Lec 10.

Region of absolute stability (RAS)

$\{ z : \text{sol does not grow} \}$.

LMM. $\rho(z)$, $\sigma(z)$

$$\begin{cases} \dot{u} = \lambda u \\ u(0) = 1 \end{cases} \Rightarrow \rho(\omega) - \lambda h \sigma(\omega) = 0.$$

$$\Longleftrightarrow \quad \rho(\omega) - z\sigma(\omega) = 0.$$

check root condition (for each z)

$$RAS = \{ z : \rho(\omega) - z\sigma(\omega) \text{ satisfies root condition} \}.$$

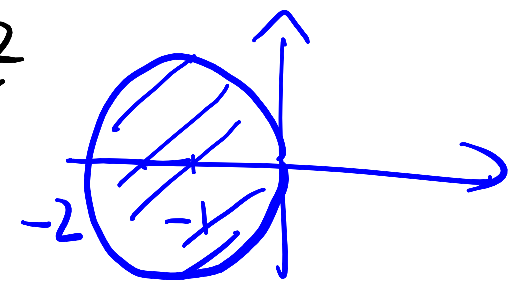
① all $|\omega_k| < 1$

② $|\omega_k| = 1 \quad \omega_k$ simple.

Ex. Forward Euler.

$$\rho(\omega) = \omega - 1, \quad \sigma(\omega) = 1.$$

$$(\omega - 1) - z = 0 \Rightarrow \omega = 1 + z$$



$$RAS = \{ z : |1+z| \le 1 \}.$$

Ex. Backward Euler.

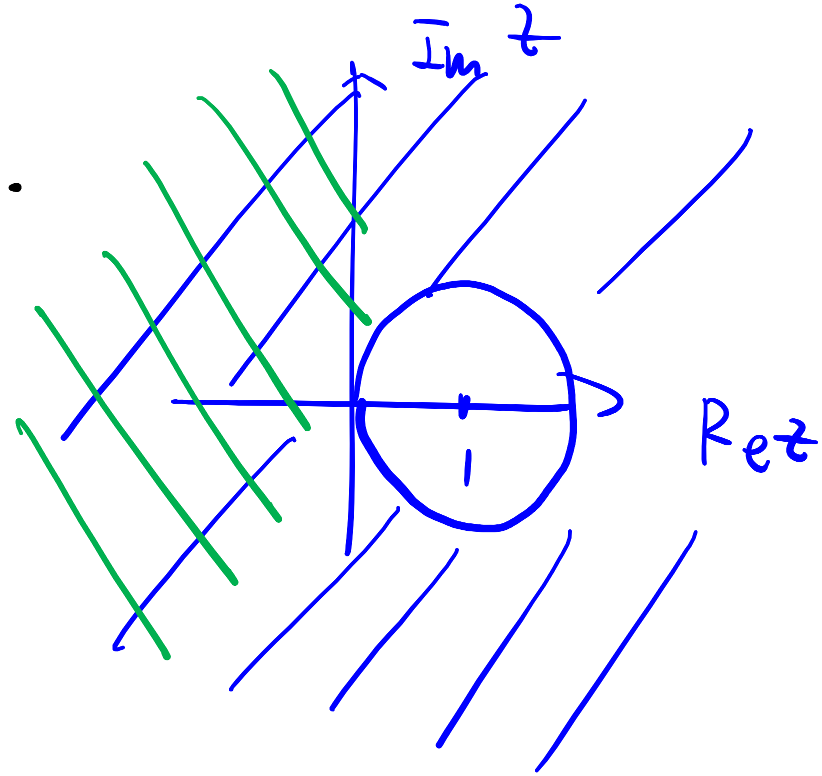
$$\rho(\omega) = \omega - 1, \quad \sigma(\omega) = \omega$$

$$(\omega - 1) - z \cdot \omega = 0 \Rightarrow \omega = \frac{1}{1-z}$$

$$RAS = \{ z : |1-z| \geq 1 \}.$$

$$\supseteq \{ z : \mathrm{Re}\, z \leq 0 \}.$$

A - stable.



Ex. Trapezoidal.

$$U_{n+1} = U_n + \frac{h}{2} (f_n + f_{n+1})$$

$$\rho(\omega) = \omega - 1, \quad \sigma(\omega) = \frac{1}{2} + \frac{1}{2}\omega$$

$$(\omega - 1) - z \cdot \frac{1+\omega}{2} = 0.$$

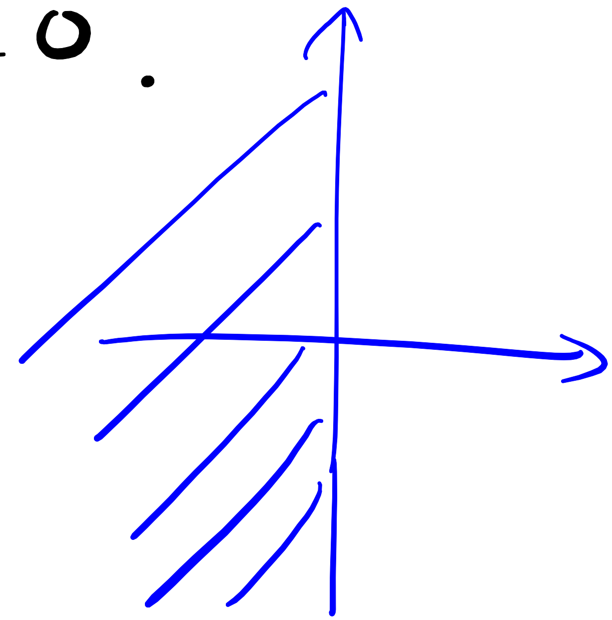
$$\Rightarrow \quad \omega = \frac{1 + \frac{z}{2}}{1 - \frac{z}{2}}$$

$$|\omega| \le 1. \Rightarrow \left|1 + \frac{z}{2}\right|^2 \le \left|1 - \frac{z}{2}\right|^2$$

$$z = a + ib, \quad a, b \in \mathbb{R}$$

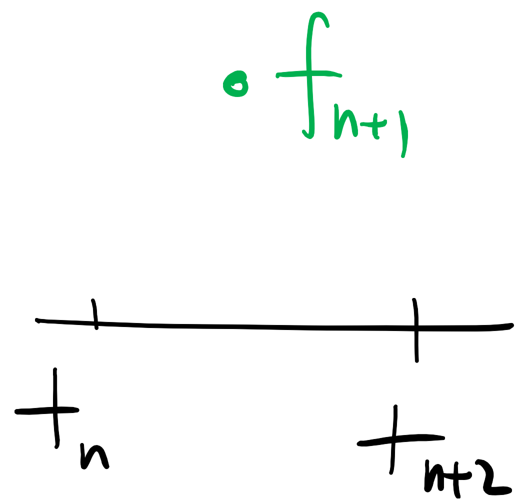
$$\underbrace{\left(1 + \frac{a}{2}\right)^2} + \cancel{\left(\frac{b}{2}\right)^2} \le \underbrace{\left(1 - \frac{a}{2}\right)^2} + \cancel{\left(\frac{b}{2}\right)^2}$$

$a \leq 0 \implies \mathrm{Re}\, z \leq 0.$



Ex. Leap frog.

$$u_{n+2} - u_n = 2h\, f_{n+1}$$



zero stable. 2nd order.

$\rho(\omega) = \omega^2 - 1$, $\sigma(\omega) = 2\omega$

$$\omega^2 - 2z\omega - 1 = 0$$

$$\omega_1 \cdot \omega_2 = -1$$

$\Rightarrow |\omega_1| \cdot |\omega_2| = 1$, and $|\omega_1| \le 1$, $|\omega_2| \le 1$

$$\Rightarrow |\omega_1| = |\omega_2| = 1$$

say $\omega_1 = e^{i\theta}$, $\omega_2 = -e^{-i\theta}$

$$\omega_1 \neq \omega_2 \Rightarrow e^{2i\theta} \neq -1, \quad \theta \neq \left\{ \frac{\pi}{2}, \frac{3\pi}{2} \right\}$$

$$z = \frac{1}{2}\left(e^{i\theta} - e^{-i\theta}\right) = i \sin\theta$$