Runge-Kutta order condition [Hai II.2]

Table 2.1. Number of trees up to order 10

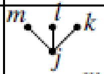
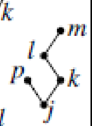
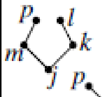
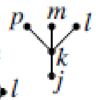
q	1	2	3	4	5	6	7	8	9	10
card(T_q)	1	1	2	4	9	20	48	115	286	719

Table 2.3. Number of order conditions

order p	1	2	3	4	5	6	7	8	9	10
no. of conditions	1	2	4	8	17	37	85	200	486	1205

It is difficult to design high order RK schemes!

Table 2.2. Trees and elementary differentials up to order 5

q	t	graph	$\gamma(t)$	$\alpha(t)$	$F^J(t)(y)$	$\Phi_j(t)$
0	$\emptyset$	$\emptyset$	1	1	y^J	
1	τ	$\bullet^{j}$	1	1	f^J	1
2	t_{21}		2	1	$\sum_K f^J_K f^K$	$\sum_k a_{jk}$
3	t_{31}		3	1	$\sum_{K,L} f^J_{KL} f^K f^L$	$\sum_{k,l} a_{jk} a_{jl}$
	t_{32}		6	1	$\sum_{K,L} f^J_K f^K_L f^L$	$\sum_{k,l} a_{jk} a_{kl}$
4	t_{41}		4	1	$\sum_{K,L,M} f^J_{KLM} f^K f^L f^M$	$\sum_{k,l,m} a_{jk} a_{jl} a_{jm}$
	t_{42}		8	3	$\sum_{K,L,M} f^J_{KM} f^K_L f^L f^M$	$\sum_{k,l,m} a_{jk} a_{kl} a_{jm}$
	t_{43}		12	1	$\sum_{K,L,M} f^J_K f^K_{LM} f^L f^M$	$\sum_{k,l,m} a_{jk} a_{kl} a_{km}$
	t_{44}		24	1	$\sum_{K,L,M} f^J_K f^K_L f^L_M f^M$	$\sum_{k,l,m} a_{jk} a_{kl} a_{lm}$
5	t_{51}		5	1	$\sum f^J_{KLMP} f^K f^L f^M f^P$	$\sum a_{jk} a_{jl} a_{jm} a_{jp}$
	t_{52}		10	6	$\sum f^J_{KMP} f^K_L f^L f^M f^P$	$\sum a_{jk} a_{kl} a_{jm} a_{jp}$
	t_{53}		15	4	$\sum f^J_{KP} f^K_{ML} f^L f^M f^P$	$\sum a_{jk} a_{kl} a_{km} a_{jp}$
	t_{54}		30	4	$\sum f^J_{KP} f^K_L f^L_M f^M f^P$	$\sum a_{jk} a_{kl} a_{lm} a_{jp}$
	t_{55}		20	3	$\sum f^J_{KM} f^K_L f^L f^M_P f^P$	$\sum a_{jk} a_{kl} a_{jm} a_{mp}$
	t_{56}		20	1	$\sum f^J_K f^K_{LMP} f^L f^M f^P$	$\sum a_{jk} a_{kl} a_{km} a_{kp}$
	t_{57}		40	3	$\sum f^J_K f^K_{LP} f^L_M f^M f^P$	$\sum a_{jk} a_{kl} a_{lm} a_{kp}$
	t_{58}		60	1	$\sum f^J_K f^K_L f^L_{MP} f^M f^P$	$\sum a_{jk} a_{kl} a_{lm} a_{lp}$
	t_{59}		120	1	$\sum f^J_K f^K_L f^L_M f^M_P f^P$	$\sum a_{jk} a_{kl} a_{lm} a_{mp}$

Butcher barriers for explicit Runge-Kutta method

1. For $p \geq 5$, no explicit Runge-Kutta method exists of order p with $s = p$ stages.

2. Any explicit Runge-Kutta scheme must have
   - $p + 1$ stages if $p = 5,6$
   - $p + 2$ stages if $p = 7$
   - $p + 3$ stages if $p > 7$

Dormand–Prince Runge-Kutta pair

0							
1/5	1/5						
3/10	3/40	9/40					
4/5	44/45	−56/15	32/9				
8/9	19372/6561	−25360/2187	64448/6561	−212/729			
1	9017/3168	−355/33	46732/5247	49/176	−5103/18656		
1	35/384	0	500/1113	125/192	−2187/6784	11/84	
	35/384	0	500/1113	125/192	−2187/6784	11/84	0
	5179/57600	0	7571/16695	393/640	−92097/339200	187/2100	1/40

7-stage method with 4th order propagator and 5th order estimator
designed to be close to have the minimal local truncation error (under the restraint of also having the minimal number of steps to achieve order 5). The extra steps are needed for the order 5 estimator.