Lec 2.

Euler's method.

convergence / convergence order.

linear multistep method.

Exer: $f: \mathbb{R} \to \mathbb{R}$. $I \subseteq \mathbb{R}$ interval.

$f \in C^1(I)$. $\sup_{x \in I} |f'(x)| = M < \infty$

$\Rightarrow f$ LC on I. (what is the Lipschitz constant and why?)

Def $f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is LC.

if $\exists L > 0$, $\forall t \in I$, $\forall x, y \in \mathbb{R}^n$.

$$\|f(x,t) - f(y,t)\|_2 \leq L \|x-y\|_2$$

Thm. (Picard-Lindlöf)

$f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ L.C.

$\Rightarrow$ Sol exists & unique

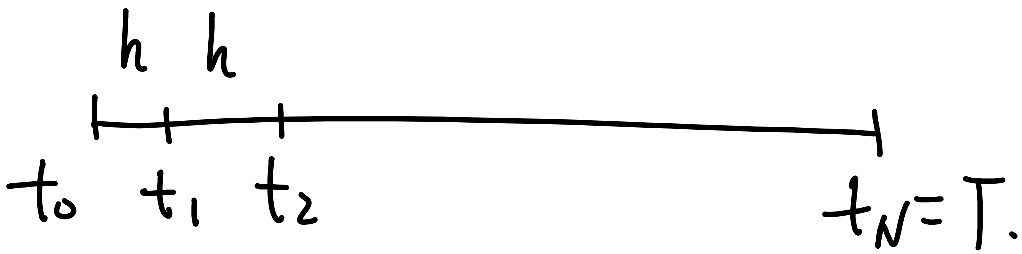
Pf: Ref. [Hai] I. 8

from now on we always assume f is L.C.

Forward Euler.

$0 = t_0 < t_1 < \cdots < t_N = T$

Uniform step size $t_n = nh$, $h = t_2 - t_1$



$u(t_n)$: true sol of IVP at t_n

$U_n \equiv U_h(t_n)$: numerical sol " " .

$$\dot{u}(t_n) \approx \frac{u_{n+1} - u_n}{h} = f(u_n, t_n) := f_n$$

starting from u_0

$$u_{n+1} - u_n = h\, f(u_n, t_n).$$

Linear multistep method (LMM). step-r.

$$\sum_{j=0}^{r} \alpha_j\, u_{n+j} = h \left(\sum_{k=0}^{r} \beta_k\, f(u_{n+k}, t_{n+k}) \right)$$

Forward Euler, $r=1$.

$$\alpha_1 = 1, \quad \alpha_0 = -1.$$

$$\beta_1 = 0, \quad \beta_0 = 1$$

Convergence:

error: $e_n = u(t_n) - U_n \in \mathbb{R}^n$

Goal:

$$h \to 0. \qquad \max_{0 \le t_n \le T} \|e_n\|_2 \to 0$$

Def. An LMM is convergent if for all IVPs.

$$\begin{cases} \dot{u}(t) = f(u(t), t) \\ u(0) = u_0 \end{cases}, \quad 0 \le t \le T.$$

$$\| u(t) - u_h(t) \| \to 0 \quad \text{as } h \to 0, \ 0 \le t_n = nh \le T.$$

whenever the initial values satisfy

$$\| u(t_n) - u_h(t_n) \| \to 0 \quad \text{as } h \to 0.$$

for $n = 0, \cdots, k$

k is the number of initial values to start LMM.

<u>Def</u> An LMM is convergent of order p

if $\exists\, h_0 > 0,\ C > 0$ s.t.

$$\|u(t) - u_h(t)\| \le C h^p, \quad 0 \le h \le h_0, \quad 0 \le t = nh \le T.$$

whenever initial condition satisfy

$$\|u(t_n) - u_h(t_n)\| \le \tilde{C} h^p, \quad 0 \le h \le h_0,$$

$n = 0, \dots, k$.

Thm. Forward Euler is convergent of order 1.

Sketch of proof:

$$u_{n+1} = u_n + h f(u_n, t_n)$$

$$u(t_{n+1}) = u(t_n) + h f(u(t_n), t_n) + \tau_n$$

local truncation error.

$$e_{n+1} = e_n + h\left[f(u(t_n), t_n) - f(u_n, t_n) \right] + \tau_n .$$

$$\|e_{n+1}\| \le \|e_n\| + h\|f(u(t_n),t_n) - f(u_n,t_n)\| + \|\tau_n\|$$

$$\le \|e_n\| + hL\|e_n\| + \|\tau_n\|$$

$$= (1+hL)\|e_n\| + \|\tau_n\|$$

$$\le (1+hL)^2\|e_{n-1}\| + (1+hL)\|\tau_{n-1}\| + \|\tau_n\|$$

$$\vdots$$

$$\le (1+hL)^{n+1}\|e_0\| + \left[(1+hL)^n\|\tau_0\| + \cdots + \|\tau_n\|\right]$$

Bound LTE.

$$\tau_n = u(t_{n+1}) - u(t_n) - h f(u(t_n), t_n)$$

$$= \left[\cancel{u(t_n)} + \cancel{u'(t_n)\, h} + \underbrace{\int_{t_n}^{t_{n+1}} (t_{n+1} - s)\, u''(s)\, ds}_{} \right]$$

$$- \cancel{u(t_n)} - \cancel{h f(u(t_n), t_n)}$$

$$\|\tau_n\| \le \int_{t_n}^{t_{n+1}} (t_{n+1} - s)\, ds \cdot \underbrace{\left(\sup_{0 \le t \le T} \|u''(s)\| \right)}_{M}$$

$$= \frac{h^2 M}{2}$$