Lec 7.

Order condition of RK.

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

Ex. Forward. Euler.

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}$$

Ex. Modified Euler

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

Ex. Trapezoidal.

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

Ex. RK4.

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

$$
\begin{cases}
\vec{k} = (k_1, \cdots, k_r)^T, \quad \vec{e} = (1, \cdots, 1)^T \\
\vec{k} = f(u_n \vec{e} + h A \vec{k}, t_n \vec{e} + \vec{c} h) \leftarrow f \text{ acts on each component.} \\
u_{n+1} = u_n + h \vec{b}^T \vec{k}
\end{cases}
$$

Explicit : All upper triangular (including diagonal) entries of $A = 0$.

implicit : otherwise.

Advantages of RK:

① Easy to start (all needed are u_0)

② adaptive time step.

③ Relatively good stability properties

very good.

Disadvantage

① No reuse of history

② high order methods are tricky to design.

autonomization (remove explicit time dependence)

$$\begin{cases} \dot{u}(t) = f(\overbrace{u(t), t}^{\xi(t)}) \\ \dot{u}(0) = u_0 \end{cases} \qquad \begin{cases} \dot{\xi} = \begin{pmatrix} f(\xi(t)) \\ 1 \end{pmatrix} \\ \xi(0) = \begin{pmatrix} u_0 \\ 0 \end{pmatrix} \end{cases}$$

Apply RK to last component of ξ.

$$t_n + h \sum_{j=1}^{r} a_{ij} = t_n + c_i h.$$

$$\sum_{j=1}^{r} a_{ij} = c_i$$ Kutta's condition. $\begin{array}{c|c} & A \\ \hline & b^T \end{array}$

2-stage explicit RK

$$\begin{array}{c|cc} & 0 & 0 \\ & a & 0 \\ \hline & b_1 & b_2 \end{array}$$

$$\begin{cases} k_1 = f(u_n) \\ k_2 = f(u_n + h a k_1) \end{cases}$$

$$u_{n+1} = u_n + h(b_1 k_1 + b_2 k_2)$$

LTE. $\tau_n = \underbrace{u(t_{n+1}) - u(t_n)} - h\Big[b_1 f(u(t_n))$

$\qquad + b_2 f\big(u(t_n) + h a f(u(t_n))\big)\Big]$

$u := u(t_n)$, $u' = u'(t_n) \equiv f(u(t_n)) \equiv f$

$f_u \equiv f_u(u(t_n))$

$$u'' = f' = f_u f$$

$$u''' = f_{uu} f^2 + (f_u)^2 \cdot f$$

$$\tau_n = hf + \frac{h^2}{2} f_u f + \frac{h^3}{6}\left(f_{uu} f^2 + f_u^2 f\right) + O(h^4)$$
$$\quad - h(b_1 + b_2) f - h^2 b_2 a f_u f - \frac{h^3}{2} b_2 a^2 f_{uu} f^2 + O(h^4)$$

$$= hf\left[1 - (b_1 + b_2)\right] + h^2 f_u f \left[\frac{1}{2} - b_2 a\right]$$
$$\quad + h^3 f_{uu} f^2 \left[\frac{1}{6} - \frac{1}{2} b_2 a^2\right] + \frac{h^3}{6} f_u^2 f + O(h^4)$$

$$\begin{cases} 1 = b_1 + b_2 \\ \frac{1}{2} = b_2 a \end{cases} \rightarrow \text{not unique.}$$

at most second order.

could add $\frac{1}{6} = \frac{1}{2} b_2 a^2$

Heun's method

$$\begin{array}{c|cc} & 0 & 0 \\ & \frac{2}{3} & 0 \\ \hline & \frac{1}{4} & \frac{3}{4} \end{array}$$

General 2nd RK

$$\begin{array}{c|cc} & a_{11} & a_{12} \\ & a_{21} & a_{22} \\ \hline & b_1 & b_2 \end{array}$$

$$\tau_n = h f \Big(\underbrace{1 - \sum_i b_i}\Big) + h^2 f_u f \Big(\underbrace{\tfrac{1}{2} - \sum_{ij} b_i a_{ij}}\Big)$$

$$+ h^3 f_{uu} f^2 \Big(\underbrace{\tfrac{1}{6} - \tfrac{1}{2}\sum_{ijk} b_i a_{ij} a_{ik}}\Big)$$

$$+ h^3 f_u^2 f \Big(\underbrace{\tfrac{1}{6} - \sum_{ijk} b_i a_{ij} a_{jk}}\Big) + O(h^4)$$