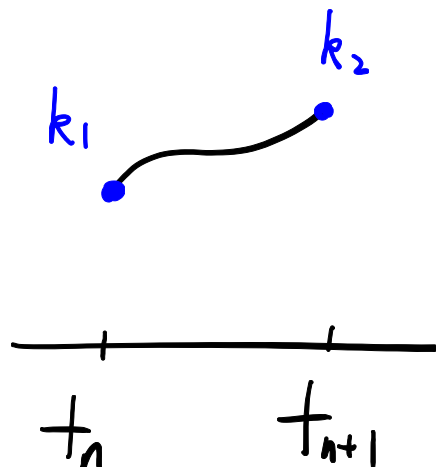
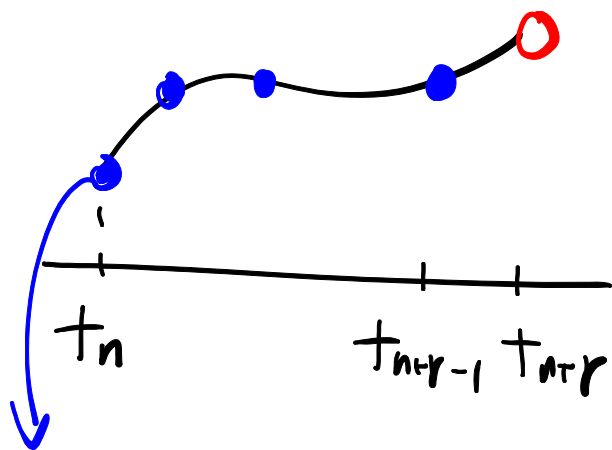
“co-locate”

$$\begin{cases} \vec{k} = f \circ (u_n \vec{e} + h A \vec{k}, t_n \vec{e} + \vec{c} h) \\ u_{n+1} = u_n + h \vec{b}^T \vec{k} \end{cases}$$

Ex. trapezoidal rule.

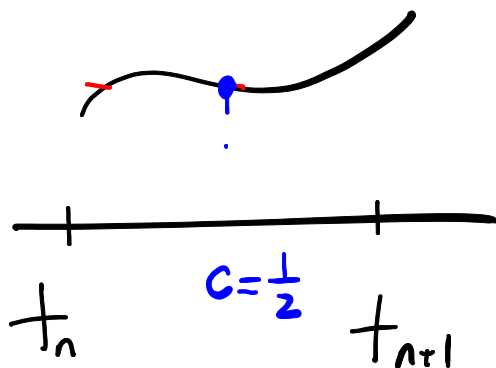
$$\begin{cases} k_1 = f(u_n) \\ k_2 = f(u_n + \frac{h}{2}(k_1 + k_2)) \\ u_{n+1} = u_n + \frac{h}{2}(k_1 + k_2) \end{cases}$$

Recall LMM



already know

$$\text{Ex. } \begin{cases} k = f\left(u_n + \frac{h}{2}k\right) \\ u_{n+1} = u_n + h k \end{cases}$$



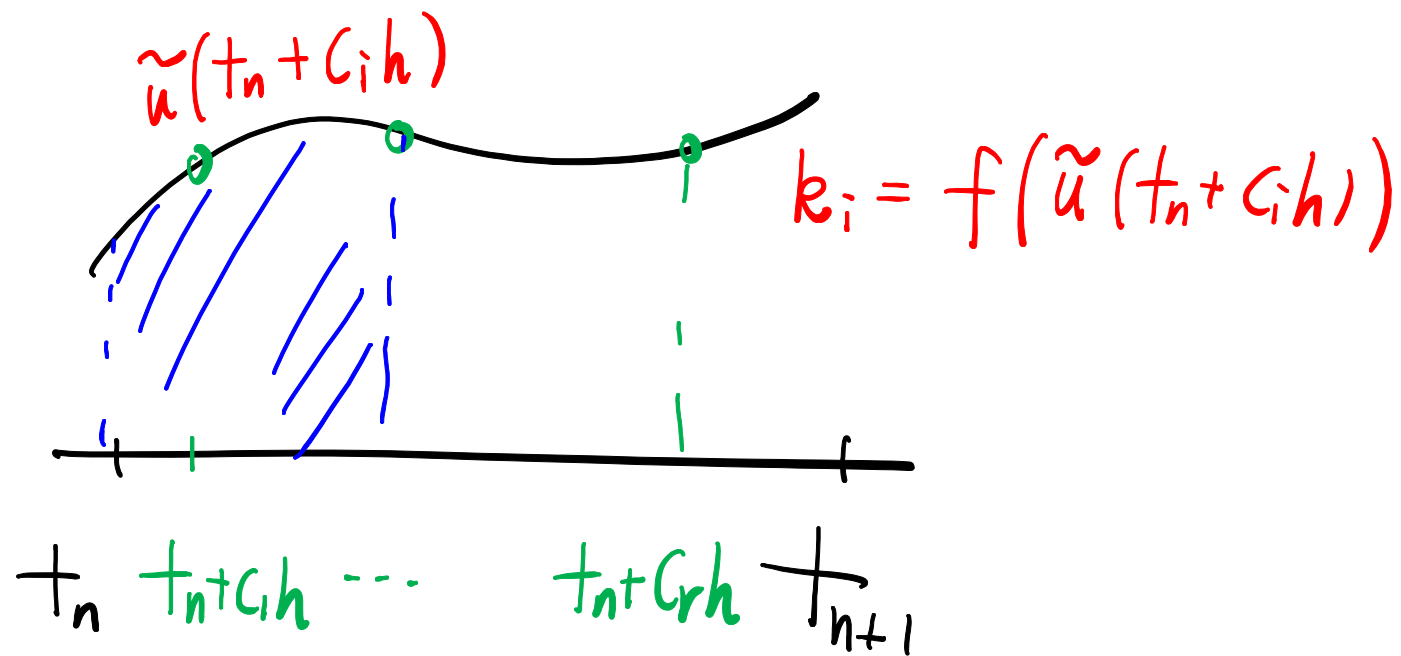
$$\begin{aligned} u(t_n + ch) &\leftarrow \text{unknown } \tilde{u} \\ &= u(t_n) + \int_{t_n}^{t_n+ch} f(u(s)) ds \\ &\approx u(t_n) + \underbrace{\frac{h}{2} f(u(t_n + ch))}_{k} \end{aligned}$$

alternatively

$$\tilde{u} = u_n + \frac{h}{2} \underbrace{f(\tilde{u})}_k$$

Collocation:

“Co-locate” all functions on the same set of points.



$$0 \leq C_1 < C_2 < \dots < C_r \leq 1.$$

Lagrange interpolation for f

$$\tilde{u}(t_n + C_i h) = \underbrace{\tilde{u}(t_n)}_{u_n} + \int_{t_n}^{t_n + C_i h} f(\tilde{u}(s)) ds$$

Lag-
use ALL
r points

$$u_n + \sum_{j=1}^r \int_{t_n}^{t_n + c_j h} p_j(s) k_j$$

$$\left(p_j(t_n + c_j h) = \delta_{ij} \quad p_j \in \mathbb{P}_{r-1} \right)$$

$$= u_n + h \sum_{j=1}^r A_{ij} k_j$$

$$\Rightarrow k_i = f \left(u_n + h \sum_{j=1}^r A_{ij} k_j \right)$$

$$u_{n+1} = u_n + \int_{t_n}^{t_{n+1}} f(\tilde{u}(s)) ds$$

$$\rightarrow u_n + h \sum_{j=1}^r b_j k_j$$

$$\left\{ \begin{array}{l} b_j = \frac{1}{h} \int_{t_n}^{t_{n+1}} p_j(s) ds \\ A_{ij} = \frac{1}{h} \int_{t_n}^{t_n + c_i h} p_j(s) ds \end{array} \right.$$

\rightarrow RK
 COMPLETELY
 determined
 by
 \vec{c}

(exer) Butta's condition is.
always satisfied.

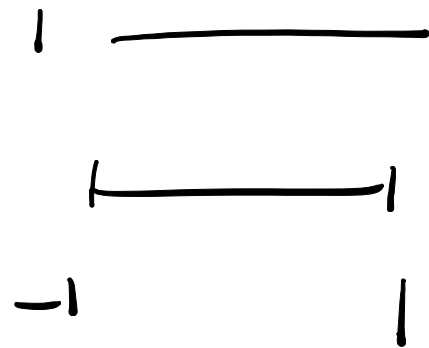
$$\sum_j A_{ij} = c_i$$

Gauss quadrature.

r pts. $2r$ order.

Gauss-Legendre.

Orthogonal poly.



A diagram showing a horizontal line segment representing the interval from -1 to 1. The left endpoint is labeled -1 and the right endpoint is labeled 1. Above the line, there is a horizontal bar spanning the entire interval. To the right of the diagram, the text $w(x) \equiv 1.$ is written.

$$\int_{-1}^1 f(x) dx$$

$$\int_{-1}^1 P_l(x) P_k(x) dx = 0 \text{ if } l \neq k.$$

$$P_l(1) = 1.$$

Gram-Schmit $\rightarrow P_l$

$$\left\{ \begin{array}{l} P_0(x) = 1 \\ P_1(x) = x \\ P_2(x) = \frac{3x^2}{2} - \frac{1}{2} \\ \dots \end{array} \right.$$

Quadrature pts. roots of $P_r(x)$

$$\{x_i\}_{i=1}^r \quad P_r(x_i) = 0.$$

Weights Lagrange interpolation on $\{x_i\}_{i=1}^r$

$$w_i = \int_{-1}^1 p_i(x) dx.$$

